Contents

0.1 Basic Facts

1. DO NOT BLINDLY APPLY powers and roots across expressions that have $+$ or $-$ signs.

2. As in comment 1, $\sqrt{9x^2+16}$ is something that can NOT be simplified!!

3. As in comment 1, $(2x+5)^2$ can not be done without care.

The square formula applies: $(a+b)^2 = a^2 + 2ab + b^2$. Notice the $2ab$ term. This means when you square you will have a term that looks like twice the product of the terms in parentheses. You get this from FOIL.

4. In particular, $(2x+5)^2 = 4x^2 + 20x + 25$. Do NOT forget the middle term. Note that you can get this quickly by multiplying $2x$ and 5 and doubling.

0.2 Factoring Formulas

A. Formulas

Perfect Square Factoring: $a^2 \pm 2ab + b^2 = (a \pm b)^2$

Difference of Squares: $a^2 - b^2 = (a + b)(a - b)$

Difference and Sum of Cubes: $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$

B. Comments

1. There is no "sum of squares" formula, i.e. no formula for $a^2 + b^2$ (over the real numbers).

2. With $\pm$ and $\mp$ in the same equation, you get one equation when you take the "top" signs, and you get another when you take the "bottom" signs.

Thus you get $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ and $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

3. The easy way to remember the Difference and Sum of Cubes Formula is to remember that the first factor looks like you just remove the cubes. Then the second factor looks like you "square" the first factor, except rather than doubling the middle term, you take the negative of the middle term.

C. Examples

Example 1: Factor $4x^2 - 12x + 9$

Solution

$$4x^2 - 12x + 9 = (2x)^2 - 2(2x)(3) + 3^2$$

Now use the Perfect Square Formula (with minus):

Ans $\boxed{(2x-3)^2}$

Example 2: Factor $4x^2 - 49$

Solution

$$4x^2 - 49 = (2x)^2 - 7^2$$

Now use the Difference of Squares Formula:

Ans $\boxed{(2x+7)(2x-7)}$

Example 3: Factor $27x^3 + 125$

Solution

$$27x^3 + 125 = (3x)^3 + 5^3$$

Now use the Sum of Cubes Formula:

Ans $\boxed{(3x+5)(9x^2 - 15x + 25)}$

Example 4: Factor $8x^3 - 27y^3$

Solution

$$8x^3 - 27y^3 = (2x)^3 - (3y)^3$$

Now use the Difference of Cubes Formula:

Ans $\boxed{(2x-3y)(4x^2 + 6xy + 9y^2)}$

Exercises

1. Expand $(3x + 2)^2$.

2. Expand $(4x - 5)^2$.

3. Factor $4x^2 + 20x + 25$.

4. Factor $9x^2 - 24x + 16$.

5. Factor $9x^2 - 16$.

6. Factor $8x^3 + 27$.

7. Factor $64x^3 - 125$.

Chapter 1

Review of Functions

1.1 Functions

A. Definition of a Function

Every valid input, x, produces **exactly one** output, y; no more, no less

B. Explicit vs. Implicit Functions

1. **Explicit Functions:** function whose defining equation is solved for y.

2. **Implicit Functions:** function whose defining equation is **not** solved for y.

C. Examples

Determine if the following equations define functions of x; if so, state whether they are explicit or implicit.

Example 1: $y = 3x^2 + 1$

Solution

Plug in some x-values, see how many y-values you get:

$x = -2 \Rightarrow y = 3(-2)^2 + 1 = 3 \cdot 4 + 1 = 13$

$x = 1 \Rightarrow y = 3(1)^2 + 1 = 3 \cdot 1 + 1 = 4$

$x = 2 \Rightarrow y = 3(2)^2 + 1 = 3 \cdot 4 + 1 = 13$

For each x, we only get one y, so this a function of x.

Ans $\boxed{\text{This is an explicit function of } x.}$

Example 2: $y^2 = 3x^2 + 1$

Solution

$x = 0 \Rightarrow y^2 = 3(0)^2 + 1 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1.$

We have two y's. So this is not a function of x.

Ans $\boxed{\text{This is not function of } x.}$

Example 3: $2x^2 + 3y^3 = 10$

Solution

$$x = 0 \Rightarrow 2(0)^2 + 3y^3 = 10 \Rightarrow 3y^3 = 10 \Rightarrow y^3 = \tfrac{10}{3} \Rightarrow y = \sqrt[3]{\tfrac{10}{3}}.$$

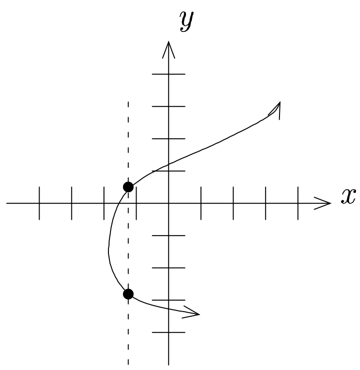
$$x = 2 \Rightarrow 2(2)^2 + 3y^3 = 10$$

$$\Rightarrow 8 + 3y^3 = 10 \Rightarrow 3y^3 = 2 \Rightarrow y^3 = \tfrac{2}{3} \Rightarrow y = \sqrt[3]{\tfrac{2}{3}}.$$

For each x, we only get one y, so this a function of x.

Ans $\boxed{\text{This is an implicit function of } x.}$

Note: If we have a graph, we may determine if we have a function of x by using the **Vertical Line Test** (if any vertical line hits the graph more than once, it is not a function of x). For example:



not a function of x

D. Notation and Comments

$$y = f(x)$$

1. The function operator f is written in **cursive** to distinguish it from a variable.

2. **Interpretation:**

x	input
f	function operator; **represents** the function; "eats" x to spit back y
y	output
$f(x)$	same as y; output of the function

Note: $f(x)$ is **not** the function, f represents it. $f(x)$ is a y-value. For instance, if 3 is an input, $f(3)$ is the output (y-value).

3. In terms of x, $f(x)$ is the **formula for the output**.

E. Evaluation Examples

Consider $f(x) = 3x^2 - 2x$

Example 1: Find $f(3)$

Solution

We want the output when $x = 3$. Use formula for the output, $f(x)$, and plug in 3.

Now $f(x) = 3x^2 - 2x$, so $f(3) = 3(3)^2 - 2(3) = 3 \cdot 9 - 6 = 27 - 6 = 21$.

Ans $\boxed{21}$

Example 2: Find $f(2a + 3b)$ and simplify

Solution

We want the output when the input is $2a + 3b$. Plug "$2a + 3b$" into formula for output $f(x)$ where you see x:

Now $f(x) = 3x^2 - 2x$, so $f(2a + 3b) = 3(2a + 3b)^2 - 2(2a + 3b)$.

Then simplify: $3(4a^2 + 12ab + 9b^2) - 4a - 6b$

Ans $\boxed{12a^2 + 36ab + 27b^2 - 4a - 6b}$

Example 3: Simplify the **difference quotient** $\frac{f(x+h)-f(x)}{h}$

Solution

Now $f(x) = 3x^2 - 2x$ so $f(x+h) = 3(x+h)^2 - 2(x+h)$.

Then $f(x+h) - f(x) = [3(x+h)^2 - 2(x+h)] - [3x^2 - 2x]$

$$
\begin{aligned}
\text{Thus } \frac{f(x+h)-f(x)}{h} &= \frac{[3(x+h)^2 - 2(x+h)] - [3x^2 - 2x]}{h} \\
&= \frac{3(x^2 + 2xh + h^2) - 2x - 2h - 3x^2 + 2x}{h} \\
&= \frac{3x^2 + 6xh + 3h^2 - 2x - 2h - 3x^2 + 2x}{h} \\
&= \frac{6xh + 3h^2 - 2h}{h} \\
&= \frac{h(6x + 3h - 2)}{h}
\end{aligned}
$$

Ans $\boxed{6x + 3h - 2, \text{ if } h \neq 0}$

Exercises

1. Determine if the following equations define functions of x; if so, state whether they are explicit or implicit.

a. $x^2 + y^2 = 9$

b. $3x^2 - y = 5$

c. $y = \sqrt{x + 5}$

d. $x = 4$

e. $y = |x^2 - 7|$

f. $y = 6 \pm x$

g. $x + y^3 = 9$

2. Let $f(x) = 2x^2 - 6$. Find and simplify:

a. $f(-2)$

b. $f(0)$

c. $f(h) - f(3)$

d. $f(h - 3)$ *Note: This is different than part c.*

e. $f(x + h)$

f. $\frac{f(x+h) - f(x)}{h}$

3. Let $f(x) = \frac{1}{x-3}$. Find and simplify:

a. $f(2)$

b. $f(-1)$

c. $f(x+h)$

d. $\frac{f(x+h)-f(x)}{h}$

4. Find the difference quotient $\frac{f(x+h)-f(x)}{h}$ and simplify for $f(x) = 3x^2 - x + 5$.

5. Find the difference quotient $\frac{f(x+h)-f(x)}{h}$ and simplify for $f(x) = \sqrt{x}$.

6. Find $\frac{f(x+h)-f(x-h)}{2h}$ and simplify for $f(x) = x^3$.

1.2 Domain and Range of Functions

A. Domain

$\mathrm{dom}\, f$ all valid inputs

B. Range

$\mathrm{rng}\, f$ all outputs

C. Finding Domain

We “throw” away all problem values.
In particular, we don’t allow division by zero or complex numbers.

Three things to check:

1. **Denominators**: Throw away values making the denominator zero.

2. **Even Roots**: Set inside ≥ 0, and solve inequality.

3. **Logarithms**: Set inside > 0, and solve inequality.

D. Domain Finding Examples

Example 1: Given $f(x) = x^2 + 1$, find $\operatorname{dom} f$.

Solution

Nothing in checklist, so domain is all real numbers.

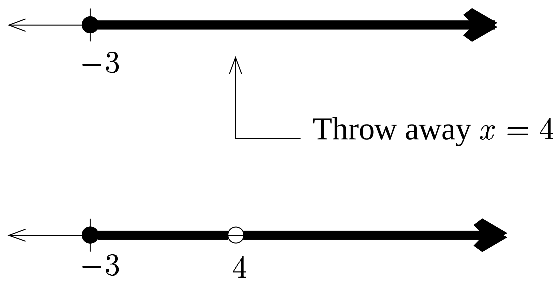
Ans $\boxed{\operatorname{dom} f = (-\infty, \infty)}$

Example 2: Given $f(x) = \frac{\sqrt{x+3}}{x-4}$, find $\operatorname{dom} f$.

Solution

1. **Denominator:** Throw away $x = 4$.

2. **Even Root:** Set $x + 3 \geq 0 \Rightarrow x \geq -3$



Ans $\boxed{\operatorname{dom} f = [-3, 4) \cup (4, \infty)}$

E. Finding Range

This is more difficult.

Methods

1. By plugging in different x-values, try to see what y-values you get back. What is the smallest y-value? What is the largest y-value? Are any y-values missed?

   Heuristic: Expressions that are raised to even powers or even roots of expressions have smallest y-value equal to 0.

2. Graph it, and read off the y-values from the graph.

3. See if you can apply "HSRV transformations" to a known base graph (reviewed later in Section 1.3)

4. For a quadratic function, find the vertex. Depending on whether the parabola opens up or down, the y-value of the vertex will give you the minimum or the maximum value of the range, respectively.

5. Odd degree polynomials have range $(-\infty, \infty)$.

There are other methods, such as the Back Door method, which will not be reviewed here.

F. Range Finding Examples

Example 1: Given $f(x) = x^2 + 1$, find $\mathrm{rng}\, f$.

Solution

Using Method 1:

The smallest y-value possible is 1 (since x^2 has smallest y-value 0).

What is the largest y-value possible? There is no upper limit!
(The y-values go to ∞.)

We see that 1 and everything larger gets hit (nothing missed).

Ans $\boxed{\mathrm{rng}\, f = [1, \infty)}$

Example 2: Given $f(x) = \sqrt{3x - 2} - 4$, find $\mathrm{rng}\, f$.

Solution

Using Method 1:

The smallest y-value possible is -4 (since the square root has smallest y-value 0.)

largest y-value possible? no upper limit

no values larger than -4 are missed

Ans $\boxed{\mathrm{rng}\, f = [-4, \infty)}$

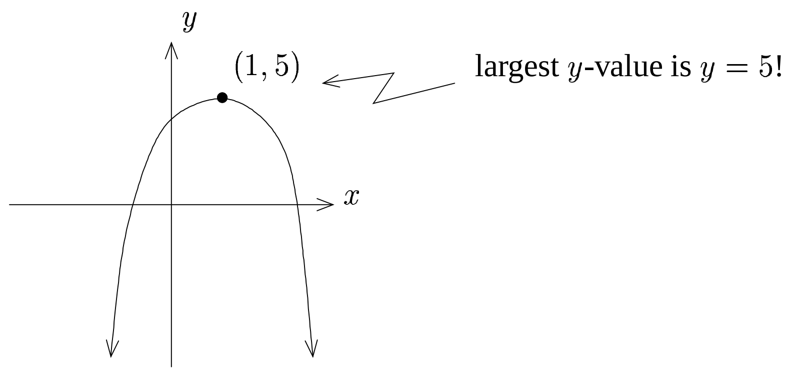
Example 3: Given $f(x) = -3x^2 + 6x + 2$, find $\operatorname{rng} f$.

Solution

Using Method 4:

This is a quadratic function. The parabola opens down since the leading coefficient is negative. Now find the vertex.

Vertex Formula: $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right) = \left(-\frac{6}{2(-3)}, f\left(-\frac{6}{2(-3)}\right)\right) = (1, f(1)) = (1, 5)$.



Ans $\boxed{\operatorname{rng} f = (-\infty, 5]}$

Exercises

1. Find $\mathscr{dom}\, f$ and $\mathscr{rng}\, f$ for f where

   a. $f(x) = x^2 + 3$

   b. $f(x) = \sqrt[3]{x-1}$

   c. $f(x) = \sqrt{x+3} - 2$

2. Find $\mathscr{dom}\, f$ for f where

   a. $f(x) = \frac{6}{x+5}$

   b. $f(x) = \frac{\sqrt{x+5}}{x-2}$

   c. $f(x) = \frac{x+5}{\sqrt{x-2}}$

   d. $f(x) = \sqrt{\frac{x+5}{x-2}}$

   e. $f(x) = \frac{\sqrt{x+3}}{(x-4)(x+1)}$

   f. $f(x) = \frac{\mathscr{log}_7\,(2x+3)}{x-1}$

3. Determine which of the following are true or false:

   a. Every polynomial $f(x)$ satisfies $\mathscr{dom}\, f = (-\infty, \infty)$

   b. Every polynomial $f(x)$ satisfies $\mathscr{rng}\, f = (-\infty, \infty)$

   c. No quadratic function has $\mathscr{rng}\, f = (-\infty, \infty)$

4. Construct a function f that satisfies $\mathscr{dom}\, f = [-1, 3]$ and $\mathscr{rng}\, f = [0, 2]$

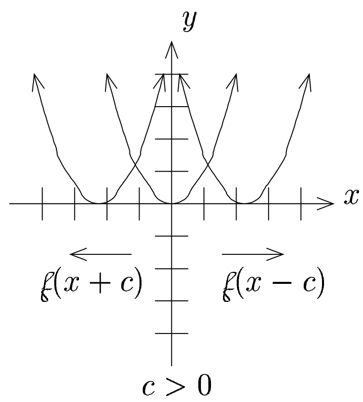
1.3 HSRV Transformations

A. Summary of Transformations

1. **Horizontal Translation:** Add/Subtract Number INSIDE of f (Left/Right Respectively)

    a. y-values fixed

    b. x-values change



2. **Stretching/Shrinking (Vertical):** Multiply OUTSIDE of f By Positive Number a

    a. x-values fixed

    b. y-values multiplied by positive number a

3. **Reflections:**

a. Outside $-$ sign: x-axis reflection, x-values fixed, y-values times -1

b. Inside $-$ sign (next to x): y-axis reflection, y-values fixed, x-values times -1

4. **Vertical Translation:** Add/Subtract Number OUTSIDE of f (Up/Down Respectively)

a. x-values fixed

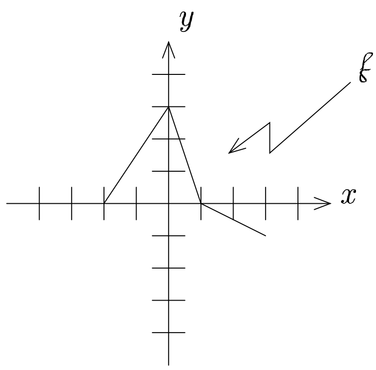
b. y-values change

We always perform transformations in the order HSRV.

NOTE: When the output formula is not given, we identify "key points" on the graph, and then move those according to the rules given. In this case, when g is obtained from f by transforming f, then g is called the "transformed function" and f is called the "base function".

B. An Example

The graph of f is given by



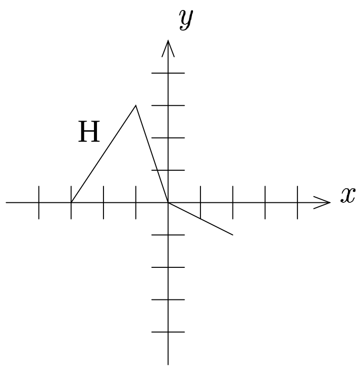
Graph g, where $g(x) = 2 - \frac{1}{2}f(x+1)$. Also determine $\operatorname{dom} g$ and $\operatorname{rng} g$.

Solution

Perform HSRV:

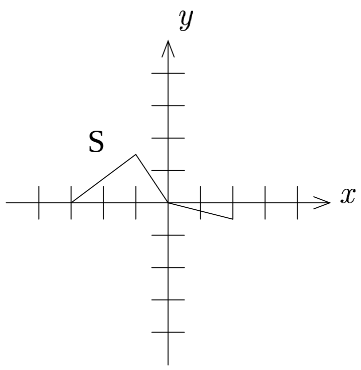
1. H: Add 1 inside: move graph left by 1; y-values fixed, x-values move

$$\begin{aligned}(-2,0) &\mapsto (-3,0)\\ (0,3) &\mapsto (-1,3)\\ (1,0) &\mapsto (0,0)\\ (3,-1) &\mapsto (2,-1)\end{aligned}$$



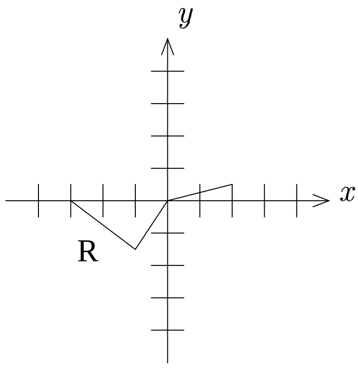
2. S: Multiply by $\frac{1}{2}$ outside: shrink by factor of $\frac{1}{2}$; x-values fixed, y-values times $\frac{1}{2}$

$$\begin{aligned}(-3,0) &\mapsto (-3,0)\\ (-1,3) &\mapsto (-1,\tfrac{3}{2})\\ (0,0) &\mapsto (0,0)\\ (2,-1) &\mapsto (2,-\tfrac{1}{2})\end{aligned}$$



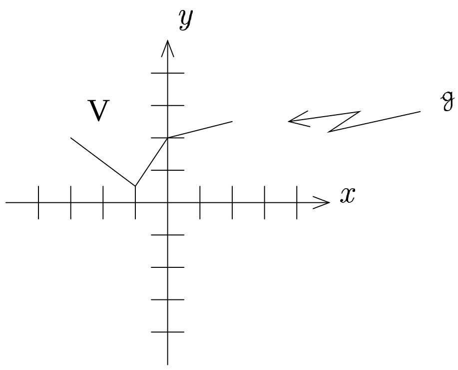
3. R: Outside $-$ sign: x-axis reflection; x-values fixed, y-values times -1

$$\begin{array}{lcl} (-3,0) & \mapsto & (-3,0) \\ (-1,\frac{3}{2}) & \mapsto & (-1,-\frac{3}{2}) \\ (0,0) & \mapsto & (0,0) \\ (2,-\frac{1}{2}) & \mapsto & (2,\frac{1}{2}) \end{array}$$



4. V: Add 2 outside: move graph up 2; x-values fixed, y-values move

$$\begin{array}{lcl} (-3,0) & \mapsto & (-3,2) \\ (-1,-\frac{3}{2}) & \mapsto & (-1,\frac{1}{2}) \\ (0,0) & \mapsto & (0,2) \\ (2,\frac{1}{2}) & \mapsto & (2,\frac{5}{2}) \end{array}$$



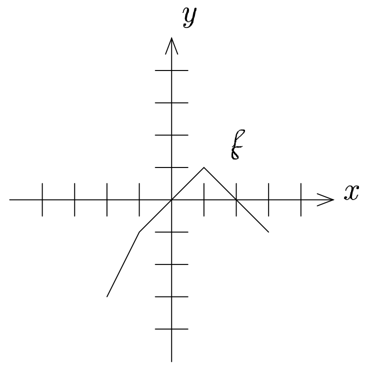
From the graph of g, we see that

$$\mathit{dom}\ g = [-3,2]$$

$$\mathit{rng}\ g = [\tfrac{1}{2},\tfrac{5}{2}]$$

Exercises

1. Let f be given by the following graph:



a. Graph g, where $g(x) = f(x-2)$

b. Graph h, where $h(x) = f(x) - 2$

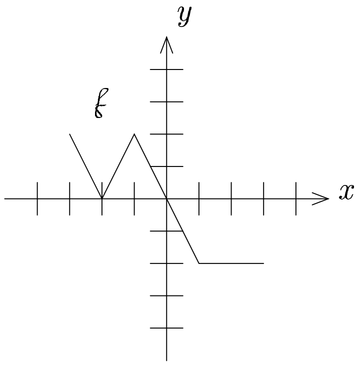
c. Graph k, where $k(x) = -f(x)$

d. Graph r, where $r(x) = f(-x)$

e. Graph s, where $s(x) = 2f(x)$

f. Graph v, where $v(x) = 2f(x+1) + 3$

2. Let f be given by the following graph:



a. Graph g, where $g(x) = 4 - \frac{1}{2}f(x+1)$. Then determine $\mathrm{dom}\ g$ and $\mathrm{rng}\ g$.

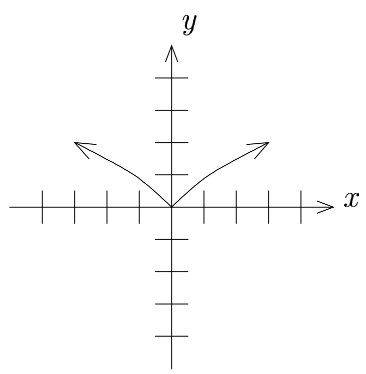
b. Graph h, where $h(x) = f(-x-2)+3$. Then determine $\mathrm{dom}\ h$ and $\mathrm{rng}\ h$.

3. Suppose f is a function with $\mathrm{dom}\ f = [-2,3]$ and $\mathrm{rng}\ f = [0,5]$.
Find $\mathrm{dom}\ g$ and $\mathrm{rng}\ g$, where $g(x) = 2f(-x+2)-1$.

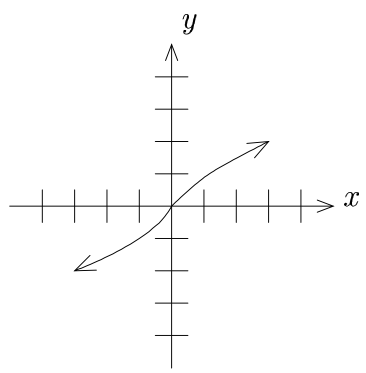
1.4 Symmetry of Functions; Even and Odd

A. Symmetry of Functions

1. A function with y-axis symmetry is called **even**.



2. A function with origin symmetry is called **odd**.



Note: A nonzero function may not have x-axis symmetry.

B. Even/Odd Tests

1. A function is **even** if $f(-x) = f(x)$.

2. A function is **odd** if $f(-x) = -f(x)$.

To use these tests, we compute $f(-x)$ and $-f(x)$ and compare to $f(x)$. Then we compare to see if any of them are equal, as in the test above. If none of them are equal, the function is **neither** even nor odd.

Note: When testing, you must use **generic x**, not just one number. For example, if $f(x) = x^3 - x - 2$, then $f(1) = -2$ and $f(-1) = -2$, but f is **not** even!

C. Examples

Determine if f is even, odd, or neither:

Example 1: $f(x) = x^3$

Solution

$$f(-x) = (-x)^3 = (-1)^3x^3 = -x^3$$

$$f(x) = x^3$$

$$-f(x) = -x^3$$

Ans $\boxed{\text{Since } f(-x) = -f(x),\ f \text{ is } \textbf{odd}.}$

Example 2: $f(x) = |x|$

Solution

$$f(-x) = |-x| = |-1|\,|x| = |x|$$

$$f(x) = |x|$$

$$-f(x) = -|x|$$

Ans $\boxed{\text{Since } f(-x) = f(x),\ f \text{ is } \textbf{even}.}$

Example 3: $f(x) = (x-1)^2$

Solution

$$f(-x) = (-x-1)^2 = x^2 + 2x + 1$$

$$f(x) = (x-1)^2 = x^2 - 2x + 1$$

$$-f(x) = -(x-1)^2 = -(x^2 - 2x + 1) = -x^2 + 2x + 1$$

Ans $\boxed{\text{Since } f(-x) \text{ is neither } f(x) \text{ nor } -f(x),\ f \text{ is } \textbf{neither} \text{ even nor odd.}}$

D. Symmetric Domains

The domain of a function is **symmetric** if the domain contains the same values to the right of the origin as to the left.

Examples of symmetric domains:

$$(-\infty, \infty), \qquad [-3, 3], \qquad (-4, 4), \qquad [-6, -2) \cup (-1, 1) \cup (2, 6]$$

Examples of domains that are not symmetric:

$$[0, 4], \qquad (1, \infty), \qquad [-2, 1), \qquad (-5, 5]$$

E. Decomposition of Functions Into Even and Odd Parts

1. **Statement:** Any function (even/odd/neither) with a symmetric domain can be decomposed into the sum of an even function and an odd function.

2. **Formulas:**

a. $f_{\text{even}}(x) = \frac{1}{2}\left[f(x) + f(-x)\right]$

b. $f_{\text{odd}}(x) = \frac{1}{2}\left[f(x) - f(-x)\right]$

3. **Comments:**

a. Note, by definition, f_{even} is an even function and f_{odd} is an odd function.

b. Note, also that $f_{\text{even}}(x) + f_{\text{odd}}(x) = f(x)$.

F. Decomposition Examples

Example 1: Given $f(x) = (x-1)^2$, decompose f into even and odd parts.

Solution

Note: $\mathrm{dom}\, f = (-\infty, \infty)$, so the domain is symmetric.

Now use the formulas:

$$\begin{aligned} f_{\text{even}}(x) &= \frac{1}{2}\left[f(x) + f(-x)\right] \\ &= \frac{1}{2}\left[(x-1)^2 + (-x-1)^2\right] \\ &= \frac{1}{2}\left[x^2 - 2x + 1 + x^2 + 2x + 1\right] \\ &= \frac{1}{2}(2x^2 + 2) = x^2 + 1. \end{aligned}$$

$$\begin{aligned} f_{\text{odd}}(x) &= \frac{1}{2}\left[f(x) - f(-x)\right] \\ &= \frac{1}{2}\left[(x-1)^2 - (-x-1)^2\right] \\ &= \frac{1}{2}\left[x^2 - 2x + 1 - (x^2 + 2x + 1)\right] \\ &= \frac{1}{2}(-4x) = -2x. \end{aligned}$$

Ans $\boxed{\begin{array}{l} f_{\text{even}}(x) = x^2 + 1 \\ f_{\text{odd}}(x) = -2x \end{array}}$

Note: $f_{\text{even}}(x) + f_{\text{odd}}(x) = x^2 + 1 - 2x = (x-1)^2 = f(x)$.

Example 2: Given $g(x) = x^3$, decompose g into even and odd parts.

Solution

Note: $\operatorname{dom} g = (-\infty, \infty)$, so the domain is symmetric.

Now use the formulas:

$$\begin{aligned} g_{\text{even}}(x) &= \frac{1}{2}\left[\, g(x) + g(-x)\right] \\ &= \frac{1}{2}\left[x^3 + (-x)^3\right] \\ &= \frac{1}{2}\left[x^3 + (-1)^3 x^3\right] \\ &= \frac{1}{2}(x^3 - x^3) = 0. \end{aligned}$$

$$\begin{aligned} g_{\text{odd}}(x) &= \frac{1}{2}\left[\, g(x) - g(-x)\right] \\ &= \frac{1}{2}\left[x^3 - (-x)^3\right] \\ &= \frac{1}{2}\left[x^3 - (-1)^3 x^3\right] \\ &= \frac{1}{2}(x^3 + x^3) = x^3. \end{aligned}$$

Ans $\boxed{\begin{array}{l} g_{\text{even}}(x) = 0 \\ g_{\text{odd}}(x) = x^3 \end{array}}$

This was no surprise, really. g was already odd.

Note: This is another even/odd test. To test a function, do the decomposition . . .

If the even part is 0, the function is **odd**. If the odd part is 0, the function is **even**.

If neither are 0, the function is **neither** even nor odd.

Exercises

1. Determine if f is even, odd, or neither where

   a. $f(x) = 3x^2 + 2$

   b. $f(x) = 2x^2 - x + 1$

   c. $f(x) = x^3 - 2x$

   d. $f(x) = \sqrt{|x|}$

   e. $f(x) = x^{\frac{1}{3}}$

   f. $f(x) = \frac{1}{1+x^2}$

   g. $f(x) = \frac{x}{1+x^2}$

2. Given $f(x) = (x-3)^2$, decompose f into even and odd parts.

3. Explain why a nonzero function can not have x-axis symmetry.

4. Explain why the domain of a function must be symmetric in order to be able to decompose it into even and odd parts.

1.5 Combinations of Functions

A. Definitions of $(+, -, \cdot, \div, \circ)$ for Functions

1. $(f+g)(x) = f(x) + g(x)$
2. $(f-g)(x) = f(x) - g(x)$
3. $(fg)(x) = f(x)g(x)$
4. $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$
5. $(f \circ g)(x) = f(g(x))$ $\quad$ $f \circ g$ is called f **composed with** g

Note: $f \circ g$, $f - g$, etc. are functions; whereas, $(f \circ g)(x)$ etc. are the **outputs**.

Warning: The above are **definitions** of new functions, having nothing to do with the "distributive" property for variables.

B. Examples

Example 1: Find the output formulas for $f+g$, $f-g$, fg, $\frac{f}{g}$, $f \circ g$, $g \circ f$ where $f(x) = \sqrt{x+1}$ and $g(x) = \frac{2}{x}$.

Solution

$$(f+g)(x) = f(x) + g(x) = \boxed{\sqrt{x+1} + \frac{2}{x}}$$

$$(f-g)(x) = f(x) - g(x) = \boxed{\sqrt{x+1} - \frac{2}{x}}$$

$(fg)(x) = f(x)g(x) = (\sqrt{x+1})\left(\frac{2}{x}\right) = \boxed{\frac{2\sqrt{x+1}}{x}}$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x+1}}{\frac{2}{x}} = \boxed{\frac{x\sqrt{x+1}}{2}}$

$(f \circ g)(x) = f(g(x)) = f\left(\frac{2}{x}\right) = \boxed{\sqrt{\frac{2}{x}+1}}$

$(g \circ f)(x) = g(f(x)) = g(\sqrt{x+1}) = \boxed{\frac{2}{\sqrt{x+1}}}$

Example 2: Given $f(x) = 3 + x^2$ and $g(x) = x - 1$. Find:

a) $(fg)(-1)$

b) $(g \circ f)(0)$

c) $(g - f)(2)$

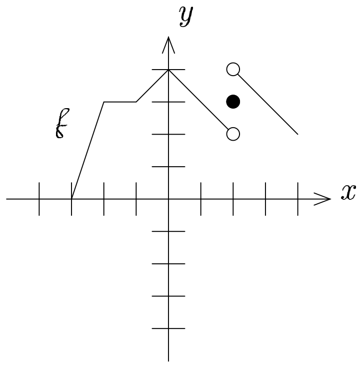
Solution

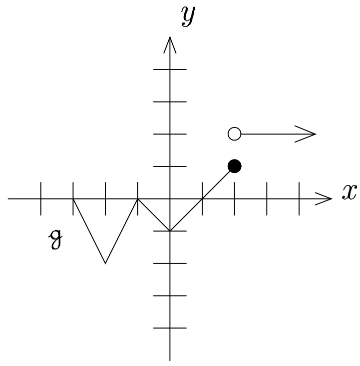
a) $(fg)(-1) = f(-1)g(-1) = (4)(-2) = \boxed{-8}$

b) $(g \circ f)(0) = g(f(0)) = g(3) = \boxed{2}$

c) $(g - f)(2) = g(2) - f(2) = 1 - 7 = \boxed{-6}$

Example 3: Let f and g be given by the graphs below:





Find:

a) $(f-g)(2)$ b) $(f\circ g)(2)$ c) $(g\circ f)(-3)$ d) $\left(\frac{g}{f}\right)(0)$

Solution

a) $(f-g)(2) = f(2) - g(2) = 3 - 1 = \boxed{2}$

by graphs of f and g

b) $(f\circ g)(2) = f(g(2)) = f(1) = \boxed{3}$

graph of g graph of f

c) $(g\circ f)(-3) = g(f(-3)) = g(0) = \boxed{-1}$

d) $\left(\frac{g}{f}\right)(0) = \frac{g(0)}{f(0)} = \boxed{-\frac{1}{4}}$

C. Comments on Domain

1. The domains of $f+g$, $f-g$, fg, $\frac{f}{g}$, $f \circ g$, etc. can **not** be found by just looking at their output formulas!

2. **Reason output formula gives the wrong domain:**

Example:

If $f(x) = x - \sqrt{x}$ and $g(x) = \sqrt{x}$, then $(f+g)(x) = x$.

However, $\mathit{dom}(f+g) \neq (-\infty, \infty)$ as suggested by the output formula.

Note that $(f+g)(-2)$ is undefined:

$(f+g)(-2) = f(-2) + g(-2)$, but $f(-2)$ and $g(-2)$ are undefined!

3. We will investigate how to find the **correct** domains of these new functions in the subsequent sections.

Exercises

1. Find the output formulas for $f + g$, $f - g$, fg, $\frac{f}{g}$, $f \circ g$, $g \circ f$ where $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{x-3}$.

2. Let $f(x) = \frac{\sqrt{x+5}}{x}$ and $g(x) = \frac{3x-1}{x+2}$. Find and simplify $(f \circ g)(x)$ and $(g \circ f)(x)$.

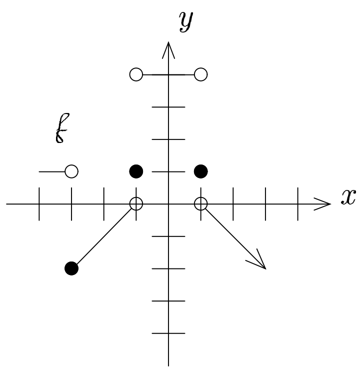
3. Given $f(x) = x^2 - 2$ and $g(x) = 3 - x$. Find:

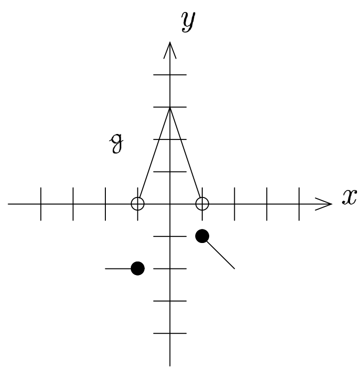
a. $(fg)(-2)$

b. $(f \circ g)(0)$

c. $(g \circ f)(1)$

4. Let f and g be given by the graphs below:





Find:

a. $(g - f)(-1)$

b. $\left(\frac{f}{g}\right)(0)$

c. $(f \circ g)(2)$

d. $(g \circ f)(1)$

1.6 Domain of Combined Functions

A. Introduction

In this section, we discuss how to find $dom(f+g)$, $dom(f-g)$, $dom(fg)$, and $dom\left(\frac{f}{g}\right)$. We will leave the discussion of $dom(f \circ g)$ for the next two sections. Recall that just looking at the output formula will yield the wrong result.

B. Method

To find $dom(f+g)$, $dom(f-g)$, $dom(fg)$, and $dom\left(\frac{f}{g}\right)$:

1. Find $dom\, f$ and $dom\, g$.

2. Intersect them in AND, i.e. find where they overlap.

3. In the $dom\left(\frac{f}{g}\right)$ case, additionally throw out any x's where $g(x) = 0$
(since the denominator can't be zero)

C. Examples

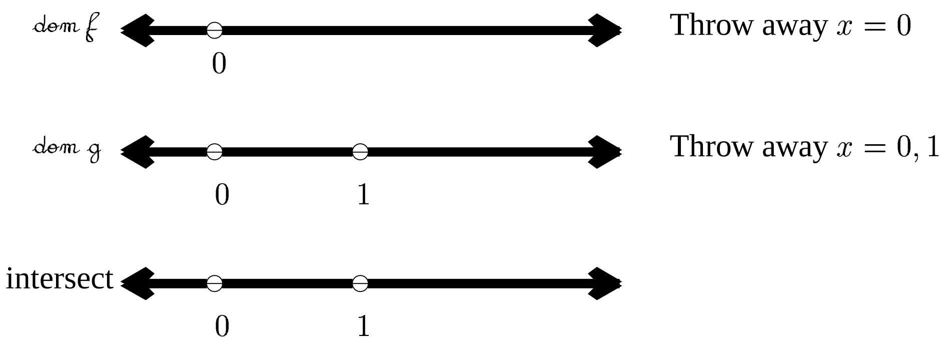
Example 1: $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{1-x} - \frac{1}{x}$. Find $(f+g)(x)$ and $\mathrm{dom}(f+g)$.

Solution

1. $(f+g)(x) = f(x) + g(x) = \frac{1}{x} + \left(\frac{1}{1-x} - \frac{1}{x}\right) = \frac{1}{1-x}$

Note: By the output formula, you would get the **incorrect** domain of $(-\infty, 1) \cup (1, \infty)$. Now we'll do it correctly.

2.



Ans $\boxed{\mathrm{dom}(f+g) = (-\infty, 0) \cup (0, 1) \cup (1, \infty)}$

Example 2: $f(x) = \frac{x+1}{\sqrt{x+3}}$ and $g(x) = \frac{x-2}{\sqrt{x+3}}$. Find $\left(\frac{f}{g}\right)(x)$ and $\mathrm{dom}\left(\frac{f}{g}\right)$.

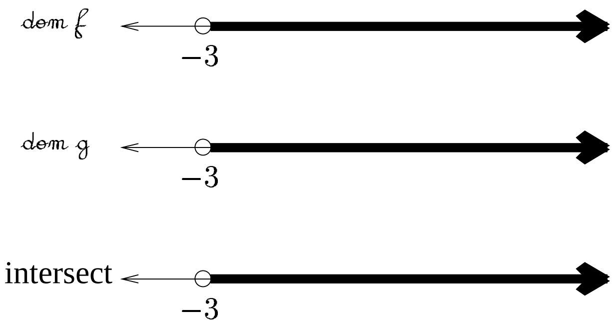
Solution

1. $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\frac{x+1}{\sqrt{x+3}}}{\frac{x-2}{\sqrt{x+3}}} = \frac{x+1}{x-2}$.

Note: As we'll see the domain is not $(-\infty, 2) \cup (2, \infty)$

2. $\mathrm{dom}\, f$: require $x + 3 \geq 0$ (root) and $x + 3 \neq 0$ (denominator)

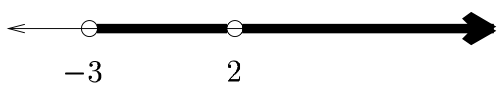
$\mathrm{dom}\, g$: require $x + 3 \geq 0$ (root) and $x + 3 \neq 0$ (denominator)



Since we have $\frac{f}{g}$, here we need to **also** throw out where $g(x) = 0$,

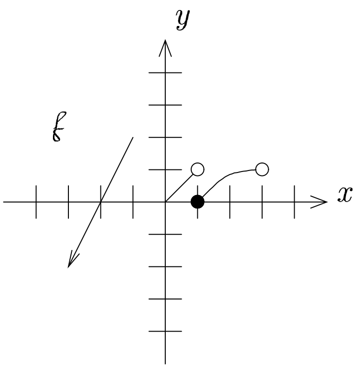
i.e. $\frac{x-2}{\sqrt{x+3}} = 0 \;\Rightarrow\; x - 2 = 0 \;\Rightarrow\; x = 2$.

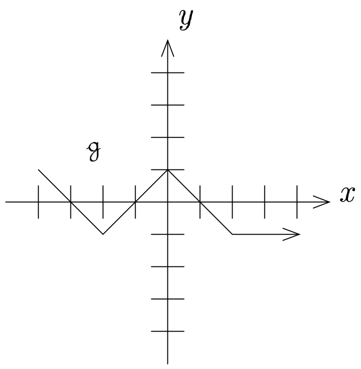
Throw out $x = 2$:



Ans $\boxed{\mathrm{dom}\left(\frac{f}{g}\right) = (-3, 2) \cup (2, \infty)}$

Example 3: Find $dom\left(\frac{f}{g}\right)$ when f and g are given by the graphs below:

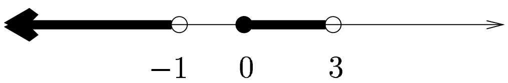




Solution

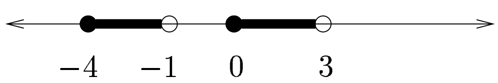
From the graphs,

$dom\ f$



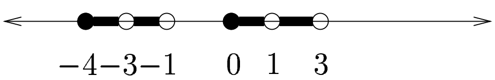
$dom\ g$

intersect



Now throw out where $g(x) = 0$ (that is, where g crosses the x-axis):

Thus throw out $x = -3, -1, 1$.



Ans $\boxed{dom\left(\frac{f}{g}\right) = [-4, -3) \cup (-3, -1) \cup [0, 1) \cup (1, 3)}$

1.7 Domain of Composition: From Output Formulas

A. Introduction

In this section, we will discuss how to find $\mathrm{dom}(f \circ g)$ from output formulas.

B. Motivation

Since $(f \circ g)(x) = f(g(x))$, the domain must consist of

1. values that g accepts $[\mathrm{dom}\, g]$

2. also, $g(x)$ must be accepted by f; this occurs when $f(g(x))$ is defined

   $[\mathrm{dom}(f \circ g)_{incorrect}]$

C. Method

1. Find $(f \circ g)(x)$.

   Look at the output formula and find the **incorrect** domain: $\mathrm{dom}(f \circ g)_{incorrect}$.

2. Find $\mathrm{dom}\, g$.

3. Intersect $\mathrm{dom}(f \circ g)_{incorrect}$ with $\mathrm{dom}\, g$ (overlap with AND).

Note: $\mathrm{dom}\, f$ never gets used!

D. Examples

Example 1: Find $\mathrm{dom}(f \circ g)$ where $f(x) = \frac{1}{x^2-1}$ and $g(x) = \sqrt{x+3}$.

Solution

1. First find $(f \circ g)(x)$:

   $(f \circ g)(x) = f(g(x)) = f(\sqrt{x+3}) = \frac{1}{(\sqrt{x+3})^2 - 1} = \frac{1}{x+2}$

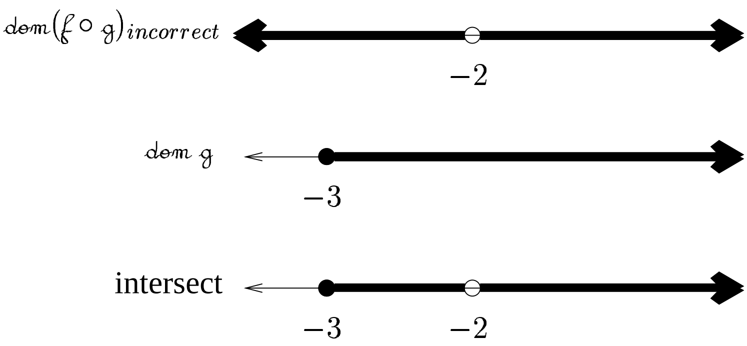
   Since $x \neq -2$, $\mathrm{dom}(f \circ g)_{incorrect} = (-\infty, -2) \cup (-2, \infty)$

2. Now find $\mathrm{dom}\, g$:

   from the even root, we require $x + 3 \geq 0 \;\Rightarrow\; x \geq -3$.

   Thus $\mathrm{dom}\, g = [-3, \infty)$.

3. Intersect:



Ans $\boxed{\mathrm{dom}(f \circ g) = [-3, 2) \cup (2, \infty)}$

Note: $\mathrm{dom}\, f$ was never used!

Example 2: Find $dom(f \circ g)$ where $f(x) = \frac{x+3}{x-1}$ and $g(x) = \frac{2x+1}{x+3}$.

Solution

1. First find $(f \circ g)(x)$:

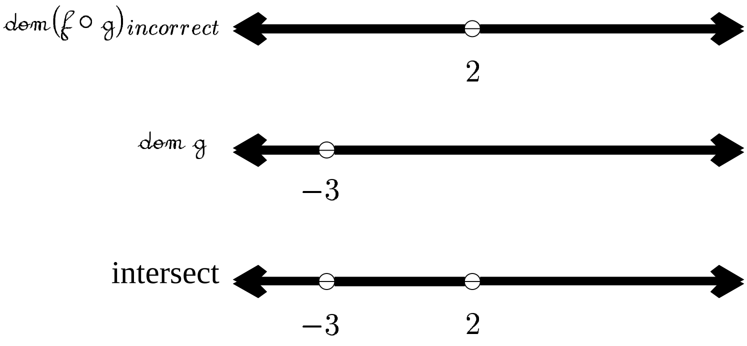
$$(f \circ g)(x) = f(g(x)) = f\left(\frac{2x+1}{x+3}\right) = \frac{\frac{2x+1}{x+3}+3}{\frac{2x+1}{x+3}-1} = \frac{\frac{2x+1}{x+3}+\frac{3(x+3)}{x+3}}{\frac{2x+1}{x+3}-\frac{x+3}{x+3}} = \frac{\frac{5x+10}{x+3}}{\frac{x-2}{x+3}} = \frac{5x+10}{x+3} \cdot \frac{x+3}{x-2}$$

Thus $(f \circ g)(x) = \frac{5x+10}{x-2}$, so $x \neq 2 \Rightarrow dom(f \circ g)_{incorrect} = (-\infty, 2) \cup (2, \infty)$

2. Now find $dom\, g$:

We see that $x \neq -3$, so $dom\, g = (-\infty, -3) \cup (-3, \infty)$.

3. Intersect:



Ans $\boxed{dom(f \circ g) = (-\infty, -3) \cup (-3, 2) \cup (2, \infty)}$

1.8 Domain of Composition: From Graphs

Note: This method is different from that used for output formulas!

A. Method

1. Find $\mathrm{dom}\, f$ from the graph.

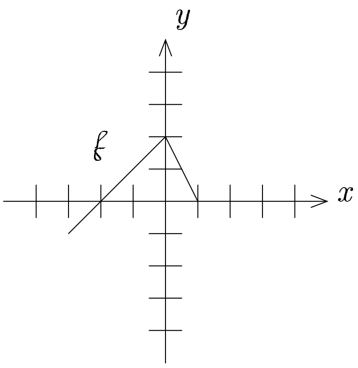
2. Draw $\mathrm{dom}\, f$ on the y-axis of the graph of g.

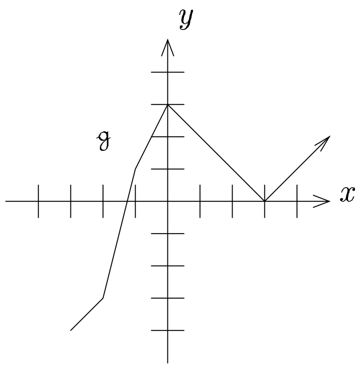
3. Throw away the parts of g that are not inside the "bands" determined by the y-axis marks.

4. Read off the domain of the new "mutilated" graph of g.

B. Examples

Example 1: Find $\mathrm{dom}(f \circ g)$ where f and g are given by the following graphs:

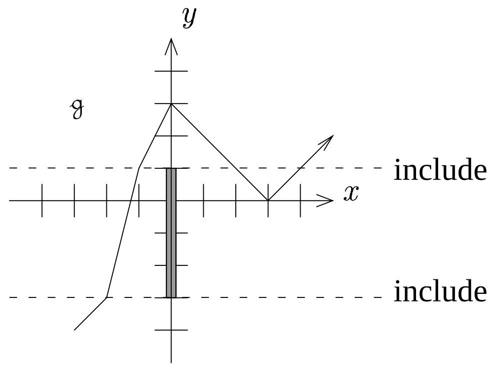




Solution

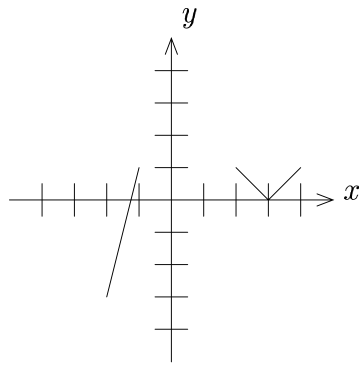
1. From the graph, $\mathrm{dom}\, f = [-3, 1]$.

2. Mark $\mathcal{dom}\, f$ on the y-axis of the graph of g:



Band determined by $\mathcal{dom}\, f$
Throw away everything outside

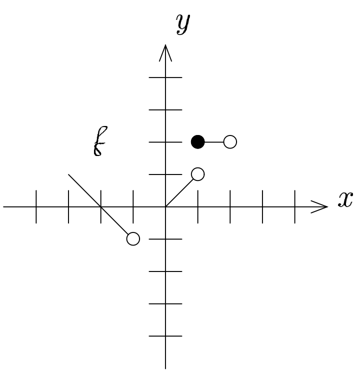
3. “Mutilated” graph of g:

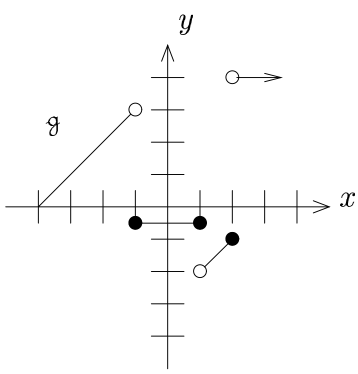


4. Now read off the domain:

Ans $\boxed{\mathcal{dom}(f \circ g) = [-2, -1] \cup [2, 4]}$

Example 2: Find $\mathcal{dom}(f \circ g)$ where f and g are given by the following graphs:

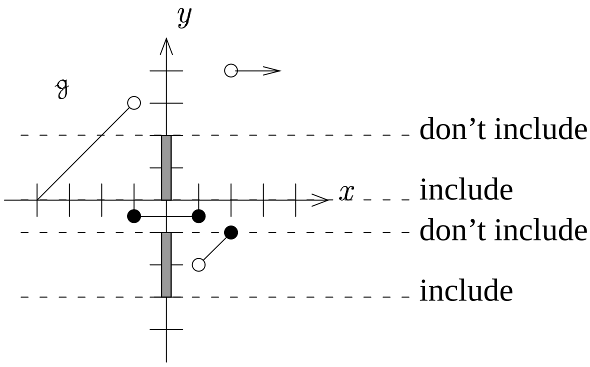




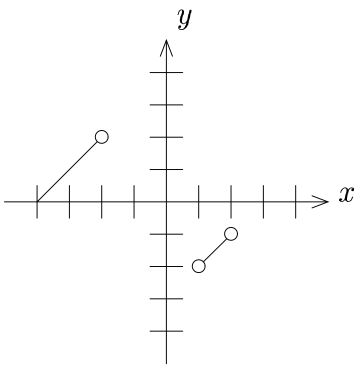
Solution

1. From the graph, $\mathrm{dom}\, f = [-3, -1) \cup [0, 2)$.

2. Mark $\mathrm{dom}\, f$ on the y-axis of the graph of g:



3. “Mutilated” graph of g:



4. Now read off domain:

Ans $\boxed{\mathrm{dom}(f \circ g) = [-4, -2) \cup (1, 2)}$

Exercises

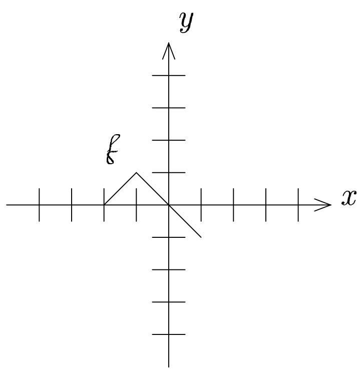
1. Let $f(x) = \frac{(x-3)(x+1)}{x+2}$ and $g(x) = \frac{(x+2)(x+1)}{x-3}$. Find $(fg)(x)$ and $\mathit{dom}(fg)$.

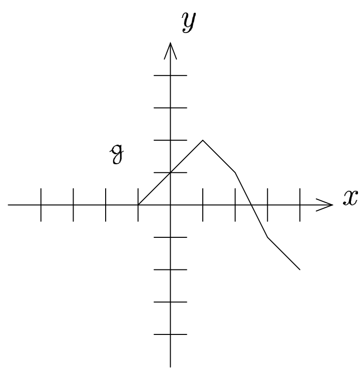
2. Let $f(x) = \frac{3x-2}{\sqrt{x+40}}$ and $g(x) = \frac{x+5}{\sqrt{x+40}}$. Find $\left(\frac{f}{g}\right)(x)$ and $\mathit{dom}\left(\frac{f}{g}\right)$.

3. Let $f(x) = \frac{1}{x^2-7}$ and $g(x) = \sqrt{x+2}$. Find $(f \circ g)(x)$ and $\mathit{dom}(f \circ g)$.

4. Let $f(x) = \frac{x+2}{x-1}$ and $g(x) = \frac{2x-3}{x+2}$. Find $(f \circ g)(x)$ and $\mathit{dom}(f \circ g)$.

5. Let f and g be given by the following graphs:





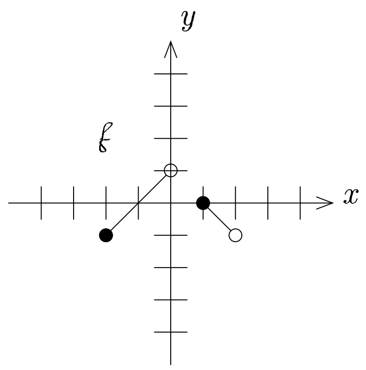
a. Find $\mathit{dom}\left(\frac{f}{g}\right)$

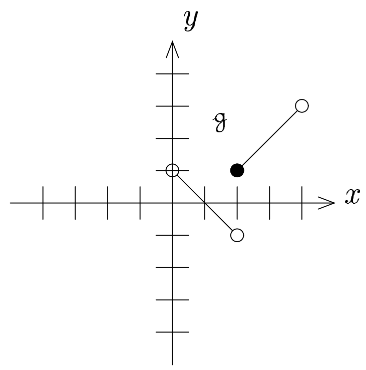
b. Find $\mathit{dom}(f \circ g)$

c. Find $\mathit{dom}\left(\frac{g}{f}\right)$

d. Find $\mathit{dom}(g \circ f)$

6. Let f and g be given by the following graphs:





a. Find $dom\left(\frac{f}{g}\right)$

b. Find $dom(f \circ g)$

c. Find $dom\left(\frac{g}{f}\right)$

d. Find $dom(g \circ f)$

1.9 Inverses

A. Definition of Inverses

Two functions f and g are called **inverses** if two conditions are met:

1. $(f \circ g)(x) = x$

2. $(g \circ f)(x) = x$

Thus f and g undo each other!

B. Examples

Example 1: Are f and g inverses, where $f(x) = x^3 + 1$ and $g(x) = \sqrt[3]{x-1}$?

Solution

Check the two conditions!

1. $(f \circ g)(x) = f(g(x)) = f(\sqrt[3]{x-1}) = (\sqrt[3]{x-1})^3 + 1 = x - 1 + 1 = x$

2. $(g \circ f)(x) = g(f(x)) = g(x^3+1) = \sqrt[3]{(x^3+1)-1} = \sqrt[3]{x^3} = x$

Ans $\boxed{\text{YES, } f \text{ and } g \text{ are inverses}}$

Example 2: Are f and g inverses, where $f(x) = x^2$ and $g(x) = \sqrt{x}$?

Solution

Check the two conditions!

1. $(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x$

2. $(g \circ f)(x) = g(f(x)) = g(x^2) = \sqrt{x^2} = |x|$

Both conditions are not met, so . . .

Ans $\boxed{f \text{ and } g \text{ are NOT inverses}}$

Exercises

1. Determine if f and g are inverses where $f(x) = 3x - 5$ and $g(x) = \frac{x+5}{3}$.

2. Determine if f and g are inverses where $f(x) = x^2 - 6$ and $g(x) = \sqrt{x+6}$.

3. Determine if f and g are inverses where $f(x) = \frac{x}{x-3}$ and $g(x) = \frac{3x}{x-1}$.

1.10 One-to-One Functions

A. Motivating Question

Let's say we have a function f. We might ask the following question:

Does there **exist** a function g, so that f and g are inverses?

B. Discussion

Well, if we can find a g, at the very least we must have that $(g \circ f)(x) = g(f(x)) = x$, i.e. g undoes f (since that is one of the conditions for two functions to be inverses).

Let's consider $f(x) = x^2$ for example.

For this particular function, we have a problem. Notice what f does to -3 and 3:

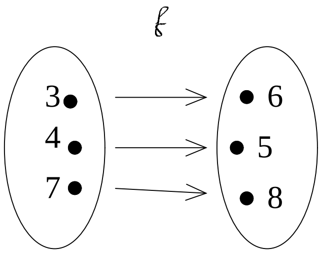
$$\begin{tikzcd}
-3 \arrow[dr, "f"] & \\
& 9 \\
3 \arrow[ur, "f"'] &
\end{tikzcd}$$

Since f sends both -3 and 3 to 9, **if** we had a g that worked, g would have to send 9 back to -3 AND 3, but functions can't do that!

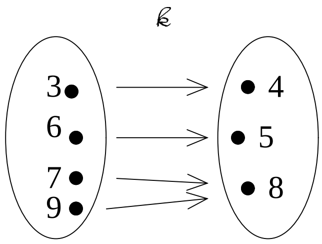
Moral: We see that for a function to have an inverse, it can **not** send two or more x's to the same number. In fact, provided the function doesn't behave badly like this, we can find an inverse (discussed later in Section 1.11).

C. Definition of a One-to-One Function

A function is called **one-to-one** if it is **impossible** for different inputs to get sent to the same output. Alternately, we see that this means that each y can only come from one x.



one-to-one

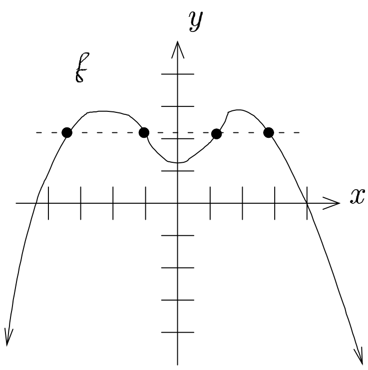


not one-to-one

D. One-to-One Tests

1. **Definition:** See if it is impossible for different x's to go to the same y.

2. **Graphical:** Horizontal Line Test
(if any horizontal line hits the graph more than once, it is not one-to-one)



not one-to-one

3. **Formal Method:**

a. Set $f(x) = f(a)$

b. Solve for x

c. If $x = a$ (only), then f is one-to-one; otherwise, it is not

E. Formal Method Examples

Example 1: Determine if f is one-to-one where $f(x) = 2x + 3$

Solution

1. Set $f(x) = f(a)$: $2x + 3 = 2a + 3$

2. Solve for x: $2x = 2a \Rightarrow x = a$

Ans Since $x = a$ (only), $\boxed{f \text{ is one-to-one}}$

Example 2: Determine if g is one-to-one where $g(x) = \frac{x}{x+1}$

Solution

1. Set $g(x) = g(a)$: $\frac{x}{x+1} = \frac{a}{a+1}$

2. Solve for x: LCD=$(x + 1)(a + 1)$, and $x \neq -1$

$$(x + 1)(a + 1)\left[\frac{x}{x+1}\right] = (x + 1)(a + 1)\left[\frac{a}{a+1}\right]$$

$$x(a + 1) = a(x + 1) \Rightarrow ax + x = ax + a \Rightarrow x = a$$

Ans Since $x = a$ (only), $\boxed{g \text{ is one-to-one}}$

Example 3: Determine if k is one-to-one where $k(x) = \frac{x^2+4}{x^2-3}$

Solution

1. Set $k(x) = k(a)$: $\frac{x^2+4}{x^2-3} = \frac{a^2+4}{a^2-3}$

2. Solve for x: LCD=$(x^2-3)(a^2-3)$, and $x \neq \sqrt{-3}, \sqrt{3}$

$$(x^2-3)(a^2-3)\left[\frac{x^2+4}{x^2-3}\right] = (x^2-3)(a^2-3)\left[\frac{a^2+4}{a^2-3}\right]$$

$$(a^2-3)(x^2+4) = (x^2-3)(a^2+4)$$

$$a^2x^2 + 4a^2 - 3x^2 - 12 = a^2x^2 + 4x^2 - 3a^2 - 12$$

$$4a^2 - 3x^2 = 4x^2 - 3a^2 \;\Rightarrow\; -7x^2 = -7a^2 \;\Rightarrow\; x^2 = a^2 \;\Rightarrow\; x = \pm a$$

Ans Since $x = \pm a$, not just $x = a$, $\boxed{k \text{ is } \textbf{not} \text{ one-to-one}}$

Exercises

Use the formal method to determine if f is one-to-one where

1. $f(x) = 2x - 7.$

2. $f(x) = x^2 + 3.$

3. $f(x) = \sqrt{5 - x}.$

4. $f(x) = \frac{2x-3}{x+1}.$

5. $f(x) = \log_3(2x + 1).$

6. $f(x) = \frac{1}{1+x^2}$

7. Show that all linear functions with nonzero slope are one-to-one.

8. Show that if f is even, then f is **not** one-to-one.

1.11 Inverse Functions

A. Notation

1. The inverse function is written as f^{-1}.

2. **Beware:** Algebra of functions is different from algebra of variables.

$$x^{-1} = \frac{1}{x} \qquad \textbf{but} \qquad f^{-1}(x) \neq \frac{1}{f(x)}$$

B. Finding Inverse Functions Algebraically

1. Verify that f is one-to-one.

2. Set output $f(x) = y$ and solve, if possible, for x (the input).

Note: $y = f(x) \ \Rightarrow \ x = f^{-1}(y)$

C. Examples of the Algebraic Method

Example 1: Find, if possible, the output formula for f^{-1} where $f(x) = 1 - 3x$

Solution

1. Check to see if f is one-to-one:

   Formal Method: Set $f(x) = f(a)$:

   $$1 - 3x = 1 - 3a \;\Rightarrow\; -3x = -3a \;\Rightarrow\; x = a$$

   Thus f is one-to-one, and has an inverse.

2. Let $y = 1 - 3x$ and solve for x:

   $$3x + y = 1 \;\Rightarrow\; 3x = 1 - y \;\Rightarrow\; x = \tfrac{1-y}{3}$$

Ans Now $x = f^{-1}(y)$, so $\boxed{f^{-1}(y) = \tfrac{1-y}{3}}$

Note: For any output y, this formula gives back the original input x.

Example 2: Find, if possible, the output formula for f^{-1} where $f(x) = \frac{3x+4}{2x-3}$

Solution

1. Check to see if f is one-to-one:

   Formal Method: Set $f(x) = f(a)$:

   $$\frac{3x+4}{2x-3} = \frac{3a+4}{2a-3}$$

   LCD=$(2x-3)(2a-3)$ and $x \neq \frac{3}{2}$, then

   $(2x-3)(2a-3)\left[\frac{3x+4}{2x-3}\right] = (2x-3)(2a-3)\left[\frac{3a+4}{2a-3}\right]$

   $(2a-3)(3x+4) = (2x-3)(3a+4)$

   $6ax + 8a - 9x - 12 = 6ax + 8x - 9a - 12 \ \Rightarrow\ -17x = -17a \ \Rightarrow\ x = a$

   Thus f is one-to-one, and has an inverse.

