4016 MATHEMATICS
TOPIC 1: NUMBERS AND ALGEBRA

SUB-TOPIC 1.11
MATRICES

CONTENT OUTLINE
1. Display of information in the form of a matrix of any order 2. Interpreting the data in a given matrix 3. Product of a scalar quantity and a matrix 4. Problems involving the calculation of the sum and product (where appropriate) of two matrices Exclude: 1. Matrix representation of geometrical transformations *2.* Solving simultaneous linear equations using the inverse matrix method

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A Introduction

Expenses in May	Sugar ($)	Flour ($)	Butter ($)
Bakery 1	400	200	600
Bakery 2	300	500	700

The table above summarises the expenses of two bakeries in the month of May on typical bakery raw materials. The data in the table represent the following:

Bakery 1 spent $400 on sugar, $200 on flour and $600 on butter.
Bakery 2 spent $300 on sugar, $500 on flour and $700 on butter.

We see that a table is an easy and convenient way to organise data. Suppose we extract the data from the table and arrange them in rows and columns within brackets as shown below:

$$\begin{pmatrix} 400 & 200 & 600 \\ 300 & 500 & 700 \end{pmatrix}$$

This rectangular array of numbers is called a matrix (plural: matrices). The numbers in a matrix are called the entries or elements of the matrix. An element is identified by its row and column positions in a matrix. For example, the element in the first row and second column is 200.

	Column 1	Column 2	Column 3
Row 1	400	200	600
Row 2	300	500	700

If a matrix has *m* rows and *n* columns, we say that the order or dimension of the matrix is *m* x *n*. For instance, the order of the above matrix is 2 x 3 (read as two by three).

We usually denote the names of matrices with capital letters. For example:

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 6 & 11 \\ 5 & 8 \\ 7 & 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 9 \\ 18 \\ 14 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 16 & 12 & 17 & 13 \end{pmatrix}, \mathbf{E} = \begin{pmatrix} 15 \end{pmatrix}$$

The above matrices **A**, **B**, **C**, **D** and **E** have an order of 2 x 2, 3 x 2, 3 x 1, 1 x 4 and 1 x 1 respectively.

A matrix having the same number of rows and columns is called a square matrix. For example, **A** is a square matrix of the order 2 x 2, or simply of order 2.

A matrix that has only one column is called a column matrix. For example, **C** is a column matrix of the order 3 x 1.

A matrix that has only one row is called a row matrix. For example, **D** is a row matrix of order 1 x 4.

Two matrices **F** and **G** are equal (i.e. **F** = **G**), if they have the same order and the corresponding elements are equal.

Consider the matrices:

$$\mathbf{A} = \begin{pmatrix} 2 & 5 \\ 6 & 8 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 & 5 \\ 6 & 9 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 2 & 5 & 3 \\ 6 & 8 & 4 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 2 & 5 \\ 2 \times 3 & 2^3 \end{pmatrix}$$

A ≠ **B** since their elements in the second row and second column are not equal (8 ≠ 9)

A ≠ **C** since the order of **A** (2 x 2) and the order of **C** (2 x 3) are not equal.

A = **D** since they have the same order and their corresponding elements are equal.

WORKED EXAMPLES

A travel agent offers 3 types of overseas tour packages to 4 cities. The number of customers in each tour in a certain month is shown in the following table:

	Hong Kong	**Jakarta**	**Kuala Lumpur**	**Bangkok**
Economic	58	70	136	62
Standard	64	92	87	75
Deluxe	23	30	40	21

(a) Represent the data by a matrix T.

$$\mathbf{T} = \begin{pmatrix} 58 & 70 & 136 & 62 \\ 64 & 92 & 87 & 75 \\ 23 & 30 & 40 & 21 \end{pmatrix}$$

(b) State the order of T.

T has an order of 3 x 4 (as it has 3 rows and 4 columns)

(c) Calculate the sum of the elements in the first column of T. What does this sum represent?

Sum of elements in 1st column = 58 + 64 + 23
= 145 (Ans)

This represents the total number of customers joining the tour package to Hong Kong in that month.

(d) Calculate the sum of elements in the second row of T. What does this sum represent?

Sum of elements in 2nd row = 64 + 92 + 87 + 75
= 318

This represents the total number of customers opting for the Standard tour package in that month.

