4038 ADDITIONAL MATHEMATICS
TOPIC 2: GEOMETRY AND TRIGONOMETRY

SUB-TOPIC 2.2
COORDINATE GEOMETRY IN TWO DIMENSIONS

CONTENT OUTLINE
1. Condition for two lines to be parallel or perpendicular 2. Mid-point of line segment 3. Finding the area of rectilinear figure given its vertices 4. Graphs of equations a. $y = ax^n$, where n is a simple rational number b. $y^2 = kx$ 5. Coordinate geometry of the circle with the equation $(x - a)^2 + (y - b)^2 = r^2$ and $x^2 + y^2 + 2gx + 2fy + c = 0$ 6. Transformation of given relationships, including $y = ax^n$ and $y = kb^x$, to linear form to determine the unknown constants from the straight line graph Exclude: 1. Finding the equation of the circle passing through three given points. 2. Intersection of two circles

SUB-TOPIC 2.2: COORDINATE GEOMETRY IN TWO DIMENSIONS

A Introduction

In Mathematics, we have learnt the above few. Let us revise:

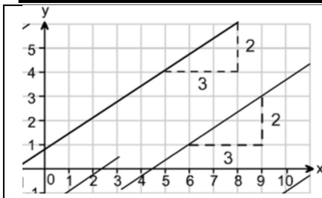
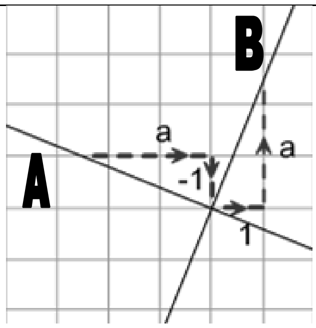
- The Length of AB, where A (x_1, y_1) and B (x_2, y_2), has a length of
  $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

- The gradient of a line refers to the vertical change as we move 1 unit across the x axis. Hence, Gradient of line L = $\frac{\text{Change in } y \text{ coordinates}}{\text{Change in } x \text{ coordinates}} = \frac{y_1 - y_2}{x_1 - x_2}$
  Whereby (x_1, y_1) and (x_2, y_2) are two points on the same line L.

- The general equation of a line segment is $y = mx + c$, whereby m is the gradient of the line and c is the y-intercept of the line.

For Additional Mathematics, you will require the knowledge of these above topics and more.

B Gradient of a Line

	<u>Parallel Lines</u> Notice that all these lines move up 2 units and across 3 units (hence the gradient is $\frac{2}{3}$). **In general, lines with same gradient are parallel and vice versa.**
	<u>Perpendicular Lines</u> Look at the two perpendicular lines on the right. Gradient of line $A = \frac{-1}{a}$ Gradient of line $B = \frac{a}{1}$ Product of gradients = $(\frac{a}{1})(\frac{-1}{a}) = -1$. **In general, if the product of the gradients of two lines is -1, they are perpendicular.** Also, the gradient of a line (m_1) is the negative reciprocal of the other line ($-\frac{1}{m_2}$).
<u>Horizontal Lines</u> As you have learnt in Mathematics, horizontal lines have a gradient of 0 as they continue to stretch to the right but do not have any change in y coordinates.	
<u>Vertical Lines</u> As you have learnt in Mathematics, vertical lines have an undefined gradient as they continue to rise upwards but do not have any change in x coordinates.	

C Equation of a Straight Line

If three points are on the same line, they are collinear. Collinear points can be identified by the fact that the gradients of any two points on the line are equal.

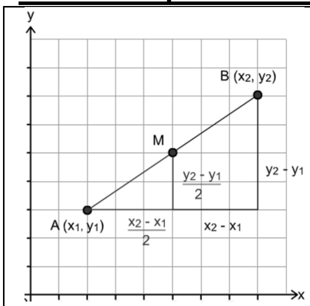
Consider the collinear points A (x_1, y_1), B (x_2, y_2) and C (x, y).