2. Let $y = \frac{3x+4}{2x-3}$ and solve for x:

   Note: $x \neq \frac{3}{2}$. Then:

   $$\begin{aligned} (2x-3)y &= 3x+4 \\ 2xy - 3y &= 3x+4 \\ 2xy - 3x &= 3y+4 \\ x(2y-3) &= 3y+4 \\ x &= \frac{3y+4}{2y-3} \end{aligned}$$

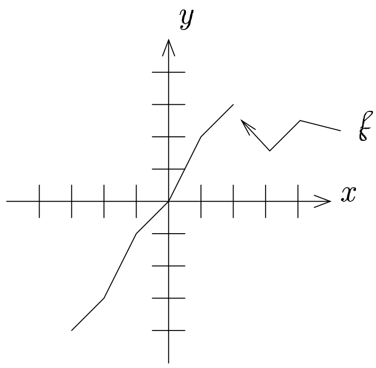
Ans Now $x = f^{-1}(y)$, so $\boxed{f^{-1}(y) = \frac{3y+4}{2y-3}}$

D. Evaluating f^{-1} on a Graph

Since $x = f^{-1}(y)$, f^{-1} takes y-values and gives back x-values. Thus, if we have the graph of f, and we want to evaluate f^{-1} at a point, we put in the y-value and take the corresponding x-value as the answer.

E. An Example

Let f be given by the following graph:



Evaluate:

a. $f^{-1}(2)$

b. $f^{-1}(-3)$

Solution

Note: f actually has an inverse, since it passes the Horizontal Line Test

a. $f^{-1}(2)$: when $y = 2$, $x = 1$, so $\boxed{f^{-1}(2) = 1}$

b. $f^{-1}(-3)$: when $y = -3$, $x = -2$, so $\boxed{f^{-1}(-3) = -2}$

Exercises

1. Find, if possible, the output formula for f^{-1} where

    a. $f(x) = 2 - 5x$

    b. $f(x) = \frac{6x-1}{3}$

    c. $f(x) = \sqrt{3x-2}$

    d. $f(x) = |x-3|$

    e. $f(x) = \frac{x+5}{2x-3}$

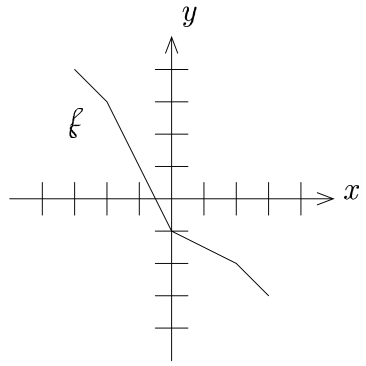
    f. $f(x) = \frac{3x-2}{5x+1}$

    g. $f(x) = 2\sqrt[3]{2x-5} - 1$

    h. $f(x) = \log_6(x+3)$

2. If f is one-to-one, find the output formula for g^{-1}, where $g(x) = af(x+b) - c$ where $a \neq 0$.

3. Let f be given by the following graph:



Evaluate:

a. $f^{-1}(3)$

b. $f^{-1}(-1)$

c. $f^{-1}(1)$

1.12 Inverse Functions II: Reflections

A. Introduction

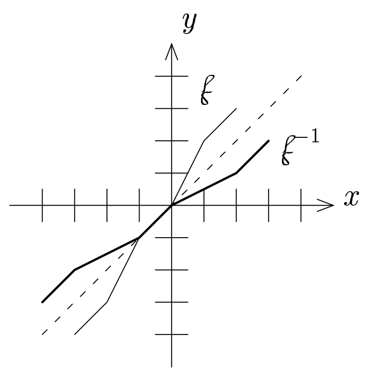
It is sometimes undesirable to examine the inverse function by looking sideways. Thus, for graphical purposes, we can get a "non-sideways" version of the graph of f^{-1} by switching the x and y coordinates. Thus to get a "non-sideways" version of the graph of f^{-1}, we would take each point on the graph of f, say $(3, 2)$ for example and plot $(2, 3)$.

Note: When we want to consider this alternate version of the graph of f^{-1}, we indicate that in our output formula as well. In this case, we switch the letter in the output formula for f^{-1} from y to x; that is, if our original output formula was $f^{-1}(y) = 3y - 2$, our new output formula is $f^{-1}(x) = 3x - 2$.

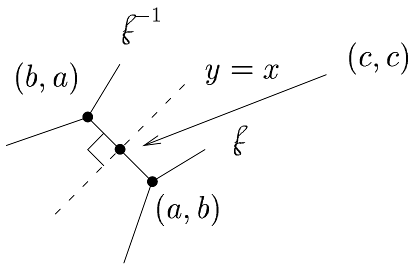
Now let's see what this means geometrically.

B. Graph of the Inverse Function

To get the graph of f^{-1}, we switch the coordinates of each point on the graph of f. Geometrically, this corresponds to **reflecting the graph about the line** $y = x$.



C. Justification of Geometric Interpretation of the Inverse



1. If (a, b) is on f, then (b, a) is on f^{-1}.

2. Connecting these two points, we cut the line $y = x$ at some point (c, c).

3. The slope of the connecting segment is $\frac{a-b}{b-a} = \frac{a-b}{-(a-b)} = -1$.

4. Since the slope of the line $y = x$ is 1, we see that the connecting segment is perpendicular to the line $y = x$.

5. To show that f^{-1} is a mirror image across $y = x$, we just need to show that (a, b) is the same distance from (c, c) as the point (b, a) is . . .

6. Distance from (a, b) to (c, c): $\sqrt{(c-a)^2 + (c-b)^2}$

7. Distance from (b, a) to (c, c): $\sqrt{(c-b)^2 + (c-a)^2} = \sqrt{(c-a)^2 + (c-b)^2}$

Thus (b, a) is the mirror image of (a, b) across the line $y = x$, so the graph of f^{-1} is the reflection of the graph of f across the line $y = x$.

1.13 Domain and Range of the Inverse Function

A. Domain and Range of f^{-1}

1. To find $dom(f^{-1})$: find $rng\, f$

2. To find $rng(f^{-1})$: find $dom\, f$

This comes from the idea that f^{-1} takes y-values back to x-values.

Warning: You have to use the above relationship. The output formula $f^{-1}(x)$ will give the **wrong** domain and range.

B. Examples

Example 1: f is invertible and $f(x) = \sqrt{x-8}$. Find $f^{-1}(x)$, $dom(f^{-1})$, and $rng(f^{-1})$.

Solution

1.

$$\begin{aligned} y &= \sqrt{x-8} \\ y^2 &= x-8 \\ x &= y^2+8 \\ f^{-1}(y) &= y^2+8 \\ f^{-1}(x) &= x^2+8 \end{aligned}$$

Hence, we have $\boxed{f^{-1}(x) = x^2+8}$.

Note: The above output formula suggests that $dom(f^{-1}) = (-\infty, \infty)$, but this is not the right answer. Now let's find it.

2. $\mathscr{dom}(f^{-1})$: Find $\mathscr{rng}\, f$

Now $\mathscr{rng}\, f = [0, \infty)$, since the square root produces outputs greater than or equal to zero.

Thus $\boxed{\mathscr{dom}(f^{-1}) = [0, \infty)}$

3. $\mathscr{rng}(f^{-1})$: Find $\mathscr{dom}\, f$

Now $\mathscr{dom}\, f = [8, \infty)$ [since we require $x - 8 \geq 0$].

Thus $\boxed{\mathscr{rng}(f^{-1}) = [8, \infty)}$

Example 2: g is invertible and $g(x) = \sqrt[4]{x} + 3$. Find $\mathscr{dom}(g^{-1})$, and $\mathscr{rng}(g^{-1})$.

Solution

1. $\mathscr{dom}(g^{-1})$: Find $\mathscr{rng}\ g$

Now $\sqrt[4]{x}$ forces $[0, \infty)$ outputs

Thus $\sqrt[4]{x} + 3$ forces outputs in $[3, \infty)$, so $\mathscr{rng}\ g = [3, \infty)$.

Hence $\boxed{\mathscr{dom}(g^{-1}) = [3, \infty)}$

2. $\mathscr{rng}(g^{-1})$: Find $\mathscr{dom}\ g$

Now $\mathscr{dom}\ g = [0, \infty)$ [since we require $x \geq 0$]

Hence $\boxed{\mathscr{rng}(g^{-1}) = [0, \infty)}$

Exercises

1. Let $f(x) = \sqrt{x+2} - 3$. Find $f^{-1}(x)$, $dom(f^{-1})$, and $rng(f^{-1})$.

2. Let $g(x) = \sqrt[4]{3x-2} + 5$. Find $g^{-1}(x)$, $dom(g^{-1})$, and $rng(g^{-1})$.

3. Let $g(x) = 2\,log_7(x+3) - 1$. Find $g^{-1}(x)$, $dom(g^{-1})$, and $rng(g^{-1})$.

1.14 Capital Functions

A. Motivation

If f is useful, but **not** invertible (not one-to-one), we create an auxiliary function $\Im$ that is similar to f, but **is** invertible.

B. Capital Functions

Given f, **not** invertible . . . , we define $\Im$ invertible.

$\Im$ must have the following properties:

1. $\Im(x) = f(x)$ [$\Im$ produces the same outputs as f, so the output formula is the same]

2. $\Im$ is given a smaller, restricted domain of f, so that

    a. $\Im$ is one-to-one

    b. $\operatorname{rng} \Im = \operatorname{rng} f$

Any such $\Im$ is called a **capital function** or **principal function**

C. Method for Constructing Capital Functions

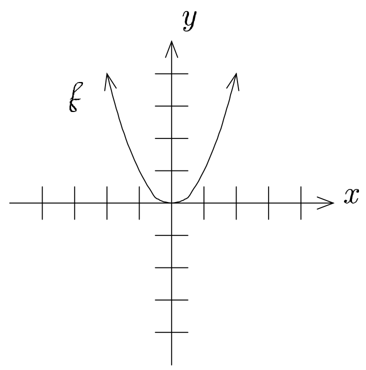
Given f, we try to determine what to cut out of $\operatorname{dom} f$ so that the result is one-to-one with the same range.

D. Examples

Example 1: f is **not** invertible, where $f(x) = x^2$. Construct a capital function $\Im$.

Solution

To see what is going on, let's look at the graph of f:



Note: $\operatorname{dom} f = (-\infty, \infty)$ and $\operatorname{rng} f = [0, \infty)$.

Cutting off either the left half or the right half makes the remaining part one-to-one, without changing the range.

Ans One solution is $\boxed{\Im(x) = x^2; x \in [0, \infty)}$

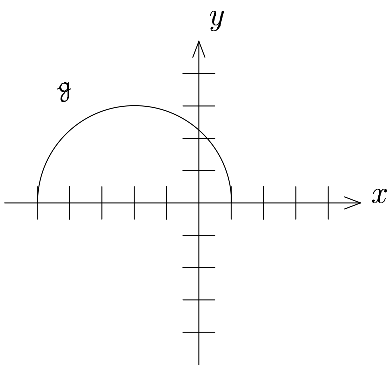
Another solution is $\boxed{\Im(x) = x^2; x \in (-\infty, 0]}$

Note: Each such solution can be loosely referred to as a "branch".

Example 2: $\mathscr{g}$ is **not** invertible, where $\mathscr{g}(x) = \sqrt{9-(x+2)^2}$. Construct a capital function $\mathscr{G}$.

Solution

To see what is going on, let's look at the graph of $\mathscr{g}$:



This is the top half of a circle with center $(-2, 0)$ and radius 3.

[The full circle would be $(x+2)^2 + y^2 = 9$]

Note: $\mathscr{dom}\ \mathscr{g} = [-5, 1]$ and $\mathscr{rng}\ \mathscr{g} = [0, 3]$.

Cutting off either the left half or the right half makes the remaining part one-to-one, without changing the range.

Ans One solution is $\boxed{\mathscr{G}(x) = \sqrt{9-(x+2)^2};\, x \in [-2, 1]}$

Another solution is $\boxed{\mathscr{G}(x) = \sqrt{9-(x+2)^2};\, x \in [-5, -2]}$

E. Comments

1. By construction, the capital functions **are** invertible.

2. Since $\operatorname{rng} \mathcal{F} = \operatorname{rng} f$, it retains the useful information from the original function.

3. The inverse of the capital function, $\mathcal{F}^{-1}$, serves as the best approximation to an inverse of the original function one can get.

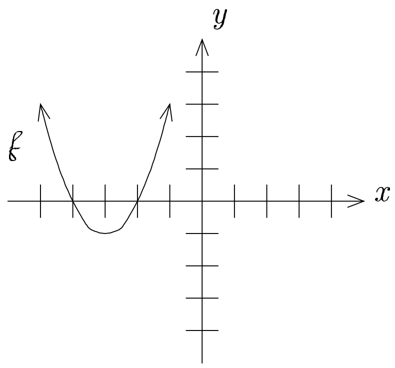
F. Inverses of the Capital Functions

Here given a function f, we examine $\mathcal{F}^{-1}$.

Example 1: Let $f(x) = (x+3)^2 - 1$. Using the left branch, determine $\mathcal{F}$ and find $\mathcal{F}^{-1}(y)$. Also, give $\operatorname{dom}(\mathcal{F}^{-1})$ and $\operatorname{rng}(\mathcal{F}^{-1})$.

Solution

Let's look at the graph of f:



Note: $\operatorname{dom} f = (-\infty, \infty)$ and $\operatorname{rng} f = [-1, \infty)$.

Using the left branch,

$$\boxed{\mathcal{F}(x) = (x+3)^2 - 1;\, x \in (-\infty, -3]}$$

Note 2: $\mathit{dom}\,\Im = (-\infty, -3]$ and $\mathit{rng}\,\Im = [-1, \infty)$

Now find $\Im^{-1}$:

$$\begin{aligned} y &= (x+3)^2 - 1 \\ (x+3)^2 &= y+1 \\ x+3 &= \pm\sqrt{y+1} \\ x &= -3 \pm \sqrt{y+1} \end{aligned}$$

Since functions only produce one value, we need to decide which of the two solutions to take. In this case, $\mathit{dom}\,\Im = (-\infty, -3]$, so we need to take the minus sign to make x smaller than -3.

Hence, $\boxed{\Im^{-1}(y) = -3 - \sqrt{y+1}}$.

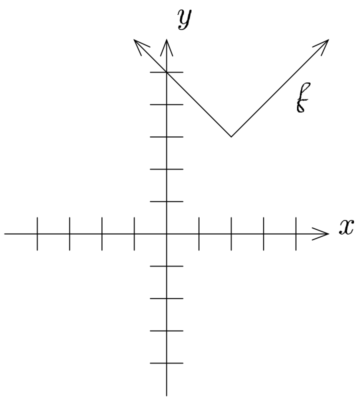
Now $\mathit{dom}(\Im^{-1}) = \mathit{rng}\,\Im$ and $\mathit{rng}(\Im^{-1}) = \mathit{dom}\,\Im$, so by Note 2, we have

$\boxed{\mathit{dom}(\Im^{-1}) = [-1, \infty)}$ and $\boxed{\mathit{rng}(\Im^{-1}) = (-\infty, -3]}$

Example 2: Let $\mathscr{f}(x) = |x-2| + 3$. Using the right branch, determine $\Im$ and find $\Im^{-1}(y)$. Also, give $\mathit{dom}(\Im^{-1})$ and $\mathit{rng}(\Im^{-1})$.

Solution

Let's look at the graph of $\mathscr{f}$:



Note: $\operatorname{dom} f = (-\infty, \infty)$ and $\operatorname{rng} f = [3, \infty)$.

Using the right branch,

$$\boxed{\mathfrak{F}(x) = |x-2| + 3;\, x \in [2, \infty)}$$

Note 2: $\operatorname{dom} \mathfrak{F} = [2, \infty)$ and $\operatorname{rng} \mathfrak{F} = [3, \infty)$

Now find $\mathfrak{F}^{-1}$:

$$\begin{aligned} y &= |x-2| + 3 \\ |x-2| &= y - 3 \end{aligned}$$

Then $\quad x - 2 = y - 3 \quad \text{OR} \quad x - 2 = -(y-3).$

$$x = y - 1 \quad \text{OR} \quad x - 2 = -y + 3$$

$$x = y - 1 \quad \text{OR} \quad x = -y + 5$$

Since functions only produce one value, we need to decide which of the two solutions to take. In this case, $\operatorname{dom} \mathfrak{F} = [2, \infty)$, so we need the equation giving x-values that are 2 or bigger. Since $y \geq 3$ [range], this happens in the **first** equation.

Hence, $\boxed{\mathfrak{F}^{-1}(y) = y - 1}$.

Now $\mathcal{dom}(\Im^{-1}) = \mathcal{rng}\,\Im$ and $\mathcal{rng}(\Im^{-1}) = \mathcal{dom}\,\Im$, so by Note 2, we have

$\boxed{\mathcal{dom}(\Im^{-1}) = [3, \infty)}$ and $\boxed{\mathcal{rng}(\Im^{-1}) = [2, \infty)}$

Exercises

1. Let $\mathscr{f}(x) = (x-3)^2 + 2$. Using the left branch, determine a capital function $\Im$ and find $\Im^{-1}(y)$. Also, determine $\mathscr{dom}(\Im^{-1})$ and $\mathscr{rng}(\Im^{-1})$.

2. Let $\mathscr{f}(x) = (x+1)^2 - 3$. Using the left branch, determine a capital function $\Im$ and find $\Im^{-1}(y)$. Also, determine $\mathscr{dom}(\Im^{-1})$ and $\mathscr{rng}(\Im^{-1})$.

3. Let $\mathscr{f}(x) = |x+4|$. Using the right branch, determine a capital function $\Im$ and find $\Im^{-1}(y)$. Also, determine $\mathscr{dom}(\Im^{-1})$ and $\mathscr{rng}(\Im^{-1})$.

4. Let $\mathscr{f}(x) = |x-1| + 2$. Using the right branch, determine a capital function $\Im$ and find $\Im^{-1}(y)$. Also, determine $\mathscr{dom}(\Im^{-1})$ and $\mathscr{rng}(\Im^{-1})$.

Chapter 2

Rational Functions

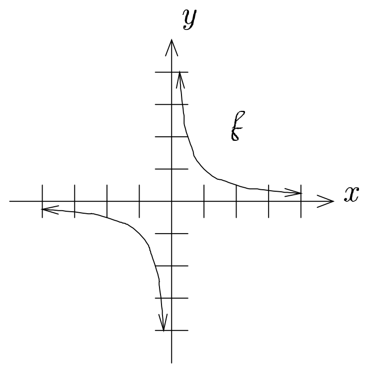
2.1 The Reciprocal Function

Let $f(x) = \frac{1}{x}$. f is called the **reciprocal function**.

We can plot this by making a table of values. Since f is undefined at $x = 0$, we pick a lot of points near 0.

x	y
-10	$-\frac{1}{10}$
-3	$-\frac{1}{3}$
-2	$-\frac{1}{2}$
-1	-1
$-\frac{1}{2}$	-2
$-\frac{1}{3}$	-3
$-\frac{1}{10}$	-10
0	undefined

x	y
0	undefined
$\frac{1}{10}$	10
$\frac{1}{3}$	3
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$
3	$\frac{1}{3}$
10	$\frac{1}{10}$



For very large numbers (like 10000) and very negative numbers (like -10000), the graph approaches the x-axis, but does not cross. Also, for numbers near 0 (like $-\frac{1}{10000}$ or $\frac{1}{10000}$), the graph approaches the y-axis, but does not cross.

These “imaginary” approaching lines are called **asymptotes**.

Note: When asymptotes don’t lie along the x or y axes, as in this function, we often draw them in for a visual aid as a dashed line.

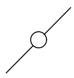
2.2 Rational Functions and Asymptotes

A. Definition of a Rational Function

f is said to be a **rational function** if $f(x) = \frac{g(x)}{h(x)}$, where g and h are polynomial functions. That is, rational functions are fractions with polynomials in the numerator and denominator.

B. Asymptotes/Holes

Holes are what they sound like:

 is a hole

Rational functions may have holes or asymptotes (or both!).

Asymptote Types:

1. vertical
2. horizontal
3. oblique ("slanted-line")
4. curvilinear (asymptote is a curve!)

We will now discuss how to find all of these things.

C. Finding Vertical Asymptotes and Holes

Factors in the denominator cause vertical asymptotes and/or holes.

To find them:

1. Factor the denominator (and numerator, if possible).

2. Cancel common factors.

3. Denominator factors that **cancel** completely give rise to **holes**. Those that don't give rise to **vertical asymptotes**.

D. Examples

Example 1: Find the vertical asymptotes/holes for f where $f(x) = \frac{(3x+1)(x-7)(x+4)}{(x-7)^2(x+4)}$.

Solution

Canceling common factors: $f(x) = \frac{3x+1}{x-7}$, $x \neq -4$

$x + 4$ factor cancels completely $\Rightarrow$ $\boxed{\textbf{hole} \text{ at } x = -4}$

$x - 7$ factor not completely canceled $\Rightarrow$ $\boxed{\textbf{vertical asymptote} \text{ with equation } x = 7}$

Example 2: Find the vertical asymptotes/holes for f where $f(x) = \frac{2x^2-5x-12}{x^2-5x+4}$.

Solution

Factor: $f(x) = \frac{(x-4)(2x+3)}{(x-4)(x-1)}$

Cancel: $f(x) = \frac{2x+3}{x-1}$, $x \neq 4$

Ans $\boxed{\textbf{Hole at } x = 4 \quad \textbf{Vertical Asymptote} \text{ with equation } x = 1}$

E. Finding Horizontal, Oblique, Curvilinear Asymptotes

Suppose $f(x) = \dfrac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0}$

If

1. degree top $<$ degree bottom: **horizontal asymptote** with equation $y = 0$
2. degree top $=$ degree bottom: **horizontal asymptote** with equation $y = \frac{a_n}{b_m}$
3. degree top $>$ degree bottom: **oblique** or **curvilinear asymptotes**

To find them: Long divide and throw away remainder

F. Examples

Example 1: Find the horizontal, oblique, or curvilinear asymptote for f where $f(x) = \frac{6x^4-x+2}{7x^5+2x-1}$.

Solution

degree top$= 4$ degree bottom$= 5$. Since $4 < 5$, we have

Ans $\boxed{\textbf{horizontal asymptote} \text{ with equation } y = 0}$

Example 2: Find the horizontal, oblique, or curvilinear asymptote for f where $f(x) = \frac{6x^3-2x^2+1}{2x^3+5}$.

Solution

degree top$=3$ degree bottom$=3$.

Since $3 = 3$, we have a horizontal asymptote of $y = \frac{6}{2} = 3$. Thus

Ans $\boxed{\textbf{horizontal asymptote} \text{ with equation } y = 3}$

Example 3: Find the horizontal, oblique, or curvilinear asymptote for f where $f(x) = \frac{2x^3-3}{x^2-1}$.

Solution

degree top$=3$ degree bottom$=2$.

Since $3 > 2$, we have an oblique or curvilinear asymptote. Now long divide:

$$\begin{array}{r|l} & 2x \\ \hline x^2+0x-1 & 2x^3+0x^2+0x-3 \\ & -\underline{(2x^3+0x^2-2x)} \\ & \qquad\qquad\quad 2x-3 \end{array}$$

Since $\dfrac{2x^3-3}{x^2-1} = 2x + \underbrace{\dfrac{2x-3}{x^2-1}}_{\text{Throw away}}$, we have that

Ans $\boxed{g(x) = 2x \text{ defines a line, and is the equation for the } \textbf{oblique asymptote}}$

Example 4: Find the horizontal, oblique, or curvilinear asymptote for f where

$$f(x) = \frac{3x^5 - x^4 + 2x^2 + x + 1}{x^2 + 1}.$$

Solution

degree top$= 5$ degree bottom$= 2$.

Since $5 > 2$, we have an oblique or curvilinear asymptote. Now long divide:

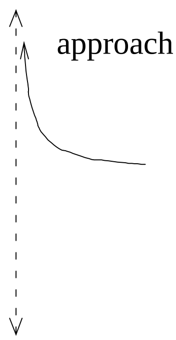
$$\begin{array}{r|l}
 & 3x^3 - x^2 - 3x + 3 \\ \hline
x^2 + 0x + 1 & 3x^5 - x^4 + 0x^3 + 2x^2 + x + 1 \\
 & -\underline{(3x^5 + 0x^4 + 3x^3)} \\
 & \quad -x^4 - 3x^3 + 2x^2 \\
 & \quad -\underline{(-x^4 + 0x^3 - x^2)} \\
 & \qquad -3x^3 + 3x^2 + x \\
 & \qquad -\underline{(-3x^3 + 0x^2 - 3x)} \\
 & \qquad\qquad 3x^2 + 4x + 1 \\
 & \qquad\qquad -\underline{(3x^2 + 0x + 3)} \\
 & \qquad\qquad\qquad 4x - 2
\end{array}$$

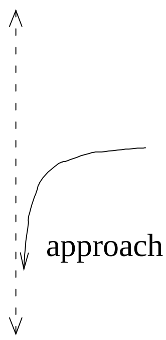
Since $\dfrac{3x^5 - x^4 + 2x^2 + x + 1}{x^2 + 1} = 3x^3 - x^2 - 3x + 3 + \underbrace{\dfrac{4x - 2}{x^2 + 1}}_{\text{Throw away}}$, we have that

Ans $\boxed{g(x) = 3x^3 - x^2 - 3x + 3 \text{ defines a } \textbf{curvilinear asymptote}}$

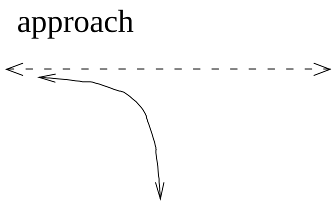
G. Asymptote Discussion for Functions

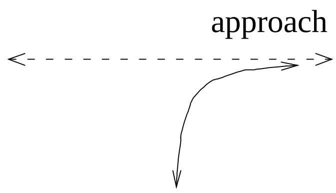
1. As the graph of a function approaches a **vertical asymptote**, it shoots up or down toward $\pm\infty$.

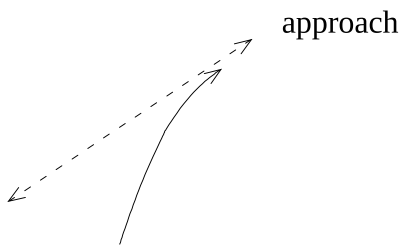


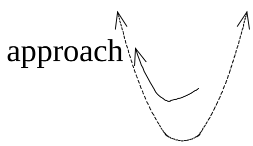


2. Graphs approach **horizontal, oblique,** and **curvilinear asymptotes** as $x \to -\infty$ or $x \to \infty$.









3. Graphs of functions **never** cross vertical asymptotes, but **may** cross other asymptote types.

Exercises

1. Find the vertical asymptotes and holes for f where

   a. $f(x) = \frac{(x-3)(x+4)}{(x+4)(2x-1)}$

   b. $f(x) = \frac{(2x-3)(x+4)(2x-1)^2}{(x+4)^2(2x-1)}$

   c. $f(x) = \frac{2x^2-5x-12}{2x^2-3x-9}$

2. Find the horizontal, oblique, or curvilinear asymptote for f where

   a. $f(x) = \frac{8x^3-6}{2x^3+1}$

   b. $f(x) = \frac{8x^3-6}{2x^4+1}$

   c. $f(x) = \frac{8x^3-6}{2x^2+1}$

   d. $f(x) = \frac{8x^3-6}{2x+1}$

   e. $f(x) = \frac{2x^4-5x^2+x+1}{3x^5-6x^3+x^2-2}$

   f. $f(x) = \frac{2x^4-5x^2+x+1}{3x^4-6x^3+x^2-2}$

3. Create a rational function with curvilinear asymptote $y = x^3 - 2x^2 + 5x - 1$.

4. Suppose a rational function has no vertical asymptotes or holes. What is the domain of the function?

5. Describe the relationship between the domain of a rational function f and its vertical asymptotes and/or holes.

2.3 Graphing Rational Functions

A. Strategy

1. Find all asymptotes (vertical, horizontal, oblique, curvilinear) and holes for the function.

2. Find the x and y intercepts.

3. Plot the x and y intercepts, draw the asymptotes, and plot "enough" points on the graph to see exactly what is going on.

Note: The graph may **not** cross vertical asymptotes, but may cross others. Also, sometimes knowledge about symmetry (even/odd/neither) can speed up the graphing.

B. Examples

Example 1: Graph f, where $f(x) = \frac{3x-1}{x+2}$

Solution

1. **Asymptotes**:

Vertical: $x = -2$

Horizontal: $y = \frac{3}{1} = 3$

2. **Intercepts**:

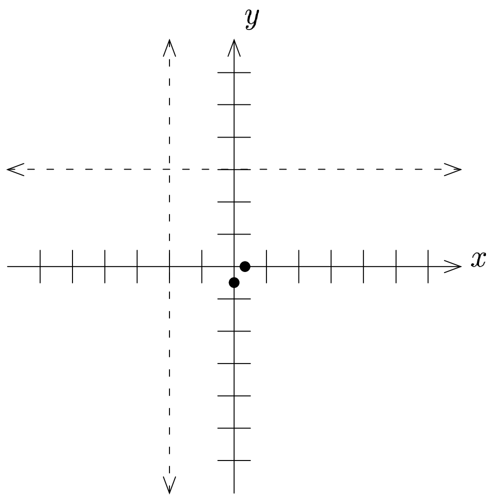
x-intercepts: set $y = 0$: $\quad 0 = \frac{3x-1}{x+2}$

Disallowed values: $x \neq -2$, and LCD=$x + 2$.

Multiplying by the LCD: $0 = 3x - 1 \;\Rightarrow\; x = \frac{1}{3}$

y-intercept: set $x = 0$: $f(0) = \frac{3(0)-1}{0+2} = -\frac{1}{2}$

Now graph an initial rough sketch:

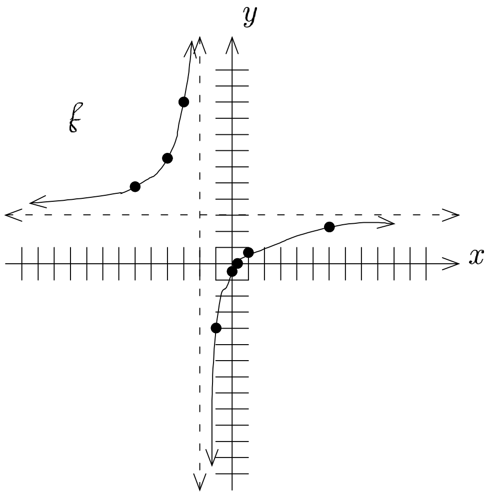


Now we need to plot enough points to see what is going on. We pick $x = -6$ and $x = 6$ to see the behavior near the horizontal asymptote, and pick $x = -3$ and $x = -1$ to see the behavior near the vertical asymptote. Then pick a few others to see what is going on:

x	y
-6	$\frac{19}{4}$
-4	$\frac{13}{2}$
-3	10
-1	-4
1	$\frac{2}{3}$
6	$\frac{17}{8}$

We plot these points on the grid we already made. Then we connect the points using the asymptote behavior.

Ans



Example 2: Graph f, where $f(x) = \dfrac{x^2 - x - 6}{x^3 - 6x^2 + 5x + 12}$

Solution

1. **Asymptotes**:

   We first have to factor the top and bottom.

   Top: $x^2 - x - 6 = (x-3)(x+2)$

   Bottom: $x^3 - 6x^2 + 5x + 12$

   Rational Root Theorem:

   $$\frac{\text{factors of } 12}{\text{factors of } 1} = \frac{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12}{\pm 1}$$

   Rational Candidates: $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$

Note: $(-1)^3 - 6(-1)^2 + 5(-1) + 12 = -1 - 6 - 5 + 12 = 0$

Thus $x + 1$ is a factor. Divide it out using synthetic division:

$$\begin{array}{r|rrrr} -1 & 1 & -6 & 5 & 12 \\ & & -1 & 7 & -12 \\ \hline & 1 & -7 & 12 & \boxed{0} \end{array}$$

$x^3 - 6x^2 + 5x + 12 = (x+1)(x^2 - 7x + 12) = (x+1)(x-3)(x-4)$

Thus, $f(x) = \frac{(x-3)(x+2)}{(x+1)(x-3)(x-4)}$.

Simplifying, $f(x) = \frac{(x+2)}{(x+1)(x-4)}$.

Thus, we have **vertical asymptotes** with equations $x = -1$ and $x = 4$.

Also, we have a **hole** at $x = 3$.

Since degree top$= 1$ and degree bottom$= 2$, and since $1 < 2$, we have a **horizontal asymptote** with equation $y = 0$.

2. **Intercepts**:

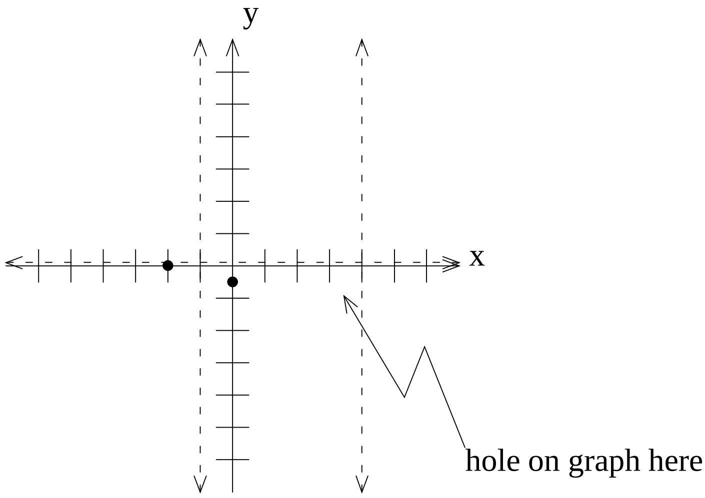
x-intercepts: set $y = 0$: $\frac{x+2}{(x+1)(x-4)} = 0$

Disallowed values: $x \neq -1, 4$, and LCD=$(x+1)(x-4)$.

Multiplying by the LCD: $x + 2 = 0 \;\Rightarrow\; x = -2$

y-intercept: set $x = 0$: $f(0) = \frac{0+2}{(0+1)(0-4)} = \frac{2}{(1)(-4)} = -\frac{1}{2}$

Now graph an initial rough sketch:



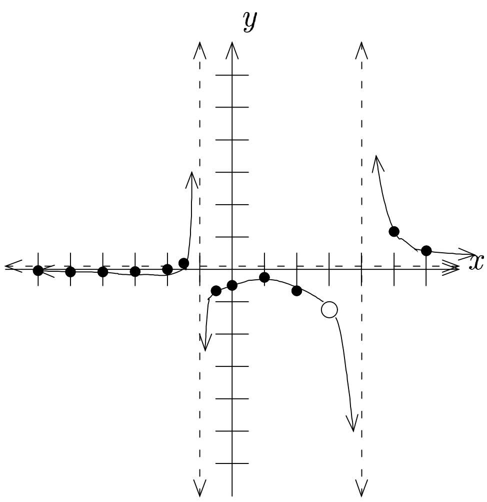
Now plot some more points using the output formula $\ell(x) = \frac{(x+2)}{(x+1)(x-4)}$:

x	y
-6	$-\frac{1}{25}$
-5	$-\frac{1}{12}$
-4	$-\frac{1}{12}$
-3	$-\frac{1}{14}$
$-\frac{3}{2}$	$\frac{2}{11}$
$-\frac{1}{2}$	$-\frac{2}{3}$
1	$-\frac{1}{2}$
2	$-\frac{2}{3}$
3	$-\frac{5}{4}$
5	$\frac{7}{6}$
6	$\frac{4}{7}$

Note: In fact, the point $(3, -\frac{5}{4})$ from the table above is the location of a hole, since we saw previously that we had a hole at $x = 3$.

We plot these points on the grid we already made. Then we connect the points using the asymptote behavior.

Ans



Example 3: Graph f, where $f(x) = \dfrac{x^3 + x^2 - 2}{x^2 + x + 1}$

Solution

1. **Asymptotes**:

   We first have to factor . . .

   Considering $x^2 + x + 1$, we can't factor it immediately, so we decide to use the quadratic formula. However $b^2 - 4ac = (1)^2 - 4(1)(1) = -3 < 0$, so the zeros are complex. Thus we have no vertical asymptotes or holes!

   Since degree top$= 3$ and degree bottom$= 2$, and since $3 > 2$, we have an oblique or curvilinear asymptote (oblique, in fact, as we see below).

Now find it by algebraic long division:

$$\begin{array}{r|l} & \quad x \\ \cline{2-2} x^2+x+1 & x^3+x^2+0x-2 \\ & -\underline{(x^3+x^2+x)} \\ & \qquad\qquad -x-2 \end{array}$$

Hence $\dfrac{x^3+x^2-2}{x^2+x+1} = x + \dfrac{-x-2}{x^2+x+1}$.

Thus $y = x$ defines an **oblique asymptote**.

2. **Intercepts**:

x-intercepts: set $y = 0$: $\qquad \dfrac{x^3+x^2-2}{x^2+x+1} = 0$

No disallowed values and LCD=x^2+x+1.

Multiplying by the LCD: $x^3+x^2-2=0$.

Rational Root Theorem:

$$\frac{\text{factors of } -2}{\text{factors of } 1} = \frac{\pm 1, \pm 2}{\pm 1}$$

Rational Candidates: $\pm 1, \pm 2$

Note: $1^3 + 1^2 - 2 = 1 + 1 - 2 = 0$

Thus $x-1$ is a factor. Divide it out using synthetic division:

$$\begin{array}{c|cccc} 1 & 1 & 1 & 0 & -2 \\ & & 1 & 2 & 2 \\ \hline & 1 & 2 & 2 & \boxed{0} \end{array}$$

Thus $(x-1)(x^2+2x+2)=0$.

Now use the quadratic formula on the remaining quadratic:

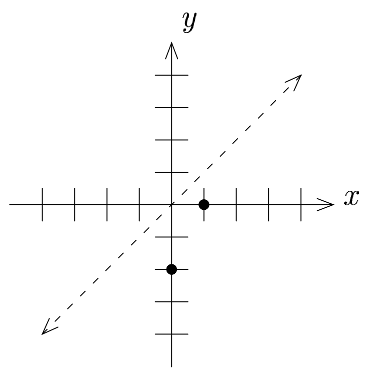
$$x = \frac{-2 \pm \sqrt{4 - 4(1)(2)}}{2(1)} = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

Since these are complex, we only get one x-intercept from the $x - 1$ factor.

Thus we have one x-intercept, $x = 1$.

y-intercept: set $x = 0$: $\quad f(0) = \frac{0^3+0^2-2}{0^{+}0+1} = -2.$

Now graph an initial rough sketch:

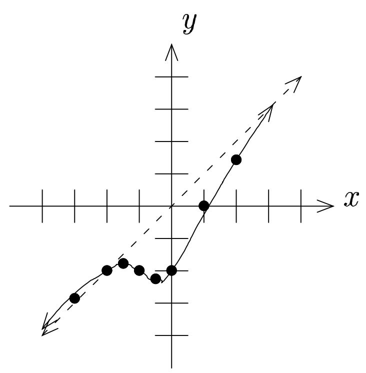


Now plot some more points using the output formula $\ell(x) = \dfrac{x^3 + x^2 - 2}{x^2 + x + 1}$:

x	y
-3	$-\frac{20}{7}$
-2	-2
$-\frac{3}{2}$	$-\frac{25}{14}$
-1	-2
$-\frac{1}{2}$	$-\frac{5}{2}$
2	$\frac{10}{7}$

We plot these points on the grid we already made. Then we connect the points using the asymptote behavior.

Ans



Example 4: Graph f, where $f(x) = \dfrac{2x^5}{x^3 - x}$

Solution

1. **Asymptotes**:

$$f(x) = \frac{2x^5}{x^3 - x} = \frac{2x^5}{x(x^2 - 1)} = \frac{2x^5}{x(x+1)(x-1)}$$

Simplifying: $f(x) = \frac{2x^4}{(x+1)(x-1)}$

Thus we have **vertical asymptotes** with equations $x = -1$ and $x = 1$.

We also have a **hole** at $x = 0$.

Since degree top= 4 and degree bottom= 2, and since $4 > 2$, we have an oblique or curvilinear asymptote (curvilinear, in fact, as we see below).

Now find it by algebraic long division:

$$\begin{array}{r|l}
 & 2x^2 + 2 \\ \hline
x^2 + 0x - 1 & 2x^4 + 0x^3 + 0x^2 + 0x + 0 \\
 & -\underline{(2x^4 + 0x^3 - 2x^2)} \\
 & \qquad\qquad 2x^2 + 0x + 0 \\
 & \qquad\qquad -\underline{(2x^2 + 0x - 2)} \\
 & \qquad\qquad\qquad\qquad 2
\end{array}$$

Hence $\dfrac{2x^4}{x^2 - 1} = 2x^2 + 2 + \dfrac{2}{x^2 - 1}$.

Thus $y = 2x^2 + 2$ defines a **curvilinear asymptote**.

2. **Intercepts**:

x-intercepts: set $y = 0$: $\dfrac{2x^4}{x^2 - 1} = 0$

Disallowed values: $x \neq -1, 1$ and LCD=$(x+1)(x-1)$.

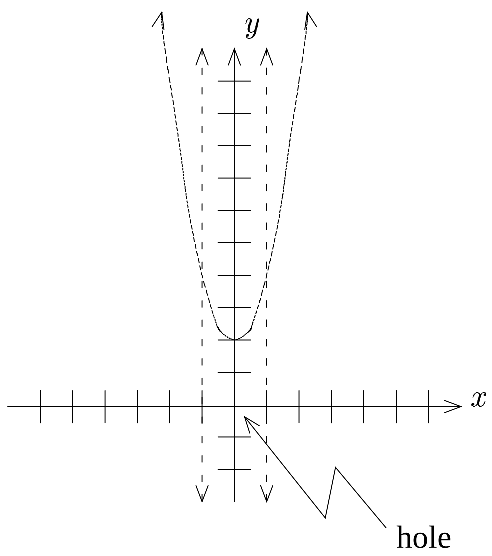
Multiplying by the LCD: $2x^4 = 0 \;\Rightarrow\; x^4 = 0 \;\Rightarrow\; x = 0$.

However, this intercept is **not** a point, since we said earlier that we had a hole at $x = 0$.

y-intercept: set $x = 0$: $f(0) = \frac{2(0)^4}{0^2-1} = \frac{0}{-1} = 0$

This intercept is not a point either, since we found it by setting $x = 0$, which is where we said we had a hole.

Now graph an initial rough sketch:



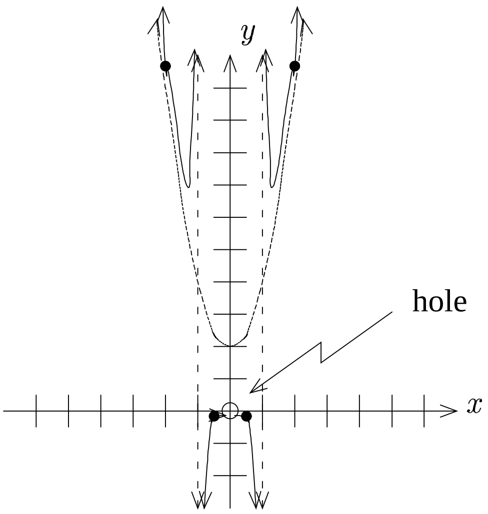
Now plot some more points using the output formula $\not{f}(x) = \dfrac{2x^4}{x^2-1}$.

Note: Since this function is **even**, we need only make our table with negative x-values. The positive x-values we get for free by reflection.

x	y
-2	$\frac{32}{3}$
$-\frac{1}{2}$	$-\frac{1}{6}$

We plot these points on the grid we already made, along with the y-axis reflected points (even function). Then we connect the points using the asymptote behavior.

Ans



Exercises

Graph the function f, where

1. $f(x) = \frac{3x+1}{2x-4}$

2. $f(x) = \frac{x-1}{x^2-4}$

3. $f(x) = \frac{(x^2-4)(x+1)}{x(x+1)}$

4. $f(x) = \frac{x^5-x^4}{(x^2-4)(x-1)}$

Chapter 3

Elementary Trigonometry

3.1 Circles and Revolutions

A. Circles

Standard Form: $(x-a)^2+(y-b)^2=r^2$

1. center: (a,b)

2. radius: r

3. circumference: $2\pi r$

B. Examples

Example 1: Find the center, radius, circumference, x and y intercepts of the circle, where $(x-1)^2 + (y+4)^2 = 12$. Then sketch the circle.

Solution

center: $\boxed{(1,-4)}$

radius: $\sqrt{12} = \boxed{2\sqrt{3}}$

circumference: $2\pi \cdot 2\sqrt{3} = \boxed{4\pi\sqrt{3}}$

x**-intercepts:** set $y = 0$:

$$\begin{aligned} (x-1)^2 + (0+4)^2 &= 12 \\ (x-1)^2 + 16 &= 12 \\ (x-1)^2 &= -4 \\ x - 1 &= \pm\sqrt{-4} \\ x - 1 &= \pm 2i \\ x &= 1 \pm 2i \end{aligned}$$

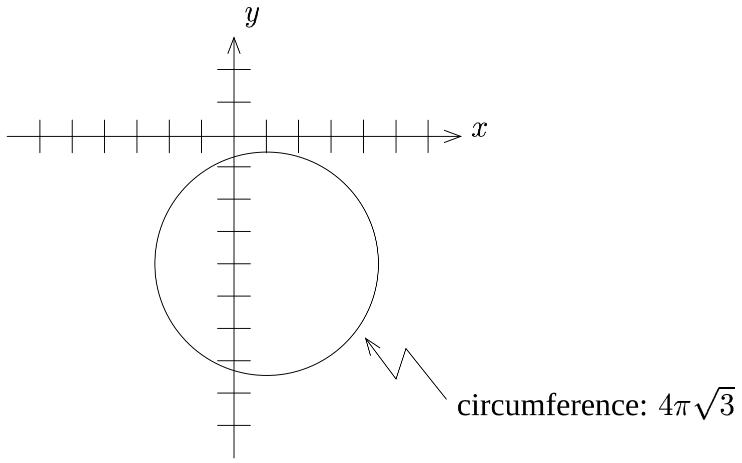
$\boxed{\text{Thus there are no } x\text{-intercepts.}}$

y**-intercepts:** set $x = 0$:

$$\begin{aligned} (0-1)^2 + (y+4)^2 &= 12 \\ 1 + (y+4)^2 &= 12 \\ (y+4)^2 &= 11 \\ y + 4 &= \pm\sqrt{11} \\ y &= -4 \pm \sqrt{11} \end{aligned}$$

$\boxed{\text{Thus the } y\text{-intercepts are } -4 \pm \sqrt{11}.}$

Graph:



Example 2: Find the center, radius, circumference, x and y intercepts of the circle, where $x^2 + y^2 = 1$. Then sketch the circle.

Solution

center: $\boxed{(0,0)}$

radius: $\sqrt{1} = \boxed{1}$

circumference: $2\pi \cdot 1 = \boxed{2\pi}$

x**-intercepts:** set $y = 0$:

$$\begin{aligned} x^2 + 0^2 &= 1 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

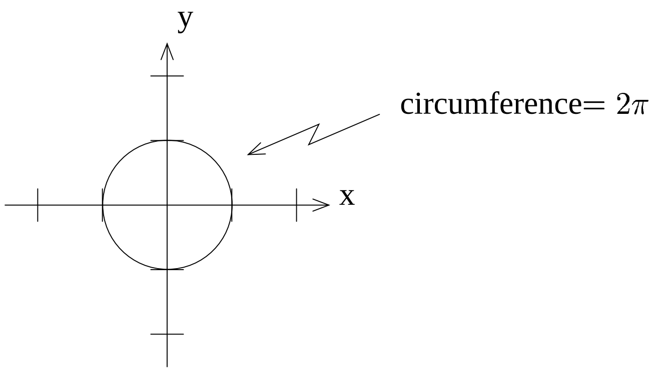
$\boxed{\text{Thus the } x\text{-intercepts are } \pm 1.}$

y**-intercepts:** set $x = 0$:

$$\begin{aligned} 0^2 + y^2 &= 1 \\ y^2 &= 1 \\ y &= \pm 1 \end{aligned}$$

$\boxed{\text{Thus the } y\text{-intercepts are } \pm 1.}$

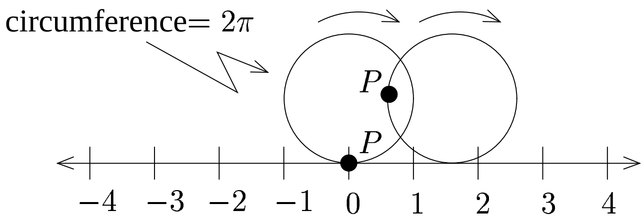
Graph:



Note: This special circle is called the **unit circle**.

C. Revolutions on the Number Line

We put a unit circle on the number line at 0, and allow the circle to "roll" along the number line:



Note: The circle will touch 2π (at the point P) on the number line when the circle rolls to the right one complete revolution.

D. Examples

Where does the circle touch the number line if the circle rolls . . .

Example 1: To the right, two complete revolutions?

Solution

Ans $\boxed{4\pi}$

Example 2: To the left, one complete revolution?

Solution

Ans $\boxed{-2\pi}$

Example 3: To the right, half a complete revolution?

Solution

Ans $\boxed{\pi}$

Example 4: To the left, one and a half complete revolutions?

Solution

Ans $\boxed{-3\pi}$

Exercises

1. Find the standard form of the equation of the specified circle:

   a. center: $(0,0)$ radius: 2

   b. center: $(-1,2)$ radius: 3

   c. center: $(3,-1)$ radius: $\sqrt{7}$

2. Find the center, radius, circumference, x and y intercepts, and sketch the circle where

   a. $x^2+y^2=9$

   b. $(x-2)^2+(y+1)^2=16$

   c. $(x+3)^2+(y-2)^2=11$

3. Why does $x^2+y^2=-1$ **not** define a circle?

4. With a unit circle sitting on the number line at 0, where does the circle touch the number line if the circle rolls . . .

   a. Right Three Complete Revolutions?

   b. Left Half a Revolution?

   c. Left One and a Half Revolutions?

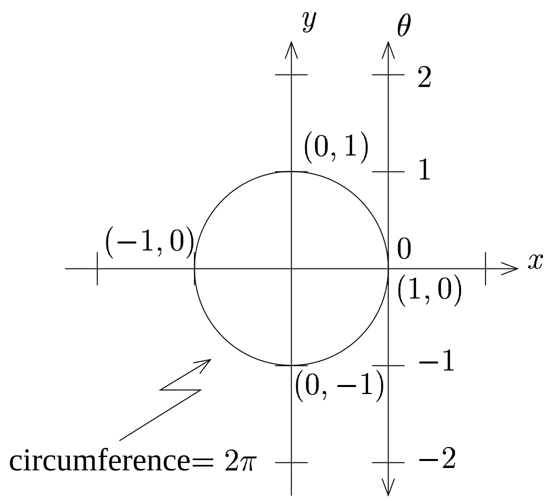
   d. Right $\frac{1}{4}$ a Revolution?

   e. Right $\frac{3}{4}$ a Revolution?

3.2 The Wrapping Function

A. Setup

Put a number line in vertical position at $(1, 0)$ on the unit circle.

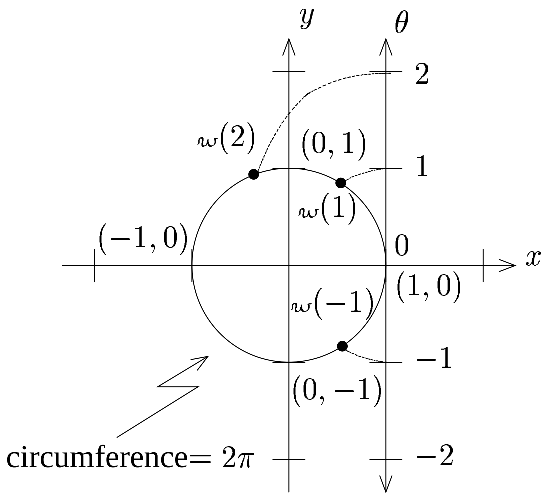


Let θ (theta) be the number line variable.

Now “wrap” the number line around the circle.

w: wrapping function

$w(\theta)$: point on the unit circle where θ on the number line wraps to



Note: $w(\theta)$ is an ordered pair in the plane and not a number.

B. Strategy

To easily and semiaccurately locate $w(\theta)$, we use the following guidelines:

1. Every π wraps halfway around the circle counterclockwise and every $-\pi$ wraps halfway around the circle clockwise.

2. To locate $w(\theta)$ when θ is a fractional multiple of π, we divide the semicircle into fractional parts. For instance, if (in lowest terms), we want to find $w(\frac{m\pi}{n})$, then we divide the semicircle into n equal sized wedges and count to the correct wedge, namely the mth one.

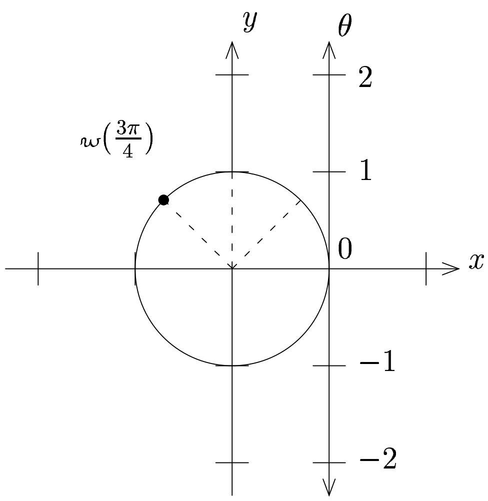
3. To locate $w(\theta)$ when θ is an integer, we use the number line as a guide and recognize that $\pi \approx 3.14$. Thus $w(3)$ is slightly above $(-1,0)$ on the unit circle.

C. Examples

Example 1: Locate and mark $w(\frac{3\pi}{4})$ on the unit circle.

Solution

We divide up the upper semicircle into four equal wedges and count over to the third wedge:

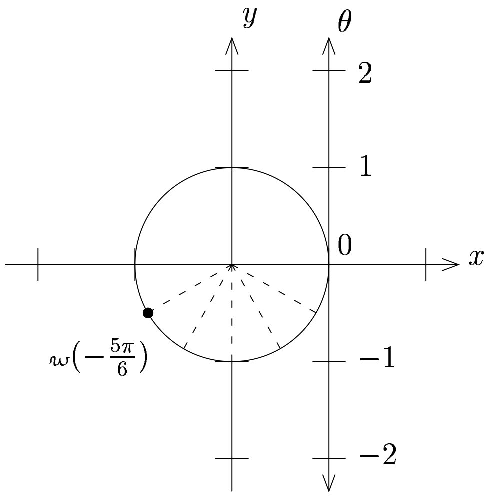


Example 2: Locate and mark $w(-\frac{5\pi}{6})$ on the unit circle.

Solution

Here we are wrapping clockwise, starting on the bottom side of the unit circle.

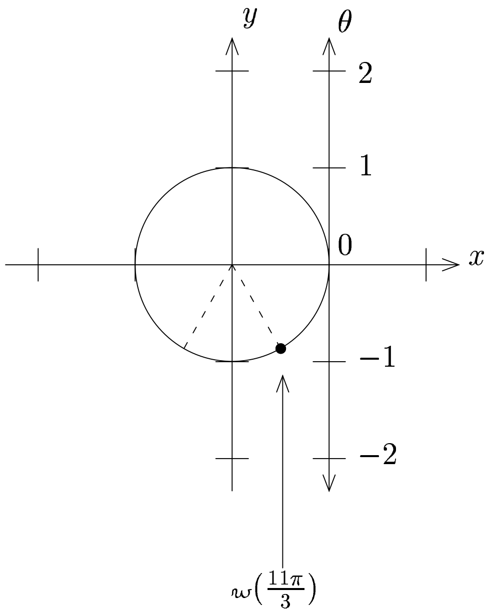
We divide up the lower semicircle into six equal wedges and count clockwise to the fifth wedge:



Example 3: Locate and mark $w(\frac{11\pi}{3})$ on the unit circle.

Solution

Here we are wrapping counterclockwise, starting on the top side of the unit circle. Note that $\frac{11\pi}{3} = 3\frac{2}{3}\,\pi$. Now 2π takes us once around, so 3π takes us halfway around to $(-1, 0)$. Then we need to go $\frac{2}{3}\pi$ more. Hence, we divide up the lower semicircle into three equal wedges and move to the second wedge.



Exercises

1. Draw a “large” unit circle and the number line at $(1,0)$ in vertical position. Locate and mark on the unit circle the following:

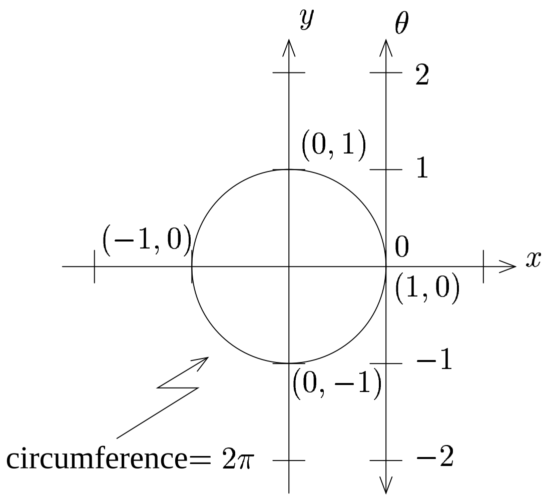
a. $\omega(2\pi)$	i. $\omega(\frac{\pi}{6})$
b. $\omega(\frac{3\pi}{4})$	j. $\omega(-\frac{2\pi}{3})$
c. $\omega(-\frac{\pi}{4})$	k. $\omega(-2\pi)$
d. $\omega(1)$	l. $\omega(3\pi)$
e. $\omega(2)$	m. $\omega(\frac{5\pi}{6})$
f. $\omega(\frac{5\pi}{4})$	n. $\omega(-\frac{\pi}{3})$
g. $\omega(-\frac{\pi}{2})$	o. $\omega(\frac{7\pi}{2})$
h. $\omega(\frac{\pi}{3})$	p. $\omega(\frac{7\pi}{6})$

2. What can you say about $\omega(\theta)$ and $\omega(\theta+2\pi)$? More generally, what can you say about $\omega(\theta+2k\pi)$, where k is an integer?

3.3 The Wrapping Function At Multiples of π and $\pi/2$

A. Setup

Unit circle and number line in vertical position at $(1, 0)$



θ: theta (number line variable)

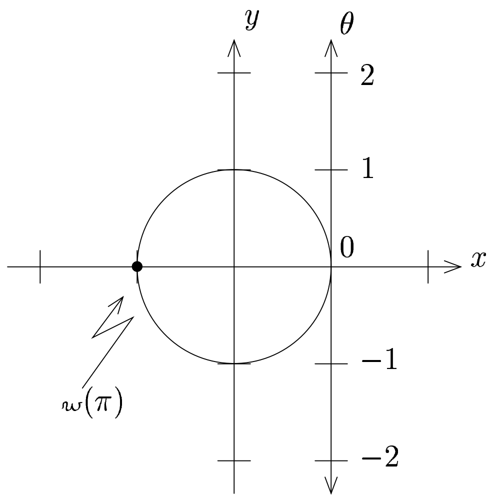
B. Evaluation

Locate $w(\theta)$ and read off the coordinates.

C. Examples

Example 1: Evaluate $w(\pi)$

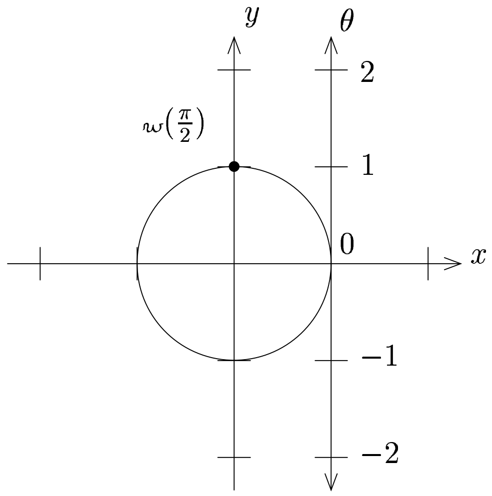
Solution



Ans $\boxed{w(\pi) = (-1, 0)}$

Example 2: Evaluate $w(\frac{\pi}{2})$

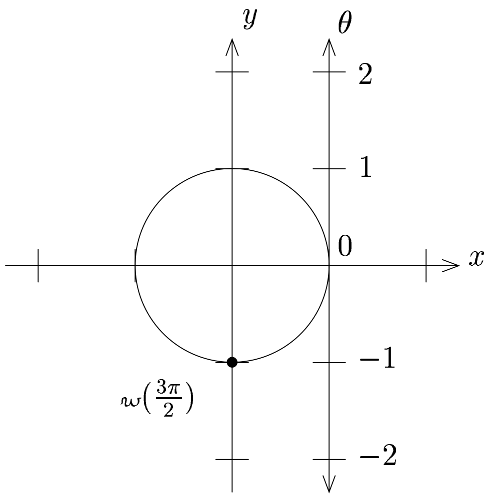
Solution



Ans $\boxed{w(\frac{\pi}{2}) = (0, 1)}$

Example 3: Evaluate $\wp(\frac{3\pi}{2})$

Solution



Ans $\boxed{\wp(\frac{3\pi}{2}) = (0, -1)}$

3.4 The Wrapping Function At Multiples of $\pi/4$

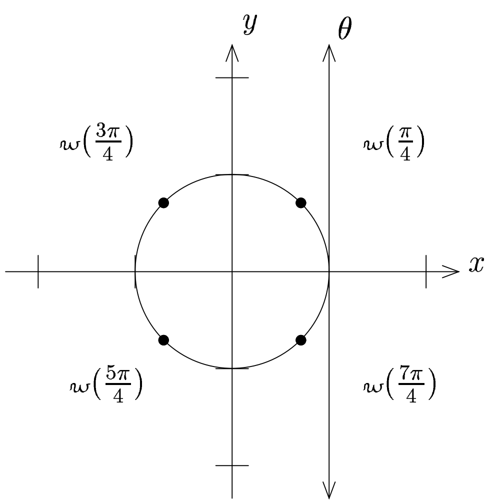
A. Introduction

Evaluating $w(\theta)$ for θ being a multiple of π or $\frac{\pi}{2}$ is direct. However, we need a rule for evaluating $w(\theta)$ when θ is a multiple of $\frac{\pi}{4}$.

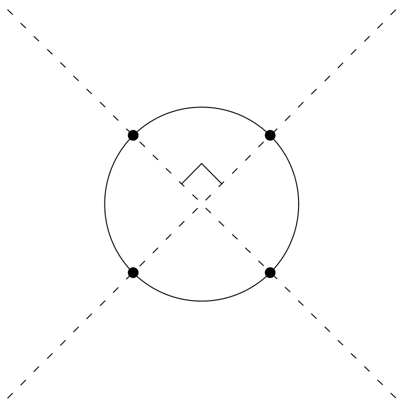
We will derive the $\frac{\pi}{4}$ rule in six easy steps.

B. Derivation of the $\frac{\pi}{4}$ Rule

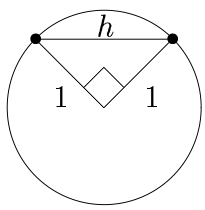
Step 1: Note that if θ is a multiple of $\frac{\pi}{4}$ (lowest terms), then $w(\theta)$ is in one of 4 spots.



Step 2: Drawing diagonal lines through the points yield perpendicular lines, since the points are vertices of a square



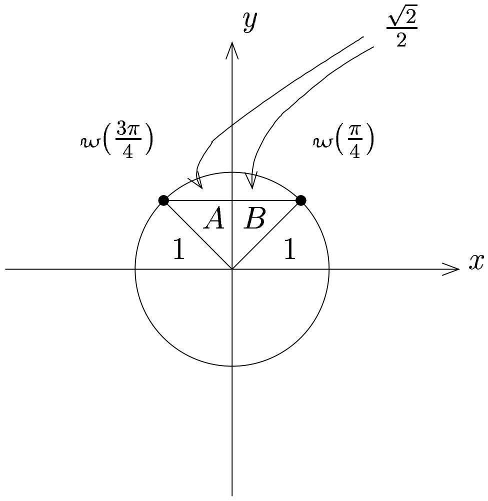
Step 3: Connecting the top two points creates a right triangle with sides having length 1



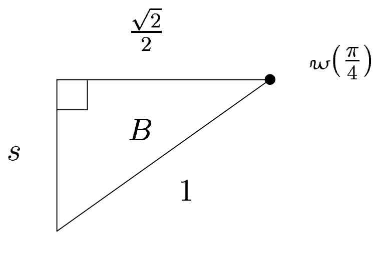
We can use the Pythagorean Theorem to find h:

$$1^2 + 1^2 = h^2 \;\Rightarrow\; h^2 = 2 \;\Rightarrow\; h = \sqrt{2} \text{ (since } h \geq 0)$$

Step 4: By symmetry, the y axis bisects the triangle into two with top edge length $\frac{\sqrt{2}}{2}$



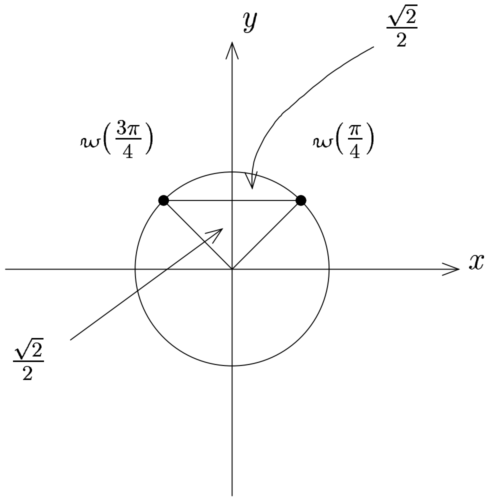
Step 5: We examine triangle B



We can use the Pythagorean Theorem again to find s:

$$s^2 + \left(\frac{\sqrt{2}}{2}\right)^2 = 1^2 \;\Rightarrow\; s^2 + \frac{2}{4} = 1 \;\Rightarrow\; s^2 = \frac{2}{4} \;\Rightarrow\; s = \frac{\sqrt{2}}{2}$$

Step 6: We now have the following picture, from which we can read off $w(\frac{\pi}{4})$



Thus $\boxed{w(\frac{\pi}{4}) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})}$. By symmetry, we get the $\frac{\pi}{4}$ rule.

