19

Angles in a Circle and Cyclic Quadrilateral

19.1 INTRODUCTION

You must have measured the angles between two straight lines, let us now study the angles made by arcs and chords in a circle and a cyclic quadrilateral.

19.2 OBJECTIVES

After studying this lesson, the learner will be able to :

- prove that angles in the same segment of a circle are equal
- cite examples of concyclic points
- define cyclic quadrilaterals
- prove that sum of the opposite angles of a cyclic quadrilateral is 180°
- use properties of a cyclic quadrilateral
- solve problems based on Theorems (proved) and solve other numerical problems based on verified properties.

19.3 EXPECTED BACKGROUND KNOWLEDGE

- Angles of a triangle
- Arc, chord and circumference of a circle
- Quadrilateral and its types

19.4 ANGLES IN A CIRCLE

Central Angle. The angle made at the centre of a circle by the radii at the end points of an arc (or a chord) is called the central angle or angle subtended by an arc (or chord) at the centre.

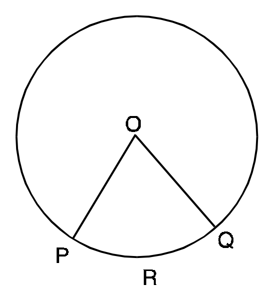


Fig.19.1

In Figure 19.1, ∠POQ is the central angle made by arc PRQ.

The length of an arc is closely associated with the central angle subtended by the arc. Let us define the "degree measure" of an arc in terms of the central angle.

The degree measure of a minor arc of a circle is the measure of its corresponding central angle.

In Figure 19.2, Degree measure of PQR = x°

The degree measure of a semicircle in 180° and that of a major arc is 360° minus the degree measure of the corresponding minor arc.

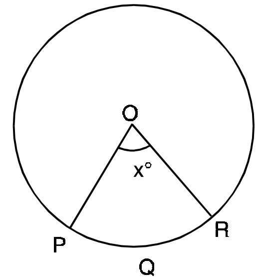


Fig.19.2

Relationship between length of an arc and its degree measure.

$$\textbf{Length of an arc} = \textbf{circumference} \times \frac{\textbf{degree measure of the arc}}{\textbf{360}^\circ}$$

If the degree measure of an arc is 40°

$$\text{then length of the arc PQR} = 2\pi r.\frac{40^\circ}{360^\circ} = \frac{2}{9}\pi r$$

Inscribed angle : The angle subtended by an arc (or chord) on any point on the remaining part of the circle is called an inscribed angle.

In Figure 19.3, ∠PAQ is the angle inscribed by arc PRQ at point A of the remaining part of the circle or by the chord PQ at the point A.

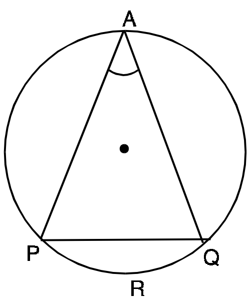


Fig.19.3

19.5. SOME IMPORTANT PROPERTIES

ACTIVITY FOR YOU :

Draw a circle with centre O. Let PAQ be an arc and B any point on the circle.

Measure the central angle POQ and an inscribed angle PBQ by the arc at remaining part of the circle. We observe that

$$\angle POQ = 2\angle PBQ$$

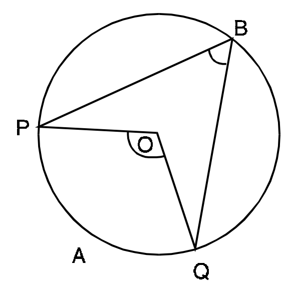


Fig.19.4

Repeat this activity taking different circles and different arcs. We observe that

The angle subtended at the centre of a circle by an arc is double the angle subtended by it on any point on the remaining part of the circle.

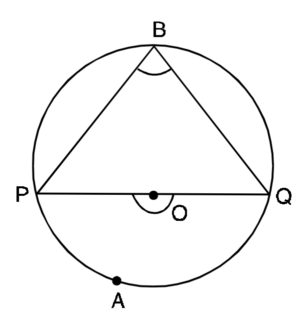


Fig.19.5

Let O be the centre of a circle. Consider a semicircle PAQ and its inscribed angle PBQ

$$\therefore \quad 2\angle PBQ = \angle POQ$$

(Since the angle subtended by an arc at the centre is double the angle subtended by it at any point on the remaining part of the circle)

But $\angle POQ = 180°$ (Since PQ is a diameter of the circle)

$2\angle PBQ = 180°$

$\therefore \angle PBQ = 90°$

Thus, we conclude the following :

Angle in a semicircle is a right angle.