Question from Discovering Mathematics 4 (2008), Chow Wai Keung, Pg 62 Example 1

If $\begin{pmatrix} 4 & x \\ 2y & 13 \\ -1 & 8 \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 7 & 13 \\ -1 & z-1 \end{pmatrix}$**, find the values of** x**,** y **and** z**.**

If two matrices are equal, then the corresponding elements are equal. Hence,

$x = 5$ (Ans)

$2y = 7$

$y = 3.5$ (Ans)

$8 = z - 1$

$z = 9$ (Ans)

Question from Discovering Mathematics 4 (2008), Chow Wai Keung, Pg 62 Example 2

B The Zero Matrix and The Identity Matrix

A zero matrix (also called a null matrix) is one in which all elements are zero. It is usually denoted by the letter **O**. For example:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

are zero matrices of the order 2 x 2 and 2 x 3 respectively.

The square matrices,

$$\mathbf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \mathbf{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which have '1's in the main diagonal (i.e. from the top left to the bottom right) and zeroes everywhere else are called identity matrices. In general, **I** or $\mathbf{I}_n$ is used to denote the identity matrix of order *n* x *n*.

We will look into them later.

C Addition of Matrices

If **A** and **B** are two matrices of the same order, the sum of **A** + **B** is the matrix obtained by adding the corresponding elements of **A** and **B**.

For example: if **C** = **A** + **B**, where:

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 5 & 8 \\ 6 & 7 \end{pmatrix}$$

Then the matrix **C** would be equal to:

$$\mathbf{C} = \begin{pmatrix} 1+5 & 3+8 \\ 2+6 & 4+7 \end{pmatrix} = \begin{pmatrix} 6 & 11 \\ 8 & 11 \end{pmatrix}$$

When adding a zero matrix to any matrix **A**, the result will still be **A**. Let's check.

$$\mathbf{A} + \mathbf{O} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1+0 & 3+0 \\ 2+0 & 4+0 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \mathbf{A}$$

The associate law of addition states that **A** + **B** = **B** + **A**. Let's check.

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 8 \\ 6 & 7 \end{pmatrix} = \begin{pmatrix} 1+5 & 3+8 \\ 2+6 & 4+7 \end{pmatrix} = \begin{pmatrix} 6 & 11 \\ 8 & 11 \end{pmatrix}$$

$$\mathbf{B} + \mathbf{A} = \begin{pmatrix} 5 & 8 \\ 6 & 7 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 5+1 & 8+3 \\ 6+2 & 7+4 \end{pmatrix} = \begin{pmatrix} 6 & 11 \\ 8 & 11 \end{pmatrix}$$

Yes, they are equal.

The commutative law of addition states that (**A** + **B**) + **C** = **A** + (**B** + **C**). Let's check.

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \left(\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 8 \\ 6 & 7 \end{pmatrix}\right) + \begin{pmatrix} 6 & 11 \\ 8 & 11 \end{pmatrix} = \begin{pmatrix} 6 & 11 \\ 8 & 11 \end{pmatrix} + \begin{pmatrix} 6 & 11 \\ 8 & 11 \end{pmatrix} = \begin{pmatrix} 12 & 22 \\ 16 & 22 \end{pmatrix}$$

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} + \left(\begin{pmatrix} 5 & 8 \\ 6 & 7 \end{pmatrix} + \begin{pmatrix} 6 & 11 \\ 8 & 11 \end{pmatrix}\right) = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} 11 & 19 \\ 14 & 18 \end{pmatrix} = \begin{pmatrix} 12 & 22 \\ 16 & 22 \end{pmatrix}$$

Yes they are equal.

Hence, we can see that the associative and commutative laws of addition apply to matrices as well. In summary,

If **A**, **B**, **C** and **O** are matrices of the same order, then:

1. **A** + **O** = **A**
2. **A** + **B** = **B** + **A**
3. (**A** + **B**) + **C** = **A** + (**B** + **C**)

D Subtraction of Matrices

If **A** and **B** are two matrices of the same order, the difference of **A** – **B** is the matrix obtained by subtracting the corresponding elements of in **B** from **A**.

For example, if **D** = **B** – **A**, where:

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 5 & 8 \\ 6 & 7 \end{pmatrix}$$

Then the matrix **D** would be equal to:

$$\mathbf{D} = \begin{pmatrix} 5 - 1 & 8 - 3 \\ 6 - 2 & 7 - 4 \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 4 & 3 \end{pmatrix}$$

E Multiplication of a Matrix by a Scalar

If k is scalar, then the scalar multiplication of a matrix **A** by k, denoted by k**A**, is obtained by multiplying every element in **A** by k.