C. $\frac{\pi}{4}$ Rule

By symmetry, the x and y coordinates of $w(\theta)$ for θ being a multiple of $\frac{\pi}{4}$ are $\frac{\sqrt{2}}{2}$ and $\frac{\sqrt{2}}{2}$ **with appropriate signs**.

D. Strategy

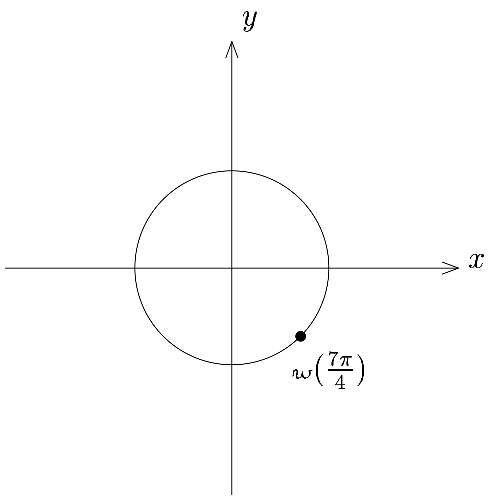
Locate the point on the unit circle, and then use the rule based on the picture.

E. Examples

Example 1: Evaluate $\mathscr{W}(\frac{7\pi}{4})$

Solution

First locate $\mathscr{W}(\frac{7\pi}{4})$ on the unit circle:



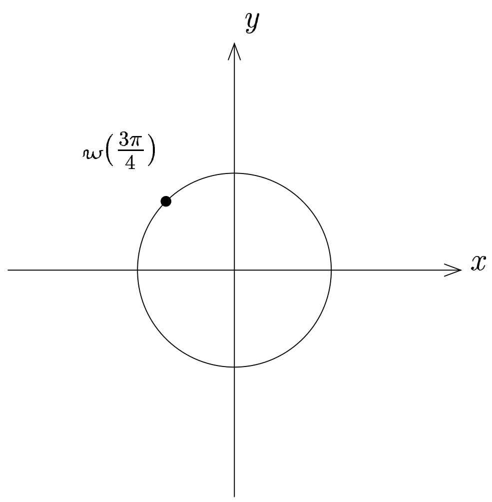
Since we see that to locate the point, we must have positive x and negative y, we have that

Ans $\boxed{\mathscr{W}(\frac{7\pi}{4}) = (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})}$

Example 2: Evaluate $w(\frac{3\pi}{4})$

Solution

First locate $w(\frac{3\pi}{4})$ on the unit circle:



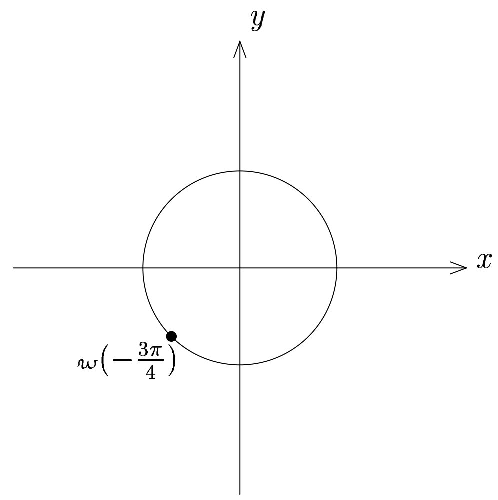
Since we see that to locate the point, we must have negative x and positive y, we have that

Ans $\boxed{w(\frac{3\pi}{4}) = (\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2})}$

Example 3: Evaluate $w(-\frac{3\pi}{4})$

Solution

First locate $w(-\frac{3\pi}{4})$ on the unit circle:



Since we see that to locate the point, we must have negative x and negative y, we have that

Ans $\boxed{w(-\frac{3\pi}{4}) = (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})}$

Exercises

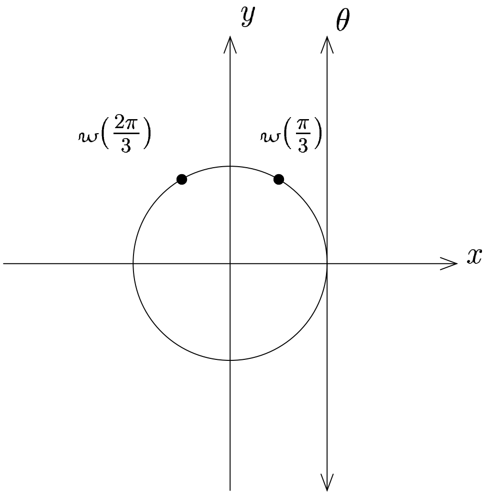
Evaluate the following exactly:

1. $w(-\frac{3\pi}{2})$
2. $w(\pi)$
3. $w(-\frac{\pi}{4})$
4. $w(\frac{5\pi}{2})$
5. $w(\frac{5\pi}{4})$
6. $w(0)$
7. $w(-\frac{7\pi}{4})$
8. $w(6\pi)$
9. $w(-\frac{3\pi}{4})$
10. $w(-3\pi)$
11. $w(\frac{11\pi}{4})$
12. $w(1023\pi)$

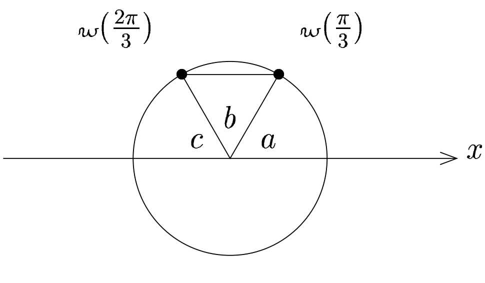
3.5 The Wrapping Function At Multiples of $\pi/3$ and $\pi/6$

A. Derivation of the $\frac{\pi}{3}$, $\frac{\pi}{6}$ Rule

Step 1: Locate $w(\frac{\pi}{3})$ and $w(\frac{2\pi}{3})$



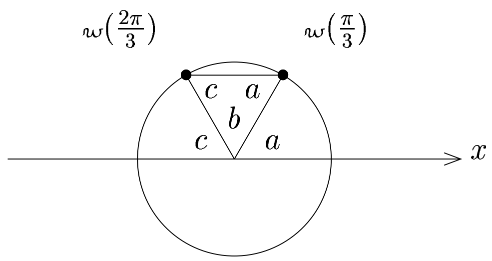
Step 2: Form a central triangle, and label the inside angles a, b, and c



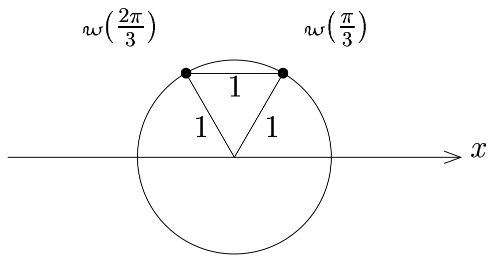
Notice that since the upper semicircle has been cut into 3 equal pieces, we have that $\angle a \cong \angle b \cong \angle c$.

Step 3: Alternate Interior Angles Of Two Parallel Lines
Cut By A Transversal Are Congruent

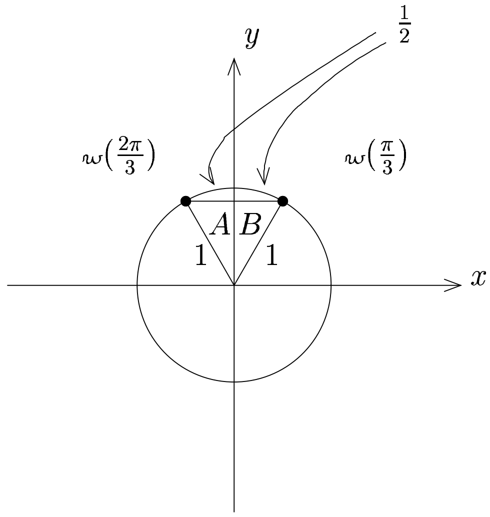
By the above geometric fact, the other internal angles of the triangle are c and a respectively, as in the diagram.



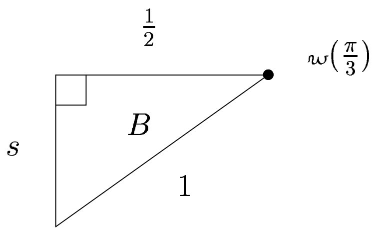
Step 4: Since $\angle a \cong \angle b \cong \angle c$, the triangle is equiangular. Thus the triangle is equilateral, and all sides have length 1.



Step 5: By symmetry, the y axis bisects the triangle into two with top edge length $\frac{1}{2}$



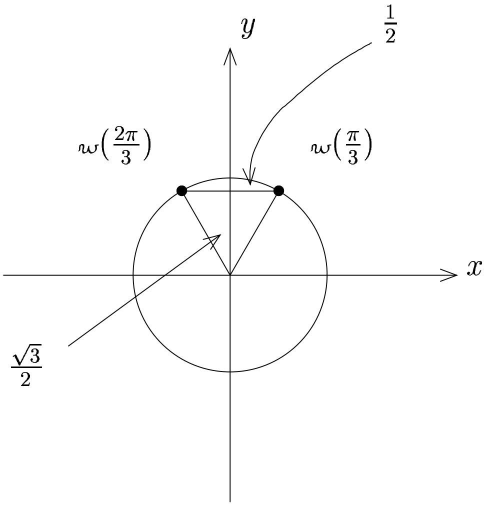
Step 6: We examine triangle B



We can use the Pythagorean Theorem to find s:

$$s^2 + \left(\tfrac{1}{2}\right)^2 = 1^2 \ \Rightarrow\ s^2 + \tfrac{1}{4} = 1 \ \Rightarrow\ s^2 = \tfrac{3}{4} \ \Rightarrow\ s = \tfrac{\sqrt{3}}{2}$$

Step 7: We now have the following picture, from which we can read off $w(\frac{\pi}{3})$



Thus $\boxed{w(\frac{\pi}{3}) = (\frac{1}{2}, \frac{\sqrt{3}}{2})}$. By symmetry, we get the $\frac{\pi}{3}$, $\frac{\pi}{6}$ rule.

B. $\frac{\pi}{3}$, $\frac{\pi}{6}$ Rule

We draw a triangle (with x-axis base) at the point, and give the longer side $\frac{\sqrt{3}}{2}$, the shorter side $\frac{1}{2}$. These give the x and y coordinates of $w(\theta)$ for θ being a multiple of $\frac{\pi}{3}$ or $\frac{\pi}{6}$ **with appropriate signs**.

C. Strategy

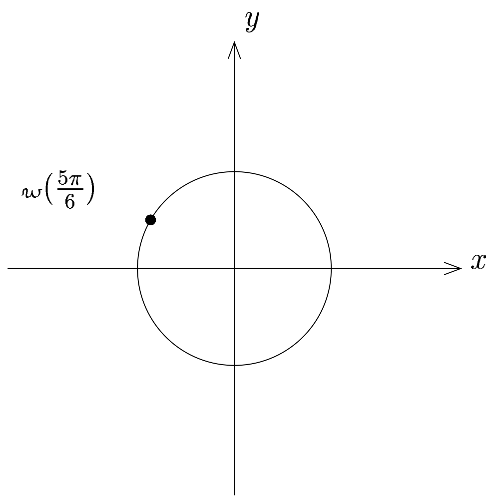
Locate the point on the unit circle, and then use the rule based on the picture.

D. Examples

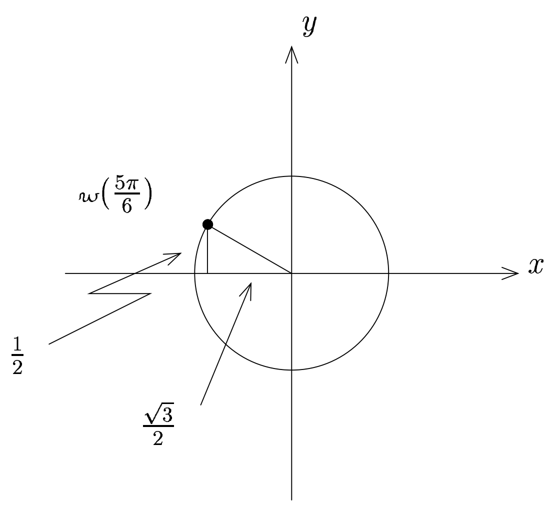
Example 1: Evaluate $\omega(\frac{5\pi}{6})$

Solution

First locate $\omega(\frac{5\pi}{6})$ on the unit circle:



Then we draw the triangle and label it:



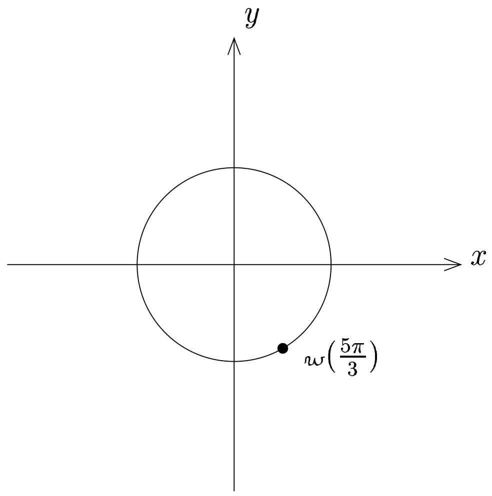
Since we see that to locate the point, we must have negative x and positive y, we have that

Ans $\boxed{\mathscr{w}\left(\frac{5\pi}{6}\right) = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)}$

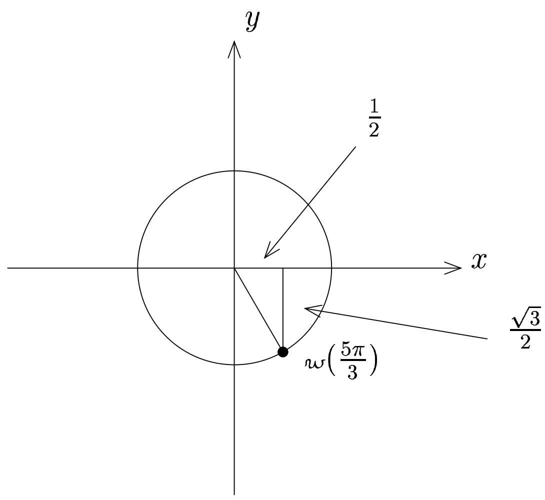
Example 2: Evaluate $\mathscr{w}\left(\frac{5\pi}{3}\right)$

Solution

First locate $\mathscr{w}\left(\frac{5\pi}{3}\right)$ on the unit circle:



Then we draw the triangle and label it:



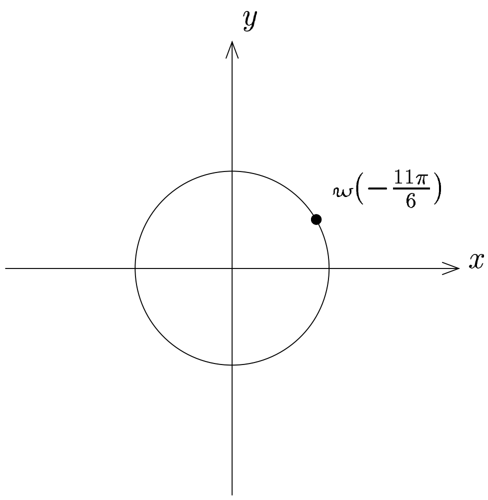
Since we see that to locate the point, we must have positive x and negative y, we have that

Ans $\boxed{\omega(\frac{5\pi}{3}) = (\frac{1}{2}, -\frac{\sqrt{3}}{2})}$

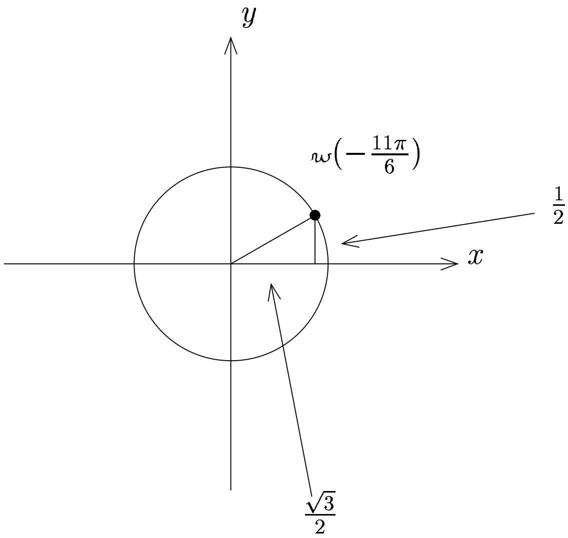
Example 3: Evaluate $\omega(-\frac{11\pi}{6})$

Solution

First locate $\omega(-\frac{11\pi}{6})$ on the unit circle:



Then we draw the triangle and label it:



Since we see that to locate the point, we must have positive x and positive y, we have that

Ans $\boxed{w\left(\frac{5\pi}{3}\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)}$

Exercises

Evaluate the following exactly:

1. $\mathscr{w}\left(\frac{4\pi}{3}\right)$
2. $\mathscr{w}\left(-\frac{\pi}{6}\right)$
3. $\mathscr{w}\left(\frac{3\pi}{4}\right)$
4. $\mathscr{w}\left(\frac{5\pi}{6}\right)$
5. $\mathscr{w}(-\pi)$
6. $\mathscr{w}\left(\frac{3\pi}{2}\right)$
7. $\mathscr{w}\left(\frac{11\pi}{6}\right)$
8. $\mathscr{w}\left(-\frac{2\pi}{3}\right)$
9. $\mathscr{w}\left(\frac{13\pi}{4}\right)$
10. $\mathscr{w}\left(\frac{7\pi}{6}\right)$
11. $\mathscr{w}\left(\frac{5\pi}{3}\right)$
12. $\mathscr{w}\left(\frac{23\pi}{6}\right)$

3.6 The Trigonometric Functions: Definitions

A. Definitions

Let $w(\theta) = (x, y)$.

Then we define

$\cos\theta = x$ "cosine of θ"

$\sin\theta = y$ "sine of θ"

$\tan\theta = \frac{y}{x}$ "tangent of θ"

$\cot\theta = \frac{x}{y}$ "cotangent of θ"

$\sec\theta = \frac{1}{x}$ "secant of θ"

$\csc\theta = \frac{1}{y}$ "cosecant of θ"

To find any of the 6 trigonometric functions, we find $w(\theta)$ and then use the definitions above.

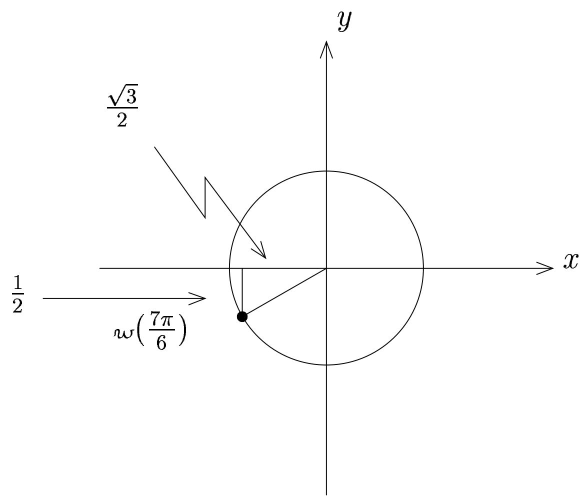
Note: For some values of θ, we may get division by zero upon evaluating a trigonometric function. In that case, the value is **undefined**. We will discuss this issue later in the next section.

B. Examples

Example 1: Find $\cos(\frac{7\pi}{6})$

Solution

We first find $w(\frac{7\pi}{6})$:



Thus $w(\frac{7\pi}{6}) = (-\frac{\sqrt{3}}{2}, -\frac{1}{2})$.

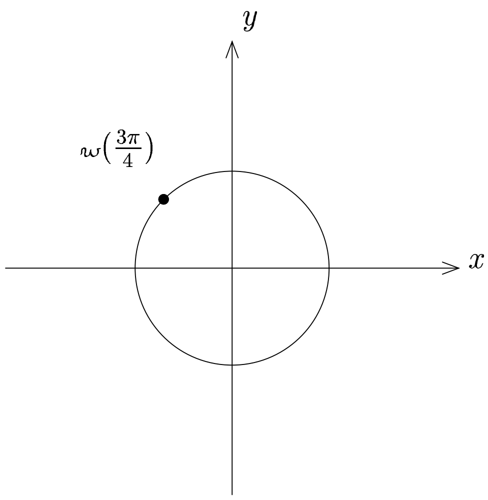
Now $\cos\theta = x$, so

Ans $\boxed{\cos(\frac{7\pi}{6}) = -\frac{\sqrt{3}}{2}}$

Example 2: Find $\tan(\frac{3\pi}{4})$

Solution

We first find $w(\frac{3\pi}{4})$:



Thus $w(\frac{3\pi}{4}) = (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.

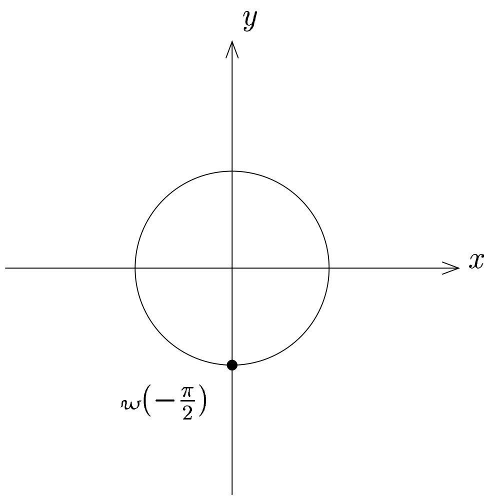
Now $\tan\theta = \frac{y}{x}$, so $\tan(\frac{3\pi}{4}) = \frac{\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = -1$

Ans $\boxed{\tan(\frac{3\pi}{4}) = -1}$

Example 3: Find $\cot(-\frac{\pi}{2})$

Solution

We first find $w(-\frac{\pi}{2})$:



Thus $w(-\frac{\pi}{2}) = (0, -1)$.

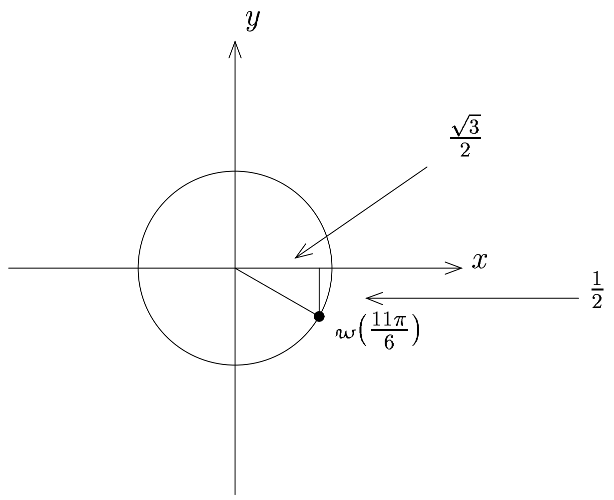
Now $\cot\theta = \frac{x}{y}$, so $\cot(-\frac{\pi}{2}) = \frac{0}{-1} = 0$

Ans $\boxed{\cot(-\frac{\pi}{2}) = 0}$

Example 4: Find $\mathscr{sec}(\frac{11\pi}{6})$

Solution

We first find $w(\frac{11\pi}{6})$:



Thus $w(\frac{11\pi}{6}) = (\frac{\sqrt{3}}{2}, -\frac{1}{2})$.

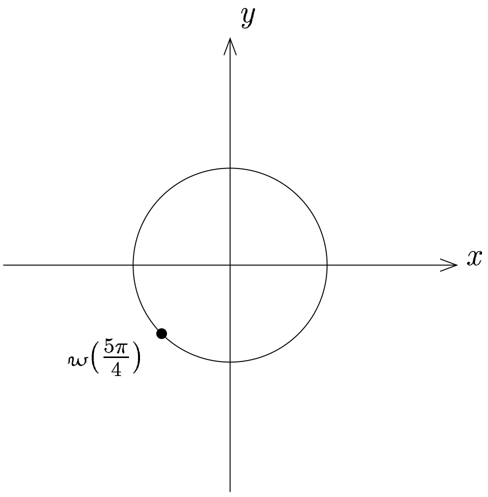
Now $\sec\theta = \frac{1}{x}$, so $\sec(\frac{11\pi}{6}) = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

Ans $\boxed{\sec(\frac{11\pi}{6}) = \frac{2\sqrt{3}}{3}}$

Example 5: Find $\csc(\frac{5\pi}{4})$

Solution

We first find $w(\frac{5\pi}{4})$:



Thus $w(\frac{5\pi}{4}) = (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$.

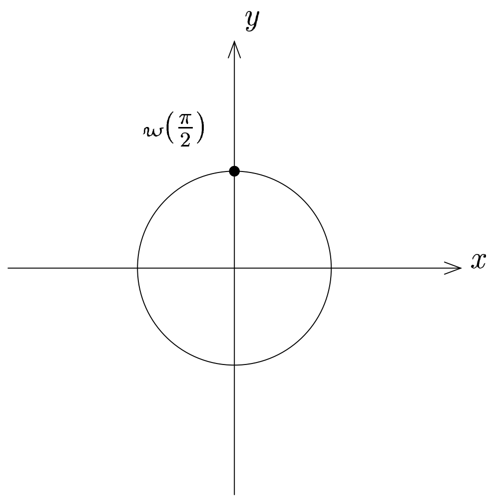
Now $\csc\theta = \frac{1}{y}$, so $\csc(\frac{5\pi}{4}) = \frac{1}{-\frac{\sqrt{2}}{2}} = -\frac{2}{\sqrt{2}} = -\frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = -\frac{2\sqrt{2}}{2} = -\sqrt{2}$

Ans $\boxed{\csc(\frac{5\pi}{4}) = -\sqrt{2}}$

Example 6: Find $\tan(\frac{\pi}{2})$

Solution

We first find $w(\frac{\pi}{2})$:



Thus $w(\frac{\pi}{2}) = (0, 1)$.

Now $\tan\theta = \frac{y}{x}$, so $\tan(\frac{\pi}{2}) = \frac{1}{0}$.

Ans $\boxed{\tan(\frac{\pi}{2}) \text{ is } \textbf{undefined}.}$

Exercises

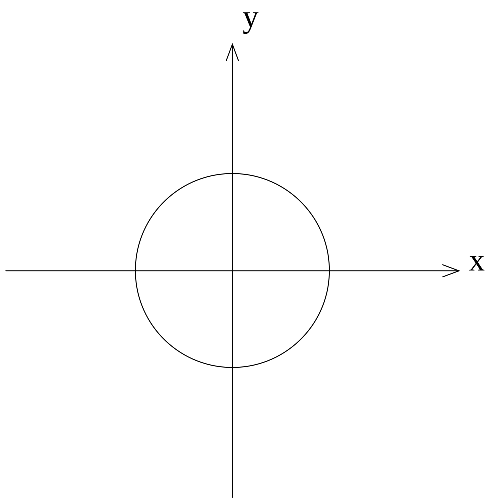
1. Evaluate the following exactly:

a. $\sin(\frac{7\pi}{4})$
b. $\tan(\frac{4\pi}{3})$
c. $\sec(\frac{\pi}{6})$
d. $\cos(-\frac{5\pi}{6})$
e. $\cot(\frac{3\pi}{4})$
f. $\csc(-\frac{5\pi}{3})$
g. $\sin(\frac{11\pi}{6})$
h. $\cot(\frac{7\pi}{6})$
i. $\sin(-\frac{3\pi}{4})$
j. $\tan(-\pi)$
k. $\sec(\frac{5\pi}{3})$
l. $\cot(-\frac{\pi}{3})$
m. $\csc(3\pi)$
n. $\cos(\frac{3\pi}{2})$
o. $\tan(-\frac{\pi}{2})$
p. $\sec(\frac{11\pi}{6})$

2. Suppose $\sin\theta = \frac{1}{5}$ and $\cos\theta = -\frac{2\sqrt{6}}{5}$. What is $w(\theta)$?

3.7 Domain and Range of the Trigonometric Functions

A. Sine and Cosine



1. **Domain:**

   Since $w(\theta)$ is defined for any θ with $\cos\theta = x$ and $\sin\theta = y$, there are no domain restrictions.

   Thus $dom(\sin) = (-\infty, \infty)$ and $dom(\cos) = (-\infty, \infty)$.

2. **Range:**

   The x-coordinate on the circle is smallest at $(-1, 0)$, namely -1; the x-coordinate on the circle is largest at $(1, 0)$, namely 1.

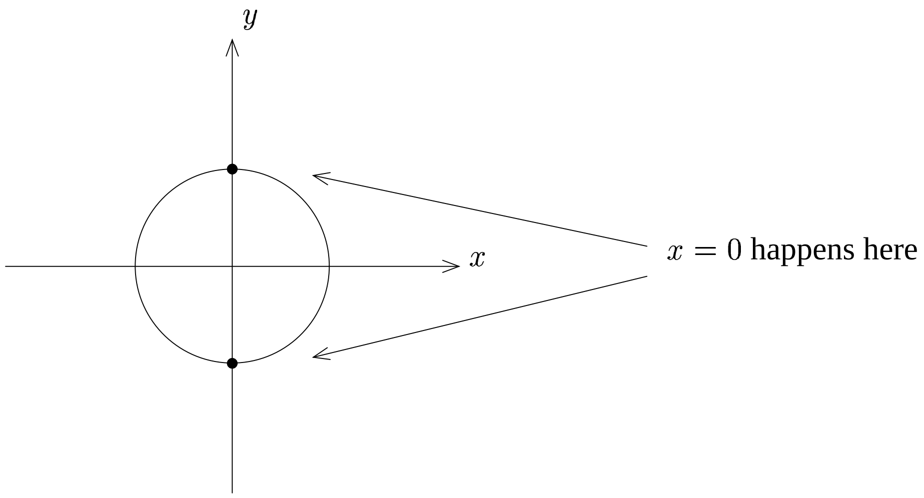
   Hence we can see that $rng(\cos) = [-1, 1]$.

   By similar reasoning, we can see that $rng(\sin) = [-1, 1]$.

B. Tangent

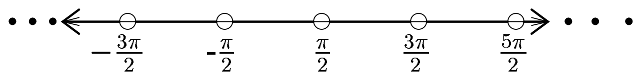
1. **Domain:**

Given $w(\theta) = (x, y)$, we have $\tan\theta = \frac{y}{x}$. Now $\frac{y}{x}$ is undefined when $x = 0$. When does this happen?



Thus $\tan\theta$ is undefined for $\theta = \ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

What is this in interval notation? To see it, let's plot the allowed values on a number line:



Thus $dom(tan)$: $\ldots \cup \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right) \cup \ldots$

Note: Each interval has an endpoint being an "odd multiple of $\frac{\pi}{2}$".

Since $2k + 1$ is the formula that generates odd numbers (for k an integer), we recognize that

$dom(tan)$: union of all intervals of the form $\left(\frac{(2k+1)\pi}{2}, \frac{(2k+3)\pi}{2}\right)$, where $k \in \mathbb{Z}$
[k is an integer]

Thus $\mathrm{dom}(\mathrm{tan}) = \bigcup_{k\in\mathbb{Z}} \left(\frac{(2k+1)\pi}{2}, \frac{(2k+3)\pi}{2}\right)$.

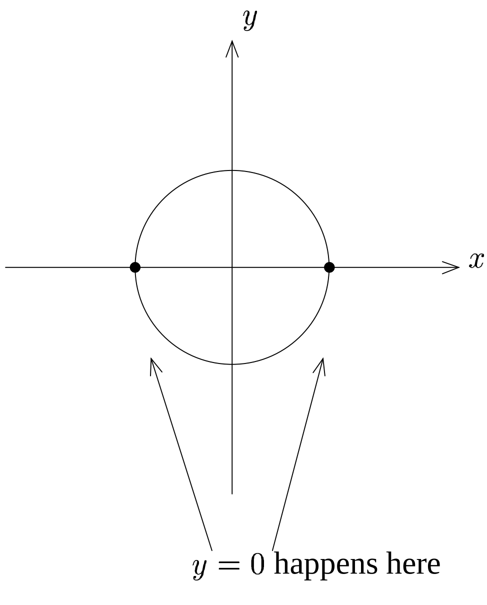
2. **Range:**

Since $\frac{y}{x}$ can be any number, $\mathrm{rng}(\mathrm{tan}) = (-\infty, \infty)$.

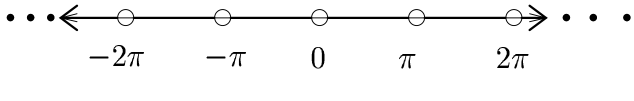
C. Cotangent

1. **Domain:**

This is similar to tangent. Given $w(\theta) = (x, y)$, we have $\cot\theta = \frac{x}{y}$. Now $\frac{x}{y}$ is undefined when $y = 0$. When does this happen?



Thus $\cot\theta$ is undefined for $\theta = \ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$



Hence $dom(cot) = \bigcup_{k\in\mathbb{Z}} (k\pi, (k+1)\pi)$.

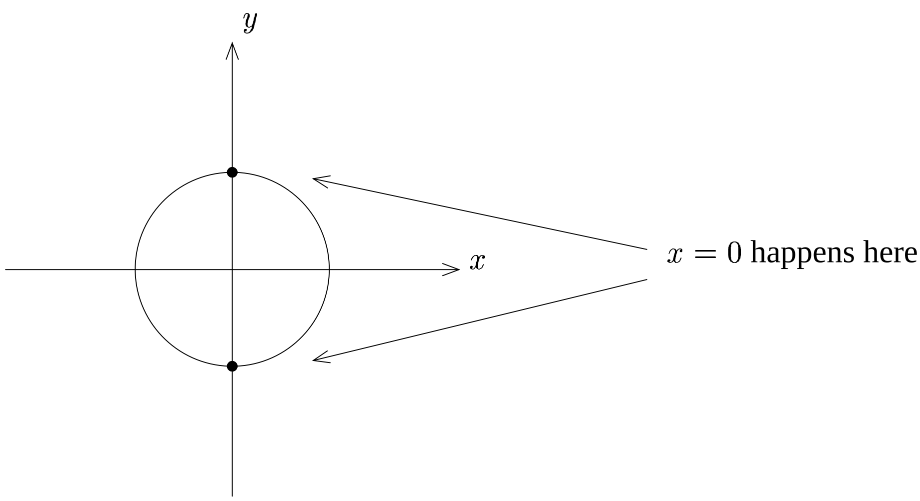
2. **Range:**

Now $\frac{x}{y}$ can be anything, so $rng(cot) = (-\infty, \infty)$.

D. Secant

1. **Domain:**

Given $w(\theta) = (x, y)$, we have $sec\,\theta = \frac{1}{x}$. Now $\frac{1}{x}$ is undefined when $x = 0$. When does this happen?



So similar to tangent, $dom(sec) = \bigcup_{k\in\mathbb{Z}} \left(\frac{(2k+1)\pi}{2}, \frac{(2k+3)\pi}{2}\right)$.

2. **Range:**

On the right semicircle, x ranges from 1 down to 0, so $\frac{1}{x}$ ranges from 1 up to ∞.

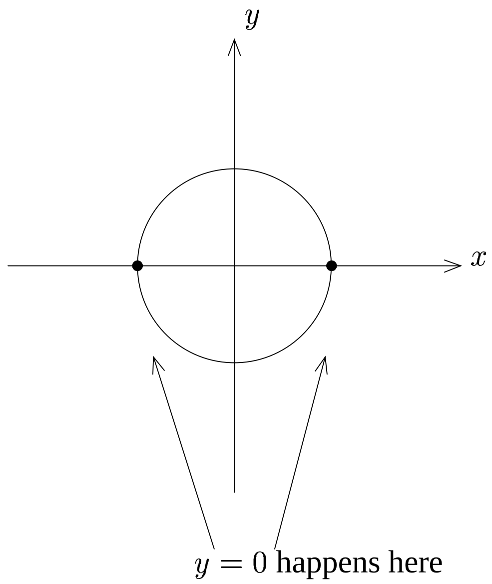
On the left semicircle, x ranges from near 0 to -1, so $\frac{1}{x}$ ranges from $-\infty$ up to -1.

Hence $\mathit{rng}(\mathit{sec}) = (-\infty, -1] \cup [1, \infty)$.

E. Cosecant

1. **Domain:**

Given $\mathit{w}(\theta) = (x, y)$, we have $\mathit{csc}\ \theta = \frac{1}{y}$. Now $\frac{1}{y}$ is undefined when $y = 0$. When does this happen?



Thus, similar to cotangent, $\mathit{dom}(\mathit{cot}) = \bigcup_{k \in \mathbb{Z}} (k\pi, (k+1)\pi)$.

2. **Range:**

By the same reasoning as for secant, we get $\mathit{rng}(\mathit{csc}) = (-\infty, -1] \cup [1, \infty)$.

F. Summary

	Domain	Range
$\sin$	$(-\infty, \infty)$	$[-1, 1]$
$\cos$	$(-\infty, \infty)$	$[-1, 1]$
$\tan$	$\bigcup_{k\in\mathbb{Z}} \left(\frac{(2k+1)\pi}{2}, \frac{(2k+3)\pi}{2}\right)$	$(-\infty, \infty)$
$\cot$	$\bigcup_{k\in\mathbb{Z}} (k\pi, (k+1)\pi)$	$(-\infty, \infty)$
$\sec$	$\bigcup_{k\in\mathbb{Z}} \left(\frac{(2k+1)\pi}{2}, \frac{(2k+3)\pi}{2}\right)$	$(-\infty, -1] \cup [1, \infty)$
$\csc$	$\bigcup_{k\in\mathbb{Z}} (k\pi, (k+1)\pi)$	$(-\infty, -1] \cup [1, \infty)$

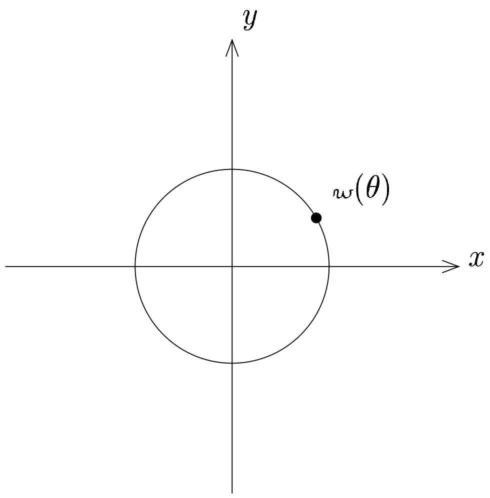
Note: To help remember the table, we remember that

1. $\tan$ and $\sec$ are undefined at odd multiples of $\frac{\pi}{2}$.

2. $\cot$ and $\csc$ are undefined at multiples of π.

3.8 Trigonometric Functions: Periodicity

A. Introduction



Since w spits back the same point every time we add 2π, we say that w is **periodic**.

B. Periodicity

Formally, a function f is said to be **periodic** if $f(x + p) = f(x)$ for some p. The smallest such value of p that makes the function periodic is called the **period**.

C. Periodicity of the Wrapping Function

By the above discussion, $w(\theta + 2\pi) = w(\theta)$, so the wrapping function is periodic. From Section 3.1C, we see that 2π is the smallest such value, so w has period 2π.

D. Periodicity of the Trigonometric Functions

Since the trigonometric functions are defined in terms of $\mathcal{w}$, they are also periodic, and repeat every 2π.

Note: If $\mathcal{w}(\theta) = (x, y)$, then $\mathcal{w}(\theta + \pi) = (-x, -y)$ so, in particular, tangent and cotangent actually repeat every π.

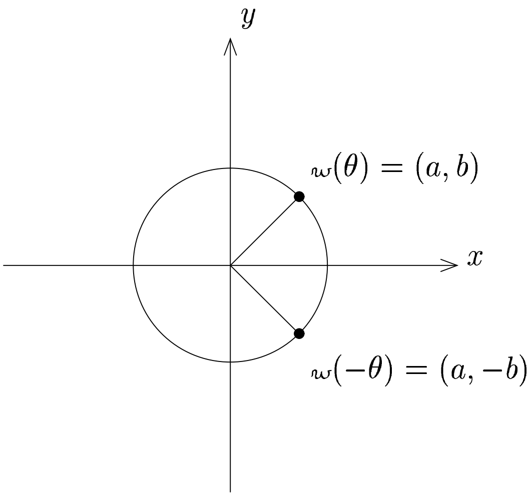
E. Summary of Periodicity

	Period
$\mathcal{sin}$	2π
$\mathcal{cos}$	2π
$\mathcal{tan}$	π
$\mathcal{cot}$	π
$\mathcal{sec}$	2π
$\mathcal{csc}$	2π

3.9 Trigonometric Functions: Even/Odd Behavior

A. Discussion

Consider $w(\theta)$ and $w(-\theta)$:



$\cos\theta = a$ $\qquad$ $\cos(-\theta) = a$

$\sin\theta = b$ $\qquad$ $\sin(-\theta) = -b$

$\tan\theta = \frac{b}{a}$ $\qquad$ $\tan(-\theta) = \frac{-b}{a}$

$\cot\theta = \frac{a}{b}$ $\qquad$ $\cot(-\theta) = \frac{a}{-b}$

$\sec\theta = \frac{1}{a}$ $\qquad$ $\sec(-\theta) = \frac{1}{a}$

$\csc\theta = \frac{1}{b}$ $\qquad$ $\csc(-\theta) = \frac{1}{-b}$

From the above facts, we can see the symmetry of the functions.

B. Symmetry

1. **Even Functions:**

$$\cos(-\theta) = \cos\theta \quad \text{even}$$

$$\sec(-\theta) = \sec\theta \quad \text{even}$$

2. **Odd Functions:**

$$\sin(-\theta) = -\sin\theta \quad \text{odd}$$

$$\tan(-\theta) = -\tan\theta \quad \text{odd}$$

$$\cot(-\theta) = -\cot\theta \quad \text{odd}$$

$$\csc(-\theta) = -\csc\theta \quad \text{odd}$$

C. Examples

Example 1: Suppose $\sin\theta = \frac{2}{5}$. Use even/odd relationships to simplify $\sin(-\theta)$.

Solution

$$\sin(-\theta) = -\sin(\theta) = -\left(\frac{2}{5}\right)$$

Ans $\boxed{-\frac{2}{5}}$

Example 2: Suppose $\cos\theta = \frac{3}{11}$. Use even/odd relationships to simplify $\cos(-\theta)$.

Solution

$$\cos(-\theta) = \cos(\theta) = \frac{3}{11}$$

Ans $\boxed{\frac{3}{11}}$

Exercises

1. Suppose $\cos\theta = -\frac{1}{4}$. Use even/odd relationships to simplify $\cos(-\theta)$.

2. Suppose $\sin\theta = \frac{2}{3}$. Use even/odd relationships to simplify $\sin(-\theta)$.

3. Suppose $f(\theta) = \cos\theta + \sin\theta$. Find $f_{\text{even}}(\theta)$ and $f_{\text{odd}}(\theta)$.

3.10 Elementary Trigonometric Relationships

A. Discussion

Given $w(\theta) = (x, y)$, we have

$$\cos\theta = x \qquad \sec\theta = \frac{1}{x}$$

$$\sin\theta = y \qquad \csc\theta = \frac{1}{y}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

We see that

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

This gives rise to some fundamental identities.

B. Reciprocal Identities

1. $\boxed{\sec\theta = \frac{1}{\cos\theta} \quad \text{and} \quad \cos\theta = \frac{1}{\sec\theta}}$

2. $\boxed{\csc\theta = \frac{1}{\sin\theta} \quad \text{and} \quad \sin\theta = \frac{1}{\csc\theta}}$

3. $\boxed{\cot\theta = \frac{1}{\tan\theta} \quad \text{and} \quad \tan\theta = \frac{1}{\cot\theta}}$

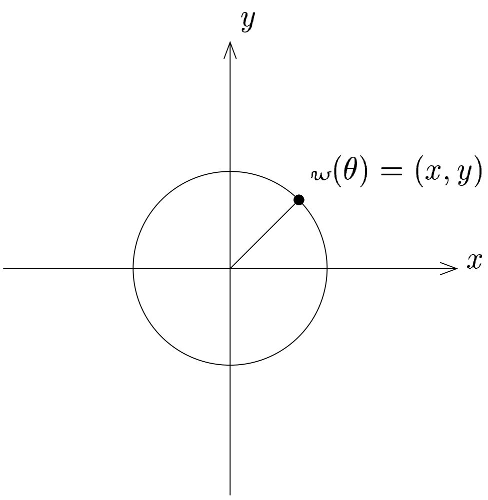
C. Quotient Identities

Using the definitions again, we get

1. $\boxed{\tan\theta = \frac{\sin\theta}{\cos\theta}}$

2. $\boxed{\cot\theta = \frac{\cos\theta}{\sin\theta}}$

D. The Pythagorean Identity



Note: $x^2 + y^2 = 1$ (because we have a unit circle)

Since we have that $\cos\theta = x$ and $\sin\theta = y$, the equation becomes

$$(\cos\theta)^2 + (\sin\theta)^2 = 1$$

Shorthand: $(\cos\theta)^2 = \cos^2\theta$

Warning: $\cos\theta^2$ **does not** mean $(\cos\theta)^2$; $\cos\theta^2$ means $\cos(\theta^2)$

E. Pythagorean Identities

Thus we have **Pythagorean I:** $\boxed{\cos^2\theta + \sin^2\theta = 1}$

Now, if we divide both sides of Pythagorean I by $\cos^2\theta$, we get:

$$1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

Thus we have **Pythagorean II:** $\boxed{1 + \tan^2\theta = \sec^2\theta}$

Dividing both sides of Pythagorean I by $\sin^2\theta$, we get:

$$\frac{\cos^2\theta}{\sin^2\theta} + 1 = \frac{1}{\sin^2\theta}$$

Thus we have **Pythagorean III:** $\boxed{\cot^2\theta + 1 = \csc^2\theta}$

G. Examples

Example 1: If $\sin\theta = \frac{2}{5}$, what are the possible values of $\cos\theta$?

Solution

We use Pythagorean I: $\cos^2\theta + \sin^2\theta = 1$

Thus,

$$\begin{aligned} \cos^2\theta + \left(\frac{2}{5}\right)^2 &= 1 \\ \cos^2\theta + \frac{4}{25} &= 1 \\ \cos^2\theta &= \frac{21}{25} \\ \cos\theta &= \pm\frac{\sqrt{21}}{5} \end{aligned}$$

Ans $\boxed{\pm\frac{\sqrt{21}}{5}}$

Example 2: If $\sec\theta = -3$, what are the possible values of $\tan\theta$?

Solution

We use Pythagorean II: $1 + \tan^2\theta = \sec^2\theta$

Thus,

$$\begin{aligned} 1 + \tan^2\theta &= (-3)^2 \\ 1 + \tan^2\theta &= 9 \\ \tan^2\theta &= 8 \\ \tan\theta &= \pm\sqrt{8} = \pm 2\sqrt{2} \end{aligned}$$

Ans $\boxed{\pm 2\sqrt{2}}$

Exercises

1. Know $\cos\theta = \frac{1}{3}$. Find $\sec\theta$.

2. Know $\sin\theta = -\frac{1}{5}$. Find $\csc(-\theta)$.

3. Know $\cos\theta = \frac{2}{7}$. Find $\sec(-\theta)$.

4. If $\sin\theta = \frac{1}{4}$, what are the possible values of $\cos\theta$?

5. If $\cos\theta = \frac{2}{3}$, what are the possible values of $\sin\theta$?

6. If $\tan\theta = 3$, what are the possible values of $\sec\theta$?

7. If $\csc\theta = -10$, what are the possible values of $\cot\theta$?

8. If $\sin\theta = 1$, what are the possible values of $\cos\theta$?

Chapter 4

Graphing Trigonometric Functions

4.1 Graphs of Sine and Cosine

A. Graph of $y = \sin x$

Since we are now familiar with the $\sin$ function, we may write $y = \sin x$ and not confuse the x and y in the equation with the x and y coordinates in the output to $w(\theta)$.

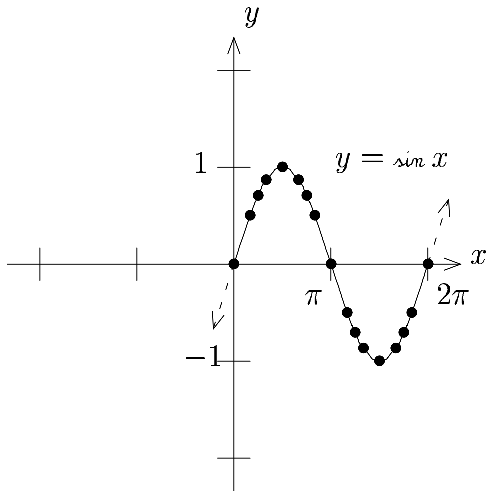
Let us graph $y = \sin x$ by making a table of values. Since we know that $\sin$ is 2π-periodic, we only need to make a table from 0 to 2π.

x	y
0	0
$\frac{\pi}{6}$	$\frac{1}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$

x	y
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$
$\frac{5\pi}{6}$	$\frac{1}{2}$
π	0
$\frac{7\pi}{6}$	$-\frac{1}{2}$
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$
$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$

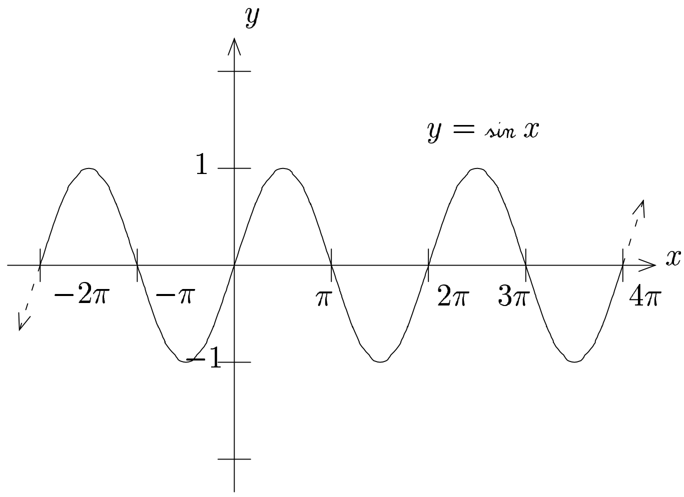
x	y
$\frac{3\pi}{2}$	-1
$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$
$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$
$\frac{11\pi}{6}$	$-\frac{1}{2}$
2π	0

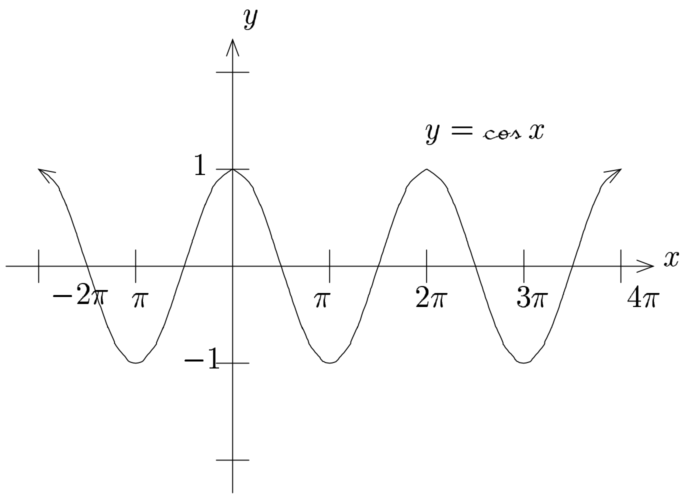
Thus the graph of $y = \sin x$ is:



We may obtain the graph of $y = \cos x$ similarly (Exercise).

B. Sine and Cosine Graphs





C. Sinusoidal Graphs

Oscillatory graphs like $y = \sin x$ and $y = \cos x$ are called **sinusoidal graphs**. They usually take the form:

$$y = d + a \sin(bx - c)$$

or

$$y = d + a \cos(bx - c)$$

D. Features

1. **Amplitude:** $|a|$

a. graph bounces between $-a$ and a instead of -1 and 1 (stretching)

b. represents half the distance from the maximum and minimum values of the function

2. **Period:** $\frac{2\pi}{|b|}$

a. horizontal distance for function to complete one cycle

b. accounts for **horizontal stretching**

3. **Phase Shift:** $\frac{c}{b}$

represents the amount of horizontal translation (to the right)

E. Examples

Example 1: Let $f(x) = 4\cos 3x$. Find the period and amplitude.

Solution

Period: $\frac{2\pi}{|b|} = \boxed{\frac{2\pi}{3}}$

Amplitude: $|a| = \boxed{4}$

Example 2: Let $f(x) = 6 - 3\sin\left(2x - \frac{\pi}{3}\right)$. Find the amplitude, period and phase shift.

Solution

Amplitude: $|a| = |-3| = \boxed{3}$

Period: $\frac{2\pi}{|b|} = \frac{2\pi}{|2|} = \frac{2\pi}{2} = \boxed{\pi}$

Phase Shift: $\frac{c}{b} = \frac{\frac{\pi}{3}}{2} = \boxed{\frac{\pi}{6}}$

In the next section, we will look at how to graph sinusoids using a “modified” HSRV strategy.

Exercises

1. Sketch the graph of $y = \cos x$ by plotting points between $x = 0$ and $x = 2\pi$.

2. Find the period and amplitude for the following:

   a. $f(x) = 3 \sin 4x$

   b. $f(x) = \frac{2}{5} \cos \pi x$

   c. $f(x) = -\frac{3}{4} \sin \frac{\pi x}{2}$

3. Find the amplitude, period, and phase shift for the following:

   a. $f(x) = 2 \cos(3x - \pi)$

   b. $f(x) = -3 \sin(\pi x + 1)$

   c. $f(x) = \frac{3}{7} \cos(\frac{\pi}{3}x - \frac{\pi}{4})$

   d. $f(x) = 2 - 5 \sin(\frac{x}{3} - 7)$

   e. $f(x) = \frac{1}{4} \sin(\frac{3\pi x}{5} + \frac{\pi}{3}) - 1$

4.2 Graphing Sinusoids

A. Strategy

Given $y = d + a \sin(bx - c)$ or $y = d + a \cos(bx - c)$

1. Set $bx - c = 0$. This gives the **start** of one cycle.

2. Set $bx - c = 2\pi$. This gives the **end** of one cycle.

3. Draw one cycle with amplitude $|a|$.

4. If a is negative, flip across the x-axis.

5. To get the final graph, perform the vertical shift using the parameter d.

B. Examples

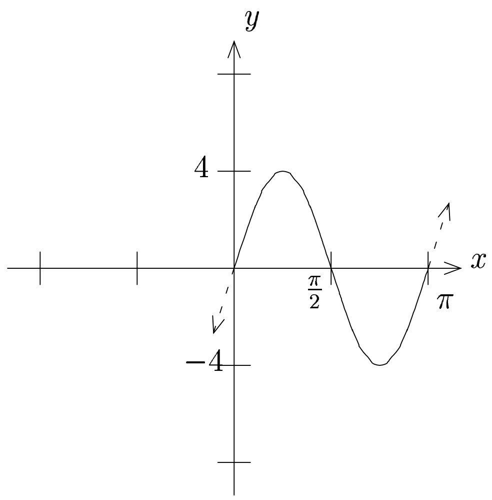
Example 1: Graph f, where $f(x) = 4 \sin 2x$.

Solution

1. $2x = 0 \;\Rightarrow\; x = 0$

2. $2x = 2\pi \;\Rightarrow\; x = \pi$

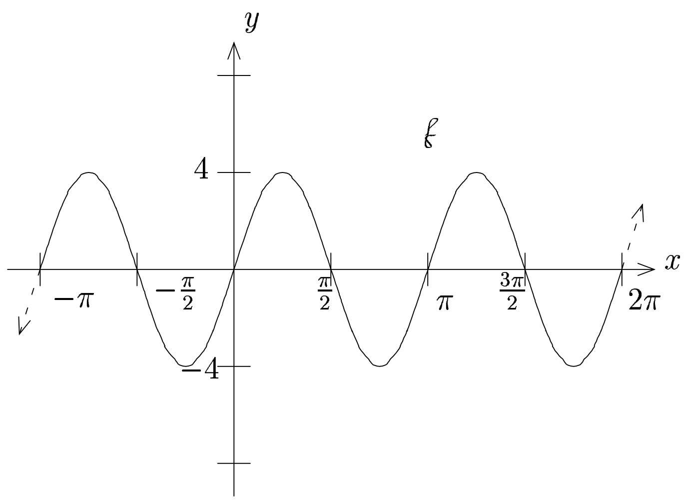
3.



4. There is no reflection.

5. There is no vertical shift.

Ans



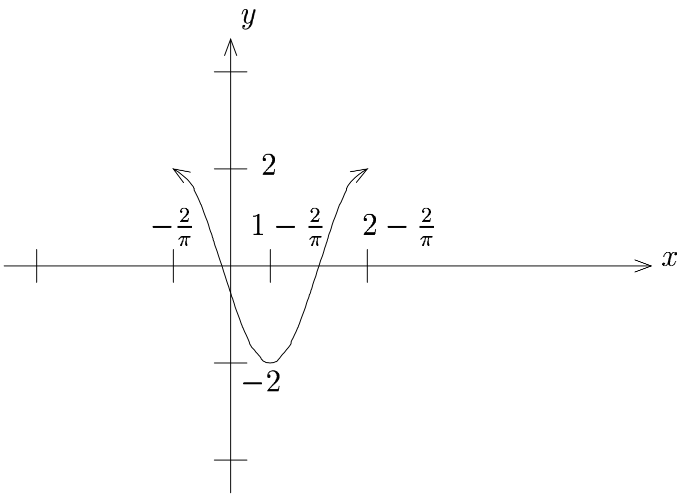
Example 2: Graph f, where $f(x) = 3 + 2\cos(\pi x + 2)$

Solution

1. $\pi x + 2 = 0 \;\Rightarrow\; x = -\frac{2}{\pi}$

2. $\pi x + 2 = 2\pi \;\Rightarrow\; \pi x = 2\pi - 2 \;\Rightarrow\; x = 2 - \frac{2}{\pi}$

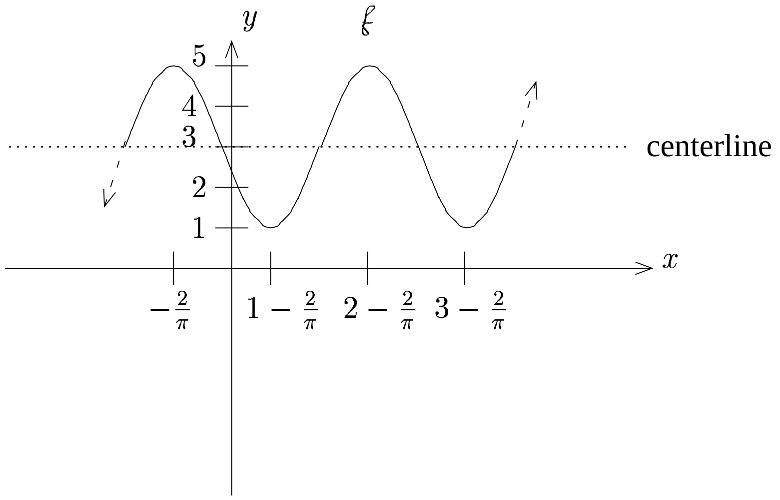
3. Note below that the y-intercept before the vertical shift, being $2\cos(2)$, is negative.



4. There is no reflection

5. Shift up 3 to get final answer

Ans



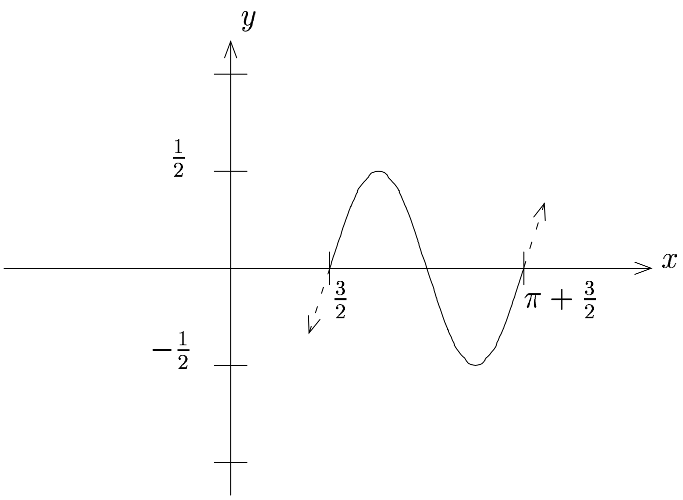
Example 3: Graph f, where $f(x) = 2 - \frac{1}{2}\sin(2x-3)$

Solution

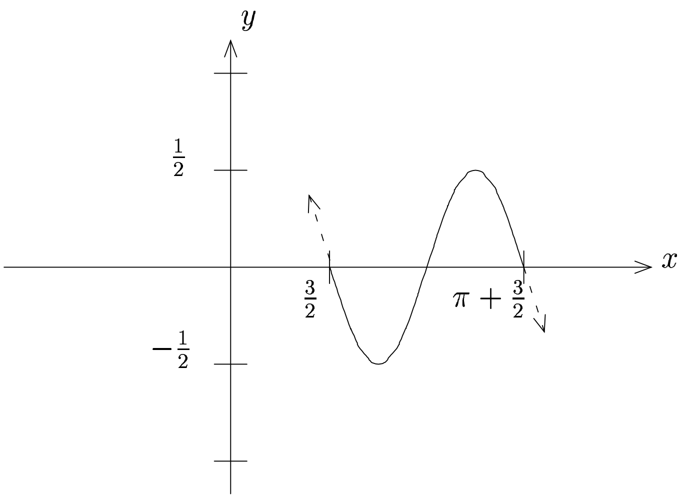
1. $2x - 3 = 0 \ \Rightarrow \ x = \frac{3}{2}$

2. $2x - 3 = 2\pi \ \Rightarrow \ 2x = 2\pi + 3 \ \Rightarrow \ x = \frac{2\pi+3}{2} = \pi + \frac{3}{2}$

3.

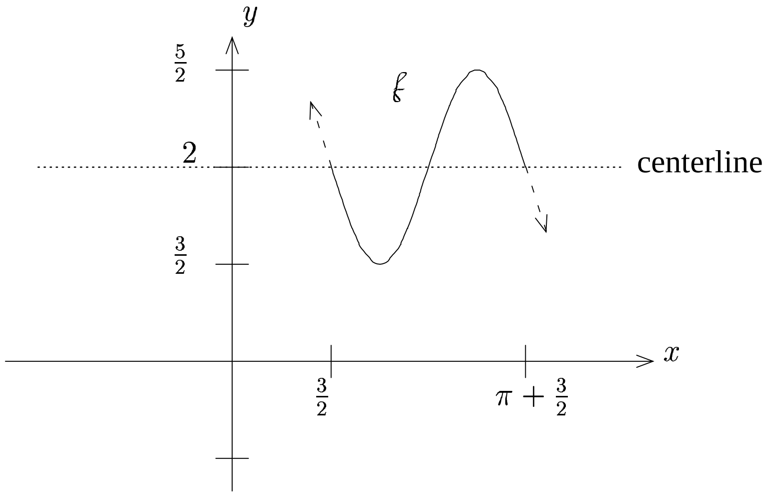


4. Reflect across the x-axis:



5. Shift up 2:

Ans



Exercises

1. Graph f, where

   a. $f(x) = 3\cos 4x$

   b. $f(x) = 2\sin(x - \frac{\pi}{4})$

   c. $f(x) = 5\cos(2x - \frac{\pi}{3})$

   d. $f(x) = -\frac{1}{2}\cos(\pi x + \frac{\pi}{2})$

   e. $f(x) = 2 + 3\sin(x + \frac{\pi}{3})$

   f. $f(x) = 4\cos(3x - 2) - 5$

   g. $f(x) = 1 - \frac{1}{3}\cos(\frac{\pi}{4}x - \frac{\pi}{3})$

   h. $f(x) = \pi + \frac{\pi}{2}\sin(\frac{3\pi}{2}x + \frac{\pi}{4})$

2. Let $f(x) = 2 + 3\cos(\pi x + 1)$. Find $\operatorname{rng} f$.

4.3 Sinusoidal Phenomena

A. Introduction

Phenomena that cycle repetitively through time can often be modeled using sinusoids.