Theorem : *Angles in the same segment of a circle are equal.*

Given : A circle with centre O and the angles $\angle PRQ$ and $\angle PSQ$ in the same segment formed by the chord PQ (or arc PAQ)

To prove : $\angle PRQ = \angle PSQ$

Construction : Join OP and OQ.

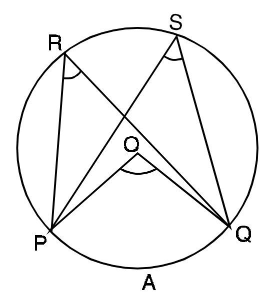


Fig.19.6

Proof : As the angle subtended by an arc at the centre is double the angle subtended by it at any point on the remaining part of the circle, therefore we have

$\angle POQ = 2\ \angle PRQ$ ...(i)

and $\angle POQ = 2\angle PSQ$ ...(ii)

From (i) and (ii), we get

$2\angle PRQ = 2\angle PSQ$

$\therefore \angle PRQ = \angle PSQ$

We take some examples using the above results

Example 19.1 : In Figure 19.7, O is the centre of the circle and $\angle AOC = 120°$. Find $\angle ABC$.

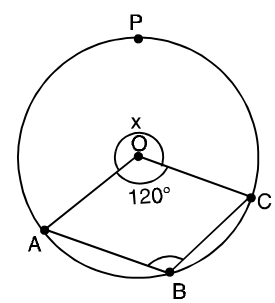


Fig.19.7

Solution : It is obvious that $\angle x$ is the central angle subtended by the arc APC and $\angle ABC$ is the inscribed angle.

$\therefore \angle x = 2\angle ABC$

But $\angle x = 360° - 120°$

$\therefore 2\angle ABC = 240°$

$\therefore \angle ABC = 120°$

Example 19.2 : In Figure 19.8 O is the centre of the circle and $\angle PAQ = 35°$. Find $\angle OPQ$.

Solution : $\angle POQ = 2\angle PAQ = 70°$ ...(i)

(Angle at the centre is double the angle on the remaining part of the circle)

Since $OP = OQ$ (Radii of the same circle)

$\therefore \quad \angle OPQ = \angle OQP$ ...(ii)

(Angles opposite to equal sides are equal)

But $\angle OPQ + \angle OQP + \angle POQ = 180°$

$\therefore \quad 2\angle OPQ = 180° - 70° = 110°$

$\therefore \quad \angle OPQ = 55°$.

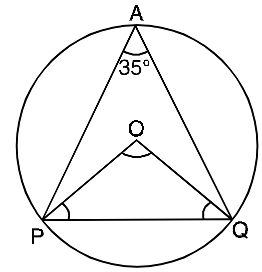


Fig.19.8

Example 19.3 : In Figure 19.9, O is the centre of the circle and AD bisects $\angle BAC$. Find $\angle BCD$.

Solution : Since BC is a diameter

$\angle BAC = 90°$

(Angle in the semicircle is a right angle)

As AD bisects $\angle BAC$

$\therefore \quad \angle BAD = 45°$

But $\angle BCD = \angle BAD$

(Angles in the same segment of a circle are equal)

$\therefore \quad \angle BCD = 45°$.

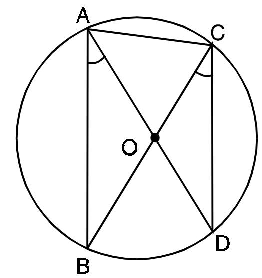


Fig.19.9

Example 19.4 : In Figure 19.10, O is the centre of the circle, $\angle POQ = 70°$ and $PS \perp OQ$. Find $\angle MQS$.

Solution :

$2\angle PSQ = \angle POQ = 70°$

(Angle subtended at the centre of a circle is twice the angle subtended by it on the remaining part of the circle)

$\therefore \quad \angle PSQ = 35°$

Since $\angle MSQ + \angle SMQ + \angle MQS = 180°$

(Sum of the angles of a triangle)

$\therefore \quad 35° + 90° + \angle MQS = 180°$

$\therefore \quad \angle MQS = 180° - 125° = 55°$.