Knowing that the gradients are equal, we gather that:

$$\text{Gradient of } AB = \text{Gradient of } AC$$
$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y - y_1}{x - x_1}$$
$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

We thus gather that the equation of a straight line is $\boldsymbol{y - y_1 = m(x - x_1)}$, where m is the gradient $\left(\frac{y_2 - y_1}{x_2 - x_1}\right)$ of the line and (x_1, y_1) are the coordinates of any given point on the line.

D Midpoint of a Line Segment



Let us find the coordinates of midpoint M of the line joining A (x_1, y_1) and B (x_2, y_2).

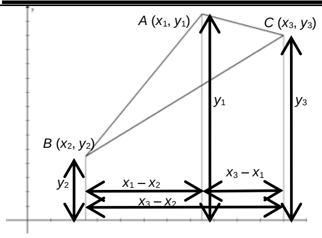
From A to B, we move $x_2 - x_1$ units across and $y_2 - y_1$ units up.

Hence, to reach M from A, we move $\frac{x_2 - x_1}{2}$ units across and $\frac{y_2 - y_1}{2}$ units up.

We hence derive that the coordinates of M are $\left(x_1 + \frac{x_2 - x_1}{2}, y_1 + \frac{y_2 - y_1}{2}\right)$ units, which is also equal to $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

In general, the midpoint of a line connecting points (x_1, y_1) and (x_2, y_2) is: $\boldsymbol{\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)}$.

E Area of Rectilinear Figures

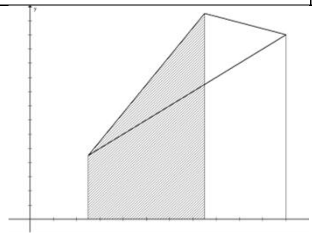


Consider the following triangle *ABC* with points *A* (x_1, y_1), *B* (x_2, y_2) and *C* (x_3, y_3).

Let's draw a few lines down from points *A*, *B* and *C* to the *x*-axis.

The points at which they touch the *x*-axis are *J*, *K* and *L* respectively.

To find its area, we have to manipulate the areas of the inscribed trapeziums. That is to say:



Area of Trapezium *ABKJ*

$(\frac{1}{2})(y_1 + y_2)(x_1 – x_2)$

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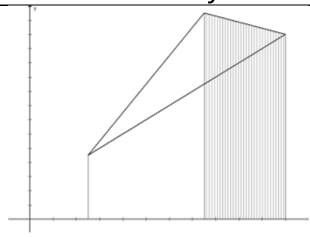

Area of Trapezium *CAJL*

$(\frac{1}{2})(y_1 + y_3)(x_3 – x_1)$

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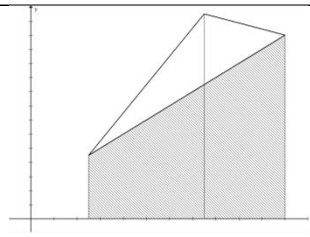


Area of Trapezium *CBKL*

$(\frac{1}{2})(y_2 + y_3)(x_3 – x_2)$

$= (\frac{1}{2})(x_1y_1 – x_2y_1 + x_1y_2 – x_2y_2) + (\frac{1}{2})(x_3y_1 – x_1y_1 + x_3y_3 – x_1y_3) - (\frac{1}{2})(x_3y_2 – x_2y_2 + x_3y_3 – x_2y_3)$

$= (\frac{1}{2})(x_1y_1 – x_2y_1 + x_1y_2 – x_2y_2 + x_3y_1 – x_1y_1 + x_3y_3 – x_1y_3 - x_3y_2 + x_2y_2 - x_3y_3 + x_2y_3)$