Examples: tides, yearly precipitation, yearly temperature

B. Strategy

Given a data set, we write down $y = d + a\cos(bx - c)$ as follows:

1. **Find a (amplitude):** Let $a = \frac{1}{2}(\max - \min)$

2. **Find d (centerline):** Let $d = \frac{1}{2}(\max + \min)$

3. **Find b:**

   a. first find the period: $p = 2 \cdot (\text{time from max to min})$

   b. then let $b = \frac{2\pi}{p}$ (since $p = \frac{2\pi}{b}$)

4. **Find c:**

   a. first find the phase shift: $\frac{c}{b} = \text{time when max occurs}$

   b. then let $c = b \cdot (\text{phase shift})$

C. Example

(Tides)

The depth of water at the end of a dock varies with the tides. The following table shows the depths (in meters) of the water:

t (time)	12 am	2 am	4 am	6 am	8 am	10 am	12 pm
y (depth)	3.5	4.2	3.5	2.1	1.4	2.1	3.5

Solution

We model using $y = d + a\cos(bx - c)$

1. **Find a:** $a = \frac{1}{2}(\text{max} - \text{min}) = \frac{1}{2}(4.2 - 1.4) = 1.4$

2. **Find d:** $d = \frac{1}{2}(\text{max} + \text{min}) = \frac{1}{2}(4.2 + 1.4) = 2.8$

3. **Find b:**

   a. period, $p = 2(6) = 12$

   b. $b = \frac{2\pi}{12} = \frac{\pi}{6}$

4. **Find c:**

   a. phase shift: $\frac{c}{b} = 2$

   b. $c = 2 \cdot b = 2(\frac{\pi}{6}) = \frac{\pi}{3}$

Ans $\boxed{y = 2.8 + 1.4\cos\left(\frac{\pi}{6}t - \frac{\pi}{3}\right)}$, where t is time in hours past midnight.

Exercises

1. The water at the end of dock varies with the tides. Measurements of the water depth were taken every 2 hours and recorded. Using the data below, construct a sinusoidal model for the water depth, y, in terms of the number of hours, t, past midnight.

Time	12 am	2 am	4 am	6 am	8 am	10 am	12 pm
Depth	3.5 m	4.7 m	5.4 m	4.7 m	3.5 m	2.8 m	3.5 m

2. The water at the end of dock varies with the tides. Measurements of the water depth were taken every 2 hours and recorded. Using the data below, construct a sinusoidal model for the water depth, y, in terms of the number of hours, t, past midnight.

Time	12 am	2 am	4 am	6 am	8 am	10 am	12 pm
Depth	3.8 m	4.4 m	3.8 m	2.6 m	2.0 m	2.6 m	3.8 m

3. The table below gives the recorded high temperature as measured on the 25th of the indicated month. Construct a sinusoidal model for temperature, T, in terms of the number of months, t, past January 25.

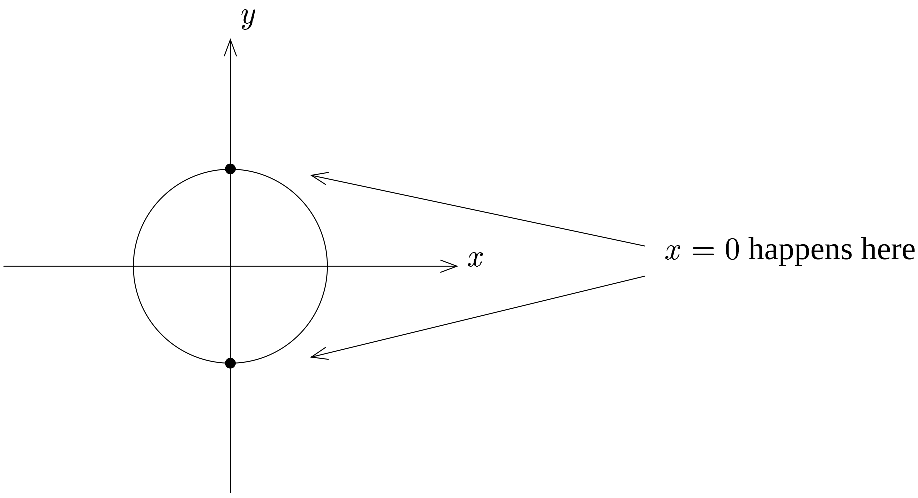
Month	JAN	FEB	MAR	APR	MAY	JUN
Temperature	$54°$F	$58°$F	$66°$F	$76°$F	$90°$F	$98°$F

Month	JUL	AUG	SEP	OCT	NOV	DEC
Temperature	$102°$F	$98°$F	$90°$F	$76°$F	$66°$F	$58°$F

4.4 Graphs of Other Trigonometric Functions

A. Graph of $y = \tan x$

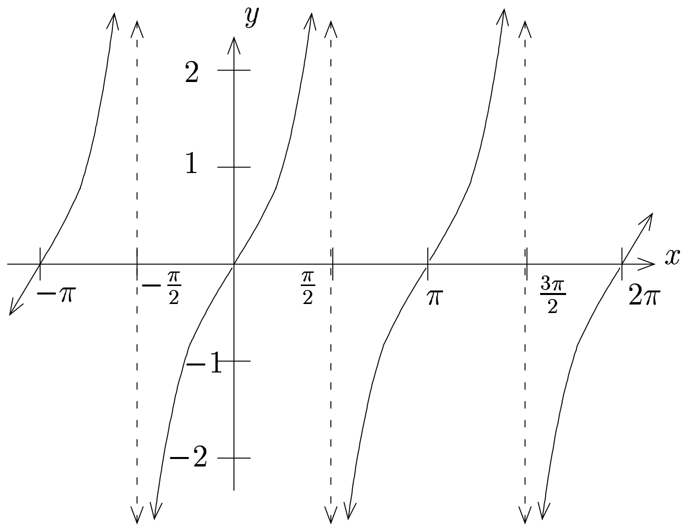
Since $\tan \theta = \dfrac{y_{coord}}{x_{coord}}$, whenever $x_{coord} = 0$ we get vertical asymptotes:



Thus the graph of $y = \tan x$ has vertical asymptotes at odd multiples of $\frac{\pi}{2}$.

Since the tangent function has period π, we only need to make a table for values 0 to π, then the graph repeats.

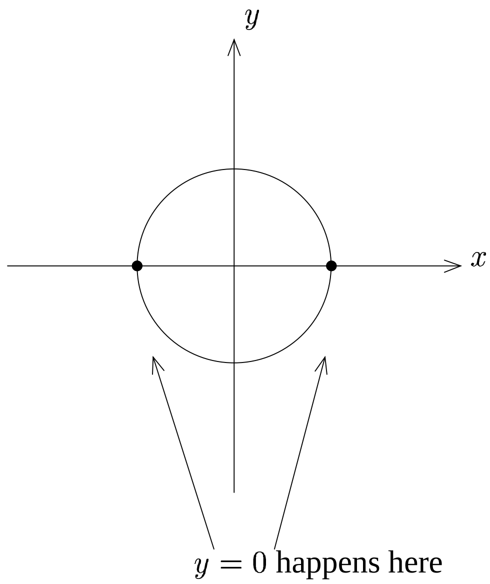
x	y
0	0
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	1
$\frac{\pi}{3}$	$\sqrt{3}$
$\frac{\pi}{2}$	undefined
$\frac{2\pi}{3}$	$-\sqrt{3}$
$\frac{3\pi}{4}$	-1
$\frac{5\pi}{6}$	$-\frac{\sqrt{3}}{3}$
π	0



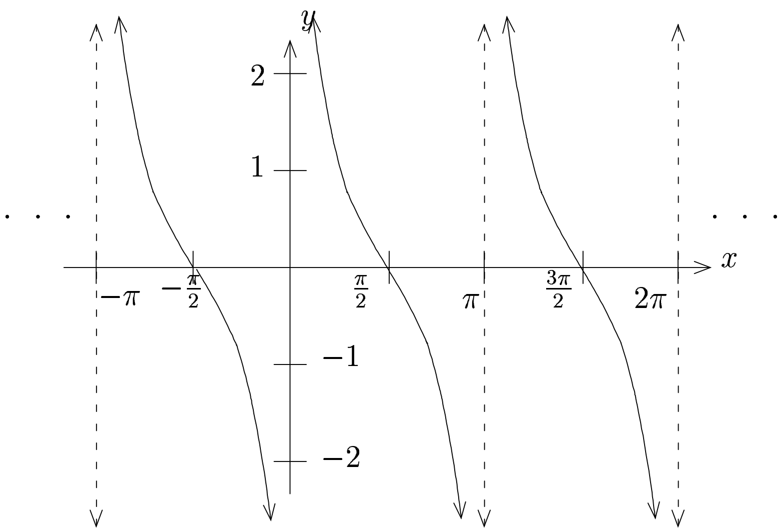
$y = \tan x$

B. Graph of $y = \cot x$

Similarly, since $\cot\theta = \dfrac{x_{coord}}{y_{coord}}$, we get vertical asymptotes when $y_{coord} = 0$.



This occurs at multiples of π, and making a similar table of values, the graph looks like:



$y = \cot x$

C. Graphs of $y = \sec x$ and $y = \csc x$

Since $\sec\theta = \dfrac{1}{x_{coord}}$, as in $y = \tan x$, we get vertical asymptotes at odd multiples of $\frac{\pi}{2}$.

Also, since $\csc\theta = \dfrac{1}{y_{coord}}$, as in $y = \cot x$, we get vertical asymptotes at multiples of π.

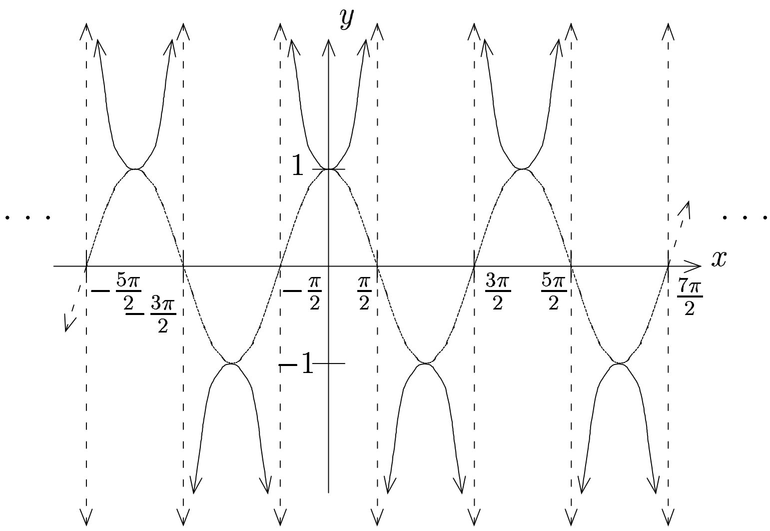
Furthermore, since $\sec\theta = \frac{1}{\cos\theta}$ and $\csc\theta = \frac{1}{\sin\theta}$, we obtain an easy way to graph these functions.

Easy Way to Graph $y = \sec x$ or $\csc x$

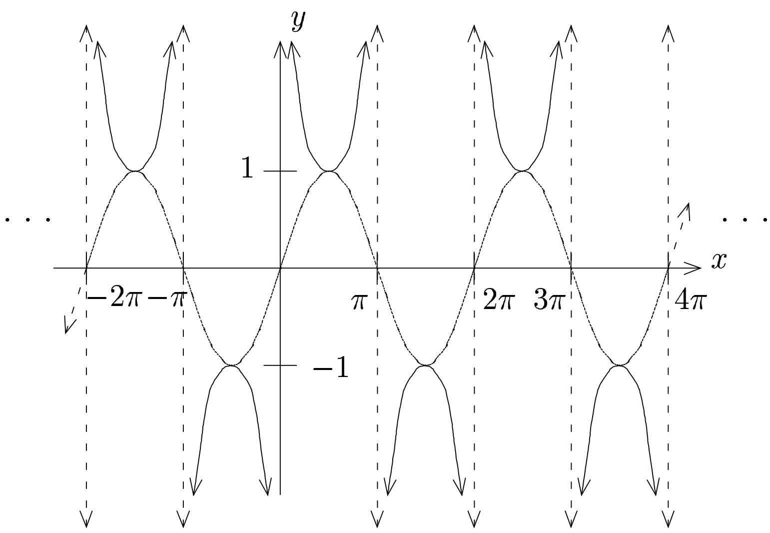
1. Graph $y = \cos x$ or $y = \sin x$.

2. Put in the vertical asymptotes at the x-intercepts and take reciprocals of the y-values.

Doing this for $y = \sec x$ and $y = \csc x$ yields:


y
1
. . .
. . .
x
$-\frac{5\pi}{2}$
$-\frac{3\pi}{2}$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
$\frac{3\pi}{2}$
$\frac{5\pi}{2}$
$\frac{7\pi}{2}$
-1

$y = \sec x$


y
1
. . .
. . .
x
-2π
$-\pi$
π
2π
3π
4π
-1

$y = \csc x$

4.5 Graphing General Tangent and Cotangent

A. General Tangent

Here we use the idea that $y = \tan x$ has vertical asymptotes at $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$ with x-intercept halfway in between, and we use the fact that $\tan$ is π-periodic.

Strategy

Given $y = d + a\tan(bx - c)$

1. Set $bx - c = -\frac{\pi}{2}$ and $bx - c = \frac{\pi}{2}$ to find the location of two vertical asymptotes.

2. Put an x-intercept halfway in between the two asymptotes.

3. Draw in a "copy of $y = \tan x$" and repeat with asymptotes to make periodic.

4. If $a < 0$, flip about x-axis.

5. Shift centerline up d units.

B. General Cotangent

This has the same strategy as general tangent, except the asymptotes for cotangent are at $x = 0$ and $x = \pi$, so for $y = d + a\cot(bx - c)$, we set $bx - c = 0$ and $bx - c = \pi$ in the above strategy.

C. Examples

Example 1: Graph f, where $f(x) = 3\tan(2x - \pi)$

Solution

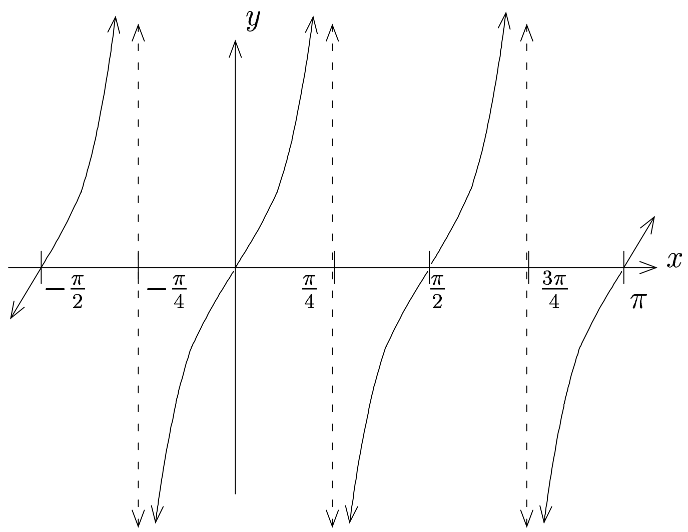
1. $2x - \pi = -\frac{\pi}{2} \;\Rightarrow\; 2x = \frac{\pi}{2} \;\Rightarrow\; x = \frac{\pi}{4}$

   $2x - \pi = \frac{\pi}{2} \;\Rightarrow\; 2x = \frac{3\pi}{2} \;\Rightarrow\; x = \frac{3\pi}{4}$

2. x-intercept (one of): halfway in between $\frac{\pi}{4}$ and $\frac{3\pi}{4}$

   $\frac{1}{2}[\frac{\pi}{4} + \frac{3\pi}{4}] = \frac{\pi}{2}$

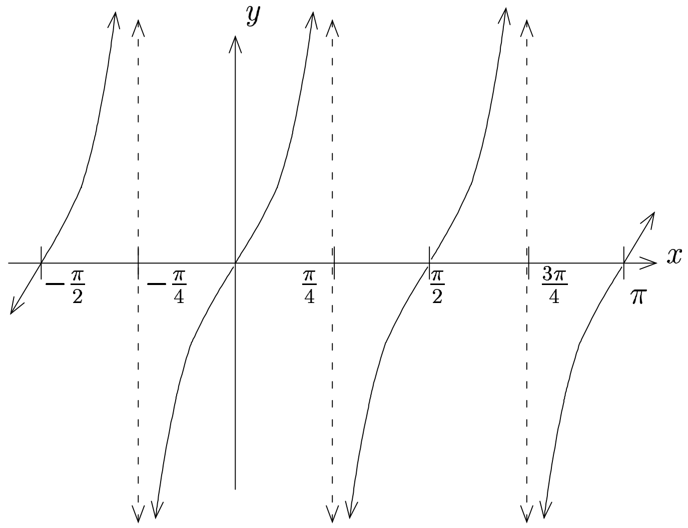
3. Now draw in the "copy of $y = \tan x$" and make periodic:



4. No reflections

5. No vertical shift

Ans



Example 2: Graph f, where $f(x) = 2 + \cot\left(\frac{x}{2} - 1\right)$

Solution

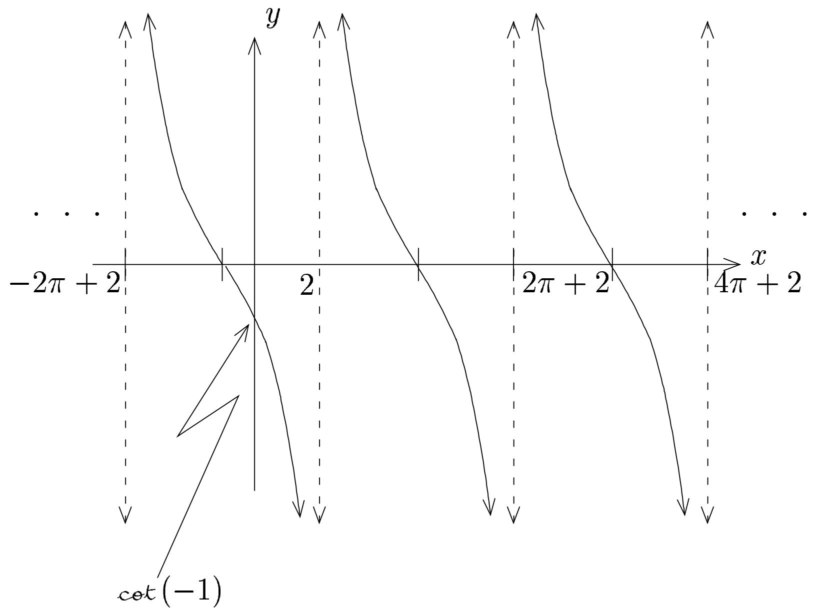
1. $\frac{x}{2} - 1 = 0 \;\Rightarrow\; \frac{x}{2} = 1 \;\Rightarrow\; x = 2$

   $\frac{x}{2} - 1 = \pi \;\Rightarrow\; \frac{x}{2} = \pi + 1 \;\Rightarrow\; x = 2\pi + 2$

2. x-intercept (one of): halfway in between 2 and $2\pi + 2$

   $\frac{1}{2}[2 + (2\pi + 2)] = \frac{1}{2}(2\pi + 4) = \pi + 2$

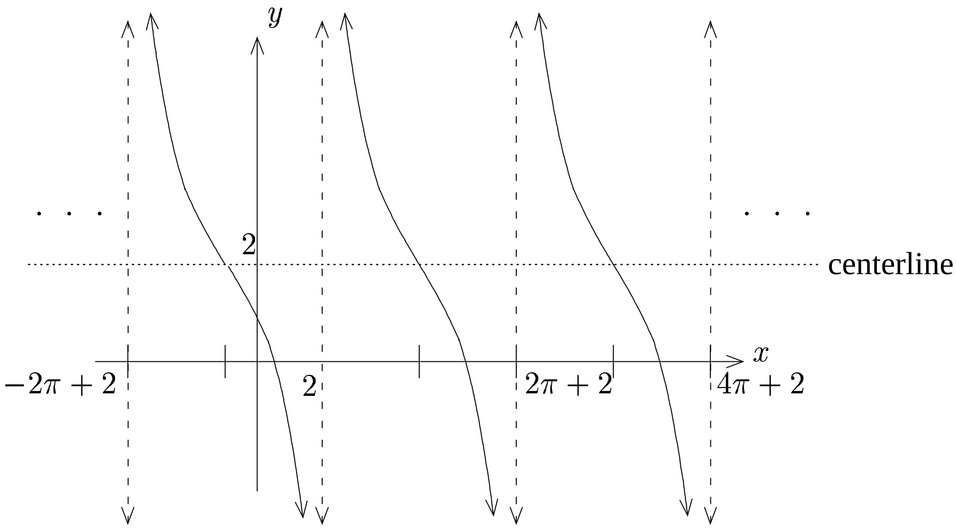
3. Now draw in the "copy of $y = \cot x$" and make periodic:



4. No reflections

5. Move Up 2.

Ans



Example 3: Graph f, where $f(x) = -4\tan\left(\frac{\pi x}{3} + \pi\right)$

Solution

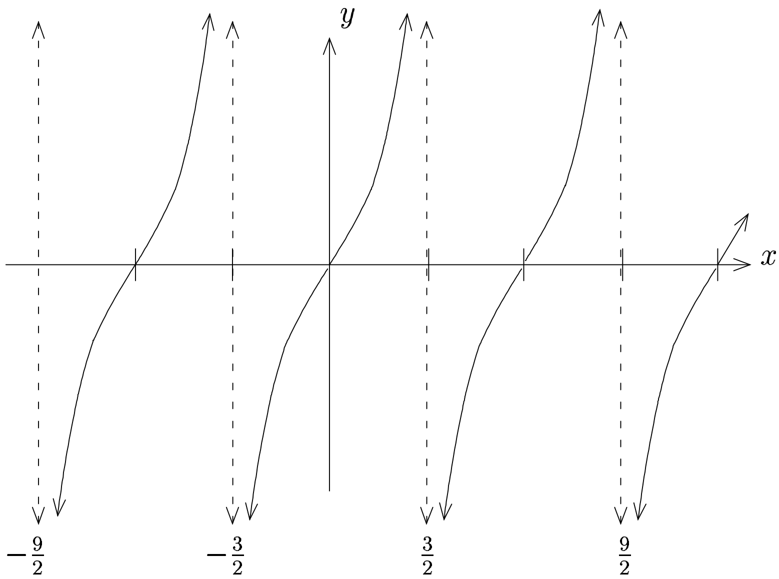
1. $\frac{\pi x}{3} + \pi = -\frac{\pi}{2} \;\Rightarrow\; \frac{\pi x}{3} = -\frac{3\pi}{2} \;\Rightarrow\; x = -\frac{3\pi}{2} \cdot \frac{3}{\pi} \;\Rightarrow\; x = -\frac{9}{2}$

   $\frac{\pi x}{3} + \pi = \frac{\pi}{2} \;\Rightarrow\; \frac{\pi x}{3} = -\frac{\pi}{2} \;\Rightarrow\; x = -\frac{\pi}{2} \cdot \frac{3}{\pi} \;\Rightarrow\; x = -\frac{3}{2}$

2. x-intercept (one of): halfway in between $-\frac{9}{2}$ and $-\frac{3}{2}$

   $[\frac{1}{2}(-\frac{9}{2} + -\frac{3}{2}) = -3]$

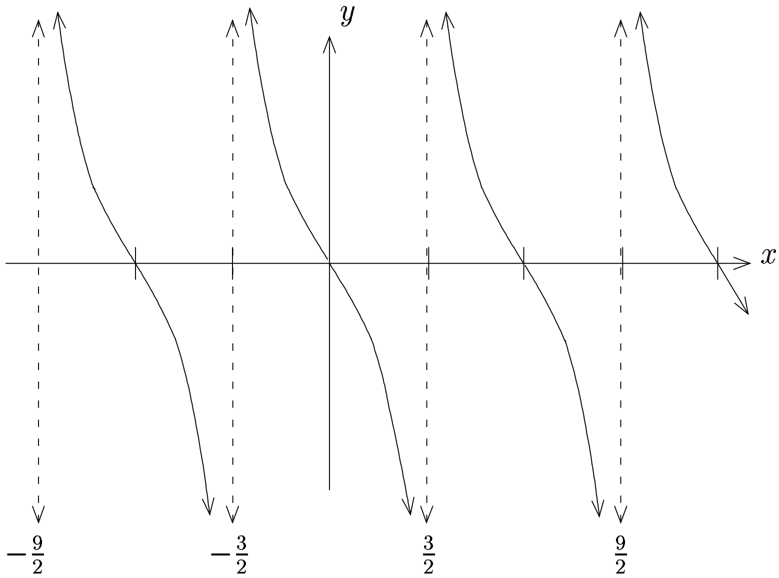
3. Now draw in the "copy of $y = \tan x$" and make periodic:



4. Now flip across the x-axis.

5. No vertical shift

Ans



D. Comments

1. For general tangent/cotangent, amplitude is undefined.

2. **Period:** $\frac{\pi}{|b|}$

4.6 Graphing General Secant and Cosecant

A. General Secant

Strategy

Given $y = d + a\sec(bx - c)$

1. Graph $y = a\cos(bx - c)$

2. At each x-intercept, put in a vertical asymptote, and "reciprocate" the graph to get $y = a\sec(bx - c)$.

3. Shift the centerline up d units.

B. General Cosecant

For $y = d + a\csc(bx - c)$, we use the same strategy as above, except we first graph $y = a\sin(bx - c)$.

C. Examples

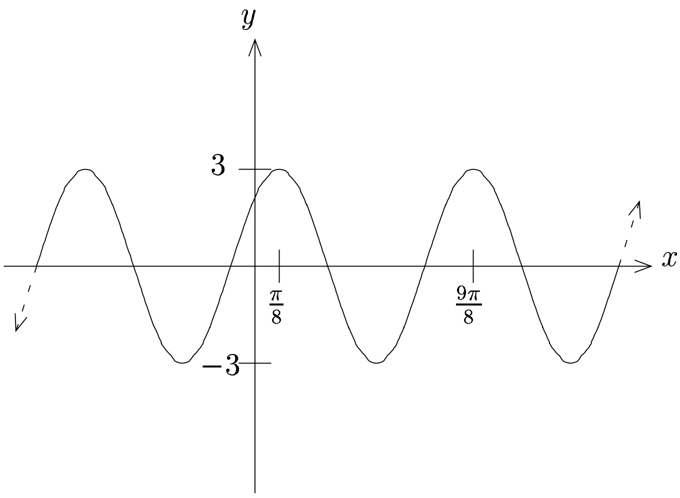
Example 1: Graph f, where $f(x) = 2 + 3\sec(2x - \frac{\pi}{4})$

Solution

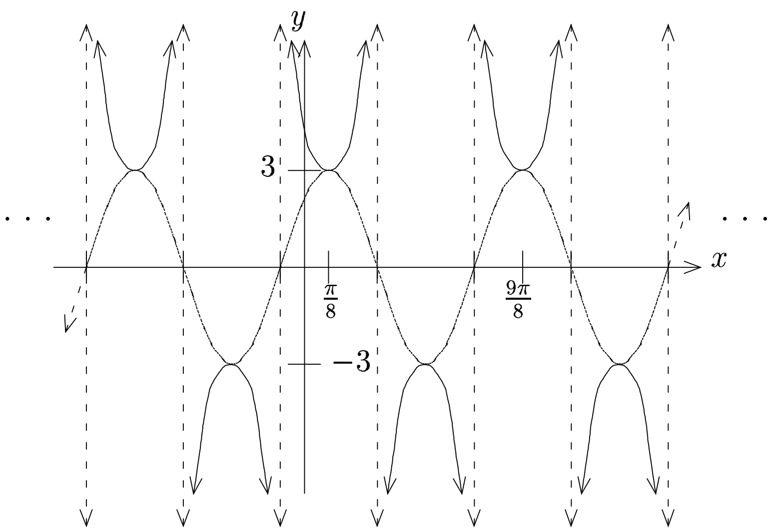
1. First graph $y = 3\cos(2x - \frac{\pi}{4})$:

a. $2x - \frac{\pi}{4} = 0 \Rightarrow 2x = \frac{\pi}{4} \Rightarrow x = \frac{\pi}{8}$

b. $2x - \frac{\pi}{4} = 2\pi \Rightarrow 2x = \frac{9\pi}{4} \Rightarrow x = \frac{9\pi}{8}$

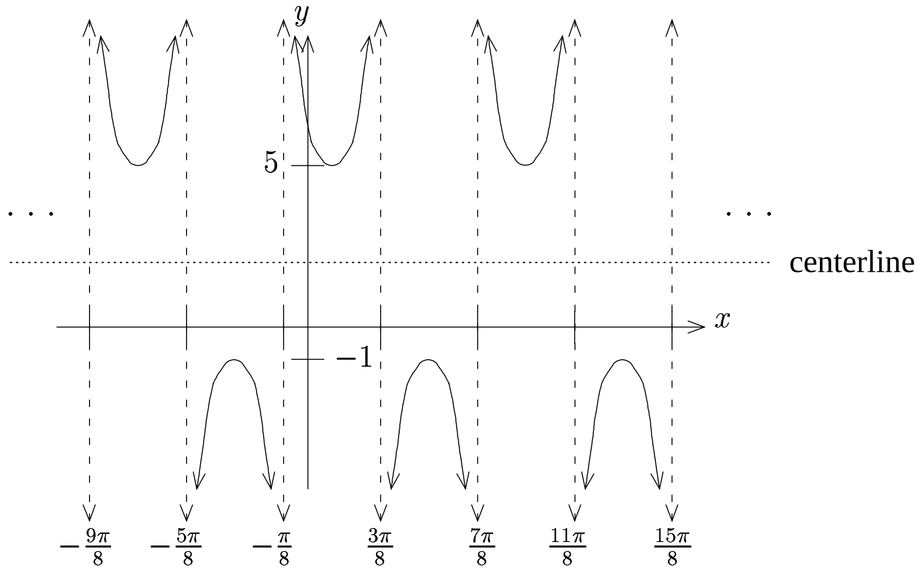


2. Put in the vertical asymptotes, and reciprocate . . .



3. Move up 2

Ans



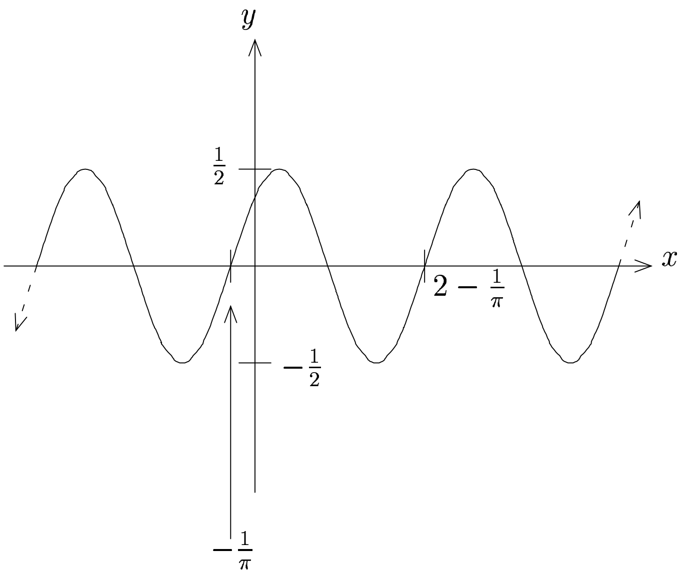
Example 2: Graph f, where $f(x) = -1 + \frac{1}{2} \csc(\pi x + 1)$

Solution

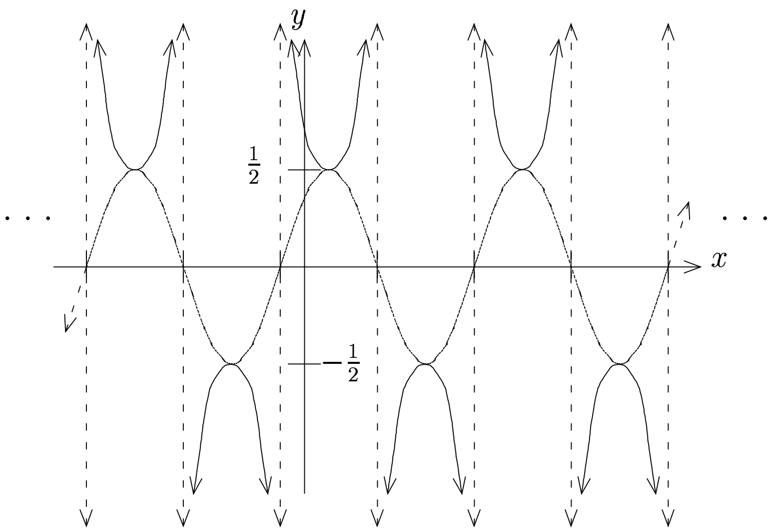
1. First graph $y = \frac{1}{2} \sin(\pi x + 1)$

   a. $\pi x + 1 = 0 \;\Rightarrow\; \pi x = -1 \;\Rightarrow\; x = -\frac{1}{\pi}$

   b. $\pi x + 1 = 2\pi \;\Rightarrow\; \pi x = 2\pi - 1 \;\Rightarrow\; x = 2 - \frac{1}{\pi}$

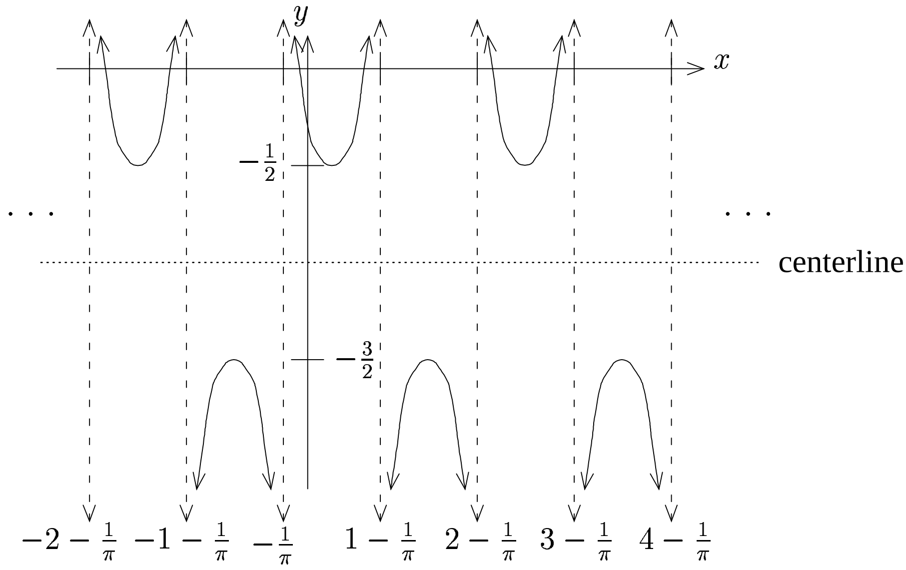


2. Put in vertical asymptotes and reciprocate:



3. Move Down 1

Ans



D. Comments

1. For general secant/cosecant, amplitude is undefined.

2. **Period:** $\frac{2\pi}{|b|}$

Exercises

1. Find the period for the following:

   a. $f(x) = 4\tan(5x+3)+1$

   b. $f(x) = -3\cot(\pi x - \frac{\pi}{2})$

   c. $f(x) = \frac{4}{9}\sec(3x - \frac{\pi}{3}) - 2$

   d. $f(x) = -\sqrt{2}\csc(\frac{\pi}{4}x + \frac{\pi}{6})$

2. Graph f, where

   a. $f(x) = 2\tan(3x - \frac{\pi}{2})$

   b. $f(x) = -3\sec(2x+\pi)$

   c. $f(x) = \frac{1}{2}\cot(\frac{\pi}{2}x - \frac{\pi}{3})$

   d. $f(x) = -\frac{2}{3}\csc(\frac{\pi}{3}x + \frac{\pi}{2}) - 1$

   e. $f(x) = \sqrt{3}\sec(\pi x - 1) + 2$

   f. $f(x) = 5\tan(2x-7)+3$

   g. $f(x) = 1 - \frac{\pi}{4}\cot(2x - \frac{\pi}{6})$

   h. $f(x) = 2 - 3\csc(5x+1)$

4.7 Damped Trigonometric Functions

A. Introduction

Sometimes a trigonometric function gets multiplied by another function called a **damping factor**

i.e. $y = x^2 \cos x$ here $f(x) = x^2$ is the damping factor

Damped trigonometric functions involving sine and cosine are straightforward to graph.

Since $-1 \leq \sin(bx - c) \leq 1$ or $-1 \leq \cos(bx - c) \leq 1$, we have that

$$-f(x) \leq f(x)\sin(bx - c) \leq f(x) \quad \text{or} \quad -f(x) \leq f(x)\cos(bx - c) \leq f(x)$$

for $f(x) \geq 0$.

Considering all possible $f(x)$ in the same way, we see that the function oscillates between $\pm f(x)$ (the damping factor).

Note: In applications, often the damping factor serves to "damp" the amplitude as time progresses (i.e. amplitude diminishes). However, in this section, we consider all viable factors $f(x)$.

B. Graphing Strategy

1. Graph the undamped trigonometric function.

2. The damped trigonometric function will oscillate between $-f(x)$ and $f(x)$ instead of -1 and 1.

C. Examples

Example 1: Graph f, where $f(x) = x^2 \sin 2x$

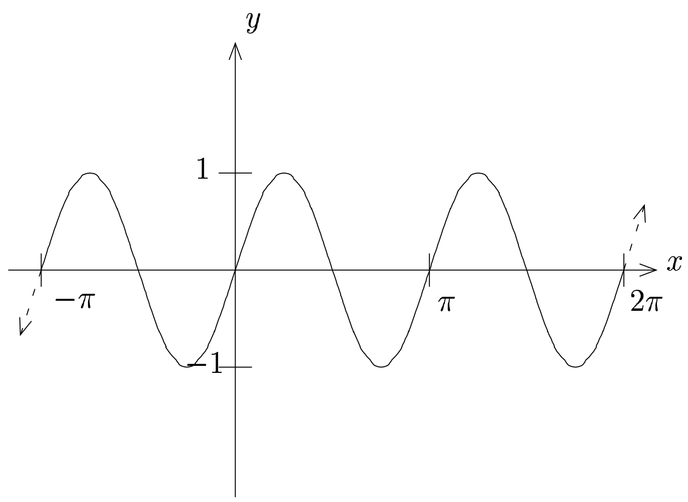
Solution

First graph $y = \sin 2x$:

a. $2x = 0 \Rightarrow x = 0$

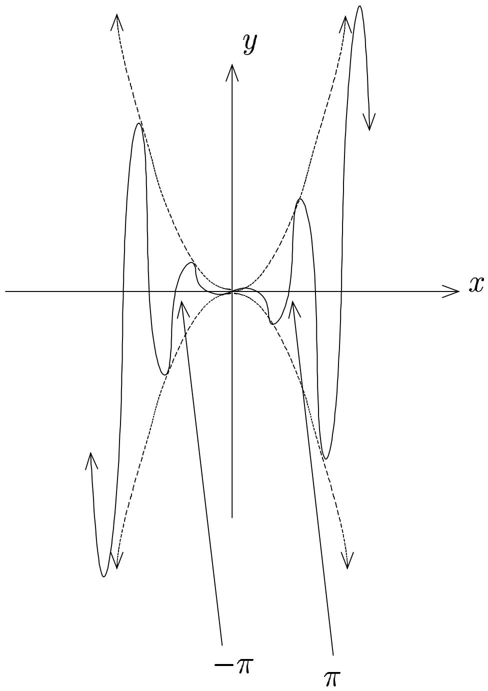
b. $2x = 2\pi \Rightarrow x = \pi$

c.



Now draw in the damping curves $y = x^2$ and $y = -x^2$, then modify:

Ans



Example 2: Graph f, where $f(x) = e^x \cos 3x$

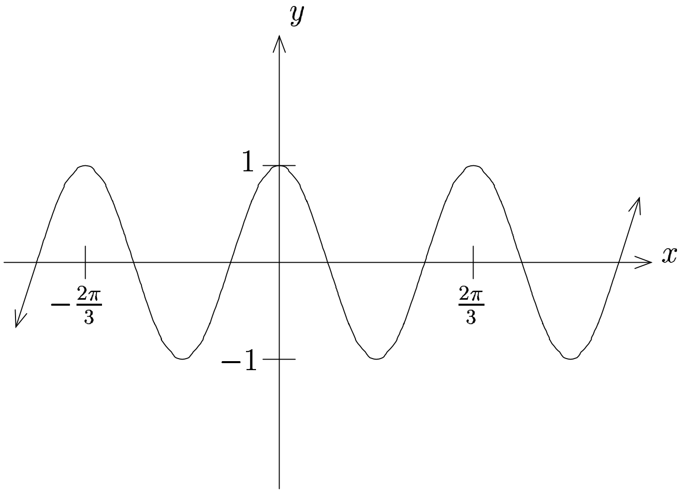
Solution

First graph $y = \cos 3x$:

a. $3x = 0 \;\Rightarrow\; x = 0$

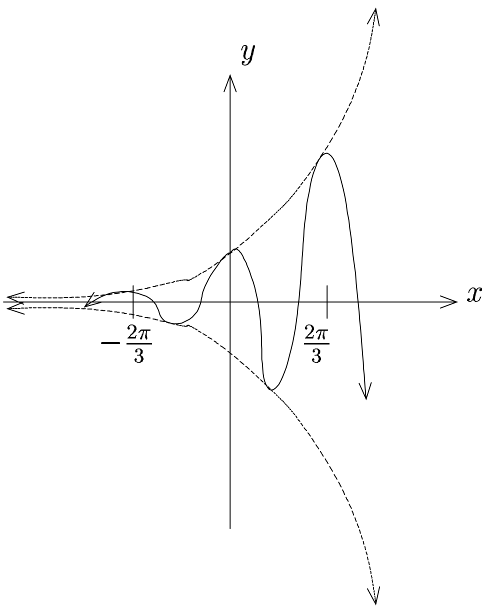
b. $3x = 2\pi \;\Rightarrow\; x = \frac{2\pi}{3}$

c.



Now draw in the damping curves $y = e^x$ and $y = -e^x$, then modify:

Ans



Exercises

Graph f, where

1. $f(x) = x \sin x$

2. $f(x) = x^2 \cos(3x)$

3. $f(x) = e^{-x} \sin(2x)$

4. $f(x) = \sqrt[3]{x} \cos x$

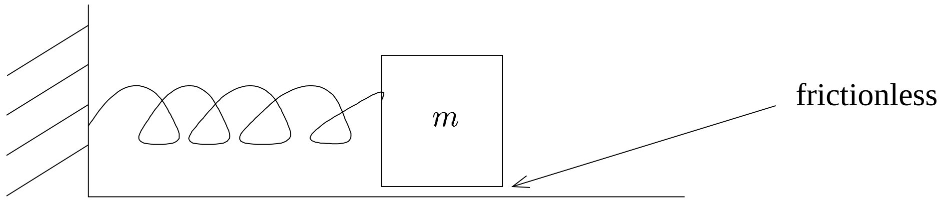
5. $f(x) = (x - \pi)^2 \cos(2x)$

4.8 Simple Harmonic Motion and Frequency

A. Simple Harmonic Motion

An object that oscillates in time uniformly is said to undergo **simple harmonic motion**.

Example: Spring-Mass System



Here $d = a \sin(\omega t)$ or $d = a \cos(\omega t)$, where

d: displacement from equilibrium position

a: maximum displacement

ω: angular frequency

B. Frequency

1. **Period,** T: $T = \frac{2\pi}{\omega}$ time to undergo one complete cycle

   Units: units of time, typically seconds (s)

2. **Frequency,** ν: $\nu = \frac{1}{T}$ "oscillation speed" (how many cycles per time)

   Units: inverse units of time, typically s^{-1}, also called Hertz (Hz)

3. **Angular Frequency,** ω: $\quad \omega = 2\pi\nu$

Units: inverse units of time–typically s^{-1}, but not usually written as Hz

C. Examples

Example 1: An object in simple harmonic motion is described by $d = 3\sin\left(\frac{\pi}{2}t\right)$. Find the period, frequency, angular frequency, and maximum displacement. Time is measured in seconds and displacement is measured in meters.

Solution

$T = \frac{2\pi}{\frac{\pi}{2}} = 2\pi \cdot \frac{2}{\pi} = 4$

$\nu = \frac{1}{T} = \frac{1}{4}$

$\omega = 2\pi\nu = 2\pi \cdot \frac{1}{4} = \frac{\pi}{2}$

Note: This checks, as we could have read ω directly from the formula above.

Ans Period: $\boxed{4\mathrm{s}}$

Frequency: $\boxed{\frac{1}{4}\ \mathrm{Hz}}$

Angular Frequency: $\boxed{\frac{\pi}{2}\ \mathrm{s}^{-1}}$

Maximum Displacement: $\boxed{3\mathrm{m}}$

Example 2: Find a model for simple harmonic motion satisfying the conditions:

- Period: 6s
- Maximum Displacement: 2m
- Displacement at $t = 0$: 2m

Solution

Since the object starts at maximum displacement, we use the cosine model:

$$d = a \cos(\omega t)$$

Now $a = 2$, and $T = 6 = \frac{2\pi}{\omega}$, so $\omega = \frac{2\pi}{6} = \frac{\pi}{3}$.

Ans $\boxed{d = 2 \cos(\frac{\pi}{3} t)}$

Exercises

1. Consider an object in simple harmonic motion. Time is measured in seconds and displacement in meters. Find the period, frequency, angular frequency, and maximum displacement, when the motion is described by:

a. $d = 2\sin(5t)$

b. $d = 4\cos(6\pi t)$

c. $d = 3\cos(10t)$

d. $d = 5\sin(\frac{\pi}{6}t)$

2. Find a model for simple harmonic motion satisfying the conditions:

a.
- Period: 4s
- Maximum Displacement: 1m
- Displacement at $t = 0$: 1m

b.
- Period: 3s
- Maximum Displacement: 3m
- Displacement at $t = 0$: 3m

c.
- Period: 5s
- Maximum Displacement: 2m
- Displacement at $t = 0$: 0m

d.
- Period: 2s
- Maximum Displacement: 3m
- Displacement at $t = 0$: 0m

3. Describe physically the motion of a spring whose displacement from equilibrium is described by $d = 2e^{-x}\, \mathit{sin}(2\pi t)$. Explain what the factor e^{-x} does to the motion and use this to explain why the term "damping" is appropriate. What physical condition might give rise to such a damping factor?

Chapter 5

Trigonometric Identities

5.1 Using Trigonometric Relationships

A. Review

1. **Reciprocal Identities**

a. $\boxed{\sec\theta = \frac{1}{\cos\theta} \qquad \text{and} \qquad \cos\theta = \frac{1}{\sec\theta}}$

b. $\boxed{\csc\theta = \frac{1}{\sin\theta} \qquad \text{and} \qquad \sin\theta = \frac{1}{\csc\theta}}$

c. $\boxed{\cot\theta = \frac{1}{\tan\theta} \qquad \text{and} \qquad \tan\theta = \frac{1}{\cot\theta}}$

2. **Quotient Identities**

a. $\boxed{\tan\theta = \frac{\sin\theta}{\cos\theta}}$

b. $\boxed{\cot\theta = \frac{\cos\theta}{\sin\theta}}$

3. **Pythagorean Identities**

Pythagorean I: $\boxed{\cos^2\theta + \sin^2\theta = 1}$

Pythagorean II: $\boxed{1 + \tan^2\theta = \sec^2\theta}$

Pythagorean III: $\boxed{\cot^2\theta + 1 = \csc^2\theta}$

4. **Even/Odd Identities**

a. **Even Functions:**

$\boxed{\cos(-\theta) = \cos\theta}$ even

$\boxed{\sec(-\theta) = \sec\theta}$ even

b. **Odd Functions:**

$\boxed{\sin(-\theta) = -\sin\theta}$ odd

$\boxed{\tan(-\theta) = -\tan\theta}$ odd

$\boxed{\cot(-\theta) = -\cot\theta}$ odd

$\boxed{\csc(-\theta) = -\csc\theta}$ odd

B. Simplifying/Factoring

1. Use the rules of algebra and the identities.

2. When factoring, if stuck, try to convert all trigonometric functions to the same trigonometric function first.

3. If stuck, as a LAST resort, convert everything to sines and cosines.

Common Theme: Look for opportunities to use the Pythagorean Identities by looking for squared trigonometric functions.

C. Examples

Example 1: Simplify $(1 - \sin^2 x) \sec x$

Solution

Use Pythagorean I: $\cos^2 x + \sin^2 x = 1$ to replace $1 - \sin^2 x$ with $\cos^2 x$ . . .

Thus $(1 - \sin^2 x) \sec x = \cos^2 x \cdot \sec x \underbrace{=}_{\text{reciprocal}} \cos^2 x \cdot \frac{1}{\cos x} = \cos x$

Ans $\boxed{\cos x}$

Example 2: Simplify $\frac{\sin x}{1+\cos x}+\frac{1+\cos x}{\sin x}$

Solution

Find a common denominator: $(1+\cos x)(\sin x)$

Thus,

$$\frac{\sin x}{1+\cos x}+\frac{1+\cos x}{\sin x}$$

$$=\frac{(\sin x)(\sin x)}{(1+\cos x)(\sin x)}+\frac{(1+\cos x)(1+\cos x)}{(\sin x)(1+\cos x)}$$

$$=\frac{\sin^2 x}{(\sin x)(1+\cos x)}+\frac{1+2\cos x+\cos^2 x}{(\sin x)(1+\cos x)}$$

$$=\frac{(\cos^2 x+\sin^2 x)+1+2\cos x}{(\sin x)(1+\cos x)}$$

$$\underbrace{=}_{\text{Pythagorean I}}\frac{1+1+2\cos x}{(\sin x)(1+\cos x)}$$

$$=\frac{2+2\cos x}{(\sin x)(1+\cos x)}$$

$$=\frac{2(1+\cos x)}{(\sin x)(1+\cos x)}$$

$$=\frac{2}{\sin x}=2\csc x$$

Ans $\boxed{2\csc x}$

Example 3: Factor $\sec^2 x + 5\tan x + 5$

Solution

Can't factor directly, so convert to same trigonometric function!

Use Pythagorean II: $1 + \tan^2 x = \sec^2 x$

Thus,

$\sec^2 x + 5\tan x + 5$

$= (1 + \tan^2 x) + 5\tan x + 5$

$= \tan^2 x + 5\tan x + 6$

$= (\tan x + 2)(\tan x + 3)$

Ans $\boxed{(\tan x + 2)(\tan x + 3)}$

Exercises

1. Simplify $(1+\sec\theta)(1-\sec\theta)$

2. Simplify $\tan\theta\cot\theta$

3. Factor and simplify $\cos^3\theta+\sin^2\theta\cos\theta$

4. Factor $2\cot^2\theta-\cot\theta-3$

5. Factor $\cos^3\theta+\sin^3\theta$

6. Simplify $\frac{1-\sin\theta}{\cos\theta}+\frac{\cos\theta}{1-\sin\theta}$

7. Factor $2\cot^2\theta-5\csc\theta-10$

8. Simplify $\frac{\csc^3\theta-8}{\csc\theta-2}$

9. Simplify $\tan\theta+\frac{\cos\theta}{1+\sin\theta}$

10. Factor $\cos^4\theta-\sin^4\theta$

11. Simplify $(\sin\theta+\cos\theta)^2$

12. Simplify $\frac{1}{\sec\theta+1}-\frac{1}{\sec\theta-1}$

13. Simplify $(5+5\cos\theta)(5-5\cos\theta)$

14. Simplify $\frac{\sin^2\theta}{1-\cos\theta}$

15. Factor and simplify $\frac{1}{1+2\tan^2\theta+\tan^4\theta}+\frac{2}{\csc^2\theta+\csc^2\theta\tan^2\theta}+\frac{1}{\csc^4\theta}$.

16. Simplify and factor $-2\cos^2\theta-3\sin(-\theta)-3$.

5.2 Verifying Trigonometric Identities

A. Identities

Identities are equations that are **always** true

Examples:

$(3x-4)^2 = 9x^2 - 24x + 16$

$\sec\theta = \frac{1}{\cos\theta}$

$\cos^2\theta + \sin^2\theta = 1$

In each equation above, the equation is **always** true. No matter what the input is, the equation works (provided the expressions are defined). This is different from conditional equations.

Conditional equations are equations that only work for a few values of x (input)

Examples:

$3x + 4 = 10$ (only the solution $x = 2$ works)

$x^2 = 9$ (only the solutions $x = -3$ and $x = 3$ work)

We **solve** (conditional) equations, but we **verify** identities.

When we solve a (conditional) equation, we do the same operation to each side of the equation to find the values of x that work.

However, when we verify identities, we have the answer already, in some sense. We just need to demonstrate that the equation is always true. The method to do so is radically different than the method for solving equations.

B. Verifying Identities

Unlike solving equations, we are **not** allowed to work with both sides of an identity at the same time to verify it.

Here is an example that demonstrates why doing so causes nonsense.

Example: "Prove/verify" that $-5 = 5$

If we square each side, we get $25 = 25$, which is true, but obviously $-5 \neq 5$, so we've done something wrong.

As you can see, operating on both sides to verify an identity is incorrect.

C. Method to Verify an Identity

1. Pick **one** side of the equation (usually the more complicated side), and ignore the other side.

2. Manipulate it, by itself, using valid laws for **expressions**. Thus randomly squaring, or randomly adding numbers, etc. is not allowed (you don't have an equation to balance out the operation!)

3. By manipulating it, through a number of steps, you try to make it "become" the other side.

Hence, unlike equations (conditional) where the goal is to solve to get an "answer", you actually know the answer to an identity already! It is the other side of the equation! Here you know the beginning and the end, and the goal is to fill in the middle–to show **how** to get from the beginning to the end.

D. Verifying Trigonometric Identities

1. Starting with one side (typically the more complicated side), try simplifying and/or factoring it. Try to implement any trigonometric identities you can think of. Often you will look for, or try to make, squared trigonometric functions, so that you can use Pythagorean Identities.

2. If you see two fractions, find a common denominator and combine them.

3. If nothing seems to work, and you are desperate, try converting everything to sines and cosines.

4. Keep your goal in mind (you know the answer already)

5. Even if you can't see immediately what to do, try something! Dead ends sometimes give you ideas that help you see the correct approach.

Note: In verifying trigonometric identities, you know the beginning and the end. You are trying to fill in the middle.

E. Examples

Example 1: Verify the identity: $(1 + \sin\theta)(1 - \sin\theta) = \cos^2\theta$

Solution

Start with the left side:

$(1 + \sin\theta)(1 - \sin\theta)$

$= 1 - \sin^2\theta$ (Now use Pythagorean I: $\cos^2\theta + \sin^2\theta = 1$)

$= (\cos^2\theta + \sin^2\theta) - \sin^2\theta$

$$= \cos^2\theta$$

Thus we reached the right side, so we are done.

Example 2: Verify the identity: $\sec^4\theta - \sec^2\theta = \tan^4\theta + \tan^2\theta$

Solution

Start with the left side:

$$\sec^4\theta - \sec^2\theta$$

$$= \sec^2\theta\left(\sec^2\theta - 1\right) \qquad \text{(factoring)}$$

$$= \left(1 + \tan^2\theta\right)\left(\left(1 + \tan^2\theta\right) - 1\right) \qquad \text{(using Pythagorean II)}$$

$$= \left(1 + \tan^2\theta\right)\left(\tan^2\theta\right)$$

$$= \tan^4\theta + \tan^2\theta \qquad \text{(multiplying)}$$

Thus we reached the right side, so we are done.

Example 3: Verify the identity: $\dfrac{1}{1+\sin\theta}+\dfrac{1}{1-\sin\theta}=2\sec^2\theta$

Solution

Start with the left side:

$$\frac{1}{1+\sin\theta}+\frac{1}{1-\sin\theta}$$

$$=\frac{1-\sin\theta}{(1+\sin\theta)(1-\sin\theta)}+\frac{1+\sin\theta}{(1+\sin\theta)(1-\sin\theta)} \qquad \text{(LCD)}$$

$$=\frac{2}{(1+\sin\theta)(1-\sin\theta)} \qquad \text{(adding)}$$

$$=\frac{2}{1-\sin^2\theta} \qquad \text{(multiply out bottom)}$$

$$=\frac{2}{(\cos^2\theta+\sin^2\theta)-\sin^2\theta} \qquad \text{(use Pythagorean I)}$$

$$=\frac{2}{\cos^2\theta}$$

$$=2\sec^2\theta \qquad \text{(use reciprocal identity)}$$

Thus we reached the right side, so we are done.

Example 4: Verify the identity: $\dfrac{\csc^2\theta}{1+\tan^2\theta} = \cot^2\theta$

Solution

Start with the left side:

$$\frac{\csc^2\theta}{1+\tan^2\theta}$$

$$= \frac{\csc^2\theta}{\sec^2\theta} \qquad \text{(use Pythagorean II)}$$

$$= \frac{\frac{1}{\sin^2\theta}}{\frac{1}{\cos^2\theta}} \qquad \text{(use reciprocal identities)}$$

$$= \frac{1}{\sin^2\theta} \cdot \frac{\cos^2\theta}{1}$$

$$= \frac{\cos^2\theta}{\sin^2\theta}$$

$$= \cot^2\theta \text{ (use reciprocal identity)}$$

Thus we reached the right side, so we are done.

Note: Sometimes multiplying top and bottom by something that causes a Pythagorean identity is a good plan, as in the next example.

Example 5: Verify the identity: $\dfrac{\cos\theta}{1+\sin\theta} = \dfrac{1-\sin\theta}{\cos\theta}$

Solution

Start with the left side:

$$\frac{\cos\theta}{1+\sin\theta}$$

$$= \frac{\cos\theta\,(1-\sin\theta)}{(1+\sin\theta)(1-\sin\theta)} \qquad \text{(chosen to make } 1-\sin^2\theta = \cos^2\theta\text{)}$$

$$= \frac{\cos\theta\,(1-\sin\theta)}{1-\sin^2\theta}$$

$$= \frac{\cos\theta\,(1-\sin\theta)}{\cos^2\theta} \qquad \text{(use Pythagorean I [modified])}$$

$$= \frac{1-\sin\theta}{\cos\theta} \qquad \text{(canceling)}$$

Thus we reached the right side, so we are done.

Example 6: Verify the identity: $\tan\theta + \cot\theta = \sec\theta\csc\theta$

Solution

Start with the left side:

$$\tan\theta + \cot\theta$$

$$= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} \qquad \text{(convert to sines and cosines)}$$

$$= \frac{\sin^2\theta}{\sin\theta\cos\theta} + \frac{\cos^2\theta}{\sin\theta\cos\theta} \qquad \text{(LCD)}$$

$$= \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{1}{\sin\theta\cos\theta} \qquad \text{(use Pythagorean I)}$$

$$= \frac{1}{\sin\theta} \cdot \frac{1}{\cos\theta}$$

$$= \csc\theta \cdot \sec\theta$$

$$= \sec\theta\csc\theta$$

Thus we reached the right side, so we are done.

Exercises

Verify the following trigonometric identities:

1. $\csc\theta\cos\theta = \cot\theta$

2. $\sin\theta(\cot\theta + \tan\theta) = \sec\theta$

3. $(\csc\theta - \cot\theta)(\csc\theta + \cot\theta) = 1$

4. $\csc^4\theta - \csc^2\theta = \cot^4\theta + \cot^2\theta$

5. $\frac{1}{1-\cos\theta} + \frac{1}{1+\cos\theta} = 2\csc^2\theta$

6. $\frac{\cos\theta}{1-\tan\theta} + \frac{\sin\theta}{1-\cot\theta} = \sin\theta + \cos\theta$

7. $\frac{\sec^2\theta-\tan^2\theta+\tan\theta}{\sec\theta} = \cos\theta + \sin\theta$

8. $\frac{\cos\theta}{1-\sin\theta} = \sec\theta + \tan\theta$

9. $\csc^4\theta - 2\csc^2\theta + 1 = \cot^4\theta$

10. $\frac{\sin^3\theta+\cos^3\theta}{\sin\theta+\cos\theta} = 1 - \sin\theta\cos\theta$

11. $\frac{\sec\theta}{1+\sec\theta} = \frac{1-\cos\theta}{\sin^2\theta}$

12. $\frac{\cot\theta}{\csc\theta-1} = \frac{\csc\theta+1}{\cot\theta}$

13. $\frac{\sin\theta-\cos\theta+1}{\sin\theta+\cos\theta-1} = \frac{\sin\theta+1}{\cos\theta}$

14. $\sec\theta - \cos\theta = \sin\theta\tan\theta$

15. $\frac{\cos^4\theta-\sin^4\theta}{(2\cos^2\theta-1)^2} = \frac{1}{1-2\sin^2\theta}$

16. $\frac{1-\cot^2\theta}{1+\cot^2\theta} = 1 - 2\cos^2\theta$

17. $\frac{\sin\theta+\cos\theta}{\sin\theta} - \frac{\cos\theta-\sin\theta}{\cos\theta} = \sec\theta\csc\theta$

18. $\frac{\sin^2\theta-\tan\theta}{\cos^2\theta-\cot\theta} = \tan^2\theta$

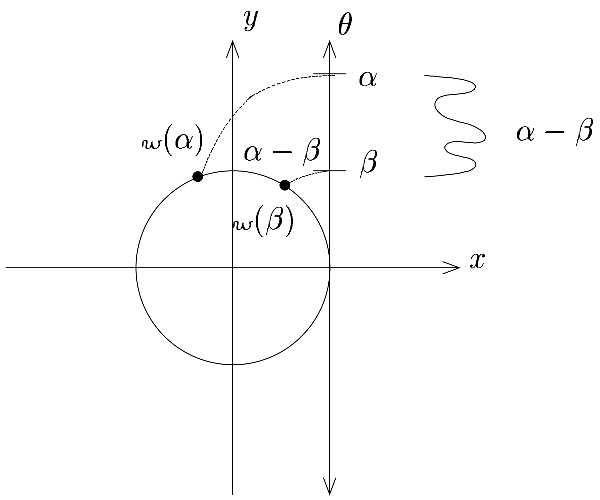
19. $\frac{\tan\theta+\sec\theta}{\cos\theta+\cot\theta} = \sec\theta\tan\theta$

20. $\frac{\csc\theta-\sec\theta}{\csc\theta\sec\theta} = \cos\theta - \sin\theta$

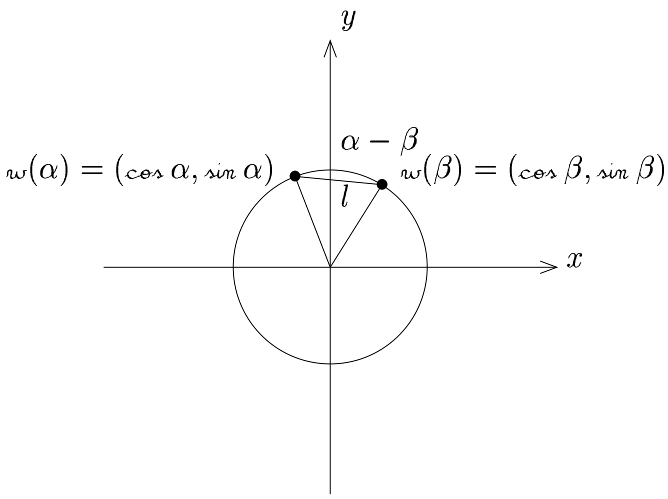
5.3 Sum and Difference Formulas I

A. Derivation of $\cos(\alpha - \beta)$

Step 1: For values α and β on the number line, identify $w(\alpha)$, $w(\beta)$, and $\alpha - \beta$

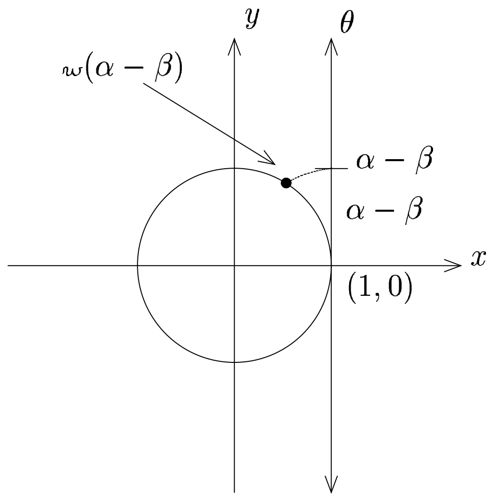


Step 2: Connect points to form triangle, and calculate length l (distance between $w(\alpha)$ and $w(\beta)$)

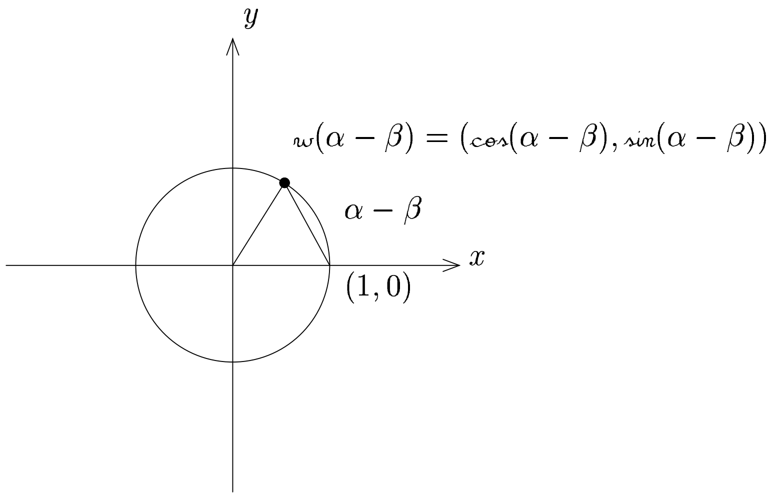


Distance Formula: $l = \sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2}$

Step 3: Reconsider $\alpha - \beta$ on the number line, and $w(\alpha - \beta)$



Step 4: Connect $w(\alpha - \beta)$ and $(1, 0)$ to form a triangle



Since the arc here of length $\alpha - \beta$ is congruent to the arc of length $\alpha - \beta$ from Step 2, the corresponding chords are congruent. Thus l as calculated in Step 2 is the same as the distance between $w(\alpha - \beta)$ and $(1, 0)$.