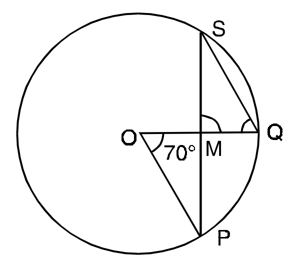


Fig.19.10

CHECK YOUR PROGRESS 19.1

1. In Figure 19.11, ADB is an arc of a circle with centre O, if $\angle ACB = 35°$, find $\angle AOB$.

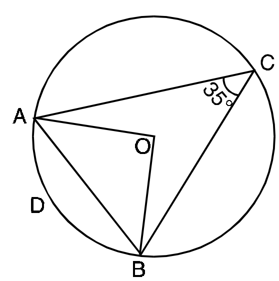


Fig. 19.11

2. In Figure 19.12, AOB is a diameter of a circle with centre O. Is ∠APB = ∠AQB = 90°? Give reasons.

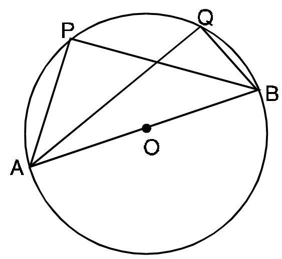


Fig. 19.12

3. In Figure 19.13, PQR is an arc of a circle with centre O. If ∠PTR = 35°, find ∠PSR.

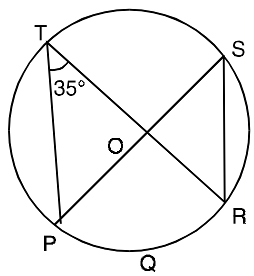


Fig. 19.13

4. In Figure 19.14, O is the centre of a circle and ∠AOB = 60°. Find ∠ADB.

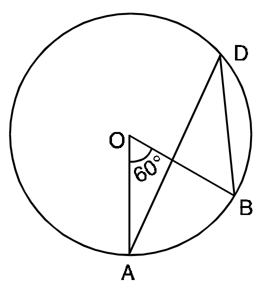


Fig. 19.14

19.6 CONCYCLIC POINTS

Definition : Points which lie on a circle are called concyclic points.

Let us now find certain conditions under which points are concyclic.

If you take a point P, you can draw not only one but many circles passing through it as in Fig. 19.15.

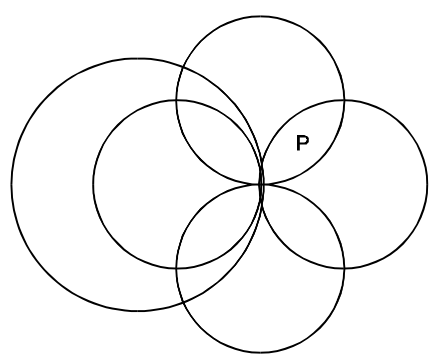


Fig.19.15

Now take two points P and Q on a sheet of a paper. You can draw as many circles as you wish, passing through the points. (Fig. 19.16).

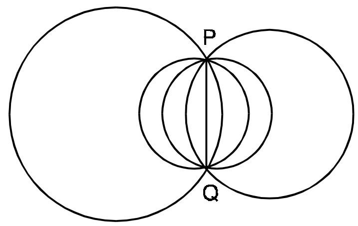


Fig. 19.16

Let us now take three points P, Q and R which do not lie on the same straight line. In this case you can draw only one circle passing through these three non-colinear points (Figure 19.17).

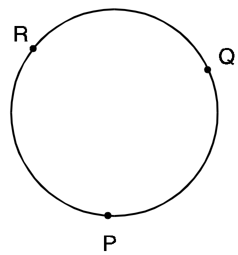
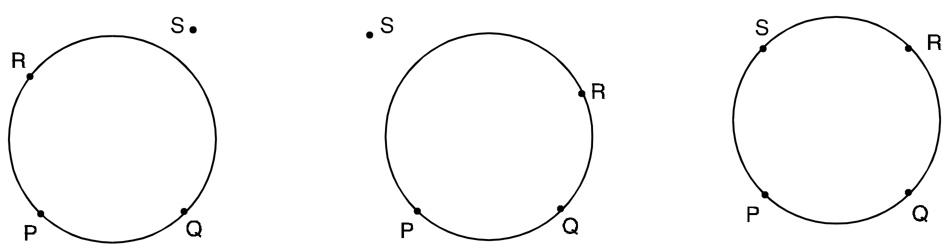


Fig. 19.17

Further let us now take four points P, Q, R, and S which do not lie on the same line. You will see that it is not always possible to draw a circle passing through four non-collinear points.