$= (\frac{1}{2})({\color{red}x_1y_1} – x_2y_1 + x_1y_2 {\color{green}– x_2y_2} + x_3y_1 {\color{red}– x_1y_1} {\color{blue}+ x_3y_3} – x_1y_3 - x_3y_2 {\color{green}+ x_2y_2} {\color{blue}- x_3y_3} + x_2y_3)$

$= (\frac{1}{2})(– x_2y_1 + x_1y_2 + x_3y_1 – x_1y_3 - x_3y_2 + x_2y_3)$

$= (\frac{1}{2})(x_1y_2 + \ + x_2y_3 + x_3y_1 – x_2y_1 – x_1y_3 – x_3y_2)$

Difficult to remember? We can rearrange this above complicated formula to become:

$$\frac{1}{2}\left|\begin{array}{cccc} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{array}\right|$$

$$= (\tfrac{1}{2})[(\text{sum of } \searrow \text{ products}) – (\text{sum of } \swarrow \text{ products})]$$

This formula can also be extended to find the areas of polygons with a specified number of sides, given the coordinates/ vertices. **For a figure with *n* sides, the formula for its area would be re-arranged like this:**

$$\frac{1}{2}\left|\begin{array}{cccccc} x_1 & x_2 & x_3 & \cdots & x_n & x_1 \\ y_1 & y_2 & y_3 & & y_n & y_1 \end{array}\right|$$

$$\mathbf{= (\tfrac{1}{2})[(\textbf{sum of } \searrow \textbf{ products}) – (\textbf{sum of } \swarrow \textbf{ products})]}$$

When applying this formula, you must take note of a few things:

- The vertices must always be arranged in an anti-clockwise manner.
- Any point can be used as the starting, and you must remember to place the point at the end as well.

F Worked Examples on Sections B, C, D, E

A line is drawn through the point $A(4, 6)$ parallel to the line $2y = x – 2$, meets the y-axis at the point B.
A line drawn through $A(4, 6)$ perpendicular to AB, meets the line $2y = x – 2$ at the point C. **[6]**

(i) Calculate the coordinates of B.

When the line meets the y-axis, $x = 0$. Hence, let $B = (0, B_y)$

$$\begin{aligned} 2y &= x - 2 \\ y &= \frac{x}{2} - 1 \\ y &= \frac{1}{2}x - 1 \end{aligned}$$

Since AB parallel to $2y = x – 2$, then gradient of $AB = \frac{1}{2}$, and equation of AB:

$$\begin{aligned} y - 6 &= \frac{1}{2}(x - 4) \\ &= \frac{1}{2}x - 2 \\ Y &= \frac{1}{2}x - 2 + 6 \\ Y &= \frac{1}{2}x + 4 \\ \text{Sub in } B\text{: } B_y &= \frac{1}{2}(0) + 4 \\ &= 4 \\ \text{Hence, } B &= (0, 4) \end{aligned}$$

(ii) Calculate the coordinates of C.

$$\text{Gradient of } AB = \frac{1}{2}$$

Since AC is perpendicular to AB, the gradient of AC is negative reciprocal of $\frac{1}{2}$, i.e.

$$\begin{aligned} \text{Gradient of } AC &= -\frac{1}{\frac{1}{2}} \\ &= -2 \end{aligned}$$

Hence equation of AC:

$$\begin{aligned} y - 6 &= -2(x - 4) \\ &= -2x + 8 \\ y &= -2x + 8 + 6 \\ &= -2x + 14 \text{ --- (1)} \\ 2y &= x - 2 \text{ --- (2)} \end{aligned}$$

Since both lines intersect, sub (1) into (2):

$$\begin{aligned} 2(-2x + 14) &= x - 2 \\ -4x + 28 &= x - 2 \\ 28 + 2 &= x + 4x \\ 30 &= 5x \\ x &= 6 \text{ (Ans)} \end{aligned}$$

Sub $x = 6$ into (1):

$$\begin{aligned} y &= -2(6) + 14 \\ &= -12 + 14 \\ &= 2 \end{aligned}$$

Hence, $C = (6, 2)$

[N97/I/2]