Thus by the distance formula, $l = \sqrt{(\cos(\alpha - \beta) - 1)^2 + (\sin(\alpha - \beta) - 0)^2}$, and this is the same l as in Step 2.

Step 5: Set the two expressions for l equal, and use algebra

$$\sqrt{(\cos(\alpha-\beta)-1)^2+\sin^2(\alpha-\beta)}=\sqrt{(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2}$$
$$[\cos(\alpha-\beta)-1]^2+\sin^2(\alpha-\beta)=(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2$$

$$\cos^2(\alpha-\beta)-2\cos(\alpha-\beta)+1+\sin^2(\alpha-\beta)=$$
$$\cos^2\alpha-2\cos\alpha\cos\beta+\cos^2\beta+\sin^2\alpha-2\sin\alpha\sin\beta+\sin^2\beta$$

$$[\cos^2(\alpha-\beta)+\sin^2(\alpha-\beta)]-2\cos(\alpha-\beta)+1=$$
$$(\cos^2\alpha+\sin^2\alpha)+(\cos^2\beta+\sin^2\beta)-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta$$

$$1-2\cos(\alpha-\beta)+1=1+1-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta$$
$$2-2\cos(\alpha-\beta)=2-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta$$
$$-2\cos(\alpha-\beta)=-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta$$
$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

Hence, we have that $\boxed{\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta}$

B. Derivation of $\cos(\alpha+\beta)$

$$\cos(\alpha+\beta)=\cos(\alpha-(-\beta))=\cos\alpha\cos(-\beta)+\sin\alpha\sin(-\beta)$$

Thus, using the even/odd identities, we get that

$$\boxed{\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta}$$

Together, the sum and difference formulas for $\cos$ are sometimes written:

$$\boxed{\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta}$$

Here, we interpret this as "take the top signs together, and take the bottom signs together."

C. Cofunction Identities I

1. $\boxed{\cos(\frac{\pi}{2} - \theta) = \sin\theta}$

   Justification:

$$\cos(\tfrac{\pi}{2} - \theta) = \cos\tfrac{\pi}{2}\cos\theta + \sin\tfrac{\pi}{2}\sin\theta = 0 \cdot \cos\theta + 1 \cdot \sin\theta = \sin\theta$$

2. $\boxed{\sin(\frac{\pi}{2} - \theta) = \cos\theta}$

   Justification:

   By the cofunction identity for $\cos$, we have that

$$\sin(\tfrac{\pi}{2} - \theta) = \cos(\tfrac{\pi}{2} - (\tfrac{\pi}{2} - \theta)) = \cos\theta$$

D. Derivation of $\sin(\alpha + \beta)$

Using the cofunction identity for $\cos$, we have that

$$\begin{aligned}
\sin(\alpha+\beta) &= \cos\left(\frac{\pi}{2} - (\alpha+\beta)\right) \\
&= \cos\left(\left(\frac{\pi}{2} - \alpha\right) - \beta\right) \\
&= \cos\left(\frac{\pi}{2} - \alpha\right)\cos\beta + \sin\left(\frac{\pi}{2} - \alpha\right)\sin\beta \\
&= \sin\alpha\cos\beta + \cos\alpha\sin\beta
\end{aligned}$$

Thus $\boxed{\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta}$

E. Derivation of $\sin(\alpha - \beta)$

$$\begin{aligned}
\sin(\alpha-\beta) &= \sin(\alpha + (-\beta)) \\
&= \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta) \\
&= \sin\alpha\cos\beta - \cos\alpha\sin\beta
\end{aligned}$$

Thus, $\boxed{\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta}$

Together, we have $\boxed{\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta}$

F. Formula for $\tan(\alpha \pm \beta)$

Writing $\tan(\alpha \pm \beta)$ as $\dfrac{\sin(\alpha \pm \beta)}{\cos(\alpha \pm \beta)}$, and then expanding and simplifying (Exercise), we get

$$\boxed{\tan(\alpha \pm \beta) = \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \tan \beta}}$$

Comments:

1. The above formula will only work when $\tan \alpha$ and $\tan \beta$ are defined!

2. If they are not defined, then you need to simplify the expression the long way, using

$$\boxed{\tan(\alpha \pm \beta) = \frac{\sin(\alpha \pm \beta)}{\cos(\alpha \pm \beta)}}$$

G. Cofunction Identities II

Using the quotient and reciprocal identities, along with the identities already established, we get the following cofunction identities (Exercise):

1. $\boxed{\tan(\frac{\pi}{2} - \theta) = \cot\theta}$

2. $\boxed{\cot(\frac{\pi}{2} - \theta) = \tan\theta}$

3. $\boxed{\sec(\frac{\pi}{2} - \theta) = \csc\theta}$

4. $\boxed{\csc(\frac{\pi}{2} - \theta) = \sec\theta}$

In fact, the cofunction identities are the reason for the prefix **co** in 3 of the trigonometric functions.

Co-sine is short for complementary sine, that is cosine is $\frac{\pi}{2}$-complementary to sine.

Similarly, cotangent is $\frac{\pi}{2}$-complementary to tangent and cosecant is $\frac{\pi}{2}$-complementary to secant.

H. Evaluation Examples

Example 1: Find $\cos(\frac{7\pi}{12})$

Solution

Write $\frac{7\pi}{12}$ as $\frac{\pi}{3} + \frac{\pi}{4}$!

Then

$$\begin{aligned}
\cos\left(\frac{7\pi}{12}\right) &= \cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right) \\
&= \cos\left(\frac{\pi}{3}\right)\,\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{3}\right)\,\sin\left(\frac{\pi}{4}\right) \\
&= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \\
&= \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}
\end{aligned}$$

Ans $\boxed{\frac{\sqrt{2}-\sqrt{6}}{4}}$

Example 2: Find $\tan(\frac{\pi}{12})$

Solution

Write $\frac{\pi}{12}$ as $\frac{\pi}{3} - \frac{\pi}{4}$!

Then

$$\tan\left(\frac{\pi}{12}\right) = \tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

$$= \frac{\tan\left(\frac{\pi}{3}\right) - \tan\left(\frac{\pi}{4}\right)}{1 + \tan\left(\frac{\pi}{3}\right)\ \tan\left(\frac{\pi}{4}\right)}$$

$$= \frac{\sqrt{3} - 1}{1 + \sqrt{3} \cdot 1}$$

$$= \frac{\sqrt{3} - 1}{\sqrt{3} + 1}$$

$$= \frac{\sqrt{3} - 1}{\sqrt{3} + 1} \cdot \frac{\sqrt{3} - 1}{\sqrt{3} - 1}$$

$$= \frac{3 - 2\sqrt{3} + 1}{3 - 1}$$

$$= \frac{4 - 2\sqrt{3}}{2}$$

$$= \frac{2(2 - \sqrt{3})}{2}$$

$$= 2 - \sqrt{3}$$

Ans $\boxed{2 - \sqrt{3}}$

Example 3: Find $\sin(\frac{\pi}{9})\cos(\frac{\pi}{18}) + \cos(\frac{\pi}{9})\sin(\frac{\pi}{18})$

Solution

Use Addition Formula for $\sin(\alpha + \beta)$ in reverse:

$$\begin{aligned}
\sin\left(\frac{\pi}{9}\right)\cos\left(\frac{\pi}{18}\right) + \cos\left(\frac{\pi}{9}\right)\sin\left(\frac{\pi}{18}\right) &= \sin\left[\left(\frac{\pi}{9}\right) + \left(\frac{\pi}{18}\right)\right] \\
&= \sin\left(\frac{2\pi}{18} + \frac{\pi}{18}\right) \\
&= \sin\left(\frac{3\pi}{18}\right) \\
&= \sin\left(\frac{\pi}{6}\right) \\
&= \frac{1}{2}
\end{aligned}$$

Ans $\boxed{\frac{1}{2}}$

I. Comments

1. From Examples 1 and 2 above, we see that we can now evaluate the wrapping function and the trigonometric functions any multiple of $\frac{\pi}{12}$, by considering sums/differences of multiples of $\frac{\pi}{3}$ and $\frac{\pi}{4}$.

2. $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$ $\quad \frac{5\pi}{12} = \frac{2\pi}{3} - \frac{\pi}{4}$ $\quad \frac{7\pi}{12} = \frac{\pi}{3} + \frac{\pi}{4}$ $\quad \frac{11\pi}{12} = \frac{2\pi}{3} + \frac{\pi}{4}$

$\frac{13\pi}{12} = \frac{4\pi}{3} - \frac{\pi}{4}$ $\quad \frac{17\pi}{12} = \frac{5\pi}{3} - \frac{\pi}{4}$ $\quad \frac{19\pi}{12} = \frac{4\pi}{3} + \frac{\pi}{4}$ $\quad \frac{23\pi}{12} = \frac{5\pi}{3} + \frac{\pi}{4}$

Exercises

1. Find $\sin(\frac{7\pi}{12})$

2. Find $\cos(\frac{5\pi}{12})$

3. Find $\tan(\frac{11\pi}{12})$

4. Find $w(\frac{23\pi}{12})$

5. Find $w(\frac{13\pi}{12})$

6. Find $\sec(\frac{13\pi}{12})$

7. Find $\cot(\frac{7\pi}{12})$

8. Find $w(\frac{17\pi}{12})$

9. Find $\cos(\frac{2\pi}{9})\cos(\frac{\pi}{18}) + \sin(\frac{2\pi}{9})\sin(\frac{\pi}{18})$

10. Find $\sin(\frac{2\pi}{9})\cos(\frac{\pi}{18}) - \cos(\frac{2\pi}{9})\sin(\frac{\pi}{18})$

11. Find $\cos(\frac{3\pi}{16})\cos(\frac{\pi}{16}) - \sin(\frac{3\pi}{16})\sin(\frac{\pi}{16})$

12. Find $\sin(\frac{\pi}{16})\cos(\frac{3\pi}{16}) + \cos(\frac{\pi}{16})\sin(\frac{3\pi}{16})$

13. Simplify $\tan(\alpha \pm \beta) = \dfrac{\sin(\alpha \pm \beta)}{\cos(\alpha \pm \beta)}$ to obtain the formula $\dfrac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$

14. Establish the cofunction identities for tangent, cotangent, secant, and cosecant as stated in this section.

5.4 Sum and Difference Formulas II

A. Summary

1. **Sum and Difference Formulas**

$$\boxed{\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta}$$

$$\boxed{\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta}$$

$$\boxed{\tan(\alpha \pm \beta) = \frac{\sin(\alpha \pm \beta)}{\cos(\alpha \pm \beta)} = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}}$$

Note: The second formulation above for $\tan(\alpha + \beta)$ only works when $\tan\alpha$ and $\tan\beta$ are defined.

2. **Cofunction Identities**

$$\boxed{\sin(\tfrac{\pi}{2} - \theta) = \cos\theta}$$

$$\boxed{\cos(\tfrac{\pi}{2} - \theta) = \sin\theta}$$

$$\boxed{\tan(\tfrac{\pi}{2} - \theta) = \cot\theta}$$

$$\boxed{\cot(\tfrac{\pi}{2} - \theta) = \tan\theta}$$

$$\boxed{\sec(\tfrac{\pi}{2} - \theta) = \csc\theta}$$

$$\boxed{\csc(\tfrac{\pi}{2} - \theta) = \sec\theta}$$

We will now use these formulas, in addition to the ones we already know, for simplifying and verifying trigonometric identities.

B. Examples

Example 1: Simplify $\sin(\theta + \frac{3\pi}{2})$

Solution

$$\sin(\theta + \tfrac{3\pi}{2}) = \sin\theta\cos\tfrac{3\pi}{2} + \cos\theta\sin\tfrac{3\pi}{2} = (\sin\theta)\cdot 0 + (\cos\theta)\cdot -1 = -\cos\theta$$

Ans $\boxed{-\cos\theta}$

Example 2: Simplify $\tan(\theta + \frac{\pi}{4})$

Solution

$$\tan\left(\theta + \frac{\pi}{4}\right) = \frac{\tan\theta + \tan\frac{\pi}{4}}{1 - \tan\theta\cdot\tan\frac{\pi}{4}} = \frac{\tan\theta + 1}{1 - (\tan\theta)\cdot 1} = \frac{1 + \tan\theta}{1 - \tan\theta}$$

Ans $\boxed{\dfrac{1 + \tan\theta}{1 - \tan\theta}}$

Example 3: Verify the identity: $\dfrac{\sin(\alpha+\beta)}{\sin\alpha\cos\beta} = 1 + \cot\alpha\tan\beta$

Solution

$$\frac{\sin(\alpha+\beta)}{\sin\alpha\cos\beta}$$

$$= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\sin\alpha\cos\beta}$$

$$= 1 + \frac{\cos\alpha\sin\beta}{\sin\alpha\cos\beta}$$

$$= 1 + \cot\alpha\tan\beta$$

Example 4: Verify the identity: $\sin(x+y)\sec x\sec y = \tan x + \tan y$

Solution

$$\sin(x+y)\sec x\sec y$$

$$= \left(\sin x\cos y + \cos x\sin y\right)\sec x\sec y$$

$$= \sin x\cos y\sec x\sec y + \cos x\sin y\sec x\sec y$$

$$= \sin x\cos y\cdot\frac{1}{\cos x}\cdot\frac{1}{\cos y} + \cos x\sin y\cdot\frac{1}{\cos x}\cdot\frac{1}{\cos y}$$

$$= \frac{\sin x}{\cos x} + \frac{\sin y}{\cos y}$$

$$= \tan x + \tan y$$

Exercises

1. Simplify $\sin(\theta + \pi)$

2. Simplify $\cos(\theta + \frac{3\pi}{4})$

3. Simplify $\tan(\theta + \frac{5\pi}{6})$

4. Simplify $\cos(\theta - \frac{2\pi}{3})$

5. Simplify $\tan(\theta - \frac{4\pi}{3})$

6. Simplify $\tan(\theta + \frac{\pi}{2})$

7. Show that $\sin(\theta + 2n\pi) = \sin\theta$ and $\cos(\theta + 2n\pi) = \cos\theta$, where n is an integer.

8. Verify the following trigonometric identities:

    a. $\sin(\frac{\pi}{2} + \theta) + \cos(\pi - \theta) + \sin(\frac{\pi}{2} - \theta) = \cos\theta$

    b. $\tan(\pi + \theta) - \tan(\pi - \theta) = 2\tan\theta$

    c. $\sin(\alpha - \beta)\cos\beta + \cos(\alpha - \beta)\sin\beta = \sin\alpha$

    d. $\frac{\tan(\alpha+\beta) - \tan\alpha}{1 + \tan(\alpha+\beta)\tan\alpha} = \frac{\sin\beta}{\cos\beta}$

    e. $\sin(\alpha + \beta)\sin(\alpha - \beta) = \sin^2\alpha - \sin^2\beta$

    f. $\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2\cos\alpha\sin\beta$

5.5 Double-Angle Formulas

A. Derivations

1. $\sin(2\theta) = \sin(\theta+\theta) = \sin\theta\cos\theta + \cos\theta\sin\theta = 2\sin\theta\cos\theta$

2. $\cos(2\theta) = \cos(\theta+\theta) = \cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - \sin^2\theta$

   a. **Alternate Sine Form:** Replacing $\cos^2\theta$ with $1-\sin^2\theta$

   $$\cos^2\theta - \sin^2\theta = (1-\sin^2\theta) - \sin^2\theta = 1 - 2\sin^2\theta$$

   b. **Alternate Cosine Form:** Replacing $\sin^2\theta$ with $1-\cos^2\theta$

   $$\cos^2\theta - \sin^2\theta = \cos^2\theta - (1-\cos^2\theta) = 2\cos^2\theta - 1$$

3. $\tan(2\theta) = \tan(\theta+\theta) = \dfrac{\tan\theta + \tan\theta}{1-\tan\theta\tan\theta} = \dfrac{2\tan\theta}{1-\tan^2\theta}$

   Note: As with the sum/difference formulas, this only holds when $\tan\theta$ is defined; otherwise $\tan(2\theta) = \dfrac{\sin(2\theta)}{\cos(2\theta)}$ must be used.

B. Summary

$$\boxed{\sin(2\theta) = 2\sin\theta\cos\theta}$$

$$\boxed{\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1}$$

$$\boxed{\tan(2\theta) = \frac{\sin(2\theta)}{\cos(2\theta)} = \frac{2\tan\theta}{1 - \tan^2\theta}}$$

Note: The second formulation above for $\tan(2\theta)$ only works when $\tan\theta$ is defined.

The above formulas are called the **double angle formulas**. The reason for the word "angle" will be explained later.

C. Examples

Example 1: Simplify $\sin(2\theta + \beta)$

Solution

$$\begin{aligned}
&\sin(2\theta + \beta) \\
&= \sin(2\theta)\cos\beta + \cos(2\theta)\sin\beta \\
&= 2\sin\theta\cos\theta\cos\beta + \left(\cos^2\theta - \sin^2\theta\right)\sin\beta
\end{aligned}$$

Ans $\boxed{2\sin\theta\cos\theta\cos\beta + \left(\cos^2\theta - \sin^2\theta\right)\sin\beta}$

Example 2: Derive a "triple-angle" formula for $\cos(3\theta)$

Solution

$$
\begin{aligned}
&\cos(3\theta)\\
&= \cos(2\theta + \theta)\\
&= \cos(2\theta)\cos\theta - \sin(2\theta)\sin\theta\\
&= \left(\cos^2\theta - \sin^2\theta\right)\cos\theta - 2\sin\theta\cos\theta\sin\theta\\
&= \cos^3\theta - \sin^2\theta\cos\theta - 2\sin^2\theta\cos\theta\\
&= \cos^3\theta - 3\sin^2\theta\cos\theta
\end{aligned}
$$

Ans $\boxed{\cos^3\theta - 3\sin^2\theta\cos\theta}$

Exercises

1. Simplify $\cos(2\theta + \beta)$

2. Simplify $\sin(\theta - 2\beta)$

3. Derive a “triple-angle” formula for $\sin(3\theta)$

4. Simplify $\cos(2\theta - 2\beta)$

5. Simplify $\frac{\cos 2\theta}{\sin \theta + \cos \theta}$

5.6 Power-Reducing Formulas

A. Derivation for $\sin^2\theta$, $\cos^2\theta$, and $\tan^2\theta$

1. $\cos(2\theta) = 1 - 2\sin^2\theta$, so $2\sin^2\theta = 1 - \cos(2\theta)$

   Then $\sin^2\theta = \dfrac{1-\cos(2\theta)}{2}$

2. $\cos(2\theta) = 2\cos^2\theta - 1$, so $2\cos^2\theta = 1 + \cos(2\theta)$

   Then $\cos^2\theta = \dfrac{1+\cos(2\theta)}{2}$

3. $\tan^2\theta = \dfrac{\sin^2\theta}{\cos^2\theta} = \dfrac{\frac{1-\cos(2\theta)}{2}}{\frac{1+\cos(2\theta)}{2}} = \dfrac{1-\cos(2\theta)}{1+\cos(2\theta)}$

B. Summary

$$\boxed{\sin^2\theta = \frac{1-\cos(2\theta)}{2}}$$

$$\boxed{\cos^2\theta = \frac{1+\cos(2\theta)}{2}}$$

$$\boxed{\tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}}$$

C. Examples

Example 1: Rewrite $\cos^4 x$ without powers

Solution

$$\cos^4 x$$

$$= (\cos^2 x)^2$$

$$= \left(\frac{1+\cos 2x}{2}\right)^2 \qquad \text{(power reducing formula)}$$

$$= \frac{(1+\cos 2x)^2}{4}$$

$$= \frac{1+2\cos 2x+\cos^2 2x}{4}$$

$$= \frac{1}{4}+\frac{1}{2}\cos 2x+\frac{1}{4}\cos^2 2x$$

$$= \frac{1}{4}+\frac{1}{2}\cos 2x+\frac{1}{4}\left[\frac{1+\cos 4x}{2}\right] \qquad \text{(power red. with } \theta = 2x\text{)}$$

$$= \frac{1}{4}+\frac{1}{2}\cos 2x+\frac{1}{4}\left[\frac{1}{2}+\frac{1}{2}\cos 4x\right]$$

$$= \frac{1}{4}+\frac{1}{2}\cos 2x+\frac{1}{8}+\frac{1}{8}\cos 4x$$

$$= \frac{3}{8}+\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x$$

Ans $\boxed{\frac{3}{8}+\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x}$

Example 2: Graph f, where $f(x) = 4\sin^2 x + 1$

Solution

Use the power reducing formula for sine!

$4\sin^2 x + 1$

$= 4(\frac{1-\cos 2x}{2}) + 1$

$= 2(1 - \cos 2x) + 1$

$= 2 - 2\cos 2x + 1$

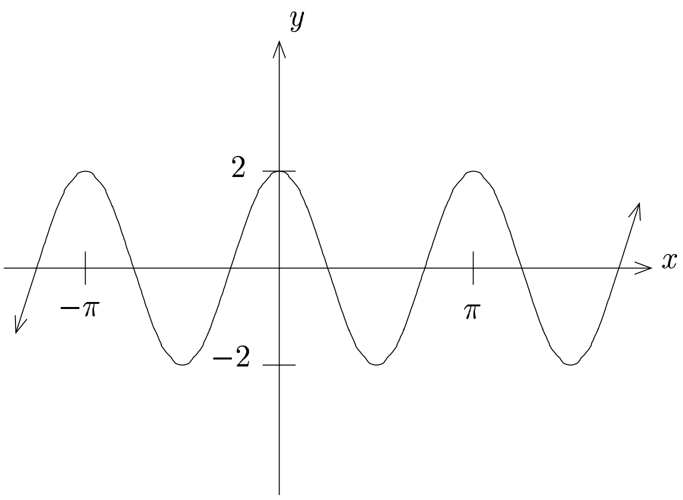
$= 3 - 2\cos 2x$

Thus, we graph f where $f(x) = 3 - 2\cos 2x$:

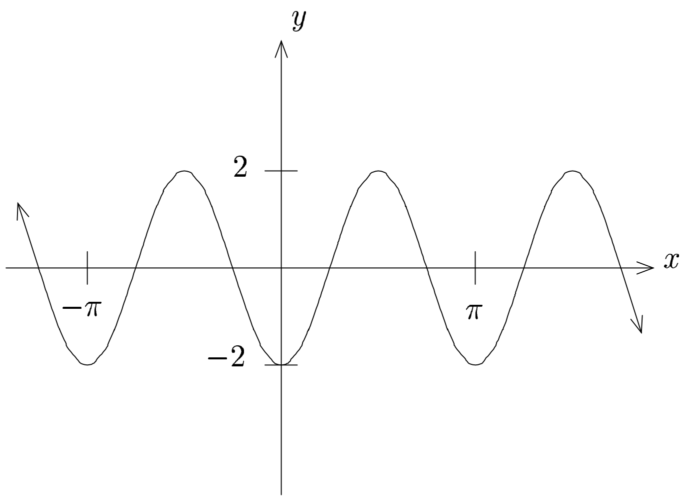
1. $2x = 0 \;\Rightarrow\; x = 0$

2. $2x = 2\pi \;\Rightarrow\; x = \pi$

3. Graph $y = 2\cos 2x$:

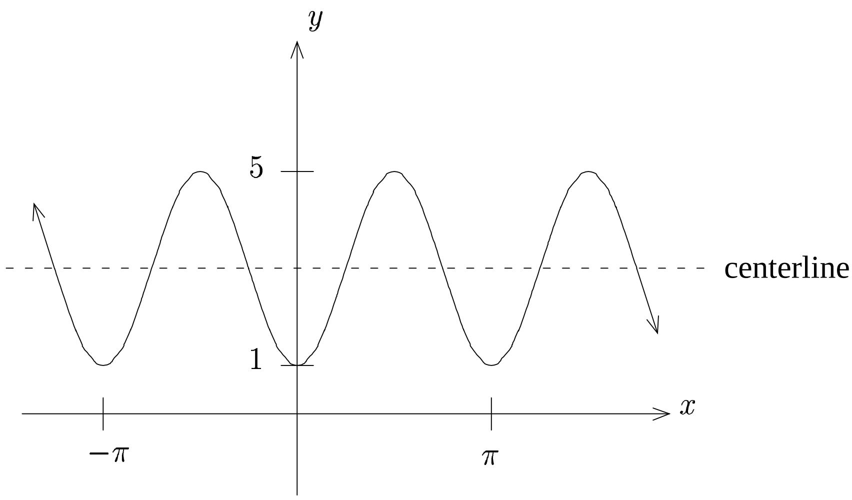


4. Reflect about the x-axis:



5. Shift up 3:

Ans



Exercises

1. Rewrite $\sin^4 x$ without powers.
2. Rewrite $\cos^4(2x)$ without powers.
3. Rewrite $\sin^4(3x)$ without powers.
4. Rewrite $\cos^4 x + \cos^2 x$ without powers.
5. Graph f, where $f(x) = \cos^2 x$
6. Graph f, where $f(x) = 3\sin^2 x - 2$

5.7 Half-Angle Relationships and Formulas

A. Half-Angle Relationships for sine and cosine

Since $\sin^2\alpha = \dfrac{1-\cos 2\alpha}{2}$ and $\cos^2\alpha = \dfrac{1-\cos 2\alpha}{2}$, if we replace α with $\frac{\theta}{2}$, we get

$$\boxed{\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{2}}$$

$$\boxed{\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos\theta}{2}}$$

Note: $\sin(\frac{\theta}{2})$ and $\cos(\frac{\theta}{2})$ will either be the positive square root or the negative square root of the above expression (but not both). The choice of root will depend on the specific value of θ that is used in a problem.

B. Half-Angle Formulas for tangent

Here we will be able to get an explicit formula for $\tan(\frac{\theta}{2})$ with no sign ambiguity.

$$\tan\left(\frac{\theta}{2}\right) = \frac{\sin(\frac{\theta}{2})}{\cos(\frac{\theta}{2})} = \frac{\sin^2(\frac{\theta}{2})}{\sin(\frac{\theta}{2})\cos(\frac{\theta}{2})} = \frac{2\sin^2(\frac{\theta}{2})}{2\sin(\frac{\theta}{2})\cos(\frac{\theta}{2})}$$

$$= \frac{2\sin^2(\frac{\theta}{2})}{\sin\left(2\cdot\frac{\theta}{2}\right)} \quad \text{(double angle formula)}$$

$$= \frac{2\sin^2(\frac{\theta}{2})}{\sin\theta}$$

$$= \frac{2(\frac{1-\cos\theta}{2})}{\sin\theta} \quad \text{(half-angle rel. for sine)}$$

$$= \frac{1-\cos\theta}{\sin\theta}$$

Now we will obtain another formulation:

$$
\begin{aligned}
\frac{1-\cos\theta}{\sin\theta} &= \frac{1-\cos\theta}{\sin\theta}\cdot\frac{\sin\theta}{\sin\theta} \\
&= \frac{(1-\cos\theta)\sin\theta}{\sin^2\theta} \\
&= \frac{(1-\cos\theta)\sin\theta}{1-\cos^2\theta} \quad \text{(Pythagorean I)} \\
&= \frac{(1-\cos\theta)\sin\theta}{(1+\cos\theta)(1-\cos\theta)} \quad \text{(diff. of squares)} \\
&= \frac{\sin\theta}{1+\cos\theta}
\end{aligned}
$$

Thus $\boxed{\tan\left(\frac{\theta}{2}\right) = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}}$

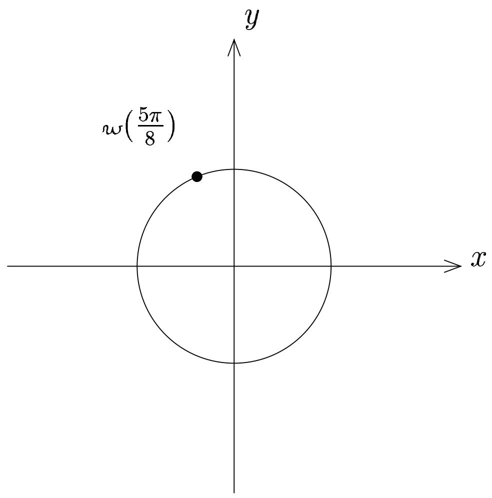
Note: The first expression $\frac{\sin\theta}{1+\cos\theta}$ is more versatile, since even multiples of π work correctly in it. However, provided these values are not used, the second expression $\frac{1-\cos\theta}{\sin\theta}$ is often simpler to use.

C. Evaluation Examples

Example 1: Find $w(\frac{5\pi}{8})$

Solution

First locate $w(\frac{5\pi}{8})$:



Use the half-angle relationships for sine and cosine:

$$\sin^2\left(\frac{5\pi}{8}\right) = \frac{1-\cos(\frac{5\pi}{4})}{2} = \frac{1-(-\frac{\sqrt{2}}{2})}{2} = \frac{1+\frac{\sqrt{2}}{2}}{2} = \frac{\frac{2+\sqrt{2}}{2}}{2} = \frac{2+\sqrt{2}}{4}$$

$$\cos^2\left(\frac{5\pi}{8}\right) = \frac{1+\cos(\frac{5\pi}{4})}{2} = \frac{1+(-\frac{\sqrt{2}}{2})}{2} = \frac{1-\frac{\sqrt{2}}{2}}{2} = \frac{\frac{2-\sqrt{2}}{2}}{2} = \frac{2-\sqrt{2}}{4}$$

Then, $x = \cos(\frac{5\pi}{8})$, and since $x < 0$, we have $x = -\sqrt{\frac{2-\sqrt{2}}{4}} = -\frac{\sqrt{2-\sqrt{2}}}{2}$.

Also $y = \sin(\frac{5\pi}{8})$, and since $y > 0$, we have $y = \sqrt{\frac{2+\sqrt{2}}{4}} = \frac{\sqrt{2+\sqrt{2}}}{2}$

Ans $\boxed{w\left(\frac{5\pi}{8}\right) = \left(-\frac{\sqrt{2-\sqrt{2}}}{2}, \frac{\sqrt{2+\sqrt{2}}}{2}\right)}$

Example 2: Find $\tan(\frac{7\pi}{8})$

Solution

Here we use the half-angle formula for tangent: $\tan\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{\sin\theta}$.

(We could have used the other half-angle formula here, and it would result in the same answer, but it would involve slightly more work)

$$\tan\left(\frac{7\pi}{8}\right) = \frac{1-\cos\left(\frac{7\pi}{4}\right)}{\sin\left(\frac{7\pi}{4}\right)} = \frac{1-\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = \frac{\frac{2-\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}}$$

$$= \frac{2-\sqrt{2}}{2} \cdot \frac{2}{-\sqrt{2}}$$

$$= \frac{2-\sqrt{2}}{-\sqrt{2}}$$

$$= \frac{2-\sqrt{2}}{-\sqrt{2}} \cdot \frac{-\sqrt{2}}{-\sqrt{2}}$$

$$= \frac{-\sqrt{2}(2-\sqrt{2})}{2}$$

$$= \frac{-2\sqrt{2}+2}{2}$$

$$= \frac{2(-\sqrt{2}+1)}{2}$$

$$= 1-\sqrt{2}$$

Ans $\boxed{1-\sqrt{2}}$

Exercises

1. Find $\cos(\frac{7\pi}{8})$

2. Find $\tan(\frac{3\pi}{8})$

3. Find $\tan(\frac{11\pi}{8})$

4. Find $\cos(\frac{9\pi}{8})$

5. Find $\cos(\frac{15\pi}{8})$

6. Find $\sin(\frac{9\pi}{16})$

5.8 Verifying More Trigonometric Identities

Examples

Example 1: Verify the identity: $\sin^2\theta\cos^2\theta = \frac{1}{8}(1-\cos 4\theta)$

Solution

$$\sin^2\theta\cos^2\theta$$

$$= (\sin\theta\cos\theta)^2$$

$$= \left(\frac{1}{2}\cdot 2\sin\theta\cos\theta\right)^2$$

$$= \left(\frac{1}{2}\cdot\sin 2\theta\right)^2 \qquad \text{(double angle formula)}$$

$$= \frac{1}{4}\sin^2 2\theta$$

$$= \frac{1}{4}\left[\frac{1-\cos 4\theta}{2}\right] \qquad \text{(power reducing formula)}$$

$$= \frac{1}{8}(1-\cos 4\theta)$$

Example 2: Verify the identity: $\cos\theta = \dfrac{1-\tan^2\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}$

Solution

Start with the right hand side:

$$\frac{1-\tan^2\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}$$

$$= \frac{1-\tan^2\left(\frac{\theta}{2}\right)}{\sec^2\left(\frac{\theta}{2}\right)} \qquad \text{(Pythagorean II)}$$

$$= \left[1-\tan^2\left(\frac{\theta}{2}\right)\right]\cdot\cos^2\left(\frac{\theta}{2}\right) \qquad \text{(reciprocal)}$$

$$= \cos^2\left(\frac{\theta}{2}\right) - \tan^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right)$$

$$= \cos^2\left(\frac{\theta}{2}\right) - \left(\frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)}\right)^2\cos^2\left(\frac{\theta}{2}\right)$$

$$= \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)$$

$$= \cos\left(2\cdot\frac{\theta}{2}\right) \qquad \text{(double angle formula)}$$

$$= \cos\theta$$

Example 3: Verify the identity: $\cot^2\left(\frac{\theta}{2}\right) = \frac{\sec\theta + 1}{\sec\theta - 1}$

Solution

$$\cot^2\left(\frac{\theta}{2}\right)$$

$$= \frac{1}{\tan^2(\frac{\theta}{2})}$$

$$= \frac{1}{(\frac{1-\cos\theta}{\sin\theta})^2} \qquad \text{(half-angle formula)}$$

$$= \frac{1}{\frac{(1-\cos\theta)^2}{\sin^2\theta}}$$

$$= \frac{\sin^2\theta}{(1-\cos\theta)^2}$$

$$= \frac{1-\cos^2\theta}{(1-\cos\theta)^2} \qquad \text{(Pythagorean I)}$$

$$= \frac{(1+\cos\theta)(1-\cos\theta)}{(1-\cos\theta)^2} \qquad \text{(diff. of squares)}$$

$$= \frac{1+\cos\theta}{1-\cos\theta}$$

$$= \frac{1+\frac{1}{\sec\theta}}{1+\frac{1}{\sec\theta}}$$

$$= \frac{\frac{\sec\theta+1}{\sec\theta}}{\frac{\sec\theta-1}{\sec\theta}}$$

$$= \frac{\sec\theta + 1}{\sec\theta - 1}$$

Exercises

Verify the following trigonometric identities:

1. $\tan(\frac{\pi}{4}+\theta)=\frac{1+\sin 2\theta}{\cos 2\theta}$

2. $4\cos^4\theta=4\cos^2\theta-\sin^2 2\theta$

3. $\sin^3\theta\sin 2\theta=\frac{3}{4}\cos\theta-\cos\theta\cos 2\theta+\frac{1}{4}\cos\theta\cos 4\theta$

4. $\frac{\sin^3\theta+\cos^3\theta}{\sin\theta+\cos\theta}=1-\frac{1}{2}\sin 2\theta$

5. $\frac{1+\sin 2\theta}{1+\cos 2\theta}=\frac{1}{2}(1+\tan\theta)^2$

6. $\frac{\sin 2\theta}{\sin\theta}-\frac{\cos 2\theta}{\cos\theta}=\sec\theta$

5.9 Product to Sum Formulas

A. Derivations

Consider the sum and difference formulas for sine:

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

Adding these two equations, we get:

$$\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta$$

Dividing by 2, we get:

$$\sin\alpha\cos\beta=\tfrac{1}{2}[\sin(\alpha-\beta)+\sin(\alpha+\beta)]$$

By similar methods, we also get:

$$\sin\alpha\sin\beta=\tfrac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$$

$$\cos\alpha\cos\beta=\tfrac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$

B. Summary

$$\boxed{\sin\alpha\cos\beta=\tfrac{1}{2}[\sin(\alpha-\beta)+\sin(\alpha+\beta)]}$$

$$\boxed{\sin\alpha\sin\beta=\tfrac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]}$$

$$\boxed{\cos\alpha\cos\beta=\tfrac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]}$$

Note: It is more useful to remember **how** to get the product-to-sum formulas than it is to memorize them.

C. Examples

Example 1: Evaluate $\sin(\frac{5\pi}{24})\cos(\frac{\pi}{24})$

Solution

Use $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha-\beta)+\sin(\alpha+\beta)]$

Thus

$$\begin{aligned}
\sin\left(\frac{5\pi}{24}\right)\cos\left(\frac{\pi}{24}\right) &= \frac{1}{2}\left[\sin\left(\frac{5\pi}{24}-\frac{\pi}{24}\right)+\sin\left(\frac{5\pi}{24}+\frac{\pi}{24}\right)\right] \\
&= \frac{1}{2}\left[\sin\left(\frac{\pi}{6}\right)+\sin\left(\frac{\pi}{4}\right)\right] \\
&= \frac{1}{2}\left[\frac{1}{2}+\frac{\sqrt{2}}{2}\right] \\
&= \frac{1}{2}\left(\frac{1+\sqrt{2}}{2}\right) \\
&= \frac{1+\sqrt{2}}{4}
\end{aligned}$$

Ans $\boxed{\frac{1+\sqrt{2}}{4}}$

Example 2: Express $\sin(2\theta)\sin(5\theta)$ as a sum or difference.

Solution

Use $\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$

Then

$$\sin(2\theta)\sin(5\theta) = \frac{1}{2}\left[\cos(2\theta - 5\theta) - \cos(2\theta + 5\theta)\right]$$

$$= \frac{1}{2}\left[\cos(-3\theta) - \cos(7\theta)\right]$$

$$= \frac{\cos(3\theta) - \cos(7\theta)}{2}$$

Ans $\boxed{\dfrac{\cos(3\theta) - \cos(7\theta)}{2}}$

Exercises

1. Evaluate $\cos\left(\frac{5\pi}{24}\right)\cos\left(\frac{\pi}{24}\right)$

2. Evaluate $\sin\left(\frac{5\pi}{24}\right)\sin\left(\frac{\pi}{24}\right)$

3. Evaluate $\sin\left(\frac{13\pi}{24}\right)\cos\left(\frac{7\pi}{24}\right)$

4. Evaluate $\sin\left(\frac{7\pi}{24}\right)\cos\left(\frac{13\pi}{24}\right)$

5. Express $\sin(3\theta)\sin(5\theta)$ as a sum or difference.

6. Express $\cos(4\theta)\cos(2\theta)$ as a sum or difference.

7. Express $\sin(7\theta)\cos\theta$ as a sum or difference.

8. Follow the derivation in Part A to verify the formulas for $\sin\alpha\sin\beta$ and $\cos\alpha\cos\beta$.

5.10 Sum to Product Formulas

A. Derivations

Consider the sum and difference formulas for sine:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

Adding these two equations, we get:

$$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$$

Let $A = \alpha + \beta$ and $B = \alpha - \beta$.

Then $A + B = 2\alpha$ and $A - B = 2\beta$.

Thus $\alpha = \frac{A+B}{2}$ and $\beta = \frac{A-B}{2}$.

Since $\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$, we have that

$$\sin A + \sin B = 2\sin(\tfrac{A+B}{2})\cos(\tfrac{A-B}{2}).$$

The derivations of the other sum to product formulas are similar.

B. Summary

$$\boxed{\sin A + \sin B = 2\sin\left(\tfrac{A+B}{2}\right)\cos\left(\tfrac{A-B}{2}\right)}$$

$$\boxed{\sin A - \sin B = 2\sin\left(\tfrac{A-B}{2}\right)\cos\left(\tfrac{A+B}{2}\right)}$$

$$\boxed{\cos A + \cos B = 2\cos\left(\tfrac{A+B}{2}\right)\cos\left(\tfrac{A-B}{2}\right)}$$

$$\boxed{\cos A - \cos B = -2\sin\left(\tfrac{A+B}{2}\right)\sin\left(\tfrac{A-B}{2}\right)}$$

Note: Again, it is more useful to remember **how** to get the sum-to-product formulas than it is to memorize them.

C. Examples

Example 1: Express $\sin 3\theta + \sin 5\theta$ as a product

Solution

Use $\sin A + \sin B = 2\sin\left(\tfrac{A+B}{2}\right)\cos\left(\tfrac{A-B}{2}\right)$:

Thus

$$\sin 3\theta + \sin 5\theta = 2\sin\left(\frac{3\theta+5\theta}{2}\right)\cos\left(\frac{3\theta-5\theta}{2}\right)$$

$$= 2\sin(4\theta)\cos(-\theta)$$

$$= 2\sin(4\theta)\cos\theta \qquad \text{(even identity for cosine)}$$

Ans $\boxed{2\sin(4\theta)\cos\theta}$

Example 2: Express $\cos 7\theta - \cos 4\theta$ as a product

Solution

Use $\cos A - \cos B = -2\sin(\frac{A+B}{2})\sin(\frac{A-B}{2})$:

Thus

$$\cos 7\theta - \cos 4\theta = -2\sin\left(\frac{7\theta + 4\theta}{2}\right)\sin\left(\frac{7\theta - 4\theta}{2}\right)$$

$$= -2\sin\left(\frac{11\,\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)$$

Ans $\boxed{-2\sin(\frac{11\,\theta}{2})\sin(\frac{3\theta}{2})}$

Exercises

1. Express $\sin(6\theta) + \sin(2\theta)$ as a product.

2. Express $\cos(5\theta) - \cos\theta$ as a product.

3. Express $\sin(3\theta) - \sin(7\theta)$ as a product.

4. Express $\cos(2\theta) + \cos(4\theta)$ as a product.

5. Follow the derivation in Part A to verify the formulas for

   $\sin A - \sin B$, $\cos A + \cos B$, and $\cos A - \cos B$.

5.11 Verifying Even More Trigonometric Identities

Examples

Example 1: Verify the identity: $\dfrac{\sin\theta + \sin 3\theta}{\cos\theta + \cos 3\theta} = \tan 2\theta$

Solution

$$\frac{\sin\theta + \sin 3\theta}{\cos\theta + \cos 3\theta}$$

$$= \frac{2\sin\left(\frac{\theta+3\theta}{2}\right)\cos\left(\frac{\theta-3\theta}{2}\right)}{2\cos\left(\frac{\theta+3\theta}{2}\right)\cos\left(\frac{\theta-3\theta}{2}\right)} \qquad \text{(sum to product formulas)}$$

$$= \frac{\sin\left(\frac{4\theta}{2}\right)}{\cos\left(\frac{4\theta}{2}\right)}$$

$$= \frac{\sin(2\theta)}{\cos(2\theta)}$$

$$= \tan 2\theta$$

Example 2: Verify the identity: $\frac{\sin\alpha + \sin\beta}{\sin\alpha - \sin\beta} = \tan\left(\frac{\alpha+\beta}{2}\right)\cot\left(\frac{\alpha-\beta}{2}\right)$

Solution

$$\frac{\sin\alpha + \sin\beta}{\sin\alpha - \sin\beta}$$

$$= \frac{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)} \qquad \text{(sum to product formulas)}$$

$$= \frac{\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)} \cdot \frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\sin\left(\frac{\alpha-\beta}{2}\right)}$$

$$= \tan\left(\frac{\alpha+\beta}{2}\right)\cot\left(\frac{\alpha-\beta}{2}\right)$$

Exercises

Verify the following trigonometric identities:

1. $\frac{\cos\theta - \cos 3\theta}{\sin\theta + \sin 3\theta} = \tan\theta$

2. $\frac{\cos 4\theta - \cos 2\theta}{2\sin 3\theta} = -\sin\theta$

3. $\frac{\cos\alpha + \cos\beta}{\cos\alpha - \cos\beta} = -\cot\left(\frac{\alpha+\beta}{2}\right)\cot\left(\frac{\alpha-\beta}{2}\right)$

4. $\frac{\cos\theta - \cos 5\theta}{\sin\theta + \sin 5\theta} = \tan 2\theta$

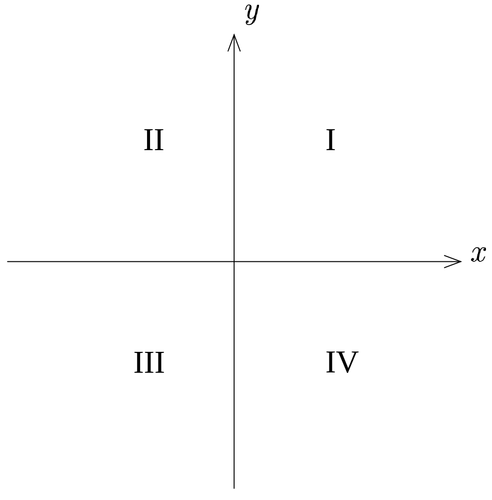
Chapter 6

Advanced Trigonometric Concepts

6.1 Capital Trigonometric Functions

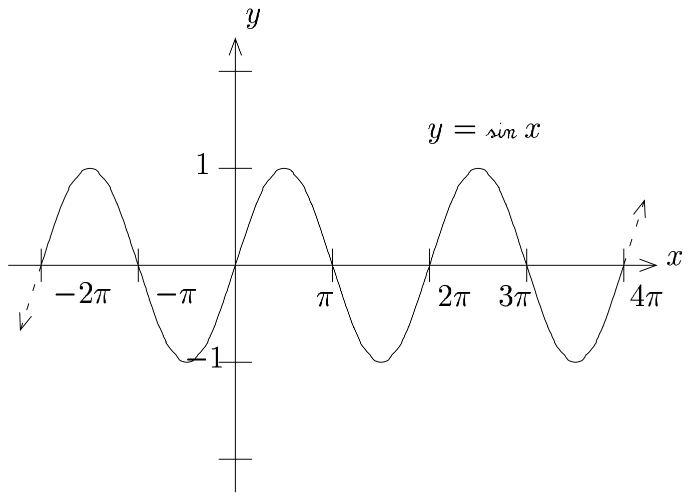
A. Quadrants in the xy-Plane

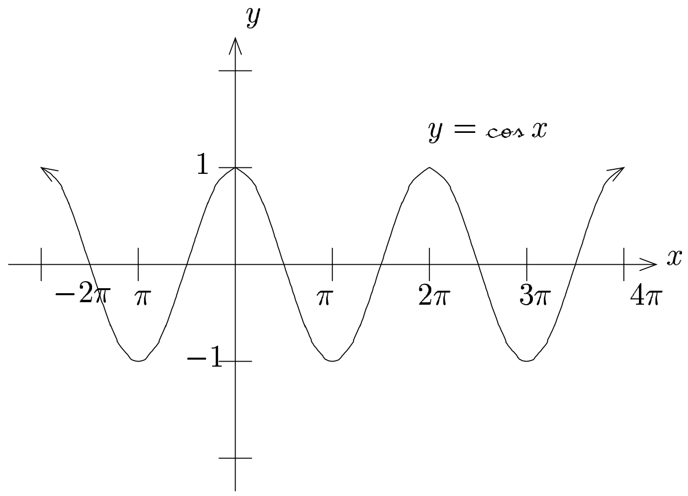
Later in this chapter it will be useful to be familiar with the idea of **quadrants in the xy-plane**. The xy-plane is divided into 4 quadrants by the x and y axes. They are named as follows:

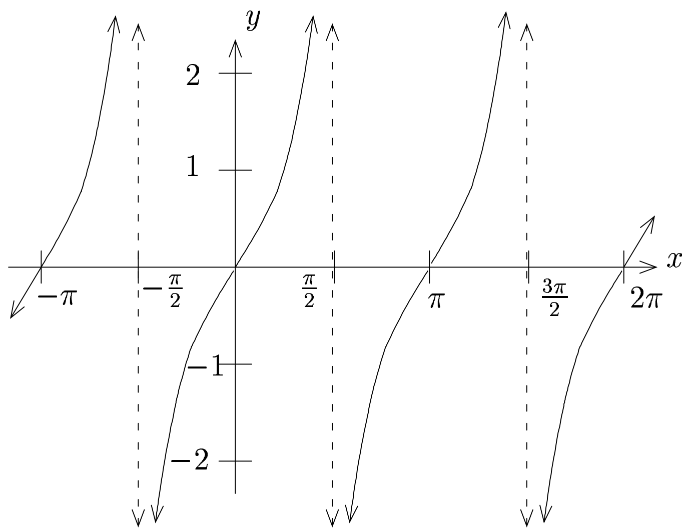


B. The Six Trigonometric Functions

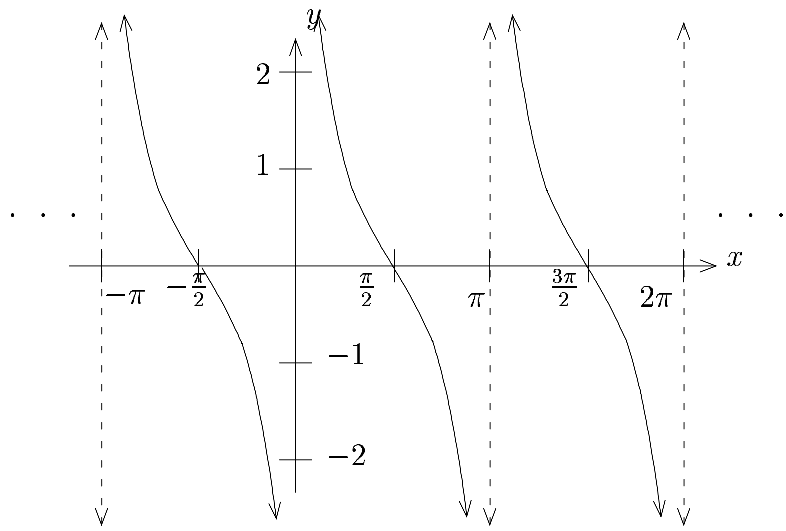
To motivate what comes next, let us first review the graphs of the six trigonometric functions.



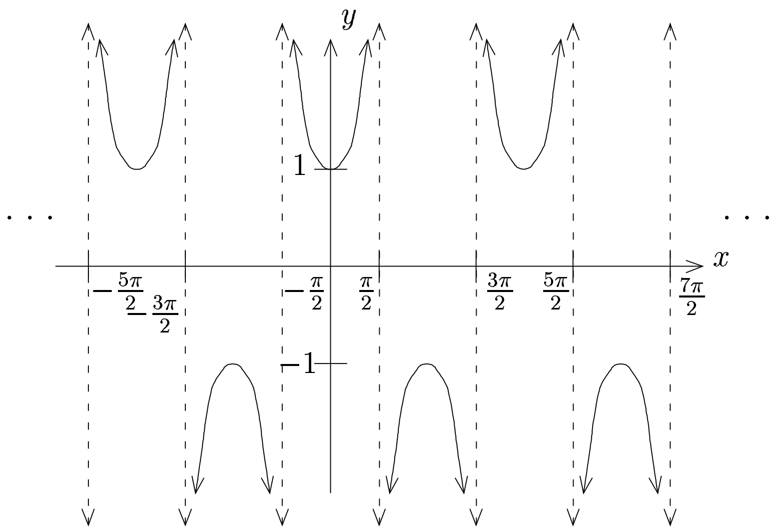




$y = \tan x$

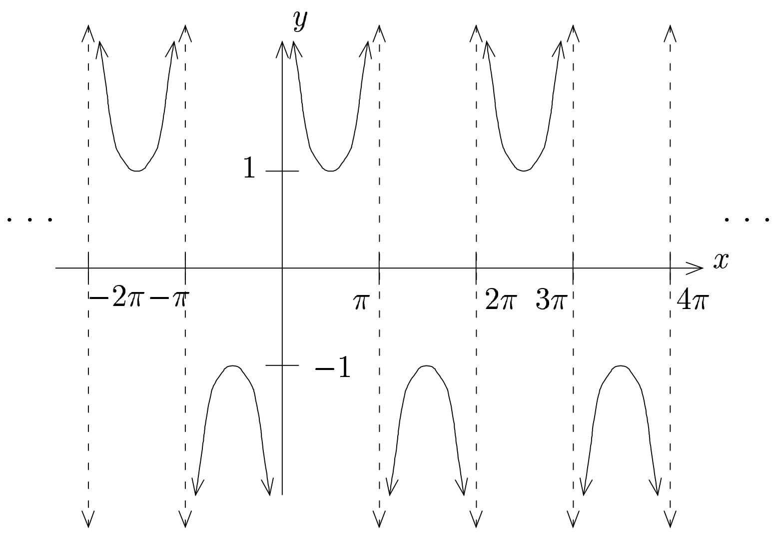


$y = \cot x$



y
1
-1
x
$-\frac{5\pi}{2}$
$-\frac{3\pi}{2}$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
$\frac{3\pi}{2}$
$\frac{5\pi}{2}$
$\frac{7\pi}{2}$
$\cdots$
$\cdots$

$y = \sec x$



y
1
-1
x
-2π
$-\pi$
π
2π
3π
4π
$\cdots$
$\cdots$

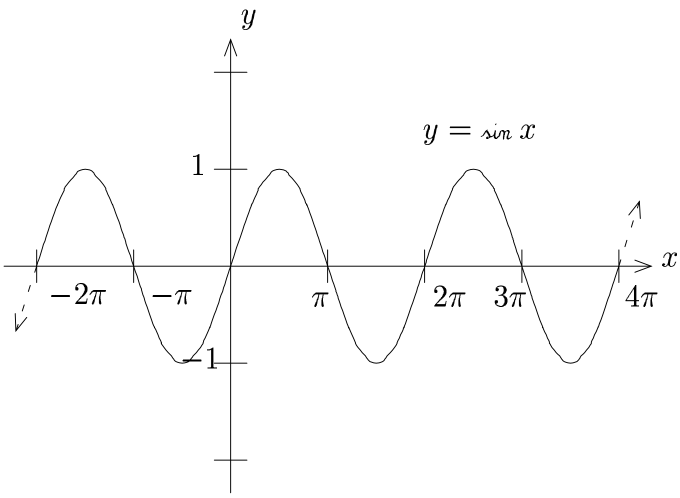
$y = \csc x$

C. Motivation

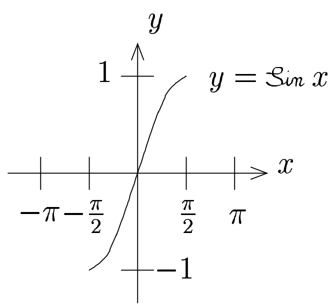
All six trigonometric functions fail the horizontal line test, so are **not** one-to-one/invertible.

We therefore define the capital trigonometric functions.

D. Capital Sine Construction



To make this function one-to-one, **without changing the range**, one choice that can be made is to throw away everything except the part of the graph between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. This is not the only choice, but it is the most obvious choice.



The residual function is a capital function. We call it $\mathcal{S}in$.

Thus $\mathcal{S}in\, x = \sin x;\ x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

E. Definitions of the Capital Trigonometric Functions

By similar considerations, we can define all of the capital trigonometric functions as well.

1. $\mathcal{S}in\, x = sin\, x;\ x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

2. $\mathcal{C}os\, x = cos\, x;\ x \in [0, \pi]$

3. $\mathcal{T}an\, x = tan\, x;\ x \in (-\frac{\pi}{2}, \frac{\pi}{2})$

4. $\mathcal{C}ot\, x = cot\, x;\ x \in (0, \pi)$

5. $\mathcal{S}ec\, x = sec\, x;\ x \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$

6. $\mathcal{C}sc\, x = csc\, x;\ x \in [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$

F. Comments

1. The only difference between the behavior of the capital trigonometric functions and the ordinary trigonometric functions is the restricted domain.

2. Like all capital functions, the capital trigonometric functions are invertible.

6.2 Capital Trigonometric Problems I

A. Solving Problems With Capital Trigonometric Functions

1. Rewrite the capital trigonometric functions as the ordinary trigonometric functions with the appropriate domain restriction.

2. Use ordinary trigonometric identities to solve the problem.

3. Use the restricted domain to remove the ambiguity in sign.

B. Examples

Example 1: You know $\mathrm{Sin}\,\theta = \frac{2}{3}$. Find $\cos\theta$.

Solution

1. $\mathrm{Sin}\,\theta = \frac{2}{3}$, so

$\sin\theta = \frac{2}{3};\ \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

2. To get $\cos\theta$ from $\sin\theta$, we use $\cos^2\theta + \sin^2\theta = 1$:

$$\cos^2\theta + \left(\frac{2}{3}\right)^2 = 1$$

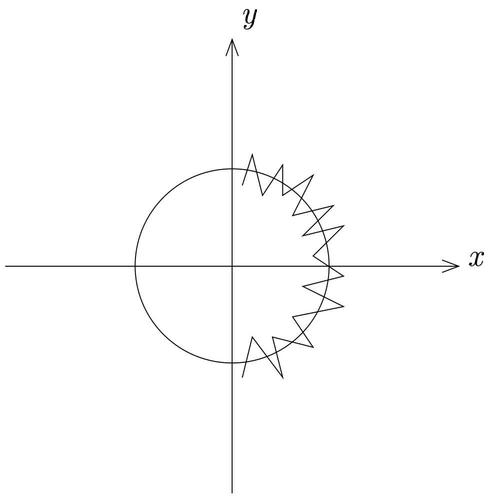
$$\cos^2\theta + \frac{4}{9} = 1$$

$$\cos^2\theta = \frac{5}{9}$$

$$\cos\theta = \pm\frac{\sqrt{5}}{3}$$

3. Use the restricted domain to try to remove the sign ambiguity:

Since $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, we are in the region marked:



Here $\cos\theta \geq 0$, so

Ans $\boxed{\cos\theta = \frac{\sqrt{5}}{3}}$

Example 2: You know $\mathrm{Cos}\,\theta = -\frac{1}{4}$. Find $\sin\theta$.

Solution

1. $\mathrm{Cos}\,\theta = -\frac{1}{4}$, so

$\cos\theta = -\frac{1}{4};\ \theta \in [0, \pi]$

2. To get $\sin\theta$ from $\cos\theta$, we use $\cos^2\theta + \sin^2\theta = 1$:

$$\left(-\frac{1}{4}\right)^2 + \sin^2\theta = 1$$

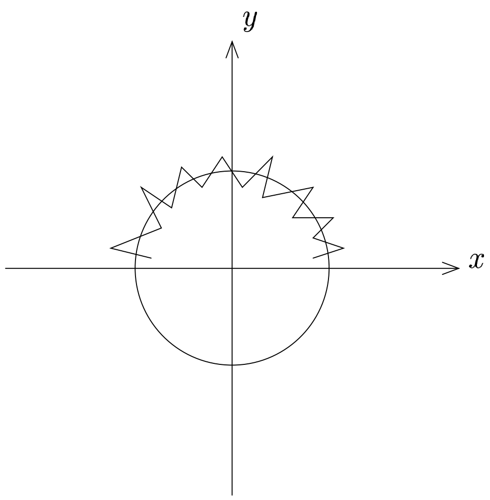
$$\frac{1}{16} + \sin^2\theta = 1$$

$$\sin^2\theta = \frac{15}{16}$$

$$\sin\theta = \pm\frac{\sqrt{15}}{4}$$

3. Use the restricted domain to try to remove the sign ambiguity:

Since $\theta \in [0, \pi]$, we are in the region marked:



Here $\sin\theta \geq 0$, so

Ans $\boxed{\sin\theta = \frac{\sqrt{15}}{4}}$

Exercises

1. You know $\mathcal{C}os\,\theta = \frac{3}{4}$. Find $sin\,\theta$.

2. You know $\mathcal{S}in\,\theta = -\frac{1}{3}$. Find $cos\,\theta$.

3. You know $\mathcal{S}in\,\theta = \frac{2}{5}$. Find $cos\,\theta$.

4. You know $\mathcal{C}os\,\theta = -\frac{5}{6}$. Find $sin\,\theta$.

5. You know $\mathcal{S}in\,\theta = \frac{2}{3}$. Find $cos\,\theta$.

6. State the values of θ for which $\mathcal{S}in^2\theta + \mathcal{C}os^2\theta = 1$.

6.3 Capital Trigonometric Problems II

We consider some more complicated examples.

Example 1: Know $\mathcal{S}in\,\theta = -\frac{3}{4}$. Find $\sin 2\theta$

Solution

1. $\mathcal{S}in\,\theta = -\frac{3}{4}$

   $\sin\theta = -\frac{3}{4};\ \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

2. We know that $\sin 2\theta = 2\sin\theta\cos\theta$, so we need $\cos\theta$.

   To get $\cos\theta$, we use $\cos^2\theta + \sin^2\theta = 1$

$$\cos^2\theta + \left(-\frac{3}{4}\right)^2 = 1$$

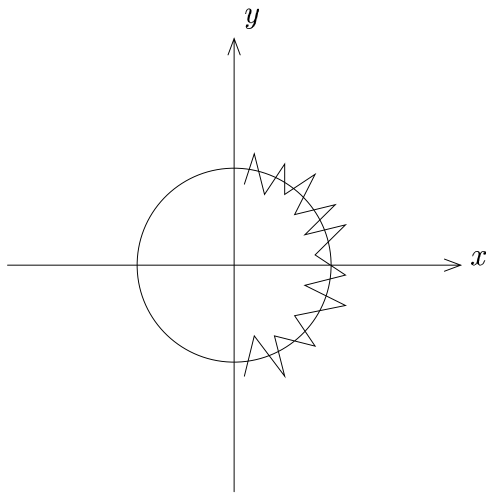
$$\cos^2\theta + \frac{9}{16} = 1$$

$$\cos^2\theta = \frac{7}{16}$$

$$\cos\theta = \pm\frac{\sqrt{7}}{4}$$

3. Use the restricted domain to try to remove the sign ambiguity:

Since $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, we are in the region marked:



Here $\cos\theta \geq 0$, so $\cos\theta = \frac{\sqrt{7}}{4}$.

However, our original problem was to find $\sin 2\theta$.

Thus,

$$\begin{aligned}
\sin 2\theta &= 2\sin\theta\cos\theta \\
&= (2)\left(-\frac{3}{4}\right)\left(\frac{\sqrt{7}}{4}\right) \\
&= -\frac{3}{2}\left(\frac{\sqrt{7}}{4}\right) \\
&= -\frac{3\sqrt{7}}{8}
\end{aligned}$$

Ans $\boxed{\sin 2\theta = -\frac{3\sqrt{7}}{8}}$

Example 2: Know $\cos\theta = -\frac{1}{5}$. Find $\sin 2\theta$

Solution

1. $\cos\theta = -\frac{1}{5}$

   $\cos\theta = -\frac{1}{5};\ \theta \in [0, \pi]$

2. We know that $\sin 2\theta = 2\sin\theta\cos\theta$, so we need $\sin\theta$.

   To get $\sin\theta$, we use $\cos^2\theta + \sin^2\theta = 1$

$$\left(-\frac{1}{5}\right)^2 + \sin^2\theta = 1$$

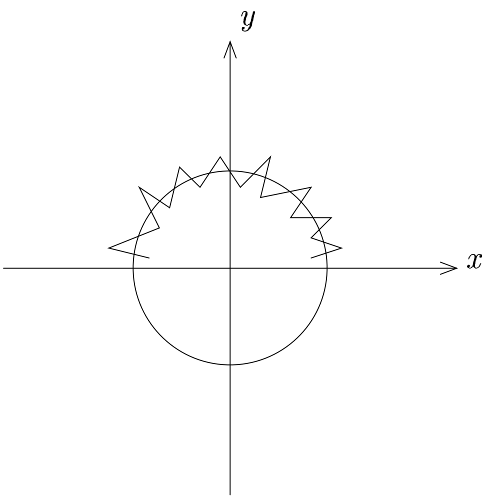
$$\frac{1}{25} + \sin^2\theta = 1$$

$$\sin^2\theta = \frac{24}{25}$$

$$\sin\theta = \pm\frac{\sqrt{24}}{5} = \pm\frac{2\sqrt{6}}{5}$$

3. Use the restricted domain to try to remove the sign ambiguity:

Since $\theta \in [0, \pi]$, we are in the region marked:



Here $\sin\theta \geq 0$, so $\sin\theta = \frac{2\sqrt{6}}{5}$.