In Fig 19.18 (a) and (b) points are noncyclic but concyclic in Fig 19.18(c).



(a) (b) (b)

Fig. 19.18

Note. *If the points P, Q and R are collinear then it is not possible to draw a circle passing through them.*

Thus we conclude

1. Given one or two points there are infinitely many circles passing through them.

2. Three non-collinear points are always concyclic and there is only one circle passing through all of them.

3. Three collinear points are not concyclic (or noncyclic).

4. Four non-collinear points may or may not be concyclic.

19.6.1 CYCLIC QUADRILATERAL

A quadrilateral is said to be a cyclic quadrilateral if there is a circle passing through all its four vertices.

For example, Fig. 19.19 shows a cyclic quadrilateral PQRS.

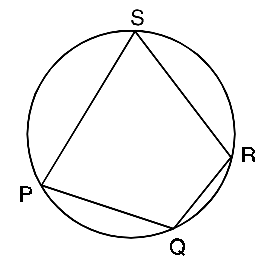


Fig.19.19

Theorem. *Sum of the opposite angles of a cyclic quadrilateral is 180°.*

Given : A cyclic quadrilateral ABCD

To prove : $\angle BAD + \angle BCD = \angle ABC + \angle ADC = 180°$

Construction : Draw AC and DB

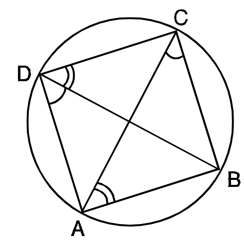


Fig.19.20

Proof : $\angle ACB = \angle ADB$

and $\angle BAC = \angle BDC$

[Angles in the same segment]

$\therefore \quad \angle ACB + \angle BAC = \angle ADB + \angle BDC = \angle ADC$

Adding $\angle ABC$ on both the sides, we get

$\angle ACB + \angle BAC + \angle ABC = \angle ADC + \angle ABC$

But $\angle ACB + \angle BAC + \angle ABC = 180°$ [Sum of the angles of a triangle]

$\therefore \quad \angle ADC + \angle ABC = 180°$

$\therefore \quad \angle BAD + \angle BCD = 360° - (\angle ADC + \angle ABC) = 180°.$

Hence proved.

Converse of this theorem is also true.

If a pair of opposite angles of a quadrilateral is supplementary, then the quadrilateral is cyclic.

Verification :

Draw a quadrilateral PQRS

Since in quadrilateral PQRS,

$\angle P + \angle R = 180°$

and $\angle S + \angle Q = 180°$

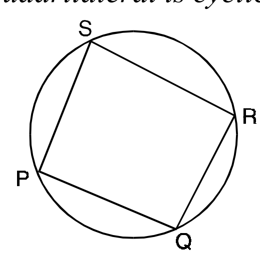


Fig.19.21

Therefore draw a circle passing through the point P, Q and R and observe that it also passes through the point S. So we conclude that quadrilateral PQRS is a cyclic quadrilateral.

We solve some examples using the above results.

Example 19.5 : ABCD is a cyclic parallelogram. Show that it is a rectangle.

Solution : $\angle A + \angle C = 180°$

(ABCD is a cyclic quadrilateral)

Since $\angle A = \angle C$

[Opposite angles of a parallelogram]

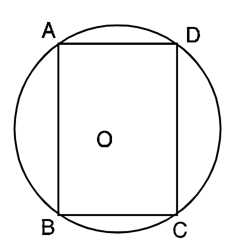


Fig.19.22

or $\angle A + \angle A = 180°$

$\therefore$ $2\angle A = 180°$

$\therefore$ $\angle A = 90°$

Thus ABCD is a rectangle.

Example 19.6 : A pair of opposite sides of a cyclic quadrilateral is equal. Prove that its diagonals are also equal (See Figure 19.23).

Solution : Let ABCD be a cyclic quadrilateral and AB = CD.