For example, if **E** = 5**A** + 4**B**, where:

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 5 & 8 \\ 6 & 7 \end{pmatrix}$$

Then the matrix **E** would be equal to:

$$\mathbf{E} = 5\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} + 4\begin{pmatrix} 5 & 8 \\ 6 & 7 \end{pmatrix} = \begin{pmatrix} 5 & 15 \\ 10 & 20 \end{pmatrix} + \begin{pmatrix} 20 & 32 \\ 24 & 28 \end{pmatrix} = \begin{pmatrix} 5 + 20 & 15 + 32 \\ 10 + 24 & 20 + 28 \end{pmatrix} = \begin{pmatrix} 25 & 47 \\ 34 & 48 \end{pmatrix}$$

F Multiplication of Matrices

Let us consider a simple problem involving the masses of three books in two bookstores and the prices of these books. The data are displayed in the two matrices below:

A =

	English	Biology	Physics
Store 1	8	9	7
Store 2	3	6	4

B =

	Price	Mass
English	15	0.6
Biology	18	1.2
Physics	20	0.9

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Suppose the matrix **A** represents the number of English, Biology and Physics books available in stores 1 and 2; the matrix **B** represents the price (in dollars) and the mass (in kg) of each copy of those books.

We see that:

Total price of books in store 1
= 8 x 15 + 9 x 18 + 7 x 20
= $422

$$\begin{pmatrix} 8 & 9 & 7 \\ 3 & 6 & 4 \end{pmatrix}\begin{pmatrix} 15 & 0.6 \\ 18 & 1.2 \\ 20 & 0.9 \end{pmatrix}$$

Total price of books in store 2
= 3 x 15 + 6 x 18 + 4 x 20
= $233

$$\begin{pmatrix} 8 & 9 & 7 \\ 3 & 6 & 4 \end{pmatrix}\begin{pmatrix} 15 & 0.6 \\ 18 & 1.2 \\ 20 & 0.9 \end{pmatrix}$$

Total mass of books in store 1
= 8 x 0.6 + 9 x 1.2 + 7 x 0.9
= 12.6 kg

$$\begin{pmatrix} 8 & 9 & 7 \\ 3 & 6 & 4 \end{pmatrix}\begin{pmatrix} 15 & 0.6 \\ 18 & 1.2 \\ 20 & 0.9 \end{pmatrix}$$

Total mass of books in store 2
= 3 x 0.6 + 6 x 1.2 + 4 x 0.9
= 12.6 kg

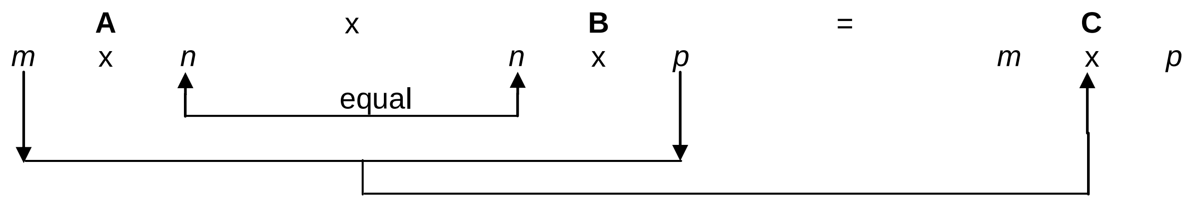
$$\begin{pmatrix} 8 & 9 & 7 \\ 3 & 6 & 4 \end{pmatrix}\begin{pmatrix} 15 & 0.6 \\ 18 & 1.2 \\ 20 & 0.9 \end{pmatrix}$$

The results can thus be summarised by:

		Total Price	Total Mass
C =	Store 1	422	21.9
	Store 2	233	12.6

We say that **C** is the product matrix **AB**, and we write **C** = **AB**.

In general, the multiplication of two matrices is defined as follows: If **A** is a matrix of order *m* x *n* and **B** is a matrix of order *n* x *p*, then the product **AB** is a matrix of order *m* x *p*, whose element at the i^{th} row and the j^{th} column is the sum of the products of the corresponding elements in the i^{th} row of **A** and the j^{th} column of **B**. That means:



Note that the matrix product **AB** is defined only when the number of columns of **A** is equal to the number of rows of **B**. In order words, the product **CD** cannot exist if matrix **C** has 3 columns, but matrix **D** has 4 rows.