However, our original problem was to find $\sin 2\theta$.

Thus,

$$\begin{aligned}
\sin 2\theta &= 2\sin\theta\cos\theta \\
&= (2)\left(\frac{2\sqrt{6}}{5}\right)\left(-\frac{1}{5}\right) \\
&= -\left(\frac{4\sqrt{6}}{5}\right)\left(\frac{1}{5}\right) \\
&= -\frac{4\sqrt{6}}{25}
\end{aligned}$$

Ans $\boxed{\sin 2\theta = -\frac{4\sqrt{6}}{25}}$

Example 3: Know $\mathcal{S}ec\,\theta = 5.$ Find $\tan\theta$

Solution

1. $\mathcal{S}ec\,\theta = 5$

   $\sec\theta = 5;\ \theta \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$

2. To get $\tan\theta$, we use $1 + \tan^2\theta = \sec^2\theta$

$$1 + \tan^2\theta = 5^2$$

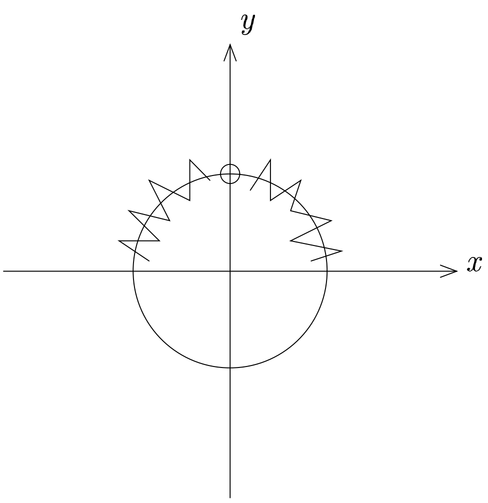
$$1 + \tan^2\theta = 25$$

$$\tan^2\theta = 24$$

$$\tan\theta = \pm\sqrt{24} = \pm 2\sqrt{6}$$

3. Use the restricted domain to try to remove the sign ambiguity:

   Since $\theta \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$, we are in the region marked:



   We see that $\tan\theta \geq 0$ in quadrant I but $\tan\theta \leq 0$ in quadrant II!

Thus we have no **initial** help.

However, we were originally given $\mathcal{S}ec\,\theta = 5$, so in particular $sec\,\theta > 0$.

This can not happen in quadrant II, so we must be in quadrant I.

Thus $tan\,\theta \geq 0$ and so we have that $tan\,\theta = 2\sqrt{6}$.

Ans $\boxed{tan\,\theta = 2\sqrt{6}}$

Example 4: Know $\mathcal{S}in\,\theta = -\frac{3}{5}$. Find $cot\,\theta$

Solution

1. $\mathcal{S}in\,\theta = -\frac{3}{5}$

   $sin\,\theta = -\frac{3}{5};\ \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

2. To get $cot\,\theta$, there are many different methods that can be used. For example:

   a. One method:

   First find $csc\,\theta = \frac{1}{sin\,\theta}$. Then use $cot^2\,\theta + 1 = csc^2\,\theta$

   b. Another method:

   Use $cos^2\,\theta + sin^2\,\theta = 1$ to find $cos\,\theta$. Then use $cot\,\theta = \frac{cos\,\theta}{sin\,\theta}$.

   Here let us arbitrarily use the first method.

   $csc\,\theta = \frac{1}{sin\,\theta} = \frac{1}{-\frac{3}{5}} = -\frac{5}{3}$.

Then

$$\cot^2\theta + 1 = \csc^2\theta$$

$$\cot^2\theta + 1 = \left(-\frac{5}{3}\right)^2$$

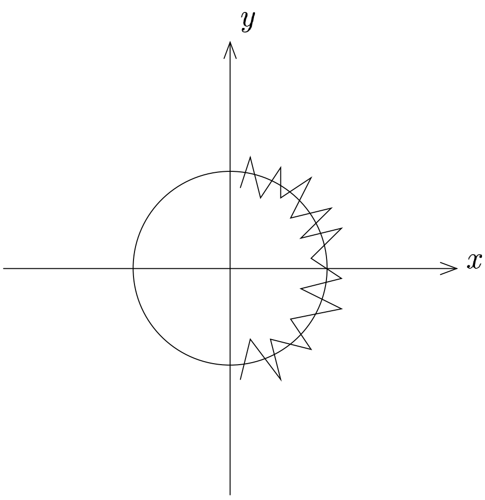
$$\cot^2\theta + 1 = \frac{25}{9}$$

$$\cot^2\theta = \frac{16}{9}$$

$$\cot\theta = \pm\frac{4}{3}$$

3. Use the restricted domain to try to remove the sign ambiguity:

Since $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, we are in the region marked:



We see that $\cot\theta \geq 0$ in quadrant I but $\cot\theta \leq 0$ in quadrant IV.

Thus we have no **initial** help!

However, since we were originally given $\sin\theta = -\frac{3}{5}$.

Thus $\sin\theta < 0$.

This can not happen in quadrant I, so we must be in quadrant IV.

Thus $\cot\theta \leq 0$ and so we have that $\cot\theta = -\frac{4}{3}$.

Ans $\boxed{\cot\theta = -\frac{4}{3}}$

Exercises

1. Know $\operatorname{Sin}\theta = -\frac{2}{5}$.   Find $\sin 2\theta$.

2. Know $\operatorname{Cos}\theta = \frac{2}{3}$.   Find $\sin 2\theta$.

3. Know $\operatorname{Sin}\theta = \frac{3}{4}$.   Find $\cos 2\theta$.

4. Know $\operatorname{Sec}\theta = 3$.   Find $\tan\theta$.

5. Know $\operatorname{Csc}\theta = -4$.   Find $\cot\theta$.

6. Know $\operatorname{Tan}\theta = -2$.   Find $\sec\theta$.

7. Know $\operatorname{Cos}\theta = -\frac{2}{3}$.   Find $\cot\theta$.

8. Know $\operatorname{Cot}\theta = 2$.   Find $\sin 2\theta$.

9. Know $\operatorname{Csc}\theta = -5$.   Find $\tan(\frac{\theta}{2})$.

10. Know $\operatorname{Sec}\theta = -3$.   Find $w(\theta)$.

11. Know $\operatorname{Tan}\theta = -\frac{1}{4}$.   Find $w(\theta)$.

12. Know $\operatorname{Tan}\theta = \frac{1}{2}$ and $\operatorname{Csc}\alpha = -6$.   Find $\sin(\theta + \alpha)$.

6.4 Inverse Trigonometric Functions

A. Introduction

Even though the ordinary trigonometric functions are **not** invertible, the capital trigonometric functions **are** (by design).

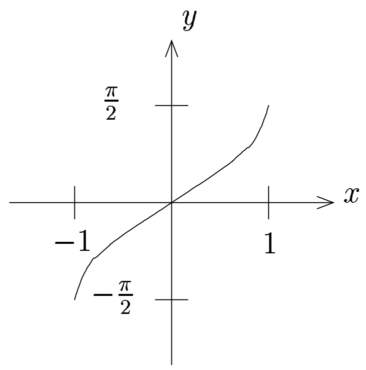
The inverses of the capital trigonometric functions are called the **inverse trigonometric functions**.

B. Domain and Range of the Inverse Trigonometric Functions

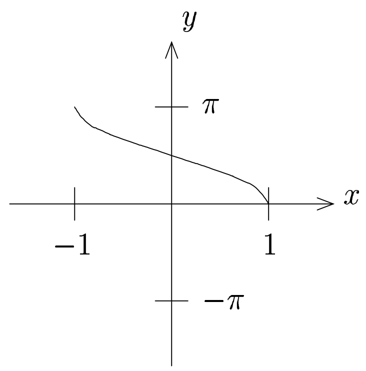
Using the techniques for finding $\mathcal{dom}(f^{-1})$ and $\mathcal{rng}(f^{-1})$ along with the properties of the capital trigonometric functions, we have:

	Domain	Range
$\mathcal{Sin}^{-1}$	$[-1,1]$	$[-\frac{\pi}{2},\frac{\pi}{2}]$
$\mathcal{Cos}^{-1}$	$[-1,1]$	$[0,\pi]$
$\mathcal{Tan}^{-1}$	$(-\infty,\infty)$	$(-\frac{\pi}{2},\frac{\pi}{2})$
$\mathcal{Cot}^{-1}$	$(-\infty,\infty)$	$(0,\pi)$
$\mathcal{Sec}^{-1}$	$(-\infty,-1]\cup[1,\infty)$	$[0,\frac{\pi}{2})\cup(\frac{\pi}{2},\pi]$
$\mathcal{Csc}^{-1}$	$(-\infty,-1]\cup[1,\infty)$	$[-\frac{\pi}{2},0)\cup(0,\frac{\pi}{2}]$

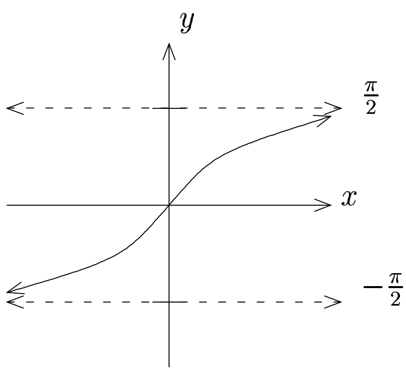
C. Graphs of the Inverse Trigonometric Functions



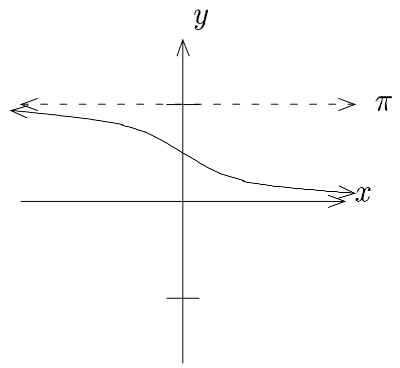
$y = \mathrm{Sin}^{-1} x$



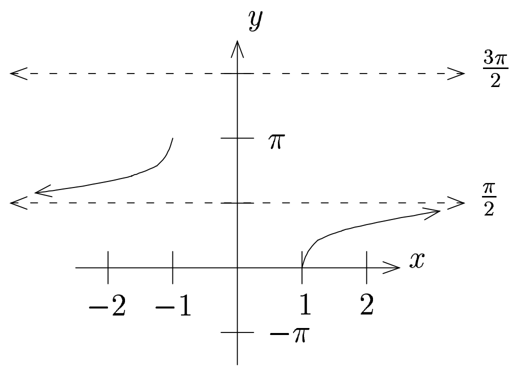
$y = \mathrm{Cos}^{-1} x$



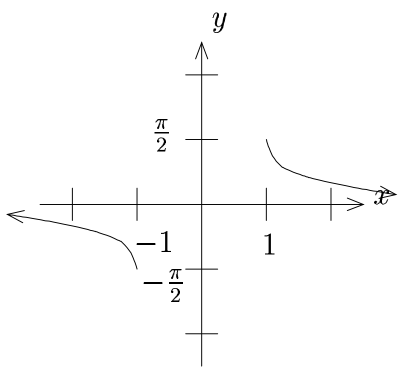
$y = \mathrm{Tan}^{-1} x$



$y = \mathcal{C}ot^{-1} x$



$y = \mathcal{S}ec^{-1} x$



$y = \mathcal{C}sc^{-1} x$

D. Comments

1. **Warnings:**

a. "$^{-1}$" means inverse function when attached to functions, **not** reciprocal

$\mathcal{S}in^{-1} x$ means inverse sine of x

$\frac{1}{\mathcal{S}in\, x}$ takes the values of cosecant

Note: These are different.

b. $\mathcal{S}in^{-1}$ and $\mathcal{sin}$ are **not** inverses! $\mathcal{sin}$ does not have an inverse!
The functions that are inverses are $\mathcal{S}in^{-1}$ and $\mathcal{S}in$. Be careful of this in problems.

c. Some authors are lazy and write $\mathcal{sin}^{-1}$, when they really mean $\mathcal{S}in^{-1}$.
To avoid confusion, write $\mathcal{S}in^{-1}$ if that is what is intended.

2. In some older books, $\mathcal{S}in^{-1}$, $\mathcal{C}os^{-1}$, $\mathcal{T}an^{-1}$, $\mathcal{C}ot^{-1}$, $\mathcal{S}ec^{-1}$, $\mathcal{C}sc^{-1}$ are sometimes written as $\mathcal{A}rcsin$, $\mathcal{A}rccos$, $\mathcal{A}rctan$, $\mathcal{A}rccot$, $\mathcal{A}rcsec$, and $\mathcal{A}rccsc$. In that context, inverse sine, $\mathcal{S}in^{-1}$, is pronounced "arc-sine" when it is written as $\mathcal{A}rcsin$.

E. Evaluation

We can evaluate inverse trigonometric functions if the output is a multiple of π, $\frac{\pi}{2}$, $\frac{\pi}{3}$, $\frac{\pi}{4}$, or $\frac{\pi}{6}$. To do so, we look for appropriate combinations/ratios of $\frac{1}{2}$, $\frac{\sqrt{3}}{2}$, $\frac{\sqrt{2}}{2}$, etc.

Remember the range of the inverse trigonometric function!

F. Examples

Example 1: Evaluate $\mathscr{C}os^{-1}(\frac{\sqrt{3}}{2})$

Solution

We ask "which $\theta \in [0, \pi]$ has $w(\theta)$ with $\frac{\sqrt{3}}{2}$ as the x-coordinate?"

Ans $\boxed{\frac{\pi}{6}}$

Example 2: Evaluate $\mathscr{S}in^{-1}(-\frac{\sqrt{2}}{2})$

Solution

We ask "which $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ has $w(\theta)$ with $-\frac{\sqrt{2}}{2}$ as the y-coordinate?"

Ans $\boxed{-\frac{\pi}{4}}$

Example 3: Evaluate $\mathscr{Tan}^{-1}(-\sqrt{3})$

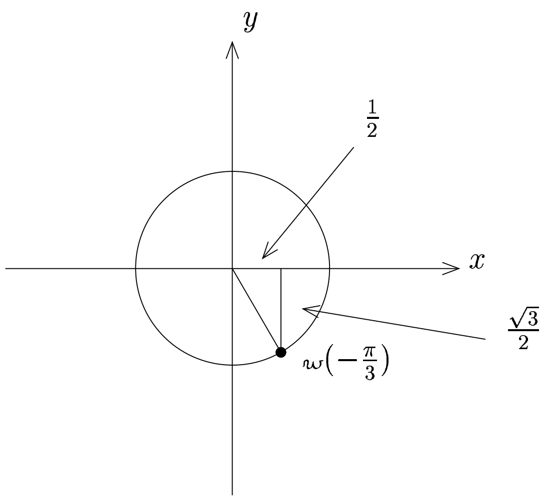
Solution

We ask "which $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ has $\mathscr{w}(\theta)$ with $\dfrac{y_{\text{coord}}}{x_{\text{coord}}} = -\sqrt{3}$?"

Clearly, one coordinate is $\pm\frac{1}{2}$ and the other is $\mp\frac{\sqrt{3}}{2}$.

To get $-\sqrt{3}$, we need $\mp\frac{\sqrt{3}}{2}$ in the numerator: $\dfrac{y_{\text{coord}}}{x_{\text{coord}}} = \dfrac{\mp\frac{\sqrt{3}}{2}}{\pm\frac{1}{2}}$

Since $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $x > 0$, so we have $x_{\text{coord}} = \frac{1}{2}$ and $y_{\text{coord}} = -\frac{\sqrt{3}}{2}$.



Thus $\mathscr{Tan}^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$.

Ans $\boxed{-\frac{\pi}{3}}$

Example 4: Evaluate $\mathcal{S}in^{-1}(sin(\frac{5\pi}{6}))$

Solution

Note: $\mathcal{S}in^{-1}$ and sin are **not** inverses. They don't cancel.

Now $\mathcal{S}in^{-1}(sin(\frac{5\pi}{6})) = \mathcal{S}in^{-1}(\frac{1}{2})$

Thus we ask "which $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ has $w(\theta)$ with $\frac{1}{2}$ as the y-coordinate?"

Ans $\boxed{\frac{\pi}{6}}$

Example 5: Evaluate $\mathcal{C}os^{-1}(cos(\frac{5\pi}{4}))$

Solution

Note: $\mathcal{C}os^{-1}$ and cos are **not** inverses. They don't cancel.

Now $\mathcal{C}os^{-1}(cos(\frac{5\pi}{4})) = \mathcal{C}os^{-1}(-\frac{\sqrt{2}}{2})$

Thus we ask "which $\theta \in [0, \pi]$ has $w(\theta)$ with $-\frac{\sqrt{2}}{2}$ as the x-coordinate?"

Ans $\boxed{\frac{3\pi}{4}}$

Example 6: Evaluate $\mathcal{S}in^{-1}(\mathcal{s}in(\frac{3\pi}{13}))$

Solution

Here we can't evaluate $\mathcal{s}in(\frac{3\pi}{13})$ directly, but we notice that $\frac{3\pi}{13} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Thus, in **this** case, $\mathcal{s}in(\frac{3\pi}{13}) = \mathcal{S}in(\frac{3\pi}{13})$.

Thus we have $\mathcal{S}in^{-1}(\mathcal{S}in(\frac{3\pi}{13})) = \frac{3\pi}{13}$, since $\mathcal{S}in^{-1}$ and $\mathcal{S}in$ **are** inverses!

Note: We couldn't do this in the previous examples, since the numbers weren't in the domain of the capital function.

Ans $\boxed{\frac{3\pi}{13}}$

Exercises

Evaluate the following:

1. $\mathcal{S}in^{-1}(\frac{1}{2})$

2. $\mathcal{T}an^{-1}(-\frac{\sqrt{3}}{3})$

3. $\mathcal{C}os^{-1}(-\frac{\sqrt{2}}{2})$

4. $\mathcal{S}ec^{-1}(-\frac{2\sqrt{3}}{3})$

5. $\mathcal{S}in^{-1}(sin\,\frac{3\pi}{4})$

6. $\mathcal{S}in^{-1}(sin\,\frac{2\pi}{9})$

7. $\mathcal{C}os^{-1}(cos\,\frac{5\pi}{6})$

8. $\mathcal{C}os^{-1}(cos\,\frac{5\pi}{11})$

9. $\mathcal{S}in^{-1}\left(sin(-\frac{2\pi}{3})\right)$

10. $\mathcal{S}in^{-1}(\frac{\sqrt{6}-\sqrt{2}}{4})$ *Hint:* $sin(\frac{\pi}{12}) =$*??*

11. $\mathcal{S}in^{-1}\left(sin(\frac{5\pi}{9})\,cos(\frac{\pi}{9}) + cos(\frac{5\pi}{9})\,sin(\frac{\pi}{9})\right)$

12. $\mathcal{C}os^{-1}\left(cos(\frac{\pi}{17})\,cos(\frac{\pi}{7}) - sin(\frac{\pi}{17})\,sin(\frac{\pi}{7})\right)$

6.5 Inverse Trigonometric Problems

A. Method of Solution

1. Define the inverse trigonometric function output to be θ.

2. Rewrite the θ definition with no inverse trigonometric function by applying the appropriate capital trigonometric function to each side.

3. Recast the problem as a capital trigonometric function problem, and solve it.

B. Examples

Example 1: Find $\sin(\operatorname{Cos}^{-1}(\frac{2}{3}))$

Solution

1. Let $\theta = \operatorname{Cos}^{-1}(\frac{2}{3})$.

2. Then $\operatorname{Cos}\theta = \frac{2}{3}$.

3. Thus we have the capital trigonometric problem:

You know $\operatorname{Cos}\theta = \frac{2}{3}$. Find $\sin\theta$.

a. $\operatorname{Cos}\theta = \frac{2}{3}$

$\cos\theta = \frac{2}{3};\ \theta \in [0, \pi]$

b.

$$\cos^2\theta + \sin^2\theta = 1$$

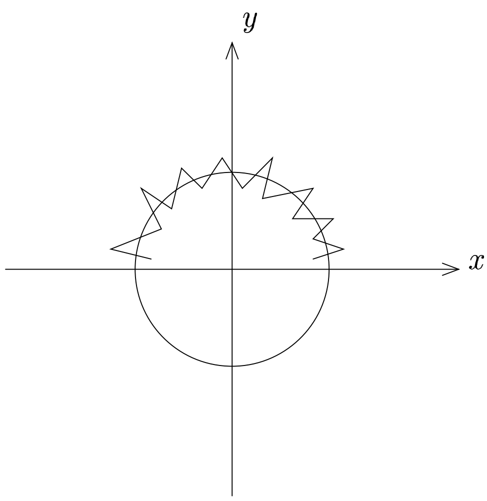
$$\left(\frac{2}{3}\right)^2 + \sin^2\theta = 1$$

$$\frac{4}{9} + \sin^2\theta = 1$$

$$\sin^2\theta = \frac{5}{9}$$

$$\sin\theta = \pm\frac{\sqrt{5}}{3}$$

c. Since $\theta \in [0, \pi]$,



Here $\sin\theta \geq 0$, so $\sin\theta = \frac{\sqrt{5}}{3}$

Ans $\boxed{\frac{\sqrt{5}}{3}}$

Example 2: Find $\mathcal{sec}(\mathcal{Sin}^{-1}(-\frac{3}{4}))$

Solution

1. Let $\theta = \mathcal{Sin}^{-1}(-\frac{3}{4})$.

2. Then $\mathcal{Sin}\,\theta = -\frac{3}{4}$.

3. Thus we have the capital trigonometric problem:

You know $\mathcal{Sin}\,\theta = -\frac{3}{4}$. Find $\mathcal{sec}\,\theta$.

a. $\mathcal{Sin}\,\theta = -\frac{3}{4}$

$\mathcal{sin}\,\theta = -\frac{3}{4};\ \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

b.

$$\mathcal{cos}^2\,\theta + \mathcal{sin}^2\,\theta = 1$$

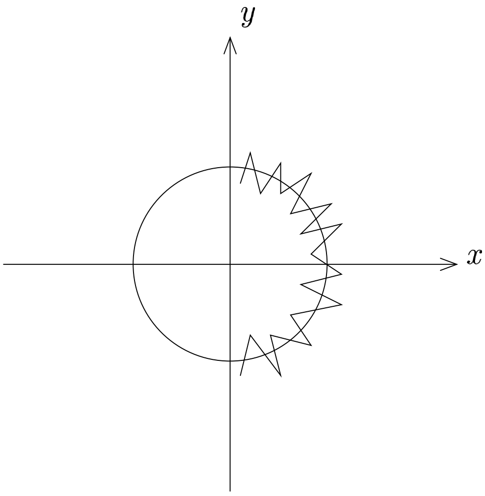
$$\mathcal{cos}^2\,\theta + \left(-\frac{3}{4}\right)^2 = 1$$

$$\mathcal{cos}^2\,\theta + \frac{9}{16} = 1$$

$$\mathcal{cos}^2\,\theta = \frac{7}{16}$$

$$\mathcal{cos}\,\theta = \pm\frac{\sqrt{7}}{4}$$

c. Since $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$,



Here $\cos\theta \geq 0$, so $\cos\theta = \frac{\sqrt{7}}{4}$

Then,

$$\begin{aligned}\sec\theta &= \frac{1}{\cos\theta} \\ &= \frac{4}{\sqrt{7}} \\ &= \frac{4}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} \\ &= \frac{4\sqrt{7}}{7}\end{aligned}$$

Ans $\boxed{\frac{4\sqrt{7}}{7}}$

Example 3: Find $\cos(2\,\operatorname{Tan}^{-1}(\frac{3}{4}))$

Solution

1. Let $\theta = \operatorname{Tan}^{-1}(\frac{3}{4})$.

2. Then $\operatorname{Tan}\theta = \frac{3}{4}$.

3. Thus we have the capital trigonometric problem:

   You know $\operatorname{Tan}\theta = \frac{3}{4}$. Find $\cos 2\theta$.

   a. $\operatorname{Tan}\theta = \frac{3}{4}$

   $\tan\theta = \frac{3}{4};\ \theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$

   b. Now $\cos(2\theta) = 2\cos^2\theta - 1$, so we need $\cos\theta$.

   However $1 + \tan^2\theta = \sec^2\theta$, so $1 + (\frac{3}{4})^2 = \sec^2\theta$.

   Thus $\sec^2\theta = 1 + \frac{9}{16} = \frac{25}{16}$.

   Then $\cos^2\theta = \frac{16}{25}$.

   In fact, we have no need for $\cos\theta$!

$$\cos(2\theta) = 2\cos^2\theta - 1$$
$$= 2\left(\frac{16}{25}\right) - 1$$
$$= \frac{32}{25} - 1$$
$$= \frac{7}{25}$$

Note: In this problem, we actually didn't have any sign ambiguity.

Ans $\boxed{\frac{7}{25}}$

Example 4: Find $\sin(2\,\mathrm{Sin}^{-1}x)$

Solution

1. Let $\theta = \mathrm{Sin}^{-1}x$.

2. Then $\mathrm{Sin}\,\theta = x$.

3. Thus we have the capital trigonometric problem:

   You know $\mathrm{Sin}\,\theta = x$. Find $\sin 2\theta$.

   a. $\mathrm{Sin}\,\theta = x$

   $\sin\theta = x;\ \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

b. Now $\sin(2\theta) = 2\sin\theta\cos\theta$, so we need $\cos\theta$.

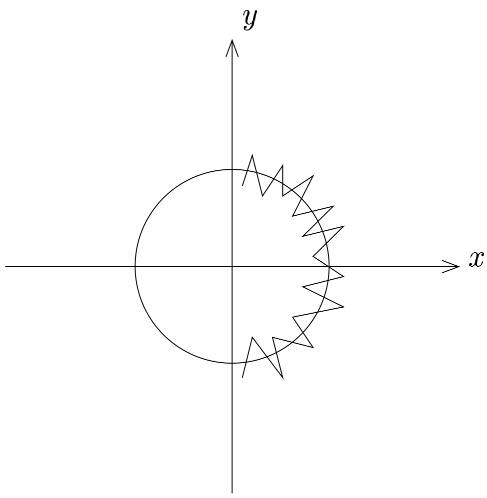
$$\cos^2\theta + \sin^2\theta = 1$$

$$\cos^2\theta + x^2 = 1$$

$$\cos^2\theta = 1 - x^2$$

$$\cos\theta = \pm\sqrt{1 - x^2}$$

c. Since $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$,



Thus $\cos\theta \geq 0$, so $\cos\theta = \sqrt{1 - x^2}$.

However we want $\sin 2\theta$, so

$$\sin 2\theta = 2\sin\theta\cos\theta = 2x\sqrt{1 - x^2}.$$

Ans $\boxed{2x\sqrt{1 - x^2}}$

Example 5: Find $\mathcal{sin}(\mathcal{Cos}^{-1}x + \mathcal{Sin}^{-1}y)$

Solution

1. Let $\theta_1 = \mathcal{Cos}^{-1}x$ and $\theta_2 = \mathcal{Sin}^{-1}y$.

2. Then $\mathcal{Cos}\,\theta_1 = x$ and $\mathcal{Sin}\,\theta_2 = y$.

3. Thus we have the capital trigonometric problem:

You know $\mathcal{Cos}\,\theta_1 = x$ and $\mathcal{Sin}\,\theta_2 = y$. Find $\mathcal{sin}(\theta_1 + \theta_2)$.

a. $\mathcal{Cos}\,\theta_1 = x$ and $\mathcal{Sin}\,\theta_2 = y$

$\mathcal{cos}\,\theta_1 = x;\ \theta_1 \in [0, \pi]$ and $\mathcal{sin}\,\theta_2 = y;\ \theta_2 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

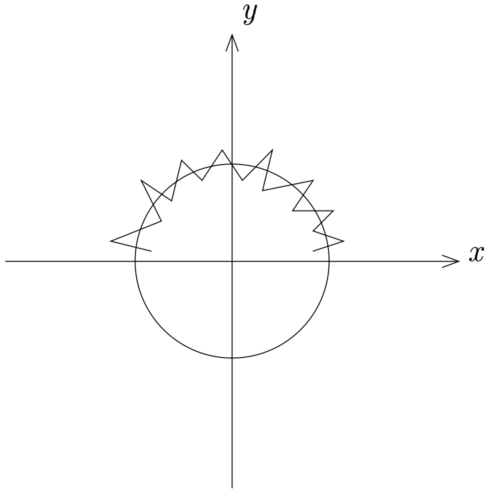
b. Now $\mathcal{sin}(\theta_1 + \theta_2) = \mathcal{sin}\,\theta_1\,\mathcal{cos}\,\theta_2 + \mathcal{cos}\,\theta_1\,\mathcal{sin}\,\theta_2$.

Thus we need $\mathcal{sin}\,\theta_1$ and $\mathcal{cos}\,\theta_2$.

I.)Find $\mathcal{sin}\,\theta_1$:

$$
\begin{aligned}
\mathcal{cos}^2\,\theta_1 + \mathcal{sin}^2\,\theta_1 &= 1 \\
x^2 + \mathcal{sin}^2\,\theta_1 &= 1 \\
\mathcal{sin}^2\,\theta_1 &= 1 - x^2 \\
\mathcal{sin}\,\theta_1 &= \pm\sqrt{1 - x^2}
\end{aligned}
$$

Since $\theta_1 \in [0, \pi]$,



Thus $\sin \theta_1 \geq 0$, so $\sin \theta_1 = \sqrt{1 - x^2}$.

II.)Find $\cos \theta_2$:

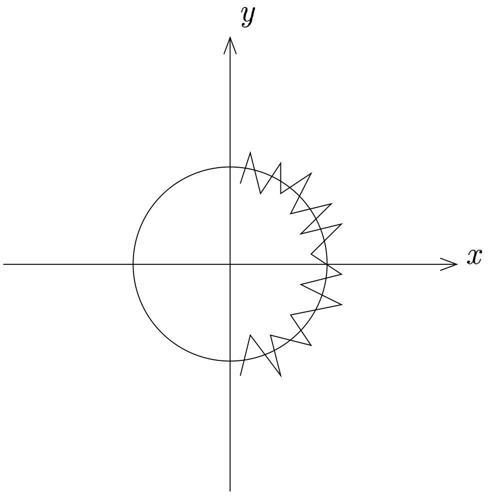
$$\cos^2 \theta_2 + \sin^2 \theta_2 = 1$$

$$\cos^2 \theta_2 + y^2 = 1$$

$$\cos^2 \theta_2 = 1 - y^2$$

$$\cos \theta_2 = \pm\sqrt{1 - y^2}$$

Since $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$,



Thus $\cos\theta_2 \geq 0$, so $\cos\theta_2 = \sqrt{1-y^2}$.

Hence,

$$\sin(\theta_1+\theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2$$

$$= \left(\sqrt{1-x^2}\right)\left(\sqrt{1-y^2}\right) + x \cdot y$$

Ans $\boxed{\sqrt{1-x^2}\sqrt{1-y^2}+xy}$

Exercises

1. Find $\cos(\mathrm{Sin}^{-1}(\frac{1}{5}))$

2. Find $\sin(\mathrm{Cos}^{-1}(-\frac{2}{3}))$

3. Find $\tan(\mathrm{Sin}^{-1}(-\frac{1}{3}))$

4. Find $\csc(\mathrm{Cos}^{-1}(-\frac{3}{4}))$

5. Find $\cos(2\,\mathrm{Sin}^{-1}(\frac{4}{5}))$

6. Find $\sin(2\,\mathrm{Sec}^{-1}(-3))$

7. Find $\cos(2\,\mathrm{Cos}^{-1}x)$

8. Find $\cos(\mathrm{Sin}^{-1}x + \mathrm{Cos}^{-1}x)$

9. Find $\tan(\frac{1}{2}\,\mathrm{Sin}^{-1}(-\frac{1}{4}))$

10. Find $\cos(2\,\mathrm{Tan}^{-1}(\frac{3}{4}))$

11. Find $\tan(2\,\mathrm{Cos}^{-1}(-\frac{3}{5}))$

12. Find $\sin(\mathrm{Sin}^{-1}(\frac{2}{5}) + \mathrm{Cos}^{-1}(-\frac{1}{3}))$

13. Find $\cos(\mathrm{Sin}^{-1}(-\frac{3}{7}) + \mathrm{Cos}^{-1}(\frac{4}{5}))$

14. Find $\sin(\mathrm{Tan}^{-1}(-4) + \mathrm{Sec}^{-1}(3))$

15. Find $\tan(\mathrm{Tan}^{-1}(\frac{1}{3}) + \mathrm{Tan}^{-1}(5))$

16. Find $w(\mathscr{S}ec^{-1}(-4))$

17. Find $w(\mathscr{C}ot^{-1}(-\frac{2}{3}))$

18. Find $w(\mathscr{S}in^{-1}(\frac{4}{5}) + \mathscr{C}os^{-1}(-\frac{1}{7}))$

6.6 Verifying Inverse Trigonometric Identities

A. Single Function Method

This method is used when each side only contains **one** inverse trigonometric function.

1. Let $\theta =$one side.

2. Manipulate this equation to get rid of the inverse trigonometric function and reduce the resulting capital trigonometric function to an ordinary trigonometric function with domain restriction.

3. Use regular trigonometric identities to simplify the resulting equation.

4. Reverse the process to get $\theta =$other side.

B. Examples

Example 1: Verify the identity: $\mathcal{S}in^{-1}(-x) = -\mathcal{S}in^{-1}x$

Solution

1. Let $\theta = \mathcal{S}in^{-1}(-x)$.

2. Then $\mathcal{S}in\,\theta = -x$, so

$sin\,\theta = -x;\ \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

3. $-sin\,\theta = x;\ \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$sin(-\theta) = x;\ \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ [odd identity for sine]

$\mathcal{sin}(-\theta) = x;\ -\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

4. $\mathcal{Sin}(-\theta) = x$

$-\theta = \mathcal{Sin}^{-1}x$

$\theta = -\mathcal{Sin}^{-1}x$

Example 2: Verify the identity: $\mathcal{Sec}^{-1}x = \mathcal{Cos}^{-1}\left(\dfrac{1}{x}\right)$

Solution

1. Let $\theta = \mathcal{Sec}^{-1}x$.

2. Then $\mathcal{Sec}\,\theta = x$, so

$\mathcal{sec}\,\theta = x;\ \theta \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$

3. $\mathcal{cos}\,\theta = \frac{1}{x};\ \theta \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$

Since $\frac{1}{x} \neq 0,\ \theta = \frac{\pi}{2}$ is impossible anyway, so

$\mathcal{cos}\,\theta = \frac{1}{x};\ \theta \in [0, \pi]$

4. $\mathcal{Cos}\,\theta = \frac{1}{x}$

$\theta = \mathcal{Cos}^{-1}\left(\frac{1}{x}\right)$

C. Sum of Functions Method

This method is used when one side contains a sum of
inverse trigonometric function outputs.

Assuming that the sum is on the left hand side of the identity . . .

1. Simplify $\sin(\text{sum})$ as an inverse trigonometric problem to get an identity for $\sin(\text{sum})$.

2. Use the domain restriction to get an appropriate identity for the original sum.

Note: If it is easier, you may use $\cos$ or $\tan$, etc. instead of $\sin$.

D. Examples

Example 1: Verify the identity: $\operatorname{Sin}^{-1} x + \operatorname{Cos}^{-1} x = \frac{\pi}{2}$

Solution

1. Simplify $\sin(\operatorname{Sin}^{-1} x + \operatorname{Cos}^{-1} x)$:

Let $\theta_1 = \operatorname{Sin}^{-1} x$ and $\theta_2 = \operatorname{Cos}^{-1} x$.

Then $\operatorname{Sin} \theta_1 = x$ and $\operatorname{Cos} \theta_2 = x$.

Hence we have the following capital trigonometric problem to solve:

You know $\operatorname{Sin} \theta_1 = x$ and $\operatorname{Cos} \theta_2 = x$. Find $\sin(\theta_1 + \theta_2)$.

Now $\operatorname{Sin} \theta_1 = x \;\Rightarrow\; \sin \theta_1 = x;\; \theta_1 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

and $\operatorname{Cos} \theta_2 = x \;\Rightarrow\; \cos \theta_2 = x;\; \theta_2 \in [0, \pi]$

Also $\sin(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2$

Thus we need to find $\cos\theta_1$ and $\sin\theta_2$:

Find $\cos\theta_1$:

$$\cos^2\theta_1 + \sin^2\theta_1 = 1$$

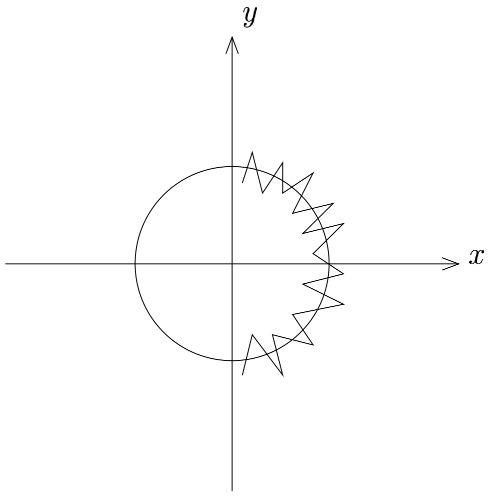
$$\cos^2\theta_1 + x^2 = 1$$

$$\cos^2\theta_1 = 1 - x^2$$

$$\cos\theta_1 = \pm\sqrt{1 - x^2}$$

Now use the domain restriction to eliminate sign ambiguity:

Since $\theta_1 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$,



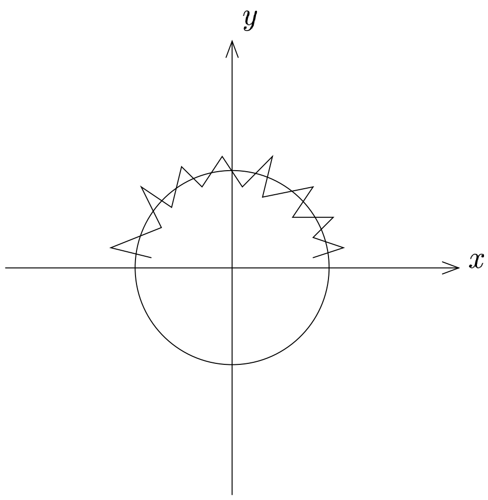
Thus $\cos\theta_1 \geq 0$, so $\cos\theta_1 = \sqrt{1 - x^2}$.

Find $\sin \theta_2$:

$$\begin{aligned}
\cos^2 \theta_2 + \sin^2 \theta_2 &= 1 \\
x^2 + \sin^2 \theta_2 &= 1 \\
\sin^2 \theta_2 &= 1 - x^2 \\
\sin \theta_2 &= \pm\sqrt{1 - x^2}
\end{aligned}$$

Now use the domain restriction to eliminate sign ambiguity:

Since $\theta_2 \in [0, \pi]$,



Thus $\sin \theta_2 \geq 0$, so $\sin \theta_2 = \sqrt{1 - x^2}$.

Thus

$$\begin{aligned}\sin(\theta_1+\theta_2) &= \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2 \\ &= x\cdot x + \sqrt{1-x^2}\cdot\sqrt{1-x^2} \\ &= x^2 + (1-x^2) \\ &= 1\end{aligned}$$

Hence our identity for $\sin(\text{sum})$ is:

$$\sin(\mathrm{Sin}^{-1}x + \mathrm{Cos}^{-1}x) = 1$$

2. Now we need to use the domain restriction to get the original identity:

Now $\theta_1 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\theta_2 \in [0, \pi]$, so $\theta_1 + \theta_2 \in [-\frac{\pi}{2}, \frac{3\pi}{2}]$.

Thus we have that $\mathrm{Sin}^{-1}x + \mathrm{Cos}^{-1}x \in [-\frac{\pi}{2}, \frac{3\pi}{2}]$.

Since $\mathrm{Sin}^{-1}x + \mathrm{Cos}^{-1}x \in [-\frac{\pi}{2}, \frac{3\pi}{2}]$ and $\sin(\mathrm{Sin}^{-1}x + \mathrm{Cos}^{-1}x) = 1$, and the only value of $\theta \in [-\frac{\pi}{2}, \frac{3\pi}{2}]$ whose sine is 1 is $\frac{\pi}{2}$, we have that

$$\boxed{\mathrm{Sin}^{-1}x + \mathrm{Cos}^{-1}x = \frac{\pi}{2}}$$

Aside: $\mathrm{Sin}^{-1}x + \mathrm{Cos}^{-1}x = \frac{\pi}{2}$ can actually be verified by a creative use of the first method, which is easier, as follows:

Let $\theta = \mathrm{Sin}^{-1}x$

Then $\mathrm{Sin}\,\theta = x$, so $\sin\theta = x;\ \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\cos(\frac{\pi}{2} - \theta) = x;\ \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ (cofunction identity)

$\cos(\frac{\pi}{2} - \theta) = x;\ -\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\mathcal{cos}(\frac{\pi}{2} - \theta) = x;\ \frac{\pi}{2} - \theta \in [0, \pi]$

$\mathcal{C}os(\frac{\pi}{2} - \theta) = x$

$\frac{\pi}{2} - \theta = \mathcal{C}os^{-1}x$

Then solving for θ, we have $\theta = \frac{\pi}{2} - \mathcal{C}os^{-1}x$.

Hence we verified that $\mathcal{S}in^{-1}x = \frac{\pi}{2} - \mathcal{C}os^{-1}x$, so

$$\boxed{\mathcal{S}in^{-1}x + \mathcal{C}os^{-1}x = \frac{\pi}{2}}$$

However, the sum of functions method is useful since it provides a way to tackle identities that you can't figure out the other way.

Example 2: Verify the identity: $\mathcal{T}an^{-1}\left(\frac{1}{4}\right) + \mathcal{T}an^{-1}\left(\frac{3}{5}\right) = \frac{\pi}{4}$

Solution

Since $\mathcal{tan}$ is more natural here . . .

1. Simplify $\mathcal{tan}\left(\mathcal{T}an^{-1}\left(\frac{1}{4}\right) + \mathcal{T}an^{-1}\left(\frac{3}{5}\right)\right)$:

Let $\theta_1 = \mathcal{T}an^{-1}\left(\frac{1}{4}\right)$ and $\theta_2 = \mathcal{T}an^{-1}\left(\frac{3}{5}\right)$.

Then $\mathcal{T}an\,\theta_1 = \frac{1}{4}$ and $\mathcal{T}an\,\theta_2 = \frac{3}{5}$.

Hence we have the following capital trigonometric problem to solve:

Know $\mathcal{T}an\,\theta_1 = \frac{1}{4}$ and $\mathcal{T}an\,\theta_2 = \frac{3}{5}$. Find $\mathcal{tan}(\theta_1 + \theta_2)$.

Now $\operatorname{Tan}\theta_1 = \frac{1}{4} \;\Rightarrow\; \tan\theta_1 = \frac{1}{4};\; \theta_1 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

and $\operatorname{Tan}\theta_2 = \frac{3}{5} \;\Rightarrow\; \tan\theta_2 = \frac{3}{5};\; \theta_2 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Also

$$\begin{aligned}
\tan(\theta_1 + \theta_2) &= \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2} \\
&= \frac{\frac{1}{4} + \frac{3}{5}}{1 - \frac{1}{4}\cdot\frac{3}{5}} \\
&= \frac{\frac{5}{20} + \frac{12}{20}}{1 - \frac{3}{20}} \\
&= \frac{\frac{17}{20}}{\frac{17}{20}} \\
&= 1
\end{aligned}$$

Hence we have the identity:

$$\tan\left(\operatorname{Tan}^{-1}\left(\tfrac{1}{4}\right) + \operatorname{Tan}^{-1}\left(\tfrac{3}{5}\right)\right) = 1$$

2. Now $\theta_1 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\theta_2 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, so $\theta_1 + \theta_2 \in (-\pi, \pi)$.

Thus we have that $\operatorname{Tan}^{-1}\left(\frac{1}{4}\right) + \operatorname{Tan}^{-1}\left(\frac{3}{5}\right) \in (-\pi, \pi)$.

Since $\operatorname{Tan}^{-1}\left(\frac{1}{4}\right) + \operatorname{Tan}^{-1}\left(\frac{3}{5}\right) \in (-\pi, \pi)$ and $\tan\left(\operatorname{Tan}^{-1}\left(\frac{1}{4}\right) + \operatorname{Tan}^{-1}\left(\frac{3}{5}\right)\right) = 1$, and the only values of $\theta \in (-\pi, \pi)$ whose tangent is 1 is $-\frac{3\pi}{4}$ and $\frac{\pi}{4}$, we have that

$$\operatorname{Tan}^{-1}\left(\tfrac{1}{4}\right) + \operatorname{Tan}^{-1}\left(\tfrac{3}{5}\right) = -\tfrac{3\pi}{4} \quad \text{or} \quad \operatorname{Tan}^{-1}\left(\tfrac{1}{4}\right) + \operatorname{Tan}^{-1}\left(\tfrac{3}{5}\right) = \tfrac{\pi}{4}.$$

It only remains to determine which of the two identities is correct.

In fact, we can do better.

Since $\tan\theta_1 = \frac{1}{4}$ with $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we actually know that $\theta_1 \in (0, \frac{\pi}{2})$ since we must be in quadrant I.

Similarly, $\theta_2 \in (0, \frac{\pi}{2})$, so

$$\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{3}{5}\right) = \theta_1 + \theta_2 \in (0, \pi).$$

Thus, we must have that

$$\boxed{\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{3}{5}\right) = \frac{\pi}{4}}$$

Exercises

Verify the following inverse trigonometric identities:

1. $\mathcal{T}an^{-1}(-x) = -\mathcal{T}an^{-1}x$

2. $\mathcal{C}sc^{-1}x = \mathcal{S}in^{-1}(\frac{1}{x})$

3. $\mathcal{C}os^{-1}(-x) = \pi - \mathcal{C}os^{-1}x$ [*Hint: Start with the right hand side*]

4. $\mathcal{T}an^{-1}x = \mathcal{S}in^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

5. $\mathcal{T}an^{-1}x + \mathcal{C}ot^{-1}x = \frac{\pi}{2}$

6. $\mathcal{S}ec^{-1}x + \mathcal{C}sc^{-1}x = \frac{\pi}{2}$

7. $\mathcal{T}an^{-1}(\frac{1}{2}) + \mathcal{T}an^{-1}(\frac{1}{3}) = \frac{\pi}{4}$

8. $\mathcal{T}an^{-1}(\frac{7}{9}) + \mathcal{T}an^{-1}(\frac{1}{8}) = \frac{\pi}{4}$

9. $\mathcal{T}an^{-1}(\frac{1}{2}) + \mathcal{T}an^{-1}(-3) = -\frac{\pi}{4}$

10. $\mathcal{T}an^{-1}(\frac{5}{3}) - \mathcal{T}an^{-1}(\frac{1}{4}) = \frac{\pi}{4}$

6.7 Inverse Trigonometric Identities

A. Summary

1. **Reciprocal Inverse Identities**

a. $\boxed{\mathcal{S}ec^{-1}x = \mathcal{C}os^{-1}\left(\frac{1}{x}\right)}$

b. $\boxed{\mathcal{C}sc^{-1}x = \mathcal{S}in^{-1}\left(\frac{1}{x}\right)}$

Warning: A similar identity for $\mathcal{C}ot^{-1}$ does **not** exist!

2. **Cofunction Inverse Identities**

a. $\boxed{\mathcal{S}in^{-1}x + \mathcal{C}os^{-1}x = \frac{\pi}{2}}$

b. $\boxed{\mathcal{T}an^{-1}x + \mathcal{C}ot^{-1}x = \frac{\pi}{2}}$

c. $\boxed{\mathcal{S}ec^{-1}x + \mathcal{C}sc^{-1}x = \frac{\pi}{2}}$

3. **Reflection Identities**

a. $\boxed{\mathcal{S}in^{-1}(-x) = -\mathcal{S}in^{-1}x}$

b. $\boxed{\mathcal{C}os^{-1}(-x) = \pi - \mathcal{C}os^{-1}x}$

c. $\boxed{\mathcal{T}an^{-1}(-x) = -\mathcal{T}an^{-1}x}$

d. $\boxed{\mathcal{C}ot^{-1}(-x) = \pi - \mathcal{C}ot^{-1}x}$

e. $\boxed{\mathcal{S}ec^{-1}(-x) = \pi - \mathcal{S}ec^{-1}x}$

f. $\boxed{\mathcal{C}sc^{-1}(-x) = -\mathcal{C}sc^{-1}x}$

B. Calculator Use

Since many calculators don't have all six inverse trigonometric functions on them, we can use the above identities to do computations in calculators.

In particular,

1. $\mathcal{C}ot^{-1}x = \frac{\pi}{2} - \mathcal{T}an^{-1}x$

2. $\mathcal{S}ec^{-1}x = \mathcal{C}os^{-1}\left(\frac{1}{x}\right)$

3. $\mathcal{C}sc^{-1}x = \mathcal{S}in^{-1}\left(\frac{1}{x}\right)$

reduces the evaluation of inverse trigonometric functions to that of inverse sine, inverse cosine, and inverse tangent.

In fact, using the identity, $\mathcal{T}an^{-1}x = \mathcal{S}in^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$, we can reduce the need to that of an **inverse sine button** only!

Then

1. $\mathcal{Cos}^{-1}x = \frac{\pi}{2} - \mathcal{Sin}^{-1}x$

2. $\mathcal{Tan}^{-1}x = \mathcal{Sin}^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

3. $\mathcal{Cot}^{-1}x = \frac{\pi}{2} - \mathcal{Sin}^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

4. $\mathcal{Sec}^{-1}x = \frac{\pi}{2} - \mathcal{Sin}^{-1}\left(\frac{1}{x}\right)$

5. $\mathcal{Csc}^{-1}x = \mathcal{Sin}^{-1}\left(\frac{1}{x}\right)$

6.8 Solving Trigonometric Equations I

A. Motivation

We know that the equation $\mathcal{S}in\; x = \frac{\sqrt{2}}{2}$ can be solved as follows:

$$\mathcal{S}in\; x = \frac{\sqrt{2}}{2}$$

$$\mathcal{S}in^{-1}\left(\mathcal{S}in\; x\right) = \mathcal{S}in^{-1}\left(\frac{\sqrt{2}}{2}\right)$$

$$x = \frac{\pi}{4}$$

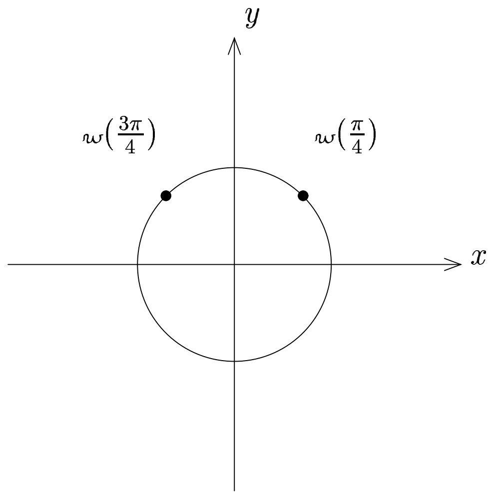
However, in typical trigonometric equations, we typically have the ordinary trigonometric functions, i.e. $\sin x = \frac{\sqrt{2}}{2}$.

B. Solving $\sin x = \frac{\sqrt{2}}{2}$

Here we have **more** solutions than just the single inverse trigonometric solution . . .

Since $\sin x = \frac{\sqrt{2}}{2}$, we have that $y_{\text{coord}} = \frac{\sqrt{2}}{2}$.

This happens in the following locations on the unit circle:



Thus we have solutions $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$.

However, more than that, by adding 2π we get two more solutions.

We can keep going, so the answer is

$$\boxed{x = \begin{cases} \frac{\pi}{4} + 2\pi k; & k \in \mathbb{Z} \\ \frac{3\pi}{4} + 2\pi k; & k \in \mathbb{Z} \end{cases}}$$

C. Strategy

1. Use algebra to isolate a trigonometric function on one side of the equation.

2. Find all solutions in $[0, 2\pi)$ through help from looking at the unit circle, and the definitions of the trigonometric functions.

3. The answer is obtained by taking each solution and adding "$2\pi k$" to get all solutions.

Note: In situations where more than one type of trigonometric function occurs in an equations, we try to either

a. separate the functions via factoring

or

b. get rid of one of the trigonometric functions via trigonometric identities.

D. Examples

Example 1: Solve $4\cos^2 x - 3 = 0$ for x

Solution

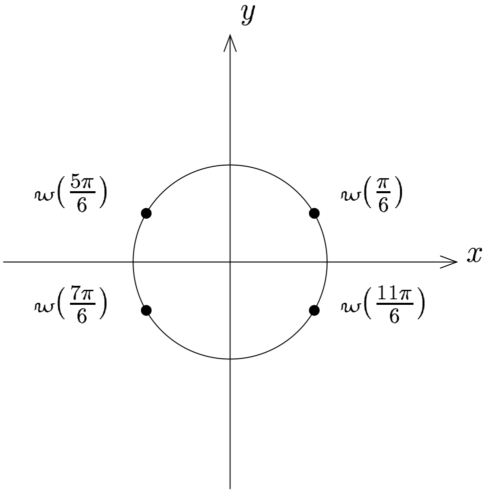
$$4\cos^2 x - 3 = 0$$

$$4\cos^2 x = 3$$

$$\cos^2 x = \frac{3}{4}$$

$$\cos x = \pm\frac{\sqrt{3}}{2}$$

$$x_{\text{coord}} = \pm\frac{\sqrt{3}}{2}$$



Ans $\boxed{x = \begin{cases} \frac{\pi}{6} + 2\pi k; & k \in \mathbb{Z} \\ \frac{5\pi}{6} + 2\pi k; & k \in \mathbb{Z} \\ \frac{7\pi}{6} + 2\pi k; & k \in \mathbb{Z} \\ \frac{11\pi}{6} + 2\pi k; & k \in \mathbb{Z} \end{cases}}$

Example 2: Solve $2\sin^2 x = 1 - \sin x$ for x

Solution

$2\sin^2 x + \sin x - 1 = 0$

Let $u = \sin x$, to make a standard quadratic equation.

Thus we have $2u^2 + u - 1 = 0$

Now factor the equation:

$$\begin{array}{ll} \quad 2u^2 + u - 1 = 0 & \boxed{-2} \quad +, - \\ \hline 2u^2 + 2u - u - 1 = 0 & -2 \ \surd \\ 2u(u+1) - 1(u+1) = 0 & \\ (u+1)(2u-1) = 0 & \end{array}$$

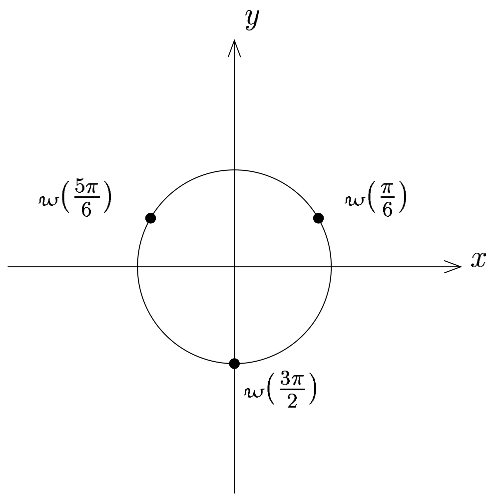
Thus $(\sin x + 1)(2\sin x - 1) = 0$.

By the Zero Product Principle:

$$\sin x + 1 = 0 \quad \text{or} \quad 2\sin x - 1 = 0$$

$$\sin x = -1 \quad \text{or} \quad \sin x = \tfrac{1}{2}$$

$$y_{\text{coord}} = -1 \quad \text{or} \quad y_{\text{coord}} = \tfrac{1}{2}$$



Ans $\boxed{x = \begin{cases} \frac{\pi}{6} + 2\pi k; & k \in \mathbb{Z} \\ \frac{5\pi}{6} + 2\pi k; & k \in \mathbb{Z} \\ \frac{3\pi}{2} + 2\pi k; & k \in \mathbb{Z} \end{cases}}$

Example 3: Solve $2\cos^2 x = 4 - 5\sin x$ for x

Solution

$2\cos^2 x + 5\sin x - 4 = 0$

Use Pythagorean I to eliminate the trigonometric function $\cos x$:

$2(1 - \sin^2 x) + 5\sin x - 4 = 0$

$2 - 2\sin^2 x + 5\sin x - 4 = 0$

$-2\sin^2 x + 5\sin x - 2 = 0$

$2\sin^2 x - 5\sin x + 2 = 0 \qquad \boxed{4} \quad -,-$

$2\sin^2 x - \sin x - 4\sin x + 2 = 0 \qquad 4 \ \surd$

$\sin x\,(2\sin x - 1) - 2(2\sin x - 1) = 0$

$(2\sin x - 1)(\sin x - 2) = 0$

By the Zero Product Principle:

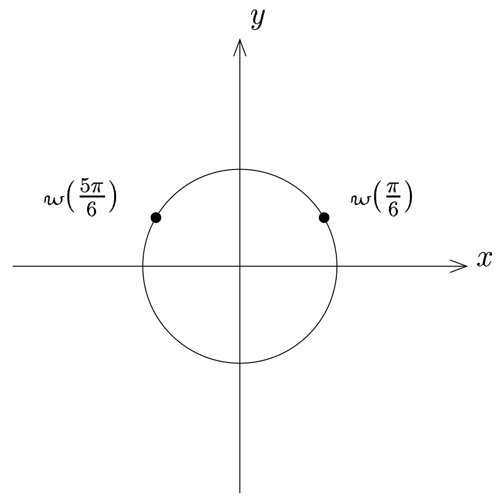
$2\sin x - 1 = 0$ or $\sin x - 2 = 0$

$2\sin x = 1$ or $\sin x = 2$

$\sin x = \frac{1}{2}$ or $\sin x = 2$

$y_{\text{coord}} = \frac{1}{2}$ or $y_{\text{coord}} = 2$

It is impossible for the y-coordinate to be 2, so $y_{\text{coord}} = \frac{1}{2}$.



Ans $\boxed{x = \begin{cases} \frac{\pi}{6} + 2\pi k; & k \in \mathbb{Z} \\ \frac{5\pi}{6} + 2\pi k; & k \in \mathbb{Z} \end{cases}}$

E. Comments

1. Sometimes solutions can be written more compactly by using multiples of something other than 2π.

For example, if we consider the answer to Example 1:

$$x = \begin{cases} \frac{\pi}{6} + 2\pi k; & k \in \mathbb{Z} \\ \frac{5\pi}{6} + 2\pi k; & k \in \mathbb{Z} \\ \frac{7\pi}{6} + 2\pi k; & k \in \mathbb{Z} \\ \frac{11\pi}{6} + 2\pi k; & k \in \mathbb{Z} \end{cases}$$

it can be written as

$$x = \begin{cases} \frac{\pi}{6} + \pi k; & k \in \mathbb{Z} \\ \frac{5\pi}{6} + \pi k; & k \in \mathbb{Z} \end{cases}$$

since $\frac{\pi}{6}$ and $\frac{7\pi}{6}$ differ by π and

since $\frac{5\pi}{6}$ and $\frac{11\pi}{6}$ differ by π.

However, we can not reduce this any further, because although $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ differ by $\frac{4\pi}{6}$, the formula $\frac{\pi}{6}+\frac{4\pi k}{6};\ k\in\mathbb{Z}$ would also give $\frac{9\pi}{6}=\frac{3\pi}{2}$ as a solution, which is not true.

2. Sometimes equations involving multiple angles occur, but we don't actually need multiple angle identities! We just solve for the inside. See the examples that follow.

F. More Examples

Example 1: Solve $(\cos 5x)\tan^2 x = \cos 5x$ for x

Solution

$$(\cos 5x)\tan^2 x - \cos 5x = 0$$

$$\cos 5x\,(\tan^2 x - 1) = 0$$

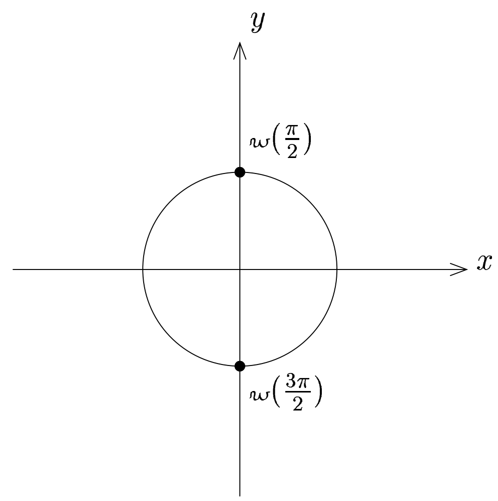
$$\cos 5x\,(\tan x + 1)(\tan x - 1) = 0 \qquad \text{(Diff. of Squares)}$$

By the Zero Product Principle:

$$\cos 5x = 0 \quad \text{or} \quad \tan x + 1 = 0 \quad \text{or} \quad \tan x - 1 = 0$$

$$\cos 5x = 0 \quad \text{or} \quad \tan x = -1 \quad \text{or} \quad \tan x = 1$$

I. Consider $\cos 5x = 0$:

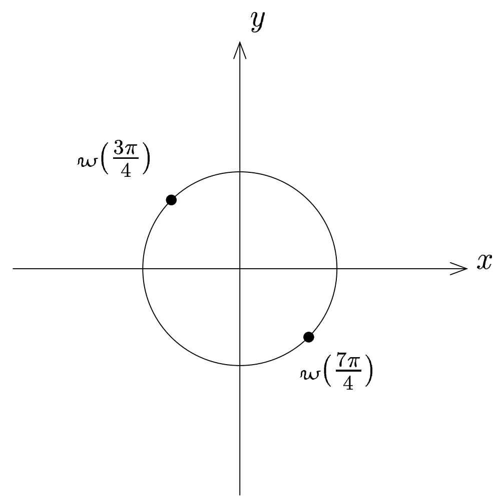


$$5x = \begin{cases} \frac{\pi}{2} + 2\pi k; & k \in \mathbb{Z} \\ \frac{3\pi}{2} + 2\pi k; & k \in \mathbb{Z} \end{cases}$$

Thus $5x = \frac{\pi}{2} + \pi k;\ k \in \mathbb{Z}$

Hence, $x = \frac{\pi}{10} + \frac{\pi k}{5};\ k \in \mathbb{Z}$

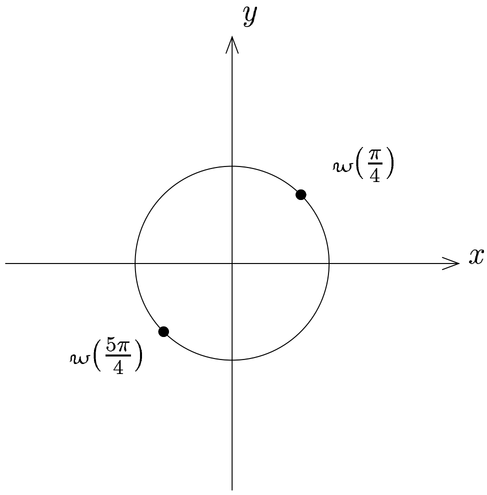
II. Consider $\tan x = -1$:



$$x = \begin{cases} \frac{3\pi}{4} + 2\pi k; & k \in \mathbb{Z} \\ \frac{7\pi}{4} + 2\pi k; & k \in \mathbb{Z} \end{cases}$$

Thus $x = \frac{3\pi}{4} + \pi k;\ k \in \mathbb{Z}$

III. Consider $\tan x = 1$:



$$x = \begin{cases} \frac{\pi}{4} + 2\pi k; & k \in \mathbb{Z} \\ \frac{5\pi}{4} + 2\pi k; & k \in \mathbb{Z} \end{cases}$$

Thus $x = \frac{\pi}{4} + \pi k;\ k \in \mathbb{Z}$

Putting all of these solutions together, we get

$$x = \begin{cases} \frac{\pi}{10} + \frac{\pi k}{5}; & k \in \mathbb{Z} \\ \frac{\pi}{4} + \pi k; & k \in \mathbb{Z} \\ \frac{3\pi}{4} + \pi k; & k \in \mathbb{Z} \end{cases}$$

Thus, we have (upon reducing),

Ans $\boxed{x = \begin{cases} \frac{\pi}{10} + \frac{\pi k}{5}; & k \in \mathbb{Z} \\ \frac{\pi}{4} + \frac{\pi k}{2}; & k \in \mathbb{Z} \end{cases}}$

Note: For $\cos 5x = 0$, we don't divide by 5 until the **very end** after we have all the solutions via adding $2\pi k$.