$\Rightarrow$ arc AB = arc CD (Corresponding arcs)

Adding arc AD to both the sides;

arc AB + arc AD = arc CD + arc AD

$\therefore$ arc BAD = arc CDA

$\Rightarrow$ Chord BD = Chord CA

$\Rightarrow$ BD = CA

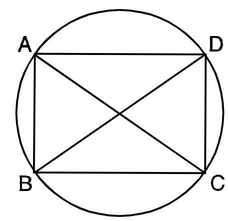


Fig.19.23

Example 19.7 : In Figure 19.24, PQRS is a cyclic quadrilateral whose diagonals intersect at A. If $\angle SQR = 80°$ and $\angle QPR = 30°$, find $\angle SRQ$.

Solution : Given $\angle SQR = 80°$

Since $\angle SQR = \angle SPR$

[Angles in the same segment]

$\therefore$ $\angle SPR = 80°$

$\therefore$ $\angle SPQ = \angle SPR + \angle RPQ$

$= 80° + 30°.$

or $\angle SPQ = 110°.$

But $\angle SPQ + \angle SRQ = 180°$ (Sum of the opposite angles of a cyclic quadrilateral is 180°)

$\therefore$ $\angle SRQ = 180° - \angle SPQ$

$= 180° - 110° = 70°$

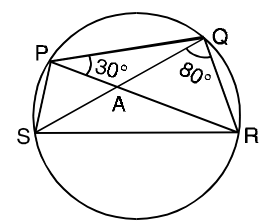


Fig.19.24

Example 19.8 : PQRS is a cyclic quadrilateral.

If $\angle Q = \angle R = 65°$, find $\angle P$ and $\angle S$.

Solution : $\angle P + \angle R = 180°$

$\therefore$ $\angle P = 180° - \angle R = 180° - 65°$

$\therefore$ $\angle P = 115°$

Similarly, $\angle Q + \angle S = 180°$

$\therefore$ $\angle S = 180° - \angle Q = 180° - 65°$

$\therefore$ $\angle S = 115°.$

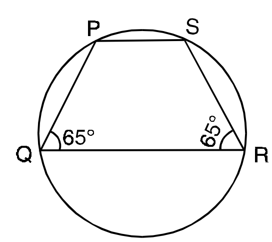


Fig.19.25

CHECK YOUR PROGRESS 19.2

1. In Figure 19.26, AB and CD are two equal chords of a circle with centre O. If $\angle AOB = 55°$, find $\angle COD$.

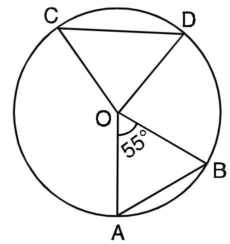


Fig. 19.26

2. In Figure 19.27, PQRS is a cyclic quadrilateral, and the side PS is extended to the point A. If $\angle PQR = 80°$, find $\angle ASR$.

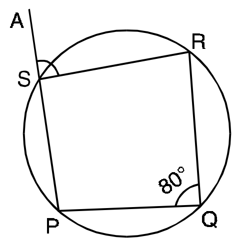


Fig. 19.27

3. In Figure 19.28, ABCD is a cyclic quadrilateral whose diagonals intersect at O. If $\angle ACB = 50°$ and $\angle ABC = 110°$, find $\angle BDC$.

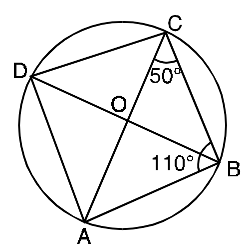


Fig. 19.28

4. In Figure 19.29, ABCD is a quadrilateral. If $\angle A = \angle BCE$, is the quadrilateral a cyclic quadrilateral ? Give reasons.