When multiplying a zero matrix to any matrix **A**, the result is the zero matrix. Let's check.

$$\mathbf{AO} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1(0) + 3(0) & 1(0) + 3(0) \\ 2(0) + 4(0) & 2(0) + 4(0) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{O}$$

Yes it is correct.

When multiply an identity matrix to any matrix **A**, the result is still **A**. Let's check.

$$\mathbf{AI} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1(1) + 3(0) & 1(0) + 3(1) \\ 2(1) + 4(0) & 2(0) + 4(1) \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \mathbf{A}$$

Yes it is correct.

Multiplication of matrices is not commutative (i.e. **AB** ≠ **BA**). Let's check.

$$\mathbf{AB} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 5 & 8 \\ 6 & 7 \end{pmatrix} = \begin{pmatrix} 1(5) + 3(6) & 1(8) + 3(7) \\ 2(5)+ 4(6) & 2(8) + 4(7) \end{pmatrix} = \begin{pmatrix} 5 + 18 & 8 + 21 \\ 10 + 24 & 16 + 28 \end{pmatrix} = \begin{pmatrix} 23 & 29 \\ 34 & 44 \end{pmatrix}$$

$$\mathbf{BA} = \begin{pmatrix} 5 & 8 \\ 6 & 7 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 5(1) + 8(2) & 5(3) + 8(4) \\ 6(1) + 7(2) & 6(3) + 7(4) \end{pmatrix} = \begin{pmatrix} 5 + 16 & 15 + 32 \\ 6 + 14 & 18 + 28 \end{pmatrix} = \begin{pmatrix} 21 & 47 \\ 20 & 46 \end{pmatrix}$$

Yes it is correct.

Multiplication of matrices is associative (i.e. (**AB**)**C** = **A**(**BC**))

(**AB**)**C**	**A**(**BC**)
AB $= \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 5 & 8 \\ 6 & 7 \end{pmatrix}$ $= \begin{pmatrix} 1(5) + 3(6) & 1(8) + 3(7) \\ 2(5)+ 4(6) & 2(8) + 4(7) \end{pmatrix}$ $= \begin{pmatrix} 5 + 18 & 8 + 21 \\ 10 + 24 & 16 + 28 \end{pmatrix}$ $= \begin{pmatrix} 23 & 29 \\ 34 & 44 \end{pmatrix}$	**BC** $= \begin{pmatrix} 5 & 8 \\ 6 & 7 \end{pmatrix}\begin{pmatrix} 6 & 11 \\ 8 & 11 \end{pmatrix}$ $= \begin{pmatrix} 5(6) + 8(8) & 5(11) + 8(11) \\ 6(6) + 7(8) & 6(11) + 7(11) \end{pmatrix}$ $= \begin{pmatrix} 30 + 64 & 55 + 88 \\ 36 + 56 & 66 + 77 \end{pmatrix}$ $= \begin{pmatrix} 94 & 143 \\ 92 & 143 \end{pmatrix}$
(AB)C $= \begin{pmatrix} 23 & 29 \\ 34 & 44 \end{pmatrix}\begin{pmatrix} 6 & 11 \\ 8 & 11 \end{pmatrix}$ $= \begin{pmatrix} 23(6) + 29(8) & 23(11) + 29(11) \\ 34(6)+ 44(8) & 34(11)+ 44(11) \end{pmatrix}$ $= \begin{pmatrix} 138 + 232 & 253 + 319 \\ 204 + 352 & 374 + 484 \end{pmatrix}$ $= \begin{pmatrix} 370 & 572 \\ 556 & 858 \end{pmatrix}$	**A**(**BC**) $= \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 94 & 143 \\ 92 & 143 \end{pmatrix}$ $= \begin{pmatrix} 1(94) + 3(92) & 1(143) + 3(143) \\ 2(94)+ 4(92) & 2(143)+ 4(143) \end{pmatrix}$ $= \begin{pmatrix} 94 + 276 & 143 + 429 \\ 188 + 368 & 286 + 572 \end{pmatrix}$ $= \begin{pmatrix} 370 & 572 \\ 556 & 858 \end{pmatrix}$

Yes, it is correct.

In summary, suppose **A**, **B** and **C** are matrices such that the following products are defined. Then

1. **AO** = **OA** = **O**
2. $\mathbf{AI}_n = \mathbf{I}_n\mathbf{A} = \mathbf{A}$ (only if the order of **A** is n x n)
3. In general, **AB** ≠ **BA**
4. (**AB**)**C** = **A**(**BC**)