Example 2: Solve $\sin x - \cos x = 1$ for x

Solution

We want to get rid of a trigonometric function, but we can't directly!

We could, if we had $\sin^2 x$ or $\cos^2 x$.

This motivates the following trick:

1. Square each side to use Pythagorean Identities.
2. Check solutions at the end to eliminate "fake" solutions.

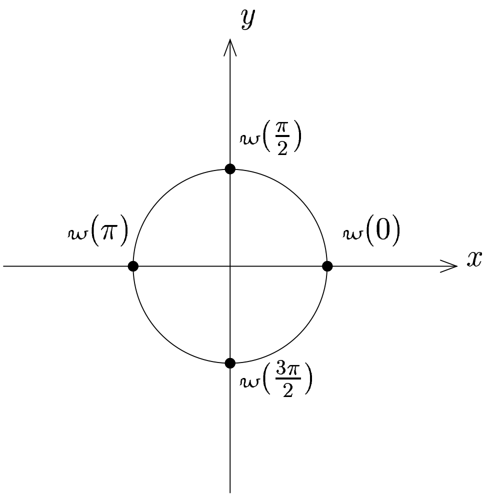
Now do it:

1.

$$\begin{aligned}
(\sin x - \cos x)^2 &= 1 \\
\sin^2 x - 2\sin x \cos x + \cos^2 x &= 1 \\
1 - 2\sin x \cos x &= 1 \qquad \text{(Pythagorean I)} \\
-2\sin x \cos x &= 0 \\
\sin x \cos x &= 0
\end{aligned}$$

By the Zero Product Principle:

$$\sin x = 0 \quad \text{or} \quad \cos x = 0$$



Check:

$$x = 0? \qquad sin(0) - cos(0) \stackrel{?}{=} 1 \;\Rightarrow\; 0 - 1 \stackrel{?}{=} 1 \qquad \text{X}$$

$$x = \tfrac{\pi}{2}? \qquad sin\left(\tfrac{\pi}{2}\right) - cos\left(\tfrac{\pi}{2}\right) \stackrel{?}{=} 1 \;\Rightarrow\; 1 - 0 \stackrel{?}{=} 1 \qquad \surd$$

$$x = \pi? \qquad sin(\pi) - cos(\pi) \stackrel{?}{=} 1 \;\Rightarrow\; 0 - (-1) \stackrel{?}{=} 1 \qquad \surd$$

$$x = \tfrac{3\pi}{2}? \qquad sin\left(\tfrac{3\pi}{2}\right) - cos\left(\tfrac{3\pi}{2}\right) \stackrel{?}{=} 1 \;\Rightarrow\; -1 - 0 \stackrel{?}{=} 1 \qquad \text{X}$$

Thus adding $2\pi k$ to the legitimate solutions, we get

Ans $\boxed{x = \begin{cases} \frac{\pi}{2} + 2\pi k; & k \in \mathbb{Z} \\ \pi + 2\pi k; & k \in \mathbb{Z} \end{cases}}$

Exercises

Solve the following equations for x:

1. $2\cos x + 1 = 0$
2. $2\sin x + \sqrt{3} = 0$
3. $\tan x - 1 = 0$
4. $2\cos^2 x - 1 = 0$
5. $2\sin^2 x - \sin x - 1 = 0$
6. $2\cos^2 x = 3\cos x - 1$
7. $\sin 3x = 1$
8. $\cos 5x = -\frac{\sqrt{3}}{2}$
9. $2\sin^2 x + 5\sin x - 3 = 0$
10. $\tan^2 3x = 3$
11. $5\tan(\frac{x}{2}) + 5 = 0$
12. $2\sin^2 x + 9\cos x + 3 = 0$
13. $(\tan^2 2x)\cos 7x - 3\cos 7x = 0$

14. $\cos^2 x + \sin x = 2$

15. $\tan^3 x - \tan^2 x + 3 \tan x - 3 = 0$

16. $\sin x = 1 - \cos x$

17. $\csc x + \cot x = 1$

6.9 Solving Trigonometric Equations II

Sometimes we need to use sum and difference formulas, double-angle formulas, etc. to solve trigonometric equations.

Examples

Example 1: Solve $\sin(x + \frac{\pi}{3}) + \sin(x - \frac{\pi}{3}) = 1$ for x

Solution

$\sin(x + \frac{\pi}{3}) + \sin(x - \frac{\pi}{3}) = 1$

Now use the sum and difference formulas:

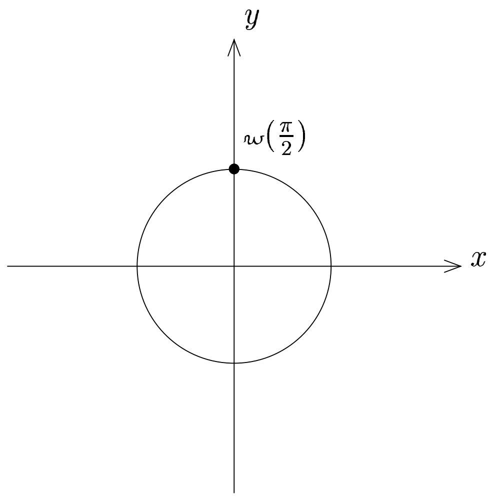
$\left(\sin x \cos \frac{\pi}{3} + \cos x \sin \frac{\pi}{3}\right) + \left(\sin x \cos \frac{\pi}{3} - \cos x \sin \frac{\pi}{3}\right) = 1$

Thus,

$$2 \sin x \cos \frac{\pi}{3} = 1$$

$$2 \sin x \cdot \frac{1}{2} = 1$$

$$\sin x = 1$$



Ans $\boxed{x = \frac{\pi}{2} + 2\pi k;\ k \in \mathbb{Z}}$

Example 2: Solve $2\cos x + \sin 2x = 0$ for x

Solution

$$2\cos x + \sin 2x = 0$$

$$2\cos x + 2\sin x \cos x = 0 \qquad \text{(double angle formula)}$$

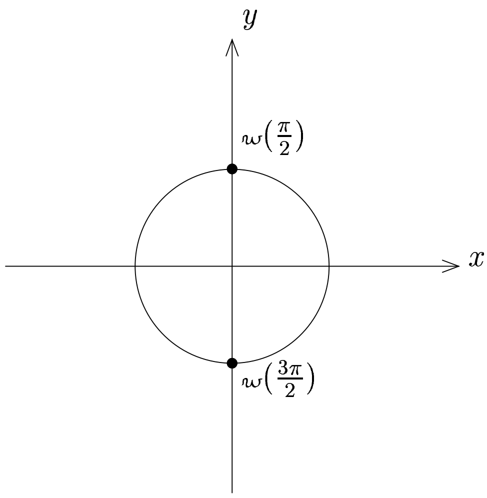
$$2\cos x(1 + \sin x) = 0$$

By the Zero-Product Principle:

$$2\cos x = 0 \quad \text{or} \quad 1 + \sin x = 0$$

$$\cos x = 0 \quad \text{or} \quad \sin x = -1$$

Both of these solutions together, lie on the unit circle in the following locations:



Ans $\boxed{x = \frac{\pi}{2} + \pi k;\ k \in \mathbb{Z}}$

Example 3: Solve $\sin x + \cos x = \frac{\sqrt{6}}{2}$ for x

Solution

Here we use the "square and check" trick:

$$(\sin x + \cos x)^2 = \frac{6}{4}$$

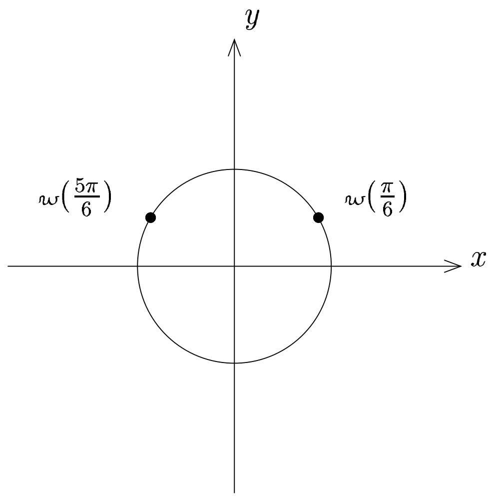
$$\sin^2 x + 2\sin x \cos x + \cos^2 x = \frac{3}{2}$$

$$1 + 2\sin x \cos x = \frac{3}{2}$$

$$2\sin x \cos x = \frac{1}{2}$$

Here we can't use the zero-product principle, because the right side is not zero. However, we can use the double angle formula.

Thus we have $\sin 2x = \frac{1}{2}$.



$$2x = \begin{cases} \frac{\pi}{6} + 2\pi k; & k \in \mathbb{Z} \\ \frac{5\pi}{6} + 2\pi k; & k \in \mathbb{Z} \end{cases}$$

Thus,

$$x = \begin{cases} \frac{\pi}{12} + \pi k; & k \in \mathbb{Z} \\ \frac{5\pi}{12} + \pi k; & k \in \mathbb{Z} \end{cases}$$

Now we need to do the check. It suffices to only check the solutions in the interval $[0, 2\pi)$.

Check:

$$x = \frac{\pi}{12}? \qquad \sin(\frac{\pi}{12}) = \sin(\frac{\pi}{3} - \frac{\pi}{4}) = \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4}$$

$$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

$$\cos(\frac{\pi}{12}) = \cos(\frac{\pi}{3} - \frac{\pi}{4}) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4}$$

$$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

Thus substituting into the original equation:

$$\sin\left(\frac{\pi}{12}\right) + \cos\left(\frac{\pi}{12}\right) \stackrel{?}{=} \frac{\sqrt{6}}{2} \Rightarrow \frac{\sqrt{6}-\sqrt{2}}{4} + \frac{\sqrt{6}+\sqrt{2}}{4} \stackrel{?}{=} \frac{\sqrt{6}}{2} \qquad \surd$$

$$x = \frac{5\pi}{12}? \qquad \sin\left(\frac{5\pi}{12}\right) = \sin\left(\frac{2\pi}{3} - \frac{\pi}{4}\right) = \sin\frac{2\pi}{3}\cos\frac{\pi}{4} - \cos\frac{2\pi}{3}\sin\frac{\pi}{4}$$

$$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \left(-\frac{1}{2}\right) \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

$$\cos\left(\frac{5\pi}{12}\right) = \cos\left(\frac{2\pi}{3} - \frac{\pi}{4}\right) = \cos\frac{2\pi}{3}\cos\frac{\pi}{4} + \sin\frac{2\pi}{3}\sin\frac{\pi}{4}$$

$$= -\frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

Thus substituting into the original equation:

$$\sin\left(\frac{5\pi}{12}\right) + \cos\left(\frac{5\pi}{12}\right) \stackrel{?}{=} \frac{\sqrt{6}}{2} \Rightarrow \frac{\sqrt{6}+\sqrt{2}}{4} + \frac{\sqrt{6}-\sqrt{2}}{4} \stackrel{?}{=} \frac{\sqrt{6}}{2} \qquad \surd$$

$$x = \frac{13\pi}{12}? \qquad \sin\left(\frac{13\pi}{12}\right) = \sin\left(\frac{4\pi}{3} - \frac{\pi}{4}\right) = \sin\frac{4\pi}{3}\cos\frac{\pi}{4} - \cos\frac{4\pi}{3}\sin\frac{\pi}{4}$$

$$= -\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \left(-\frac{1}{2}\right) \cdot \frac{\sqrt{2}}{2} = \frac{-\sqrt{6}+\sqrt{2}}{4}$$

$$\cos\left(\frac{13\pi}{12}\right) = \cos\left(\frac{4\pi}{3} - \frac{\pi}{4}\right) = \cos\frac{4\pi}{3}\cos\frac{\pi}{4} + \sin\frac{4\pi}{3}\sin\frac{\pi}{4}$$

$$= -\frac{1}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{-\sqrt{6}-\sqrt{2}}{4}$$

Thus substituting into the original equation:

$$\sin\left(\frac{13\pi}{12}\right) + \cos\left(\frac{13\pi}{12}\right) \stackrel{?}{=} \frac{\sqrt{6}}{2} \Rightarrow \frac{-\sqrt{6}+\sqrt{2}}{4} + \frac{-\sqrt{6}-\sqrt{2}}{4} \stackrel{?}{=} \frac{\sqrt{6}}{2} \qquad \text{X}$$

$x = \frac{17\pi}{12}$? $\qquad \sin(\frac{17\pi}{12}) = \sin(\frac{5\pi}{3} - \frac{\pi}{4}) = \sin\frac{5\pi}{3}\cos\frac{\pi}{4} - \cos\frac{5\pi}{3}\sin\frac{\pi}{4}$

$$= -\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2} = \frac{-\sqrt{6}-\sqrt{2}}{4}$$

$$\cos(\frac{17\pi}{12}) = \cos(\frac{5\pi}{3} - \frac{\pi}{4}) = \cos\frac{5\pi}{3}\cos\frac{\pi}{4} + \sin\frac{5\pi}{3}\sin\frac{\pi}{4}$$

$$= \frac{1}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{-\sqrt{6}+\sqrt{2}}{4}$$

Thus substituting into the original equation:

$$\sin(\frac{13\pi}{12}) + \cos(\frac{13\pi}{12}) \overset{?}{=} \frac{\sqrt{6}}{2} \Rightarrow \frac{-\sqrt{6}-\sqrt{2}}{4} + \frac{-\sqrt{6}+\sqrt{2}}{4} \overset{?}{=} \frac{\sqrt{6}}{2} \qquad \text{X}$$

Thus the only initial solutions that work in the interval $[0, 2\pi)$ are $\frac{\pi}{12}$ and $\frac{5\pi}{12}$

Ans $\boxed{x = \begin{cases} \frac{\pi}{12} + 2\pi k; & k \in \mathbb{Z} \\ \frac{5\pi}{12} + 2\pi k; & k \in \mathbb{Z} \end{cases}}$

Example 4: Solve $\sin 4x + \sin 11x = 0$ for x

Solution

Use the sum-to product formula: $\sin A + \sin B = 2\sin(\frac{A+B}{2})\cos(\frac{A-B}{2})$

$$\sin 4x + \sin 11x = 0$$

$$2\sin\left(\frac{15x}{2}\right)\cos\left(\frac{-7x}{2}\right) = 0$$

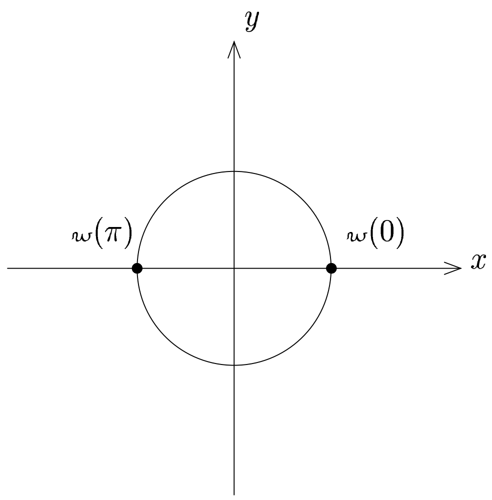
$$2\sin\left(\frac{15x}{2}\right)\cos\left(\frac{7x}{2}\right) = 0 \qquad (\cos \text{ is even})$$

$$\sin\left(\frac{15x}{2}\right)\cos\left(\frac{7x}{2}\right) = 0$$

By the Zero-Product Principle:

$$\sin(\tfrac{15x}{2}) = 0 \quad \text{or} \quad \cos(\tfrac{7x}{2}) = 0$$

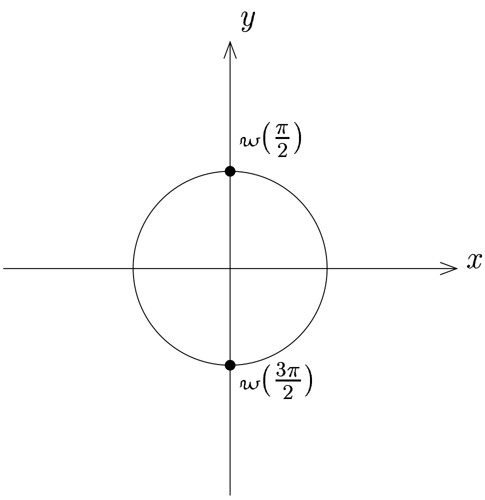
Consider first $\sin(\frac{15x}{2}) = 0$:



Here we have $\frac{15x}{2} = \pi k;\ k \in \mathbb{Z}$

Thus $x = \frac{2\pi k}{15};\ k \in \mathbb{Z}$

Now consider $\cos(\frac{7x}{2})$:



Here we have $\frac{7x}{2} = \frac{\pi}{2} + \pi k;\ k \in \mathbb{Z}$

Thus $x = \frac{\pi}{7} + \frac{2\pi k}{7};\ k \in \mathbb{Z}$

Putting all solutions together, we get

Ans $\boxed{x = \begin{cases} \frac{2\pi k}{15}; & k \in \mathbb{Z} \\ \frac{\pi}{7} + \frac{2\pi k}{7}; & k \in \mathbb{Z} \end{cases}}$

Exercises

Solve the following equations for x:

1. $\sin x \cos x = \frac{1}{2}$

2. $\sin(x + \frac{\pi}{6}) - \sin(x - \frac{\pi}{6}) = \frac{1}{2}$

3. $\cos(x + \frac{\pi}{4}) - \cos(x - \frac{\pi}{4}) = 1$

4. $2 \sin x = \sin 2x$

5. $3 - \sin x = \cos 2x$

6. $\cos^2 x = \sin^2(\frac{x}{2})$

7. $\cos \frac{x}{2} - \sin \frac{x}{2} = \sqrt{2}$

8. $\cos 6x + \cos 9x = 0$

9. $\sin 2x - \sin 5x = 0$

6.10 Harmonic Combination

A. Motivation

Let $f(\theta) = 2\sin\theta + 3\cos\theta$.

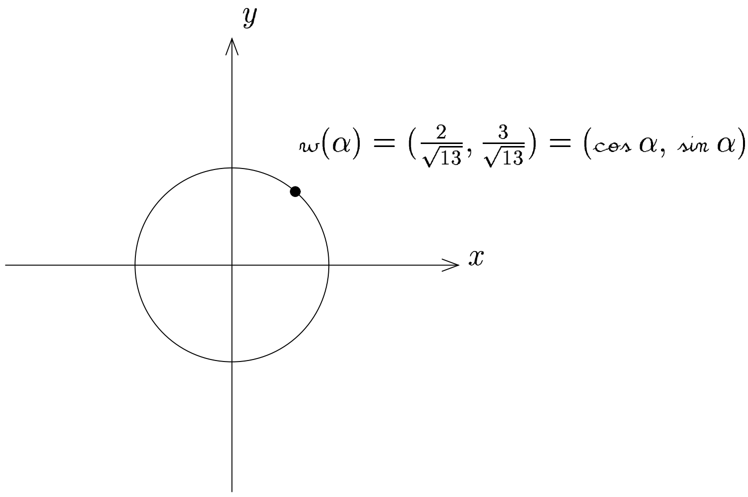
We will "reduce" this as follows:

1. Factor out $\sqrt{2^2+3^2} = \sqrt{4+9} = \sqrt{13}$:

$$f(\theta) = \sqrt{13}\left(\frac{2}{\sqrt{13}}\sin\theta + \frac{3}{\sqrt{13}}\cos\theta\right)$$

2. Since $\left(\frac{2}{\sqrt{13}}\right)^2 + \left(\frac{3}{\sqrt{13}}\right)^2 = 1, \quad \left(\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\right)$ is on the unit circle.

Thus $\left(\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\right) = w(\alpha)$ for some α:



3. Thus $f(\theta) = \sqrt{13}(\sin\theta\cos\alpha + \cos\theta\sin\alpha) = \sqrt{13}\sin(\theta+\alpha)$

4. What is α?

Note: For this problem, $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\tan\alpha = \frac{\frac{3}{\sqrt{13}}}{\frac{2}{\sqrt{13}}} = \frac{3}{2}$

Thus $\tan\alpha = \frac{3}{2} \;\Rightarrow\; \alpha = \tan^{-1}(\frac{3}{2})$

Hence, $f(\theta) = \sqrt{13}\sin(\theta + \alpha)$, where $\alpha = \tan^{-1}(\frac{3}{2})$.

B. Harmonic Combination Rule

If $f(\theta) = a\sin(n\theta) + b\cos(n\theta)$ for $a > 0$

then $f(\theta) = \sqrt{a^2 + b^2}\sin(n\theta + \alpha)$, where $\alpha = \tan^{-1}(\frac{b}{a})$.

C. Examples

Example 1: Compress the harmonic combination: $2\sin(3\theta) + 5\cos(3\theta)$

Solution

Apply the harmonic combination rule:

$\sqrt{2^2 + 5^2} = \sqrt{29}$, so

Ans $\boxed{\sqrt{29}\sin(3\theta + \alpha) \text{ where } \alpha = \tan^{-1}(\frac{5}{2})}$

Example 2: Graph f, where $f(x) = \sin x + \sqrt{3}\cos x$

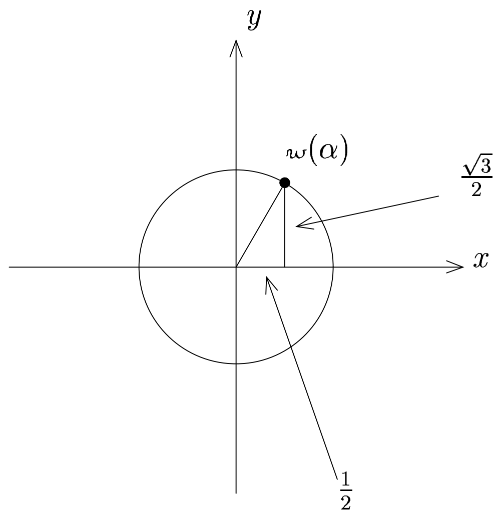
Solution

Compress the harmonic combination . . .

$$\sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = 2$$

$$f(x) = 2\sin(x+\alpha), \text{ where } \alpha = \tan^{-1}(\sqrt{3})$$

In fact, we can simplify $\tan^{-1}(\sqrt{3})$:



Thus $\alpha = \frac{\pi}{3}$.

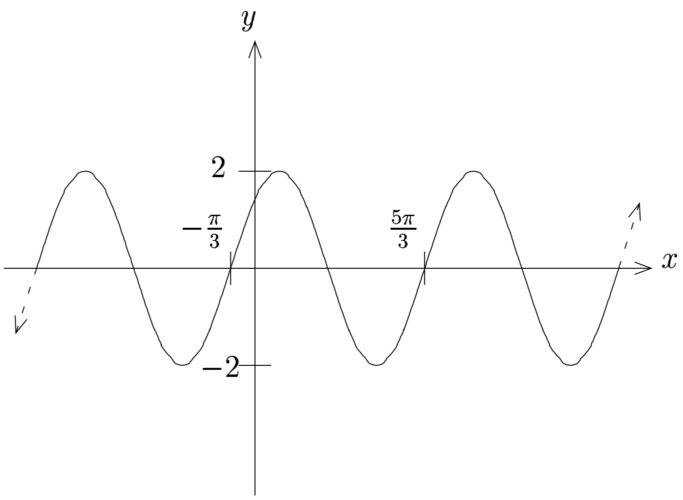
Hence, we graph f, where $f(x) = 2\sin(x + \frac{\pi}{3})$

1. $x + \frac{\pi}{3} = 0 \ \Rightarrow \ x = -\frac{\pi}{3}$

2. $x + \frac{\pi}{3} = 2\pi \ \Rightarrow \ x = \frac{5\pi}{3}$

Note that the y-intercept is $2\sin\frac{\pi}{3} = 2(\frac{\sqrt{3}}{2}) = \sqrt{3}$.

Ans



Example 3: Solve $5 \sin x - 12 \cos x = 13$ for x

Solution

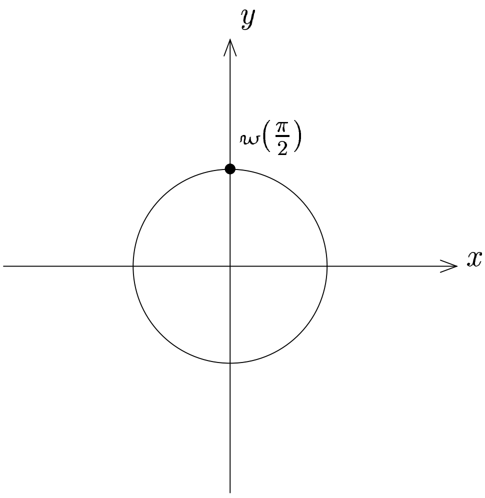
Compress the harmonic combination . . .

$$\sqrt{5^2 + (-12)^2} = \sqrt{25 + 144} = \sqrt{169} = 13$$

Thus we have $13 \sin(x + \alpha) = 13$, where $\alpha = \tan^{-1}(-\frac{12}{5})$

Then $\sin(x + \alpha) = 1$, where $\alpha = \tan^{-1}(-\frac{12}{5})$

We now locate where the y-coordinate is 1:



Thus $x + \alpha = \frac{\pi}{2} + 2\pi k;\ k \in \mathbb{Z}$.

Hence $x = \frac{\pi}{2} - \alpha + 2\pi k;\ k \in \mathbb{Z}$.

Then we have that

$$x = \tfrac{\pi}{2} - \mathcal{T}an^{-1}(-\tfrac{12}{5}) + 2\pi k;\ k \in \mathbb{Z}$$

We can simplify the minus signs, using the Reflection Identity for $\mathcal{T}an^{-1}$:

$$\mathcal{T}an^{-1}(-\theta) = -\mathcal{T}an^{-1}\,\theta.$$

Then we have

Ans $\boxed{x = \tfrac{\pi}{2} + \mathcal{T}an^{-1}(\tfrac{12}{5}) + 2\pi k;\ k \in \mathbb{Z}}$

Exercises

1. Compress the harmonic combination:

   a. $3\sin(4\theta) + 4\cos(4\theta)$

   b. $5\sin(7\theta) + 12\cos(7\theta)$

   c. $2\sin(5\theta) - 4\cos(5\theta)$

   d. $\sin(3\theta) + 5\cos(3\theta)$

2. Graph f, where

   a. $f(x) = 3\sin 2x + 3\cos 2x$

   b. $f(x) = \sqrt{3}\sin(\pi x) - \cos(\pi x)$

3. Solve the following equations for x:

   a. $3\sin 2x + 4\cos 2x = 10$

   b. $2\sin 5x - 6\cos 5x = \sqrt{30}$

Chapter 7

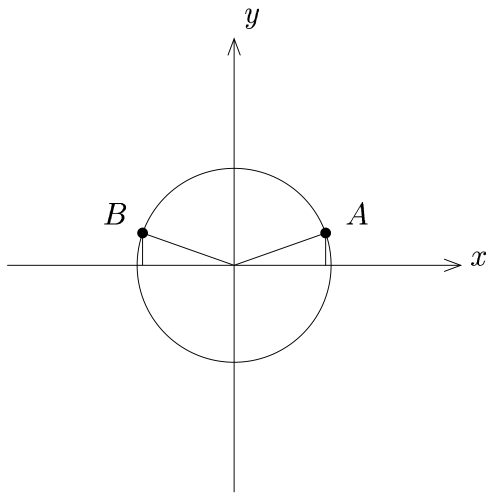
Triangle Trigonometry

7.1 General Angles

A. Motivation

Suppose we want to solve $\sin x = \frac{1}{3}$:

We want $y_{\text{coord}} = \frac{1}{3}$.

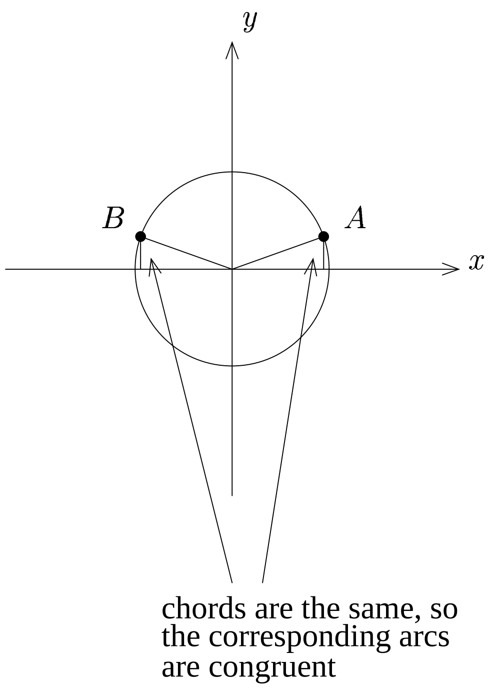


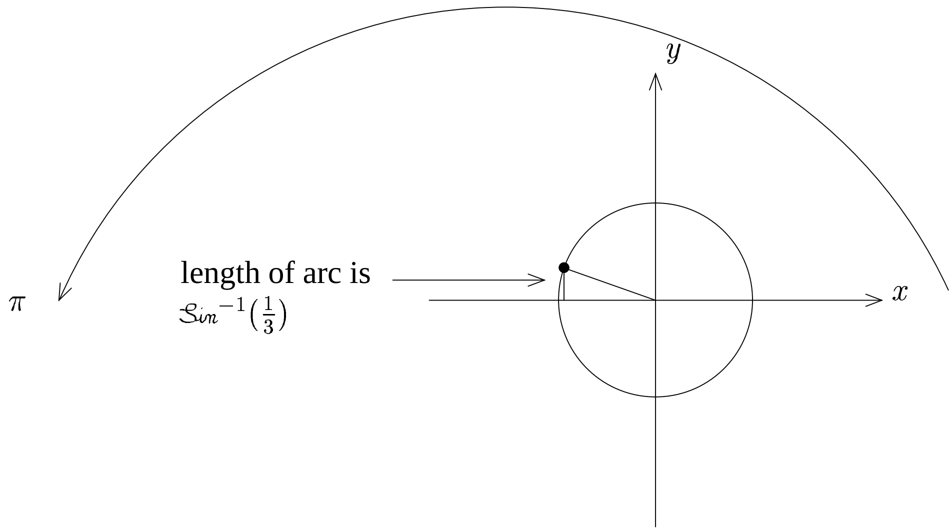
We have solutions at A and B:

A: $\mathrm{Sin}^{-1}\left(\frac{1}{3}\right) + 2\pi k;\ k \in \mathbb{Z}$

B: ?

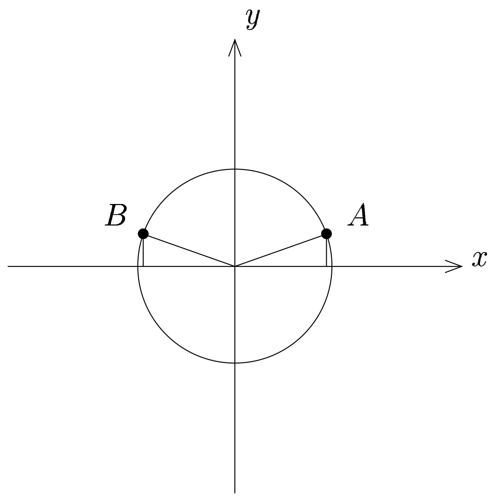
We need to find B:





Thus B: $\pi - \mathcal{S}in^{-1}(\frac{1}{3}) + 2\pi k;\ k \in \mathbb{Z}$.

This solves the problem. However, let us reconsider the original picture:



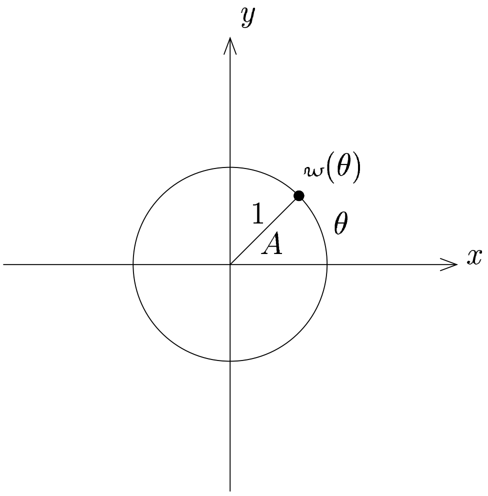
Note: If we consider the two triangles, we know that the legs of the two triangle are congruent, since both have length $\frac{1}{3}$ and the hypotenuse of the two triangles are congruent, since both have length 1 (unit circle). Thus, by the HL Postulate, the two triangles are congruent. Thus the inner **angles** of the triangles are the same.

This suggests that learning information about angles would make this problem easier.

Goal: Connect arc length to angles.

B. Radian Measure of Angles

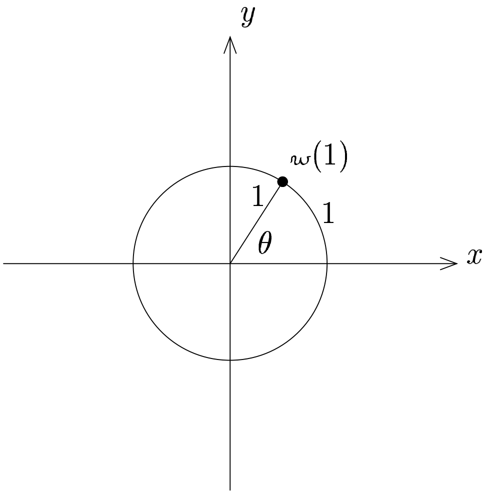
On the unit circle, we define the **radian measure** of an angle to be the signed arc length on the circle as a number (in the same fashion as the wrapping function).



Radian Measure: $m\angle A = \theta$

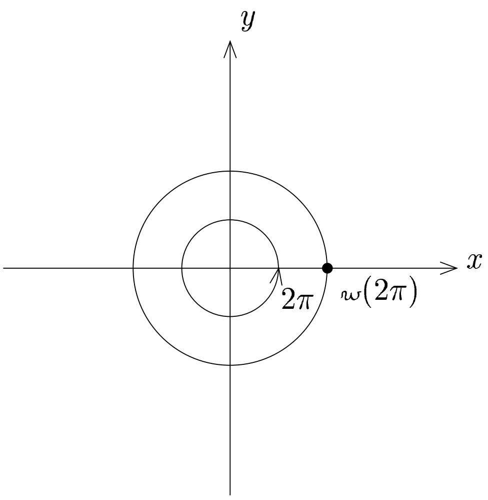
C. Comments

1. **Special Angle:** 1 radian



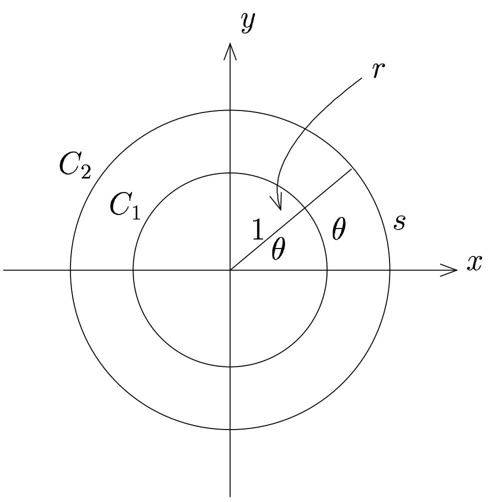
Here $m\angle\theta = 1$

2. **Special Angle:** 2π radians



D. Arc Length Formula

Consider concentric circles of radius 1 and radius r, and label the arc length cut off by a given angle θ:



The arcs on the larger circle are in the same proportion as the arcs on the smaller circle, so in particular considering arcs s and θ against the corresponding circumferences, we have

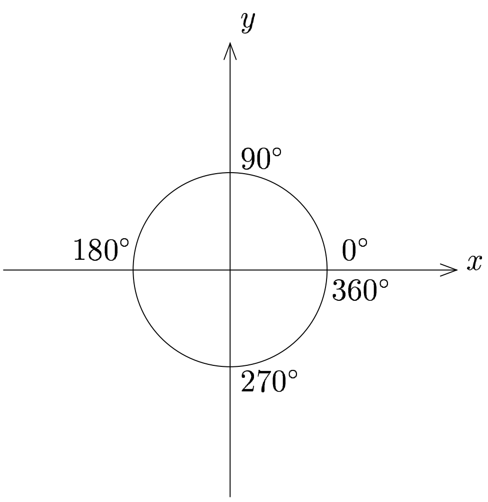
$$\frac{s}{C_2} = \frac{\theta}{C_1}$$

Thus $s = \theta(\frac{C_2}{C_1}) = \theta(\frac{2\pi r}{2\pi})$.

Hence, we have the **arc length formula**: $\boxed{s = r\theta}$

E. Degree Measure of Angles

The **degree measure** of an angle is defined by dividing up the angle of one complete revolution into $360°$.



F. Conversion

We know 2π radians $= 360°$.

Thus $\frac{2\pi}{360°} = 1 \;\Rightarrow\; \frac{\pi}{180°} = 1$.

This gives us the following conversion rules:

1. **To convert degrees to radians:** multiply by $\frac{\pi}{180°}$.

2. **To convert radians to degrees:** multiply by $\frac{180°}{\pi}$.

Note: Technically, "radians" is not a unit. Radian measure for an angle is just a number!

G. Conversion Examples

Example 1: Convert the following to radians:

a. $120°$

b. $10°$

c. $-135°$

Solution

a. $120° = 120° \cdot \frac{\pi}{180°} = \boxed{\frac{2\pi}{3}}$

b. $10° = 10° \cdot \frac{\pi}{180°} = \boxed{\frac{\pi}{18}}$

c. $-135° = -135° \cdot \frac{\pi}{180°} = -\frac{135\pi}{180} = \boxed{-\frac{3\pi}{4}}$

Example 2: Convert the following to degrees:

a. $\frac{\pi}{9}$

b. $-\frac{5\pi}{4}$

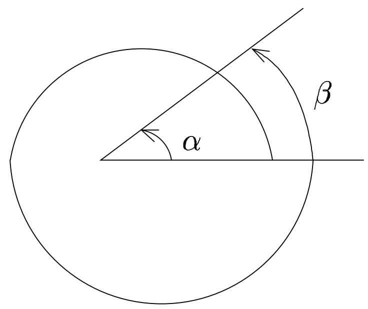
Solution

a. $\frac{\pi}{9} = \frac{\pi}{9} \cdot \frac{180°}{\pi} = \boxed{20°}$

b. $-\frac{5\pi}{4} = -\frac{5\pi}{4} \cdot \frac{180°}{\pi} = -5 \cdot 45° = \boxed{-225°}$

H. Comments on Terminology

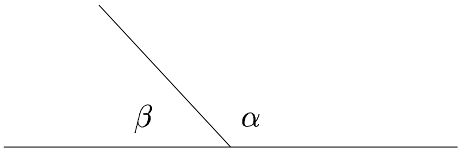
1. **Coterminal Angles:** Angles that have the same initial and terminal sides



$\angle\alpha$ and $\angle\beta$ shown above are coterminal.

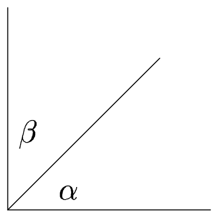
Coterminal angles differ by a multiple of 2π (or by $360°$)

2. **Supplementary Angles:** Angles that differ by π



$$\boxed{\beta = \pi - \alpha}$$

3. **Complementary Angles:** Angles that differ by $\frac{\pi}{2}$

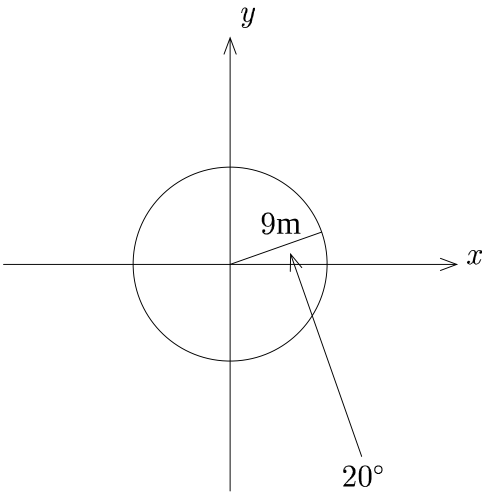


$$\boxed{\beta = \frac{\pi}{2} - \alpha}$$

I. Examples Involving Arc Length

The arc length formula $s = r\theta$ assumes that angles are measured in radians. If an angle is given in degrees, we need to convert to radians first before using the arc length formula.

Example 1: Find the arc length cut off on a circle of radius 9m by a $20°$ angle.



Solution

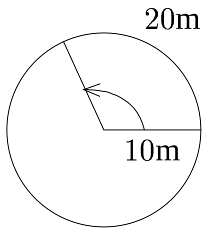
We first need to convert to radians: $20° \cdot \frac{\pi}{180°} = \frac{\pi}{9}$

Now use the arc length formula:

Since $s = r\theta$, we have that $s = (9\text{m})(\frac{\pi}{9}) = \pi$ m

Ans $\boxed{\pi \text{ m}}$

Example 2: Find the central angle of a circle of radius 10m that cuts off an arc length of 20m



Solution

$$s = r\theta \;\Rightarrow\; 20 = 10\,\theta \;\Rightarrow\; \theta = 2 \text{ (in radians)}$$

Ans $\boxed{\theta = 2}$

Exercises

1. Convert the following to radians:

    a. $150°$

    b. $-270°$

    c. $225°$

    d. $15°$

    e. $-75°$

    f. $135°$

2. Convert the following to degrees:

    a. $\frac{2\pi}{5}$

    b. $-\frac{3\pi}{8}$

    c. $-\frac{2\pi}{9}$

    d. $\frac{7\pi}{4}$

3. Find the arc length cut off on a circle of radius 3m by an angle with measure $\frac{3\pi}{4}$.

4. Find the central angle of a circle of radius 5m that cuts off an arc length of 3πm.

5. Find the arc length cut off on a circle of radius 4m by an angle having measure $40°$.

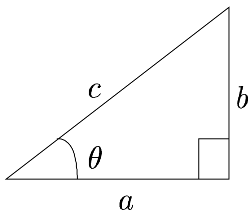
6. Find the arc length cut off on a circle of radius 3m by an angle having measure $110°$.

7.2 Right Triangle Trigonometry

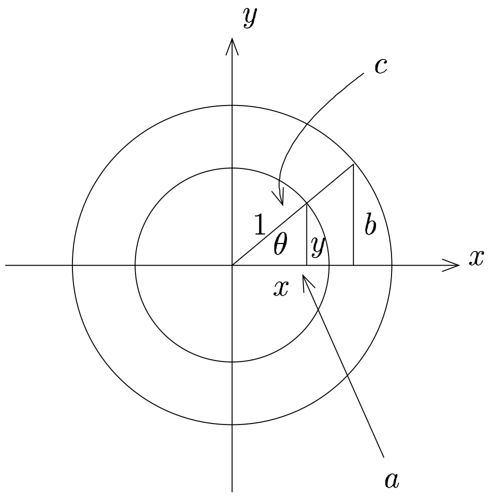
We now begin applications of trigonometry to geometry using angle ideas.

A. Development

Suppose we have a right triangle:



Draw circles of radius 1 and radius c:



Note: By similar triangles, $\frac{a}{x} = \frac{c}{1}$ and $\frac{b}{y} = \frac{c}{1}$

Thus $x = \frac{a}{c}$ and $y = \frac{b}{c}$.

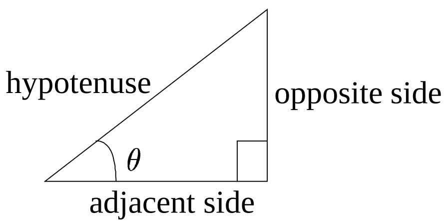
Hence $\cos\theta = \frac{a}{c}$ and $\sin\theta = \frac{b}{c}$.

Then we have that $\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$ and $\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$

Using the reciprocal and quotient identities, we get the definitions of the 6 trigonometric functions in terms of right triangle side ratios.

B. Definitions

Given a right triangle:



We have the definitions of the trigonometric functions in terms of the right triangle:

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}}$$

This can easily be remembered by the mnemonic: “SOH-CAH-TOA”

Then the other 3 trigonometric functions can be obtained by the reciprocal identities, namely

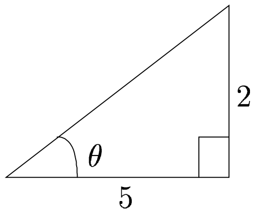
$$\cot\theta = \frac{\text{adjacent side}}{\text{opposite side}}$$

$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent side}}$$

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite side}}$$

C. Examples

Example 1: Given the right triangle:



Find $\sin\theta$, $\cos\theta$, $\tan\theta$

Solution

We first get the hypotenuse via the Pythagorean Theorem:

$$h^2 = 2^2 + 5^2 \;\Rightarrow\; h = \sqrt{4+25} = \sqrt{29}$$

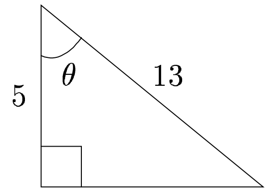
Now use the right triangle definitions:

$$\sin\theta = \frac{\text{opp.}}{\text{hyp.}} = \frac{2}{\sqrt{29}} = \boxed{\frac{2\sqrt{29}}{29}}$$

$$\cos\theta = \frac{\text{adj.}}{\text{hyp.}} = \frac{5}{\sqrt{29}} = \boxed{\frac{5\sqrt{29}}{29}}$$

$$\tan\theta = \frac{\text{opp.}}{\text{adj.}} = \boxed{\frac{2}{5}}$$

Example 2: Given the right triangle:



Find $\sin\theta$, $\cos\theta$, $\tan\theta$

Solution

We first get the third side via the Pythagorean Theorem:

$$5^2 + s^2 = 13^2 \ \Rightarrow\ s^2 = 13^2 - 5^2 = 169 - 25 = 144 \ \Rightarrow\ s = 12$$

Now use the right triangle definitions:

$$\sin\theta = \frac{\text{opp.}}{\text{hyp.}} = \boxed{\frac{12}{13}}$$

$$\cos\theta = \frac{\text{adj.}}{\text{hyp.}} = \boxed{\frac{5}{13}}$$

$$\tan\theta = \frac{\text{opp.}}{\text{adj.}} = \boxed{\frac{12}{5}}$$

D. Solving Right Triangles

Solving a right triangle means that for a triangle you want to determine all three of the side lengths and both of the **acute** angles (i.e. the angles less than $90°$).

We can solve a right triangle if we have at least the following information:

1. Two sides

or

2. One side and one acute angle.

E. Tools For Solving Right Triangles

1. Two sides determine the third side via the Pythagorean Theorem.

2. One acute angle determines the other by using the fact that the acute angles in a right triangle are complementary. Thus, if the two acute angles are $\angle\alpha$ and $\angle\beta$, we have that $\alpha + \beta = \frac{\pi}{2}$ or $\alpha + \beta = 90°$.

3. Everything else can be found by using the trigonometric ratios.

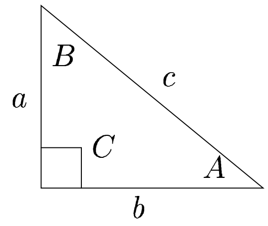
Note: We approximate appropriate trigonometric functions by using a calculator. Typically degree measure is used in triangle problems. Make sure your calculator is in the right mode.

F. Notation

We label the sides as a, b, c, where we always use c for the hypotenuse.

We label the angles opposite the sides by A, B, C.

Note: C is the right angle.

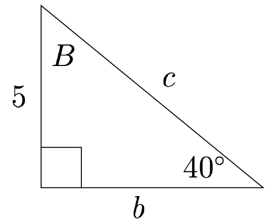


G. Examples

Example 1: Solve the right triangle with $a = 5$ and $A = 40°$

Solution

We draw a right triangle, and label it accordingly:



We can get $\angle B$, by using the fact that the acute angles are complementary:

$$B = 90° - 40° = \boxed{50°}$$

We can get b and c by using trigonometric ratios involving the $40°$ angle:

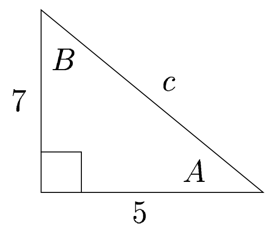
$$\tan 40° = \frac{5}{b} \;\Rightarrow\; b = \frac{5}{\tan 40°} \approx \boxed{5.96}$$

$$\sin 40° = \frac{5}{c} \;\Rightarrow\; c = \frac{5}{\sin 40°} \approx \boxed{7.78}$$

Example 2: Solve the right triangle with $a = 7$ and $b = 5$

Solution

We draw a right triangle, and label it accordingly:



We can get c, by using the Pythagorean Theorem:

$$c = \sqrt{7^2 + 5^2} = \sqrt{49 + 25} = \boxed{\sqrt{74}}$$

We can get A by using tangent:

$$\tan A = \frac{7}{5}$$

Since $A \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ [In fact, $A \in (0, \frac{\pi}{2})$], we have that

$$\tan A = \frac{7}{5} \;\Rightarrow\; A = \tan^{-1}\left(\frac{7}{5}\right) \approx \boxed{54.46°}$$

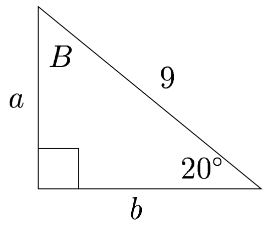
Now we can get $\angle B$, by using the fact that the acute angles are complementary:

$$B = 90° - A \approx 90° - 54.46° = \boxed{35.54°}$$

Example 3: Solve the right triangle with $c = 9$ and $A = 20°$

Solution

We draw a right triangle, and label it accordingly:



We can get $\angle B$, by using the fact that the acute angles are complementary:

$$B = 90° - 20° = \boxed{70°}$$

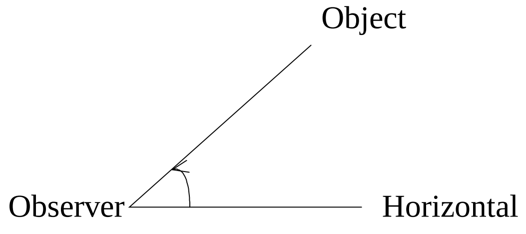
We can get a and b by using trigonometric ratios involving the $20°$ angle:

$$\sin 20° = \frac{a}{9} \Rightarrow a = 9 \sin 20° \approx \boxed{3.08}$$

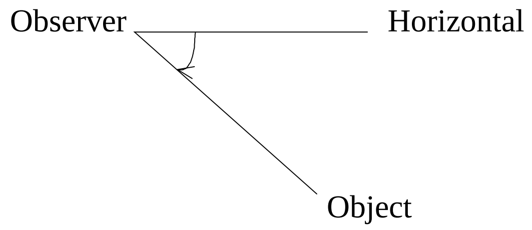
$$\cos 20° = \frac{b}{9} \Rightarrow b = 9 \cos 20° \approx \boxed{8.46}$$

H. Some Terminology

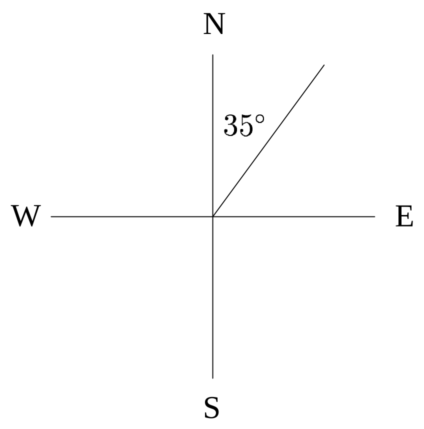
1. **Angle of Elevation**



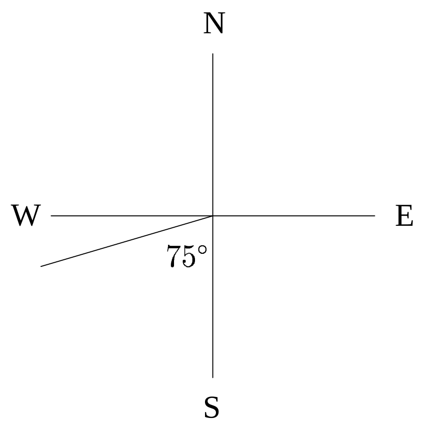
2. **Angle of Depression**



3. **Bearings:** Surveying Measurement; Angles Are Measured From North-South Line



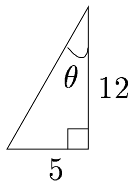
N 35° E
(35° east of north)

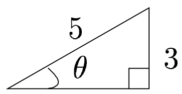


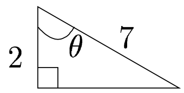
S 75° W
(75° west of south)

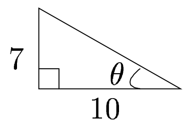
Exercises

1. Find $\sin\theta$, $\cos\theta$, and $\tan\theta$ for the given right triangle:

a. 

b. 

c. 

d. 

2. Solve the following right triangles. Use a calculator and round to two decimal places.

a. $A = 35°$, $b = 10$

b. $B = 62°$, $b = 15$

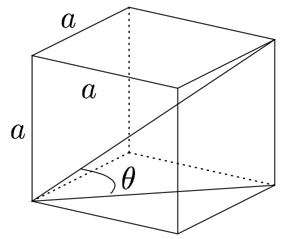
c. $a = 3$, $b = 6$

d. $a = 5$, $c = 13$

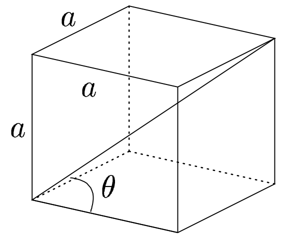
e. $A = 40°$, $c = 11$

f. $B = 51°$, $c = 7$

3. Consider a cube with edge length a. Find the angle θ as shown:



4. Consider a cube with edge length a. Find the angle θ as shown:

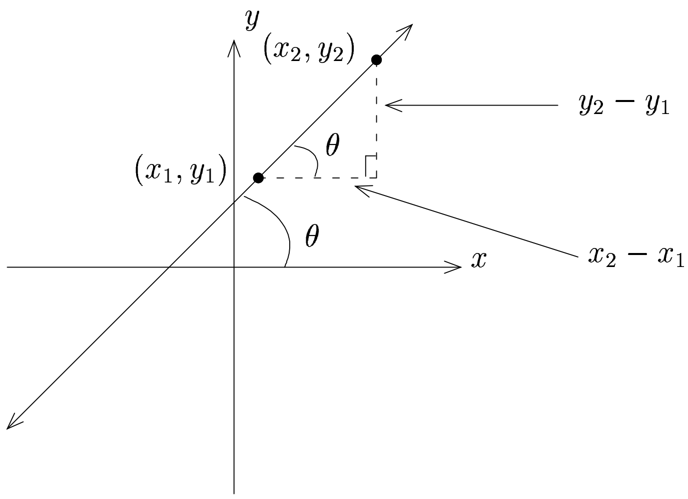


7.3 Lines and Angles

A. Using Trigonometry with the Slope of a Line

Let θ be the smaller angle between a line and the positive x-axis.

Now consider θ in relation to the slope of the line:

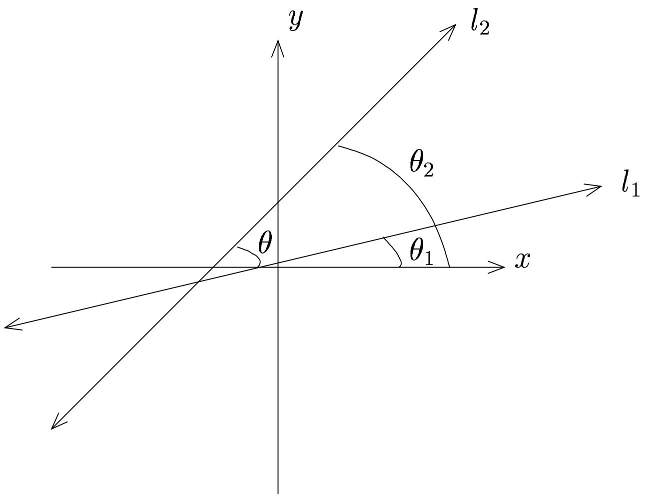


We have that $\boxed{m = \dfrac{y_2 - y_1}{x_2 - x_1} = \tan\theta}$

B. Angle Between Two Lines

Consider two nonvertical lines that are not perpendicular:

1. **Picture:**



2. **Derivation of Angle Formula:**

$$\tan\theta = \tan(\theta_2 - \theta_1) = \frac{\tan\theta_2 - \tan\theta_1}{1 + (\tan\theta_2)(\tan\theta_1)}$$

However, $m_2 = \tan\theta_2$ and $m_1 = \tan\theta_1$.

Thus, $\tan\theta = \dfrac{m_2 - m_1}{1 + m_2 m_1} = \dfrac{m_2 - m_1}{1 + m_1 m_2}$.

If the positions of l_1 and l_2 are reversed, we would have $\tan\theta = \dfrac{m_1 - m_2}{1 + m_1 m_2}$.

To ensure that the smaller angle is chosen, no matter which line is “labeled” line 1,

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$$

Since $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, we may write

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$$

3. **Angle Formula**

If two lines are not perpendicular, and neither is vertical, then the smallest angle θ between the two lines is given by:

$$\boxed{\theta = \tan^{-1}\left(\left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|\right)}$$

4. **Example:** Find the smallest angle between the two lines given by $2x - y = 4$ and $3x + y = 3$.

Solution

Slope of $2x - y = 4$: $m_1 = 2$

Slope of $3x + y = 3$: $m_2 = -3$

Then

$$\theta = \tan^{-1}\left(\left|\frac{-3 - 2}{1 + (2)(-3)}\right|\right)$$

$$= \tan^{-1}\left(\left|\frac{-5}{-5}\right|\right)$$

$$= \tan^{-1}(1)$$

$$= \frac{\pi}{4}$$

Ans $\boxed{\frac{\pi}{4}}$ (or $45°$)

Exercises

Find the smallest angle between the two lines given below:

1. $\begin{cases} 2x + 3y = 5 \\ x - y = 3 \end{cases}$

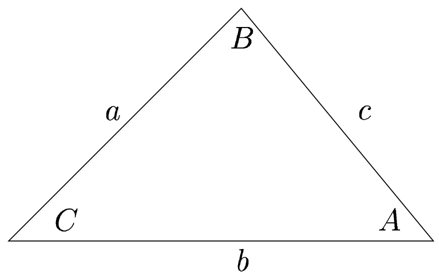
2. $\begin{cases} 2x - y = 7 \\ x + 3y = -2 \end{cases}$

3. $\begin{cases} x + 2y = 4 \\ 3x + y = 10 \end{cases}$

7.4 Oblique Triangle Formulas and Derivations

We now consider triangles that are **not** right triangles. These are called **oblique triangles**.

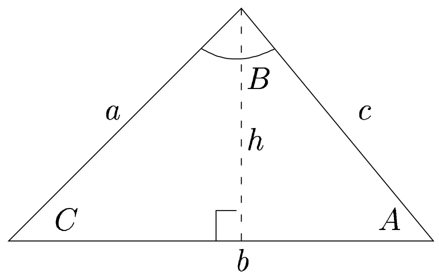
A. Law of Sines



1. **Law:** $\boxed{\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}}$

2. **Derivation:**

Here we assume that we have an **acute triangle**, i.e. all angles in the triangle are acute. If the triangle is **obtuse** (i.e. an angle whose measure is greater than $90°$ exists in the triangle), then the derivation is similar.



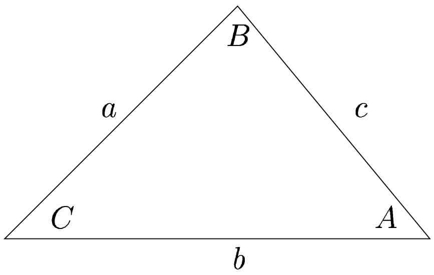
$\sin C = \frac{h}{a} \;\Rightarrow\; h = a \sin C$

$\sin A = \frac{h}{c} \;\Rightarrow\; h = c \sin A$

Thus $a \sin C = c \sin A$, so $\dfrac{\sin C}{c} = \dfrac{\sin A}{a}$.

Proceeding similarly using a triangle height drawn from $\angle A$, the result follows.

B. Law of Cosines



1. **Law:**

$$\boxed{c^2 = a^2 + b^2 - 2ab\cos C}$$

$$\boxed{a^2 = b^2 + c^2 - 2bc\cos A}$$

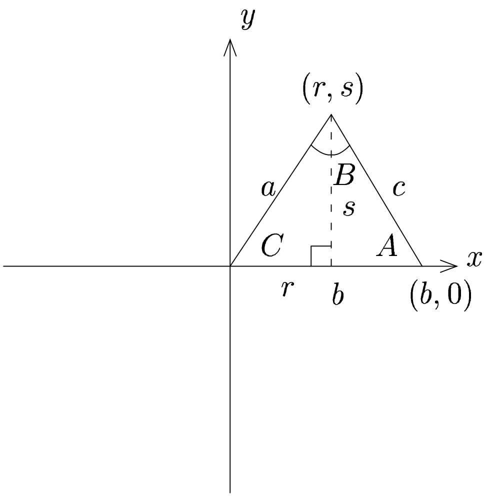
$$\boxed{b^2 = a^2 + c^2 - 2ac\cos B}$$

2. **Derivation:**

We'll derive the first one (the other two are similar).

We will assume that the triangle is acute. If the triangle is obtuse, you can use a similar argument.

We place the triangle on the x, y coordinate system as follows:



$\cos C = \frac{r}{a} \;\Rightarrow\; r = a \cos C$

$\sin C = \frac{s}{a} \;\Rightarrow\; s = a \sin C$

Now by the Pythagorean Theorem, we have that

$$c = \sqrt{(r-b)^2 + (s-0)^2}$$

$$c^2 = (r-b)^2 + s^2$$

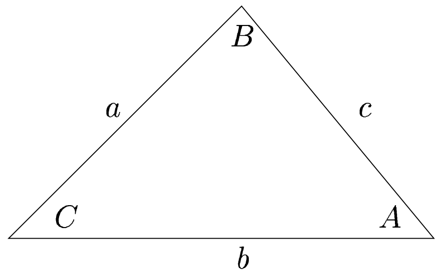
$$c^2 = (a \cos C - b)^2 + (a \sin C)^2 \qquad \text{(by above two equations)}$$

$$c^2 = a^2 \cos^2 C - 2ab \cos C + b^2 + a^2 \sin^2 C$$

$$c^2 = a^2(\cos^2 C + \sin^2 C) + b^2 - 2ab \cos C$$

$$c^2 = a^2 + b^2 - 2ab \cos C \qquad \text{(Pythagorean I)}$$

C. Mollweide's Formulas



1. **Formulas:**

$$\boxed{\frac{a+b}{c} = \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\left(\frac{C}{2}\right)}}$$

$$\boxed{\frac{a-b}{c} = \frac{\sin\left(\frac{A-B}{2}\right)}{\cos\left(\frac{C}{2}\right)}}$$

2. **Derivation:**

We'll derive the first version. The other can be derived similarly.