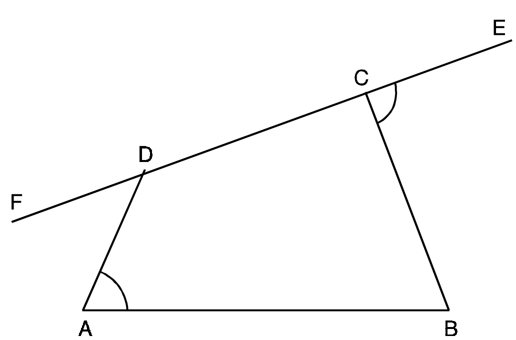


Fig. 19.29

LET US SUM UP

- The angle subtended by an arc (or chord) at the centre of a circle is called central angle and an angle subtended by it at any point on the remaining part of the circle is called inscribed angle.
- Points lying on the same circle are called concyclic points.
- The angle subtended by an arc at the centre of a circle is double the angle subtended by it at any point on the remaining part of the circle.
- Angle in a semicircle is a right angle.
- Angles in the same segment of a circle are equal.
- Sum of the opposite angles of a cyclic quadrilateral is 180°.
- If a pair of opposite angles of a quadrilateral is supplementary, then the quadrilateral is cyclic.

TERMINAL EXERCISES

1. A square PQRS is inscribed in a circle with centre O. What angle does each side subtend at the centre O ?

2. In Figure 19.30, C_1 and C_2 are two circles with centre O_1 and O_2 and intersect each other at points A and B. If O_1O_2 intersect AB at M then show that

(i) $\Delta O_1AO_2 \cong \Delta O_1BO_2$

(ii) M is the mid point of AB

(iii) $AB \perp O_1O_2$

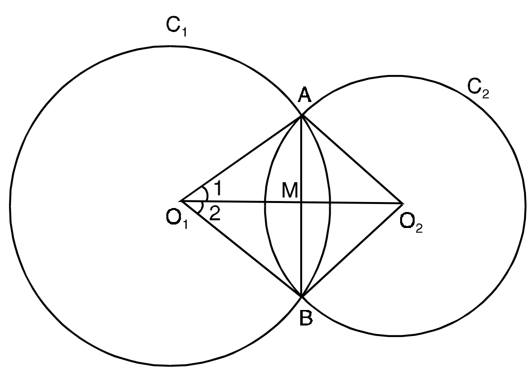


Fig. 19.30

(**Hint.** From (i) conclude that $\angle 1 = \angle 2$ and then prove that $\Delta AO_1M \cong \Delta BO_1M$ (by SAS rule)).

3. Two circle intersect in A and B. AC and AD are the diameters of the circles. Prove that C, B and D are collinear.

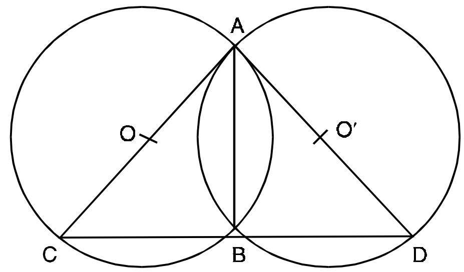


Fig. 19.31

[**Hint.** Join CB, BD and AB, Since ∠ABC = 90° and ∠ABD = 90°]

4. In Figure 19.32, AB is a chord of a circle with centre O. If ∠ACB = 40°, find ∠OAB.

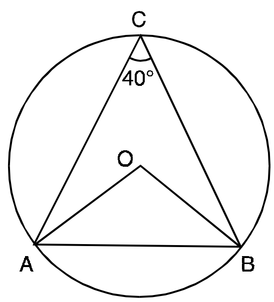


Fig. 19.32

5. In Figure 19.33, O is the centre of a circle and ∠PQR = 115°. Find ∠POR.

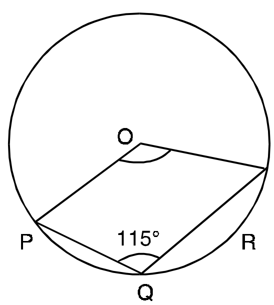


Fig. 19.33

6. In Figure 19.34, O is the centre of a circle, ∠AOB = 80° and ∠PQB = 70°. Find ∠PBQ.

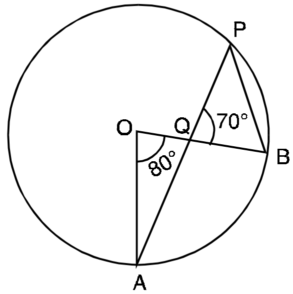


Fig. 19.34

ANSWERS

Check Your Progress 19.1

1. 70° **2.** Yes **3.** 35° **4.** 30°

Check Your Progress 19.2

1. 55° **2.** 80° **3.** 20° **4.** Yes

Terminal Exercise

1. 90° **4.** 50° **5.** 130° **6.** 70°.