Since $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$, we have that $\quad \dfrac{\sin A}{\sin C} = \dfrac{a}{c} \quad$ and $\quad \dfrac{\sin B}{\sin C} = \dfrac{b}{c}$

Thus

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c} = \frac{\sin A}{\sin C} + \frac{\sin B}{\sin C} \qquad \text{(by above formulas)}$$

$$= \frac{\sin A + \sin B}{\sin C}$$

$$= \frac{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{\sin C} \qquad \text{(sum-to-product formula)}$$

$$= \frac{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{\sin\left(2\cdot\frac{C}{2}\right)}$$

$$= \frac{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{2\sin\left(\frac{C}{2}\right)\cos\left(\frac{C}{2}\right)} \qquad \text{(double angle formula)}$$

$$= \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\left(\frac{C}{2}\right)} \cdot \frac{\sin\left(\frac{A+B}{2}\right)}{\cos\left(\frac{C}{2}\right)}$$

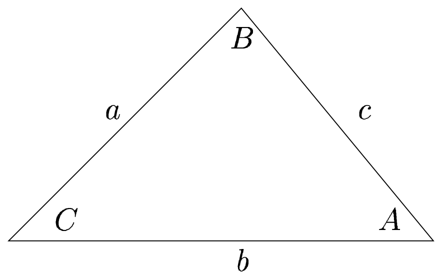
$$= \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\left(\frac{C}{2}\right)} \cdot \frac{\sin\left(\frac{\pi-C}{2}\right)}{\cos\left(\frac{C}{2}\right)} \qquad (A+B+C=\pi \quad [180^\circ])$$

$$= \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\left(\frac{C}{2}\right)} \cdot \frac{\sin\left(\frac{\pi}{2}-\frac{C}{2}\right)}{\cos\left(\frac{C}{2}\right)}$$

$$= \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\left(\frac{C}{2}\right)} \cdot \frac{\cos\left(\frac{C}{2}\right)}{\cos\left(\frac{C}{2}\right)} \qquad \text{(cofunction identity)}$$

$$= \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\left(\frac{C}{2}\right)}$$

D. Law of Tangents



1. **Law:** $\boxed{\dfrac{a-b}{a+b} = \dfrac{\tan\left(\frac{A-B}{2}\right)}{\tan\left(\frac{A+B}{2}\right)}}$

2. **Derivation:**

$$\frac{a-b}{a+b} = \frac{\frac{a-b}{c}}{\frac{a+b}{c}} = \frac{\frac{\sin\left(\frac{A-B}{2}\right)}{\cos\left(\frac{C}{2}\right)}}{\frac{\cos\left(\frac{A-B}{2}\right)}{\sin\left(\frac{C}{2}\right)}} \qquad \text{(Mollweide's Formulas)}$$

$$= \frac{\sin\left(\frac{A-B}{2}\right)}{\cos\left(\frac{C}{2}\right)} \cdot \frac{\sin\left(\frac{C}{2}\right)}{\cos\left(\frac{A-B}{2}\right)}$$

$$= \frac{\sin\left(\frac{A-B}{2}\right)}{\cos\left(\frac{\pi-(A+B)}{2}\right)} \cdot \frac{\sin\left(\frac{\pi-(A+B)}{2}\right)}{\cos\left(\frac{A-B}{2}\right)} \qquad (A+B+C=\pi \quad [180^\circ])$$

$$= \frac{\sin\left(\frac{A-B}{2}\right)}{\cos\left(\frac{\pi}{2}-\frac{A+B}{2}\right)} \cdot \frac{\sin\left(\frac{\pi}{2}-\frac{A+B}{2}\right)}{\cos\left(\frac{A-B}{2}\right)}$$

$$= \frac{\sin\left(\frac{A-B}{2}\right)}{\sin\left(\frac{A+B}{2}\right)} \cdot \frac{\cos\left(\frac{A+B}{2}\right)}{\cos\left(\frac{A-B}{2}\right)} \qquad \text{(cofunction identities)}$$

$$= \frac{\sin\left(\frac{A-B}{2}\right)}{\cos\left(\frac{A-B}{2}\right)} \cdot \frac{\cos\left(\frac{A+B}{2}\right)}{\sin\left(\frac{A+B}{2}\right)}$$

$$= \tan\left(\frac{A-B}{2}\right) \cdot \cot\left(\frac{A+B}{2}\right) \qquad \text{(quotient identities)}$$

$$= \frac{\tan\left(\frac{A-B}{2}\right)}{\tan\left(\frac{A+B}{2}\right)} \qquad \text{(reciprocal identity)}$$

7.5 Oblique Triangle Types

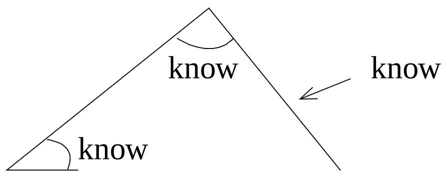
A. Introduction

We can solve oblique triangles using the Law of Sines and Law of Cosines if one side is known, along with two other parts (sides or angles).

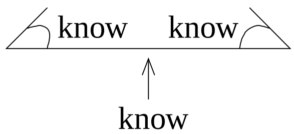
B. Cases

1. **One side:**

   a. AAS : Two angles and a nonincluded side:

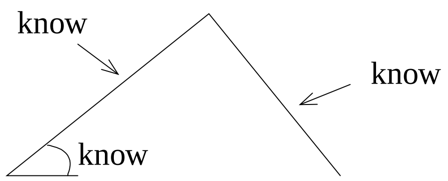


   b. ASA : Two angles and an included side:

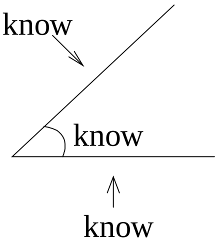


2. **Two sides:**

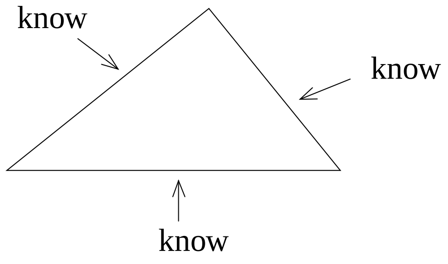
a. SSA : Two sides and a nonincluded angle:



b. SAS : Two sides and an included angle:



3. **Three sides:** SSS



C. Comments

1. AAS, ASA, SSA need Law of Sines only.

2. SAS, SSS also need the Law of Cosines

3. SSA is called the **ambiguous case**

a. possibilities:

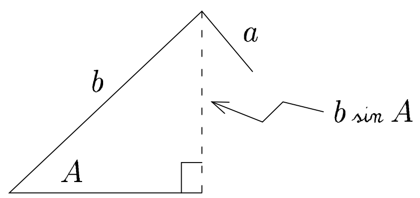
I. no triangle

II. one triangle

III. two triangles

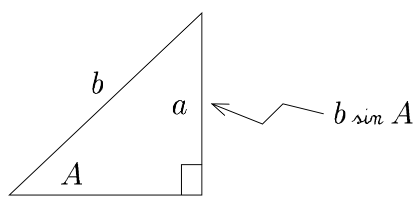
b. conditions for $\angle A$ acute:

I.



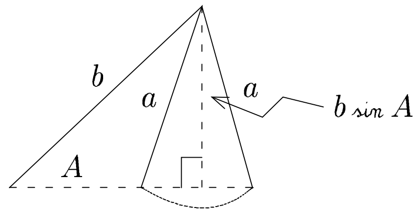
$a < b \sin A \Rightarrow$ no triangle

II.



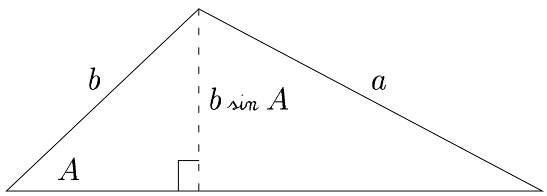
$a = b \sin A \Rightarrow$ one triangle

III.



$a > b \sin A$, but $a < b \Rightarrow$ two triangles

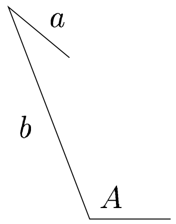
IV.



$a > b \sin A$, and $a \geq b$ also $\Rightarrow$ one triangle

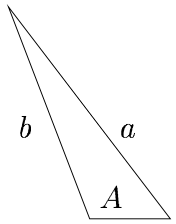
c. conditions for $\angle A$ obtuse:

I.



$a \leq b \;\Rightarrow$ no triangle

II.



$a > b \;\Rightarrow$ one triangle

7.6 Solving Oblique Triangles

A. Strategy

1. Given the side/angle data, draw a rough sketch of the triangle(s).

2. If appropriate, use Law of Sines. If not sufficient, use Law of Cosines.

3. Check your answers in one of Mollweide's Formulas (it doesn't matter which one). Some solutions may be fake, and this will tell you.

B. Tips

1. If possible, try to find the largest angle first. This is the angle opposite the longest side. This will tell you automatically that the other two angles are acute, and can help to eliminate fake solutions.

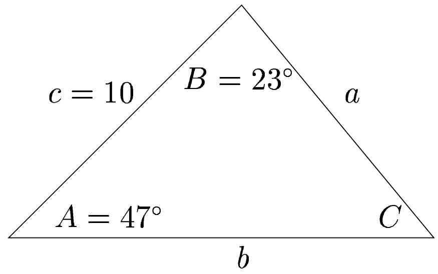
2. Remember that all three angles of a triangle add to $180°$.

C. Examples

Example 1: Solve the triangle: $A = 47°$, $B = 23°$, $c = 10$

Solution

Draw a Picture:



First find C:

$$\begin{aligned} C &= 180° - A - B \\ &= 180° - 47° - 23° \\ &= 180° - 70° \\ &= 110° \end{aligned}$$

Now find a:

Law of Sines: $\frac{\sin 110°}{10} = \frac{\sin 47°}{a}$

Thus, $a(\sin 110°) = 10 \sin 47°$

Then $a = \frac{10 \sin 47°}{\sin 110°} \approx 7.78.$

Now find b:

Law of Sines: $\frac{\sin 110°}{10} = \frac{\sin 23°}{b}$

Thus, $b(\sin 110°) = 10 \sin 23°$

Then $b = \frac{10 \sin 23°}{\sin 110°} \approx 4.16.$

Now we need to check the answer using one of Mollweide's Formulas.

Use $\dfrac{a+b}{c} = \dfrac{\cos\left(\frac{A-B}{2}\right)}{\sin\left(\frac{C}{2}\right)}$:

Check:

$$\frac{7.78 + 4.16}{10} \stackrel{?}{=} \frac{\cos\left(\frac{47°-23°}{2}\right)}{\sin\left(\frac{110°}{2}\right)}$$

$$\frac{11.94}{10} \stackrel{?}{=} \frac{\cos(12°)}{\sin(55°)}$$

$1.194 \stackrel{?}{\approx} 1.194$ (approx.) $\quad\surd$

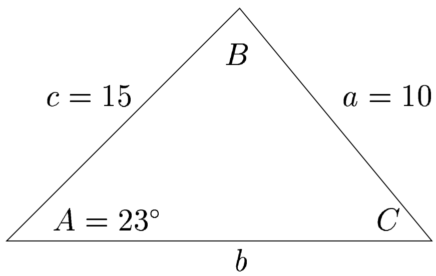
Thus

Ans $\boxed{C = 110°,\, a \approx 7.78,\, b \approx 4.16}$

Example 2: Solve the triangle: $A = 23°$, $a = 10$, $c = 15$

Solution

Draw a Picture:

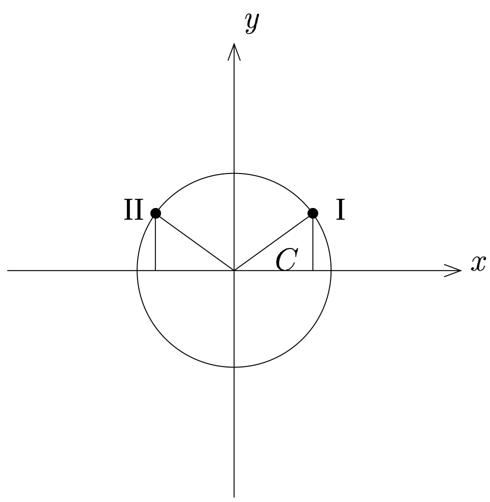


First find C:

Law of Sines: $\frac{\sin C}{15} = \frac{\sin 23°}{10}$

Thus, $10 \sin C = 15 \sin 23°$

Then $\sin C = \frac{15 \sin 23°}{10} \approx .586$



Now at I, $C \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, so we have $\operatorname{Sin} C \approx .586$,
so $C = \operatorname{Sin}^{-1}(.586) \approx 35.88°$

Now at II, the solution (as in the beginning of section 7.1), is $\pi - \mathcal{S}in\, C$, i.e. approx. $180° - 35.88° = 144.12°$.

Thus we have two cases, and two possible triangles (so far).

Case I: $C \approx 35.88°$

Then find B: $B \approx 180° - 23° - 35.88° = 121.12°$

Then find b:

Law of Sines: $\frac{\sin 121.12°}{b} = \frac{\sin 23°}{10}$

Thus, $b(\sin 23°) \approx 10 \sin 121.12°$

Then $b = \frac{10 \sin 121.12°}{\sin 23°} \approx 21.9.$

Case II: $C \approx 144.12°$

Then find B: $B \approx 180° - 23° - 144.12° = 12.88°$

Then find b:

Law of Sines: $\frac{\sin 12.88°}{b} = \frac{\sin 23°}{10}$

Thus, $b(\sin 23°) \approx 10 \sin 12.88°$

Then $b = \frac{10 \sin 12.88°}{\sin 23°} \approx 5.7.$

Now we need to check the answers using one of Mollweide's Formulas.

Use $\dfrac{a+b}{c} = \dfrac{\cos\left(\frac{A-B}{2}\right)}{\sin\left(\frac{C}{2}\right)}$:

Check:

1st triangle:

$$\frac{10+21.9}{15} \stackrel{?}{=} \frac{\cos\left(\frac{23^\circ - 121.12^\circ}{2}\right)}{\sin\left(\frac{35.88^\circ}{2}\right)}$$

$$\frac{31.9}{15} \stackrel{?}{=} \frac{\cos(-49.06^\circ)}{\sin(17.94^\circ)}$$

$2.127 \stackrel{?}{\approx} 2.127$ (approx.) $\surd$

Thus this triangle is valid.

2nd triangle:

$$\frac{10+5.7}{15} \stackrel{?}{=} \frac{\cos\left(\frac{23^\circ - 12.88^\circ}{2}\right)}{\sin\left(\frac{144.12^\circ}{2}\right)}$$

$$\frac{15.7}{15} \stackrel{?}{=} \frac{\cos(5.06^\circ)}{\sin(72.06^\circ)}$$

$1.047 \stackrel{?}{\approx} 1.047$ (approx.) $\surd$

Thus this triangle is valid.

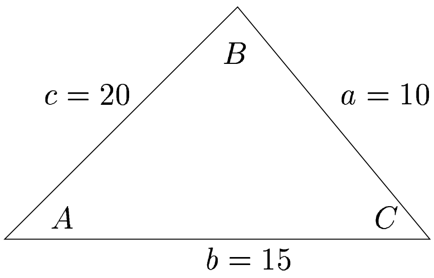
Ans $\boxed{C \approx 35.88^\circ,\ B \approx 121.12^\circ,\ b \approx 21.9}$

OR $\boxed{C \approx 144.12^\circ,\ B \approx 12.88^\circ,\ b \approx 5.70}$

Example 3: Solve the triangle: $a = 10$, $b = 15$, $c = 20$

Solution

Draw a Picture:



Law of Sines won't work (yet).

Law of Cosines: Find C (largest angle)

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$20^2 = 10^2 + 15^2 - 2(10)(15)\cos C$$

$$2(10)(15)\cos C = 10^2 + 15^2 - 20^2$$

$$\cos C = \frac{10^2 + 15^2 - 20^2}{2(10)(15)} = -.25$$

Since $C \in [0, \pi]$, we have $\text{Cos}\, C = -.25$, so $C = \text{Cos}^{-1}(-.25) \approx 104.48°$

Note: Since we found the largest angle, we know that the other two angles are acute!

Find A:

Now we can use the Law of Sines: $\frac{\sin 104.48°}{20} = \frac{\sin A}{10}$

Thus $\sin A = \frac{10 \sin 104.48°}{20} \approx .484$.

Now since we know that A is acute, $A \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Thus we have $\mathcal{S}in\, A \approx .484$, so $A \approx \mathcal{S}in^{-1}(.484) \approx 28.95°$

Find B: $B \approx 180° - 28.95° - 104.48° = 46.57°$.

Aside: Say you had forgotten or didn't know that A was acute . . .

Then the other solution to $\sin A = .484$ in the interval $[0°, 180°]$ would have been $180° - A = 180° - 28.95° = 151.05°$.

This is a problem, because then $B = 180° - 151.05° - 104.48° = -75.53°$, which is impossible!

Now we need to check the answers using one of Mollweide's Formulas.

Use $\dfrac{a+b}{c} = \dfrac{\cos\left(\frac{A-B}{2}\right)}{\sin\left(\frac{C}{2}\right)}$:

Check:

$$\frac{10+15}{20} \stackrel{?}{=} \frac{\cos\left(\frac{28.95°-46.57°}{2}\right)}{\sin\left(\frac{104.48°}{2}\right)}$$

$$\frac{25}{20} \stackrel{?}{=} \frac{\cos(-8.81°)}{\sin(52.24°)}$$

$1.250 \stackrel{?}{\approx} 1.250$ (approx.) $\qquad \surd$

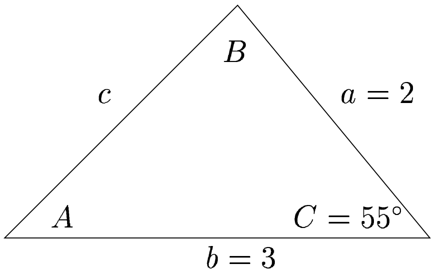
Thus

Ans $\boxed{C \approx 104.48°, A \approx 28.95°, B \approx 46.57°}$

Example 4: Solve the triangle: $a = 2$, $b = 3$, $C = 55^\circ$

Solution

Draw a Picture:



Law of Sines won't work (yet).

Find c:

Law of Cosines:

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$c^2 = 2^2 + 3^2 - 2(2)(3) \cos 55^\circ$$

$$c^2 = 4 + 9 - 12 \cos 55^\circ$$

$$c = \sqrt{13 - 12 \cos 55^\circ} \approx 2.473$$

Find B:

Law of Sines: $\frac{\sin B}{3} = \frac{\sin 55^\circ}{2.473}$

Then $\sin B = \frac{3 \sin 55^\circ}{2.473} \approx .994$.

Thus we have two solutions:

a. $B \approx \mathcal{S}in^{-1}(.994) \approx 83.72°$

b. $B \approx 180° - \mathcal{S}in^{-1}(.994) \approx 96.28°$

Case I: $B \approx 83.72°$

$A \approx 180° - 83.72° - 55° = 41.28°$

Case II: $B \approx 96.28°$

$A \approx 180° - 96.28° - 55° = 28.72°$

Now we need to check the answers using one of Mollweide's Formulas.

Use $\dfrac{a+b}{c} = \dfrac{\cos\left(\frac{A-B}{2}\right)}{\sin\left(\frac{C}{2}\right)}$:

Check:

1st triangle:

$$\frac{2+3}{2.473} \stackrel{?}{=} \frac{\cos\left(\frac{41.28° - 83.72°}{2}\right)}{\sin\left(\frac{55°}{2}\right)}$$

$$\frac{5}{2.473} \stackrel{?}{=} \frac{\cos(-21.22°)}{\sin(27.5°)}$$

$2.02 \stackrel{?}{\approx} 2.02$ (approx.) $\checkmark$

Thus this triangle is valid.

2nd triangle:

$$\frac{2+3}{2.473} \stackrel{?}{=} \frac{\cos\left(\frac{28.72^\circ - 96.28^\circ}{2}\right)}{\sin\left(\frac{55^\circ}{2}\right)}$$

$$\frac{5}{2.473} \stackrel{?}{=} \frac{\cos(-33.78^\circ)}{\sin(27.5^\circ)}$$

$2.02 \stackrel{?}{\approx} 1.80$ (approx.) X

Thus this triangle is **not** valid.

Ans $\boxed{c \approx 2.473,\ B \approx 83.72^\circ,\ A \approx 41.28^\circ}$

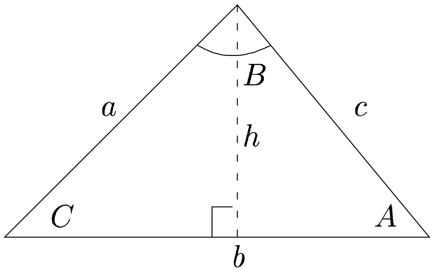
Exercises

Solve the following triangles. Use a calculator and round to two decimal places.

1. $A = 40°$, $a = 10$, $B = 35°$
2. $A = 20°$, $a = 2$, $B = 40°$
3. $A = 42°$, $a = 9$, $b = 5$
4. $A = 28°$, $a = 7$, $b = 6$
5. $A = 35°$, $a = 6$, $b = 8$
6. $A = 17°$, $a = 5$, $b = 10$
7. $A = 63°$, $a = 12$, $b = 22$
8. $A = 110°$, $a = 15$, $b = 7$
9. $A = 41°$, $B = 31°$, $c = 13$
10. $a = 6$, $b = 8$, $c = 12$
11. $A = 20°$, $b = 10$, $c = 15$
12. $a = 15$, $b = 7$, $c = 14$
13. $A = 38°$, $b = 12$, $c = 8$
14. $A = 12°$, $b = 8$, $c = 9$

7.7 Area of a Triangle

A. Oblique Triangle Formula



The area of the triangle, A_Δ, is given by $A_\Delta = \frac{1}{2}b \cdot h = \frac{1}{2}ab \cdot \frac{h}{a} = \frac{1}{2}ab \sin C$

Using the same idea for other triangle heights, we have

$$\boxed{A_\Delta = \frac{1}{2}ab \sin C = \frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B}$$

The easy way to remember this is to take "one half the product of two sides and sine of the included angle".

B. Heron's Formula

Heron's Formula for the area of a triangle is a formula that only involves the lengths of the three sides of the triangle.

$$\boxed{A_\Delta = \sqrt{s(s-a)(s-b)(s-c)}, \text{ where } s = \frac{a+b+c}{2}}$$

C. Derivation of Heron's Formula

$$\frac{a+b}{c} = \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\left(\frac{C}{2}\right)} \qquad \text{(Mollweide's Formula)}$$

$$\frac{a+b}{c} + 1 = \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\left(\frac{C}{2}\right)} + 1$$

$$\frac{a+b}{c} + \frac{c}{c} = \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\left(\frac{C}{2}\right)} + \frac{\sin\left(\frac{C}{2}\right)}{\sin\left(\frac{C}{2}\right)}$$

$$\frac{a+b+c}{c} = \frac{\cos\left(\frac{A-B}{2}\right) + \sin\left(\frac{C}{2}\right)}{\sin\left(\frac{C}{2}\right)}$$

$$\frac{a+b+c}{c} = \frac{\cos\left(\frac{A-B}{2}\right) + \sin\left(\frac{\pi-(A+B)}{2}\right)}{\sin\left(\frac{C}{2}\right)} \qquad (A+B+C=\pi \quad [180°])$$

$$\frac{a+b+c}{c} = \frac{\cos\left(\frac{A}{2}-\frac{B}{2}\right) + \sin\left(\frac{\pi}{2}-\left(\frac{A}{2}+\frac{B}{2}\right)\right)}{\sin\left(\frac{C}{2}\right)}$$

$$\frac{a+b+c}{c} = \frac{\cos\left(\frac{A}{2}-\frac{B}{2}\right) + \cos\left(\frac{A}{2}+\frac{B}{2}\right)}{\sin\left(\frac{C}{2}\right)} \qquad \text{(cofunction identity)}$$

$$\frac{a+b+c}{c} = \frac{\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right) + \sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right) + \cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right) - \sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)}{\sin\left(\frac{C}{2}\right)} \qquad \text{(sum/diff formulas)}$$

$$\frac{a+b+c}{c} = \frac{2\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)}{\sin\left(\frac{C}{2}\right)}$$

$$\frac{a+b+c}{2c} = \frac{2\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)}{2\sin\left(\frac{C}{2}\right)}$$

$$\frac{s}{c} = \frac{\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)}{\sin\left(\frac{C}{2}\right)} \qquad \text{(definition of } s\text{)}$$

Hence $\boxed{\dfrac{s}{c} = \dfrac{\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)}{\sin\left(\frac{C}{2}\right)}}$

Then

$$\frac{s}{c} - 1 = \frac{\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)}{\sin\left(\frac{C}{2}\right)} - 1$$

$$\frac{s}{c} - \frac{c}{c} = \frac{\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)}{\sin\left(\frac{C}{2}\right)} - \frac{\sin\left(\frac{C}{2}\right)}{\sin\left(\frac{C}{2}\right)}$$

$$\frac{s-c}{c} = \frac{\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right) - \sin\left(\frac{C}{2}\right)}{\sin\left(\frac{C}{2}\right)}$$

$$\frac{s-c}{c} = \frac{\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right) - \sin\left(\frac{\pi-(A+B)}{2}\right)}{\sin\left(\frac{C}{2}\right)} \qquad (A+B+C=\pi \quad [180°])$$

$$\frac{s-c}{c} = \frac{\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right) - \sin\left(\frac{\pi}{2} - \left(\frac{A}{2}+\frac{B}{2}\right)\right)}{\sin\left(\frac{C}{2}\right)}$$

$$\frac{s-c}{c} = \frac{\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right) - \cos\left(\frac{A}{2}+\frac{B}{2}\right)}{\sin\left(\frac{C}{2}\right)} \qquad \text{(cofunction identity)}$$

$$\frac{s-c}{c} = \frac{\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right) - \left[\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right) - \sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\right]}{\sin\left(\frac{C}{2}\right)} \qquad \text{(sum formula for cosine)}$$

$$\frac{s-c}{c} = \frac{\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)}{\sin\left(\frac{C}{2}\right)}$$

Hence $\boxed{\dfrac{s-c}{c} = \dfrac{\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)}{\sin\left(\frac{C}{2}\right)}}$.

By a similar argument, we get $\boxed{\dfrac{s-a}{a} = \dfrac{\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)}{\sin\left(\frac{A}{2}\right)}}$ and $\boxed{\dfrac{s-b}{b} = \dfrac{\sin\left(\frac{A}{2}\right)\sin\left(\frac{C}{2}\right)}{\sin\left(\frac{B}{2}\right)}}$

Now multiply all the boxed expressions together:

$$\frac{s}{c} \cdot \frac{s-a}{a} \cdot \frac{s-b}{b} \cdot \frac{s-c}{c}$$

$$= \frac{\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)}{\sin\left(\frac{C}{2}\right)} \cdot \frac{\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)}{\sin\left(\frac{A}{2}\right)} \cdot \frac{\sin\left(\frac{A}{2}\right)\sin\left(\frac{C}{2}\right)}{\sin\left(\frac{B}{2}\right)} \cdot \frac{\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)}{\sin\left(\frac{C}{2}\right)}$$

$$= \sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\cos\left(\frac{B}{2}\right)$$

$$= \frac{1}{4} \cdot 2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right) \cdot 2\sin\left(\frac{B}{2}\right)\cos\left(\frac{B}{2}\right)$$

$$= \frac{1}{4} \cdot \sin A \cdot \sin B \qquad \text{(double angle identities)}$$

Thus $s(s-a)(s-b)(s-c) = \frac{1}{4} \cdot ac \cdot bc \cdot \sin A \cdot \sin B$.

Then $s(s-a)(s-b)(s-c) = (\frac{1}{2}bc\sin A)(\frac{1}{2}ac\sin B) = A_\Delta \cdot A_\Delta$

Hence $A_\Delta^2 = s(s-a)(s-b)(s-c)$, so $A_\Delta = \sqrt{s(s-a)(s-b)(s-c)}$.

D. Examples

Example 1: Find the area of the triangle $a = 3$, $b = 4$, $c = 50°$

Solution

Use the oblique triangle formula!

$A = \frac{1}{2}ab\sin C = \frac{1}{2}(3)(4)\sin 50° \approx 4.60$

Ans $\boxed{4.60}$

Example 2: Find the area of the triangle $a = 5$, $b = 11$, $c = 8$

Solution

Use Heron's Formula: $A_\Delta = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{a+b+c}{2}$

Now $s = \frac{5+11+8}{2} = 12$

Then

$$\begin{aligned}
A_\Delta &= \sqrt{12(12-5)(12-11)(12-8)} \\
&= \sqrt{12 \cdot 7 \cdot 1 \cdot 4} \\
&= 2\sqrt{3 \cdot 7 \cdot 4} \\
&= 4\sqrt{21}
\end{aligned}$$

Ans $\boxed{4\sqrt{21}}$

Exercises

Find the area of the triangle:

1. $a = 5$, $b = 9$, $C = 120°$

2. $a = 13$, $b = 6$, $c = 15$

3. $a = 20$, $c = 13$, $B = 43°$

4. $a = 9$, $b = 10$, $c = 12$

Selected Answers to the Exercises

0.2

2. $16x^2 - 40x + 25$

3. $(2x+5)^2$

6. $(2x+3)(4x^2 - 6x + 9)$

1.1

1a. no

1b. yes, implicit

1c. yes, explicit

1d. no

1e. yes, explicit

1f. no

1g. yes, implicit

2a. 2

2d. $2h^2 - 12h + 12$

2f. $4x + 2h$

3d. $-\frac{1}{(x-3)(x+h-3)}$

4. $6x + 3h - 1$

6. $3x^2 + h^2$

1.2

1a. $\mathscr{dom}\, f = (-\infty, \infty)$; $\mathscr{rng}\, f = [3, \infty)$

1b. $\mathscr{dom}\, f = (-\infty, \infty)$; $\mathscr{rng}\, f = (-\infty, \infty)$

1a. $\mathscr{dom}\, f = [-3, \infty)$; $\mathscr{rng}\, f = [-2, \infty)$

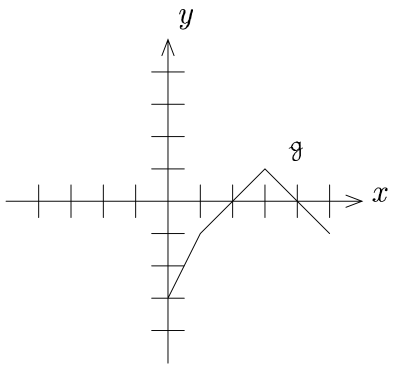
2b. $\mathscr{dom}\, f = [-5, 2) \cup (2, \infty)$

2d. $\mathscr{dom}\, f = (-\infty, -5] \cup (2, \infty)$

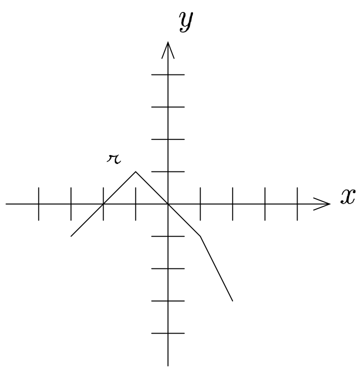
2f. $\mathscr{dom}\, f = (-\frac{3}{2}, 1) \cup (1, \infty)$

1.3

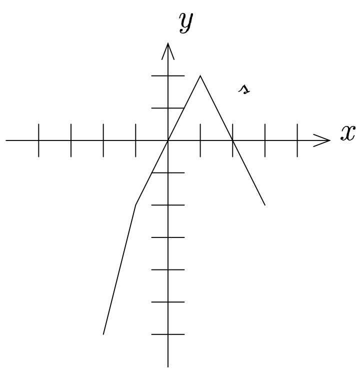
1a.



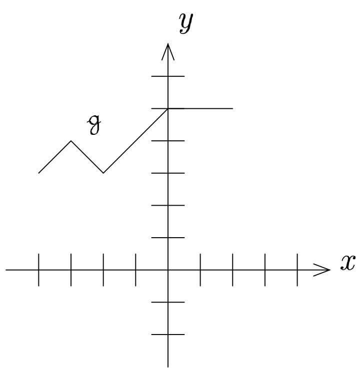
1d.



1e.



2a.



$$\mathit{dom}\ g = [-4, 2]; \quad \mathit{rng}\ g = [3, 5]$$

1.4

1b. neither

1c. odd

1d. even

2. $f_{even}(x) = x^2 + 9$; $f_{odd}(x) = -6x$

1.5

1. $(f + g)(x) = \sqrt{x} + \frac{1}{x-3}$; $(f + g)(x) = \sqrt{x} - \frac{1}{x-3}$

   $(fg)(x) = \frac{\sqrt{x}}{x-3}$; $\left(\frac{f}{g}\right)(x) = \sqrt{x}(x-3)$

   $(f \circ g)(x) = \sqrt{\frac{1}{x-3}} = \frac{\sqrt{x-3}}{x-3}$; $(g \circ f)(x) = \frac{1}{\sqrt{x}-3} = \frac{\sqrt{x}+3}{x-9}$

2. *After you fully simplify:* $(f \circ g)(x) = \frac{\sqrt{8x^2+25x+18}}{3x-1}$; $(g \circ f)(x) = \frac{7x\sqrt{x+5}-2x^2-3x-15}{4x^2-x-5}$

3b. 7

4a. -3

4b. -1

1.8

2. $\left(\frac{f}{g}\right)(x) = \frac{3x-2}{x+5}$; $dom\left(\frac{f}{g}\right) = (-40, -5) \cup (-5, \infty)$

3. $(f \circ g)(x) = \frac{1}{x-5}$; $dom(f \circ g) = [-2, 5) \cup (5, \infty)$

5a. $dom\left(\frac{f}{g}\right) = (-1, 1]$

5b. $\mathit{dom}(f \circ g) = [-1, 0] \cup [2, 4]$

1.9

2. no

3. yes

1.10

2. no

4. yes

1.11

1a. $f^{-1}(y) = \frac{2-y}{5}$

1d. **not** one-to-one; no inverse exists

1f. $f^{-1}(y) = \frac{y+2}{3-5y}$

3b. 0

1.13

1. $f^{-1}(x) = (x+3)^2 - 2;\quad \mathit{dom}(f^{-1}) = [-3, \infty);\quad \mathit{rng}(f^{-1}) = [-2, \infty)$

1.14

2. $\mathfrak{I}(x) = (x+1)^2 - 3;\ x \in (-\infty, -1]$

$\mathfrak{I}^{-1}(y) = -1 - \sqrt{y+3};\quad \mathit{dom}(\mathfrak{I}^{-1}) = [-3, \infty);\quad \mathit{rng}(\mathfrak{I}^{-1}) = (-\infty, -1]$

4. $\mathcal{F}(x) = |x-1| + 2;\ x \in [1, \infty)$

$\mathcal{F}^{-1}(y) = y - 1; \quad \mathit{dom}(\mathcal{F}^{-1}) = [2, \infty); \quad \mathit{rng}(\mathcal{F}^{-1}) = [1, \infty)$

2.2

1b. vertical asymptote: $x = 4$; hole: $x = \frac{1}{2}$

1c. vertical asymptote: $x = 3$; hole: $x = -\frac{3}{2}$

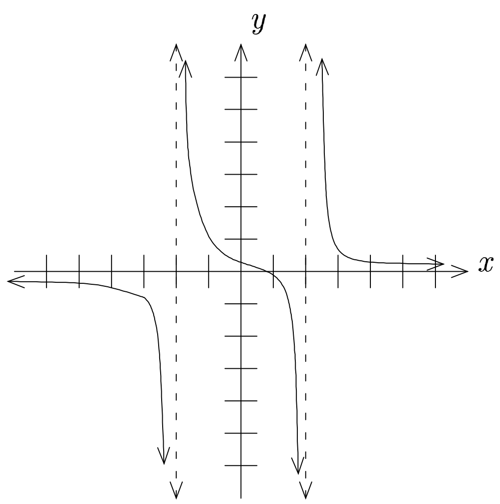
2b. horizontal asymptote: $y = 0$

2d. curvilinear asymptote: $y = 4x^2 - 2x + 1$

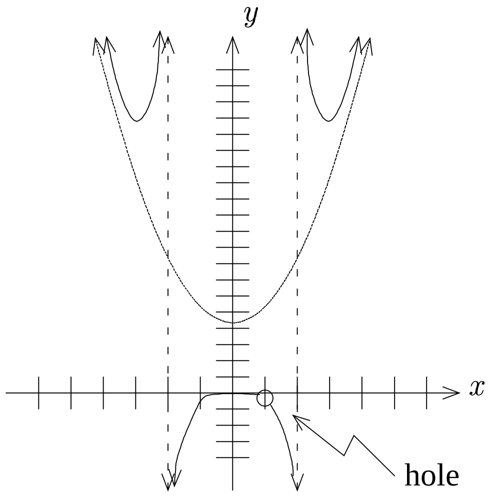
2f. horizontal asymptote: $y = \frac{2}{3}$

2.3

2.



4.



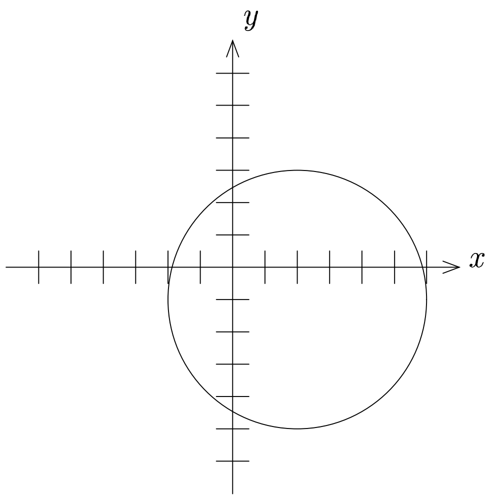
3.1

1a. $x^2 + y^2 = 4$

1c. $(x-3)^2 + (y+1)^2 = 7$

2b. center: $(2, -1)$; radius: 4; circumference: 8π

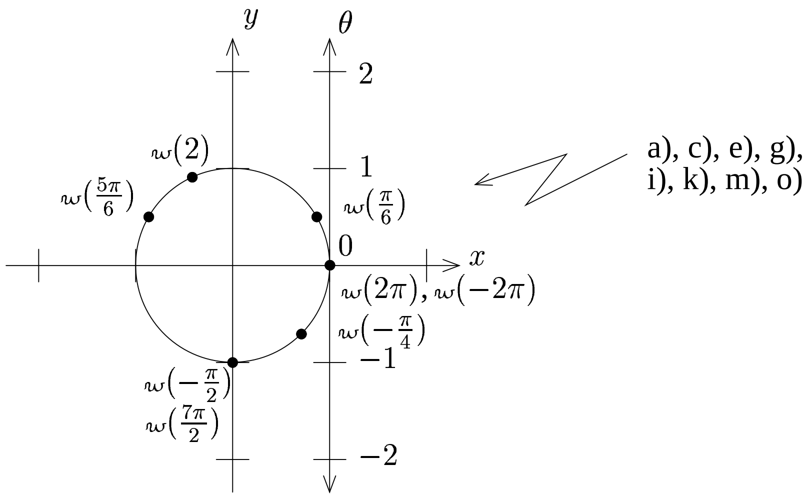
x-intercepts: $2 \pm \sqrt{15}$; y-intercepts: $-1 \pm 2\sqrt{3}$



4b. $-\pi$

4d. $\frac{\pi}{2}$

3.2



3.4

1. $(0, 1)$

3. $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$

6. $(1, 0)$

7. $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$

8. $(1, 0)$

11. $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$

3.5

1. $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$

4. $(-\frac{\sqrt{3}}{2}, \frac{1}{2})$

6. $(0, -1)$

7. $(\frac{\sqrt{3}}{2}, -\frac{1}{2})$

9. $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$

12. $(\frac{\sqrt{3}}{2}, -\frac{1}{2})$

3.6

1a. $-\frac{\sqrt{2}}{2}$

1c. $\frac{2\sqrt{3}}{3}$

1e. -1

1g. $-\frac{1}{2}$

1i. $-\frac{\sqrt{2}}{2}$

1k. 2

1m. undefined

1o. undefined

3.9

1. $-\frac{1}{4}$

3.10

1. 3

3. $\frac{7}{2}$

5. $\pm\frac{\sqrt{5}}{3}$

7. $\pm 3\sqrt{11}$

4.1

2b. period$= 2$; amplitude$= \frac{2}{5}$

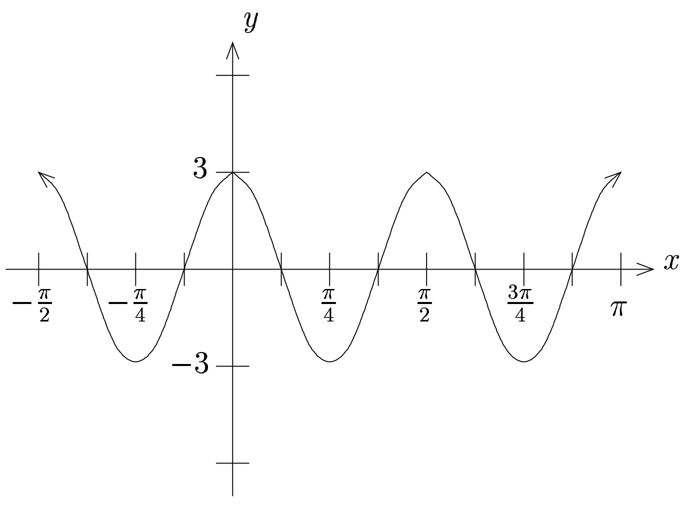
3b. amplitude$= 3$; period$= 2$; phase shift$= -\frac{1}{\pi}$

3d. amplitude$= 5$; period$= 6\pi$; phase shift$= 21$

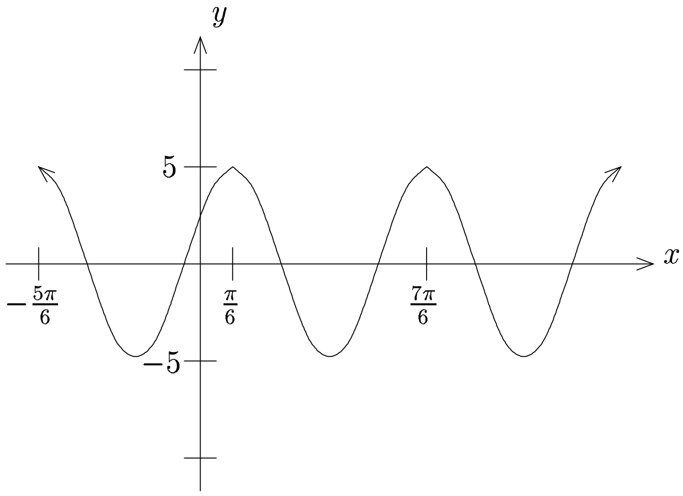
3e. amplitude$= \frac{1}{4}$; period$= \frac{10}{3}$; phase shift$= -\frac{5}{9}$

4.2

1a.

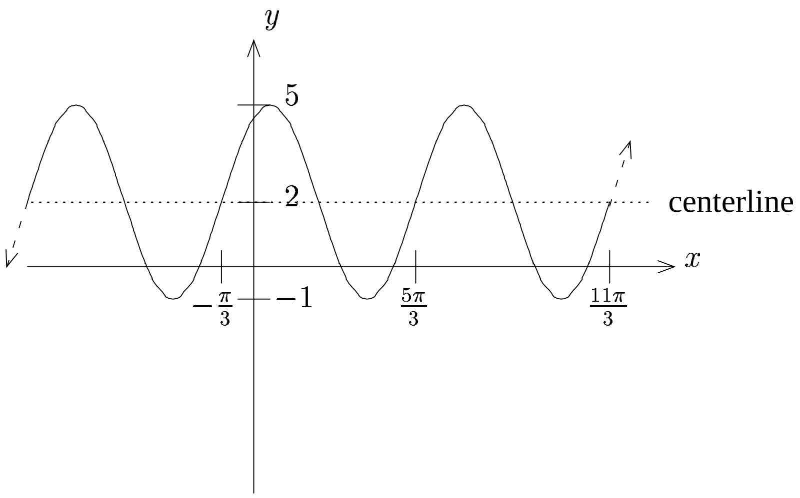


1c.



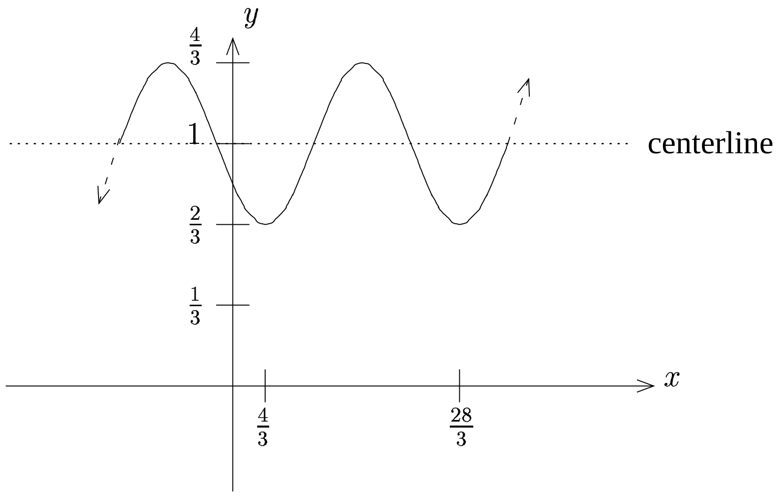
y
5
-5
x
$-\frac{5\pi}{6}$
$\frac{\pi}{6}$
$\frac{7\pi}{6}$

1e.



y
5
2
centerline
x
$-\frac{\pi}{3}$
-1
$\frac{5\pi}{3}$
$\frac{11\pi}{3}$

1g.



4.3

2. $y = 3.2 + 1.2\cos(\frac{\pi}{6}t - \frac{\pi}{3})$

3. $T = 78 + 24\cos(\frac{\pi}{6}t - \pi)$

Note: This model does not fit the data *exactly*.

For instance, when $t = 3$ above, we get $T = 78$ (rather than 76).

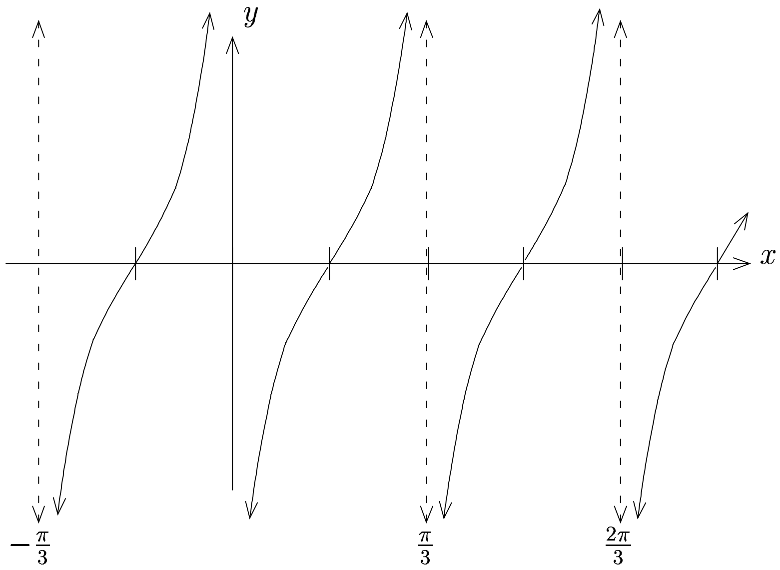
A model only gives an approximation to real world behavior.

4.6

1a. $\frac{\pi}{5}$

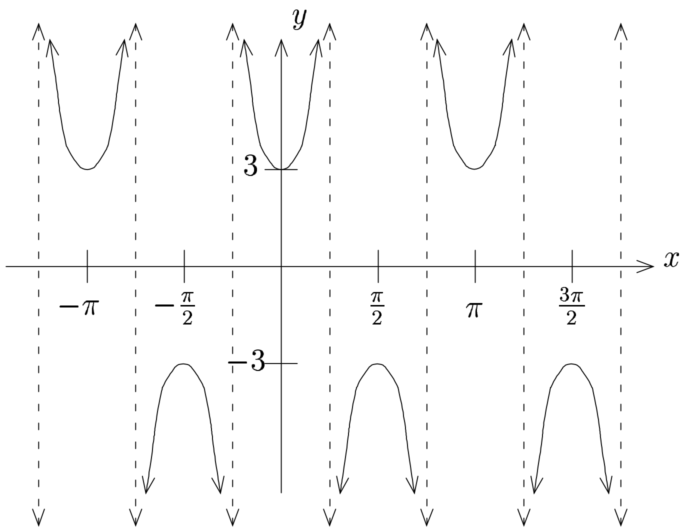
1c. $\frac{2\pi}{3}$

2a.



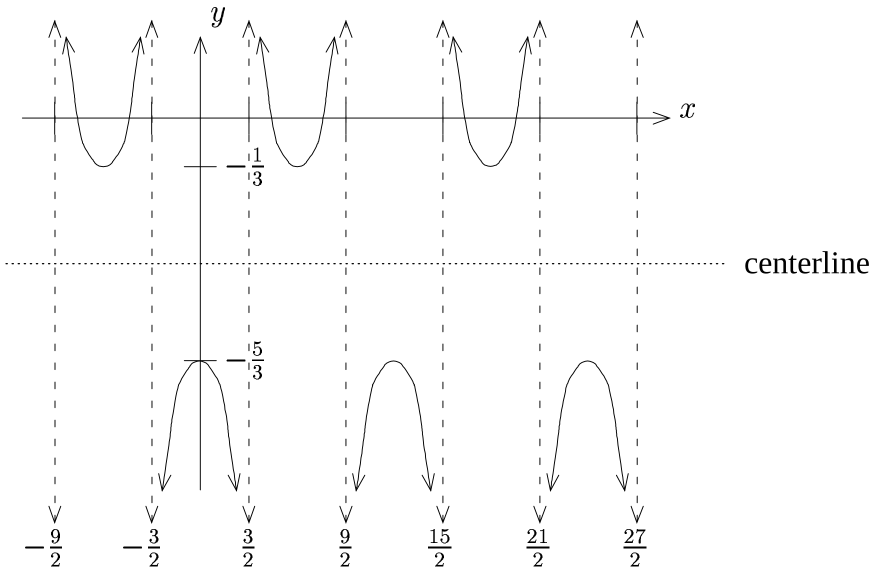
y
x
$-\frac{\pi}{3}$
$\frac{\pi}{3}$
$\frac{2\pi}{3}$

2b.



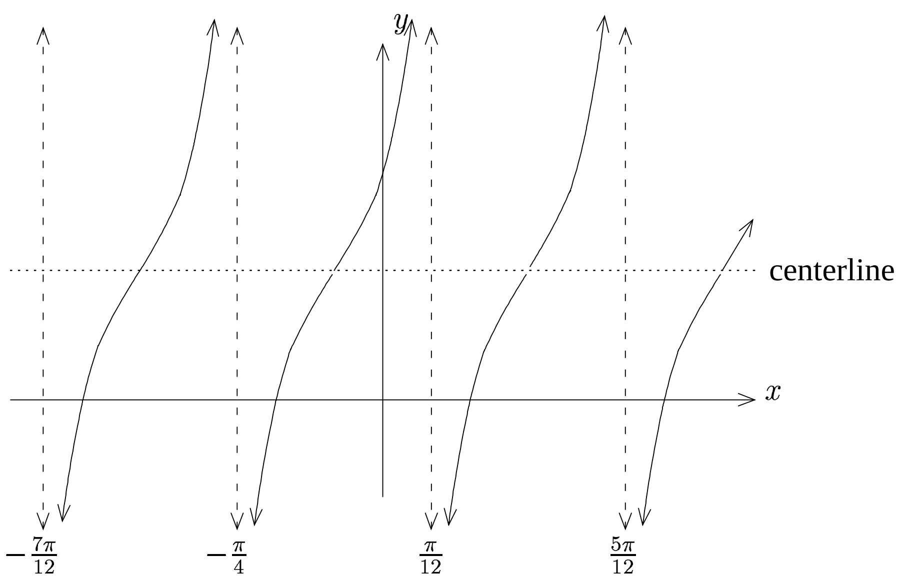
y
x
3
-3
$-\pi$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$

2d.



y
x
$-\frac{1}{3}$
centerline
$-\frac{5}{3}$
$-\frac{9}{2}$
$-\frac{3}{2}$
$\frac{3}{2}$
$\frac{9}{2}$
$\frac{15}{2}$
$\frac{21}{2}$
$\frac{27}{2}$

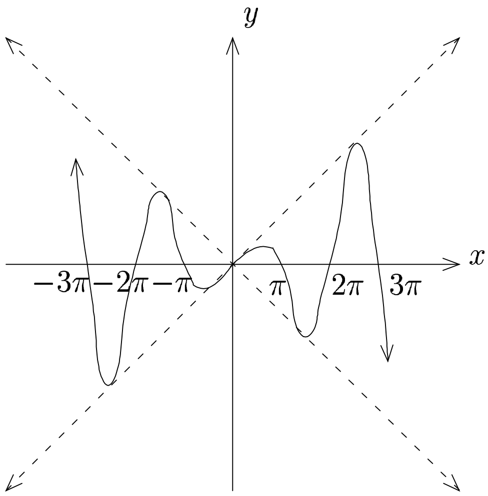
2g.



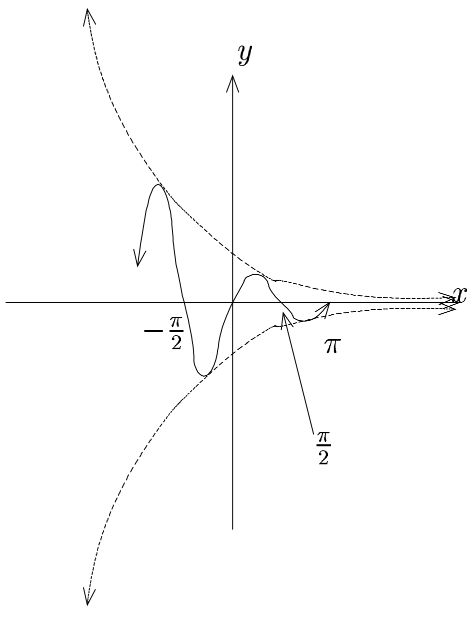
y
centerline
x
$-\frac{7\pi}{12}$
$-\frac{\pi}{4}$
$\frac{\pi}{12}$
$\frac{5\pi}{12}$

4.7

1.



3.



4.8

1b. Period: $\frac{1}{3}$ s | Frequency: 3 Hz
Angular Frequency: 6π s^{-1} | Maximum Displacement: 4 m

1d. Period: 12 s | Frequency: $\frac{1}{12}$ Hz
Angular Frequency: $\frac{\pi}{6}$ s^{-1} | Maximum Displacement: 5 m

2a. $d = \cos(\frac{\pi}{2}t)$

2c. $d = 2\sin(\frac{2\pi}{5}t)$

5.1

1. $-\tan^2\theta$

2. 1

3. $\cos\theta$

4. $(2\cot\theta - 3)(\cot\theta + 1)$

5. $(\cos\theta + \sin\theta)(1 - \sin\theta\cos\theta)$

6. $2\sec\theta$

7. $(2\csc\theta + 3)(\csc\theta - 4)$

8. $\csc^2\theta + 2\csc\theta + 4$

9. $\sec\theta$

10. $(\cos\theta + \sin\theta)(\cos\theta - \sin\theta)$

11. $1 + 2\sin\theta\cos\theta$

12. $-2\cot^2\theta$

13. $25\sin^2\theta$

14. $1+\cos\theta$

5.3

1. $\frac{\sqrt{2}+\sqrt{6}}{4}$

3. $\sqrt{3}-2$

4. $\left(\frac{\sqrt{2}+\sqrt{6}}{4}, \frac{\sqrt{2}-\sqrt{6}}{4}\right)$

6. $\sqrt{2}-\sqrt{6}$

7. $\sqrt{3}-2$

9. $\frac{\sqrt{3}}{2}$

12. $\frac{\sqrt{2}}{2}$

5.4

2. $-\frac{\sqrt{2}}{2}(\sin\theta+\cos\theta)$

3. $\frac{\tan\theta-\sqrt{3}}{1+\sqrt{3}\tan\theta}$ (unrationalized!)

6. $-\cot\theta$

5.5

2. $\sin\theta(\cos^2\beta-\sin^2\beta)-2\cos\theta\sin\beta\cos\beta$

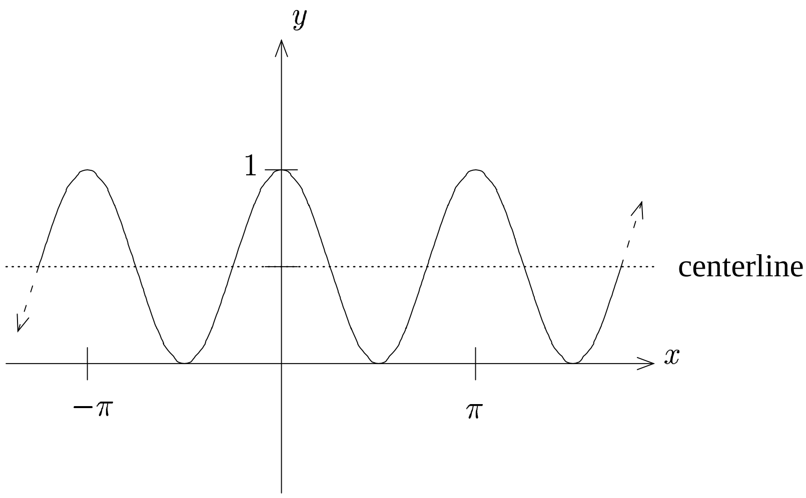
3. $3\sin\theta\cos^2\theta - \sin^3\theta$

5.6

1. $\frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$

2. $\frac{3}{8} + \frac{1}{2}\cos 4x + \frac{1}{8}\cos 8x$

5.



5.7

1. $\left(-\frac{\sqrt{2+\sqrt{2}}}{2}, \frac{\sqrt{2-\sqrt{2}}}{2}\right)$

3. $\sqrt{2}+1$

5. $\left(\frac{\sqrt{2+\sqrt{2}}}{2}, -\frac{\sqrt{2-\sqrt{2}}}{2}\right)$

5.9

1. $\frac{\sqrt{3}+\sqrt{2}}{4}$

4. $\frac{1-\sqrt{2}}{4}$

5. $\frac{cos(2\theta)-cos(8\theta)}{2}$

7. $\frac{sin(6\theta)-sin(8\theta)}{2}$

5.10

2. $-2\,sin(3\theta)\,sin(2\theta)$

4. $2\,cos(3\theta)\,cos\,\theta$

6.2

1. $\frac{\sqrt{7}}{4}$

2. $\frac{2\sqrt{2}}{3}$

4. $\frac{\sqrt{11}}{6}$

6.3

1. $-\frac{4\sqrt{21}}{125}$

3. $-\frac{1}{8}$

5. $-\sqrt{15}$

8. $\frac{4}{5}$

9. $2\sqrt{6}-5$

11. $\left(\frac{4\sqrt{17}}{17}, -\frac{\sqrt{17}}{17}\right)$

12. $\frac{5\sqrt{7}-2\sqrt{5}}{30}$

6.4

1. $\frac{\pi}{6}$

2. $-\frac{\pi}{6}$

4. $\frac{5\pi}{6}$

5. $\frac{\pi}{4}$

8. $\frac{5\pi}{11}$

11. $\frac{\pi}{3}$

6.5

1. $\frac{2\sqrt{6}}{5}$

4. $\frac{4\sqrt{7}}{7}$

5. $-\frac{7}{25}$

7. $2x^2 - 1$

8. 0

9. $\sqrt{15} - 4$

11. $\frac{24}{7}$

13. $\frac{8\sqrt{10}+9}{35}$

14. $\frac{4\sqrt{17}+2\sqrt{34}}{51}$

16. $\left(-\frac{1}{4}, \frac{\sqrt{15}}{4}\right)$

18. $\left(\frac{-3-16\sqrt{3}}{35}, \frac{12\sqrt{3}-4}{35}\right)$

6.8

1. $x = \begin{cases} \frac{2\pi}{3} + 2\pi k;\ k \in \mathbb{Z} \\ \frac{4\pi}{3} + 2\pi k;\ k \in \mathbb{Z} \end{cases}$

3. $x = \frac{\pi}{4} + \pi k;\ k \in \mathbb{Z}$

5. $x = \begin{cases} \frac{\pi}{2} + 2\pi k;\ k \in \mathbb{Z} \\ \frac{7\pi}{6} + 2\pi k;\ k \in \mathbb{Z} \\ \frac{11\pi}{6} + 2\pi k;\ k \in \mathbb{Z} \end{cases}$

8. $x = \begin{cases} \frac{\pi}{6} + \frac{2\pi k}{5};\ k \in \mathbb{Z} \\ \frac{7\pi}{30} + \frac{2\pi k}{5};\ k \in \mathbb{Z} \end{cases}$

10. $x = \begin{cases} \frac{\pi}{9} + \frac{\pi k}{3};\ k \in \mathbb{Z} \\ \frac{2\pi}{9} + \frac{\pi k}{3};\ k \in \mathbb{Z} \end{cases}$

13. $x = \begin{cases} \frac{\pi}{14} + \frac{\pi k}{7};\ k \in \mathbb{Z} \\ \frac{\pi}{6} + \frac{\pi k}{2};\ k \in \mathbb{Z} \\ \frac{\pi}{3} + \frac{\pi k}{2};\ k \in \mathbb{Z} \end{cases}$

15. $x = \frac{\pi}{4} + \pi k;\ k \in \mathbb{Z}$

16. $x = \begin{cases} 2\pi k;\ k \in \mathbb{Z} \\ \frac{\pi}{2} + 2\pi k;\ k \in \mathbb{Z} \end{cases}$

6.9

1. $x = \frac{\pi}{4} + \pi k;\ k \in \mathbb{Z}$

3. $x = \begin{cases} \frac{5\pi}{4} + 2\pi k;\ k \in \mathbb{Z} \\ \frac{7\pi}{4} + 2\pi k;\ k \in \mathbb{Z} \end{cases}$

5. $\emptyset$

6. $x = \begin{cases} \frac{\pi}{3} + 2\pi k;\ k \in \mathbb{Z} \\ \pi + 2\pi k;\ k \in \mathbb{Z} \\ \frac{5\pi}{3} + 2\pi k;\ k \in \mathbb{Z} \end{cases}$

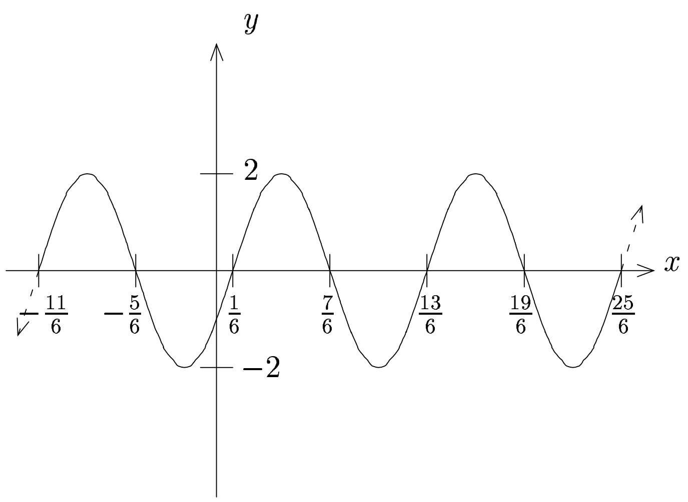
8. $x = \begin{cases} \frac{\pi}{15} + \frac{2\pi k}{15};\ k \in \mathbb{Z} \\ \frac{\pi}{3} + \frac{2\pi k}{3};\ k \in \mathbb{Z} \end{cases}$

6.10

1a. $5\sin(4\theta + \alpha)$, where $\alpha = \tan^{-1}(\frac{4}{3})$

1c. $2\sqrt{5}\sin(5\theta + \alpha)$, where $\alpha = \tan^{-1}(-2)$

2b.



3b. $x = \begin{cases} \frac{\pi}{15} + \frac{1}{5}\tan^{-1}(3) + \frac{2\pi k}{5};\ k \in \mathbb{Z} \\ \frac{2\pi}{15} + \frac{1}{5}\tan^{-1}(3) + \frac{2\pi k}{5};\ k \in \mathbb{Z} \end{cases}$

7.1

1a. $\frac{5\pi}{6}$

1c. $\frac{5\pi}{4}$

1e. $-\frac{5\pi}{12}$

2a. $72°$

2c. $-40°$

3. $\frac{9\pi}{4}$ m

4. $\frac{3\pi}{5}$

6. $\frac{11\pi}{6}$ m

7.2

1a. $\sin\theta = \frac{5}{13}$; $\cos\theta = \frac{12}{13}$; $\tan\theta = \frac{5}{12}$

1c. $\sin\theta = \frac{3\sqrt{5}}{7}$; $\cos\theta = \frac{2}{7}$; $\tan\theta = \frac{3\sqrt{5}}{2}$

2a. $B = 55°$, $a \approx 7.00$, $c \approx 12.21$

2c. $c = 3\sqrt{5}$, $A \approx 26.57°$, $B \approx 63.43°$

2e. $B = 50°$, $a \approx 7.07$, $c \approx 8.43$

3. $\theta = \tan^{-1}(\frac{\sqrt{2}}{2}) \approx 35.26°$

7.3

2. $\tan^{-1}(7) \approx 81.87°$

7.6

1. $C = 105°, \quad b \approx 8.92, \quad c \approx 15.03$

3. $B \approx 21.82°, \quad C \approx 116.18°, \quad c \approx 12.07$

5. $B \approx 49.89°, \quad C \approx 95.11°, \quad c \approx 10.42$
OR $B \approx 130.11°, \quad C \approx 14.89°, \quad c \approx 2.69$

7. $\emptyset$

10. $C \approx 117.28°, \quad A \approx 26.38°, \quad B \approx 36.34°$

13. $a \approx 7.53, \quad B \approx 101.15°, \quad C \approx 40.85°$

7.7

1. 19.49

4. 44.04