The points A and B have coordinates (2, 2) and (10, 8) respectively. Find the equation of the perpendicular bisector of AB. **[4]**

$$\begin{aligned} \text{Midpoint of } AB &= \left(\frac{2+10}{2}, \frac{2+8}{2}\right) \\ y &= \left(\frac{12}{2}, \frac{10}{2}\right) \end{aligned}$$

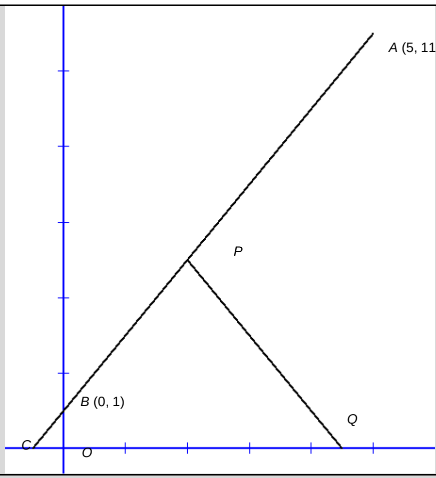
$$y = (6, 5)$$

$$\text{Gradient of} \perp \text{bisector} = -\left(\frac{1}{\text{Gradient of } AB}\right) = -\left(\frac{1}{\frac{8-2}{10-2}}\right) = -\left(\frac{10-2}{8-2}\right) = -\left(\frac{8}{6}\right) = -\left(\frac{4}{3}\right)$$

Equation of ⊥ bisector:

$$Y - 5 = -\left(\frac{4}{3}\right)(x - 6)$$
$$3y - 15 = -4(x - 6) = -4x + 24$$
$$3y = -4x + 24 + 15 = -4x + 39 \text{ (Ans)}$$

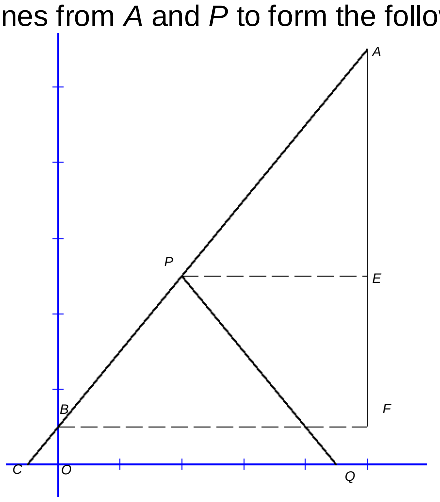
[N2000/I/2]



The line joining *A*(5, 11) and *B*(0, 1) meets the *x*-axis at *C*. The point *P* lies on *AC* such that *AP*:*PB* = 3:2. **[5]**

(i) Find the coordinates of *P*

By similar triangles, we drop lines from *A* and *P* to form the following figure:



Let P be (x, y). By similar triangles,

$$\frac{PE}{BF} = \frac{AP}{AB}$$
$$\frac{5 - x}{5 - 0} = \frac{AP}{AP + PB}$$
$$\frac{5 - x}{5} = \frac{3}{5}$$
$$5 - x = 3$$
$$x = 2$$

By similar triangles,

$$\frac{AE}{AF} = \frac{AP}{AB}$$
$$\frac{11-y}{11-1} = \frac{3}{5}$$
$$\frac{11 - y}{10} = \frac{3}{5}$$
$$\frac{11 - y}{10} = \frac{6}{10}$$
$$11 - y = 6$$
$$y = 5$$

Hence, $P = (2, 5)$

A line through *P* meets the *x*-axis at *Q* and angle *PCQ* = angle *PQC*. Find:

(ii) The equation of *PQ*

Since angle *PCQ* = angle *PQC*, then gradient of *AC* = - gradient of *PQ*

$$\text{Gradient of } PQ = -\text{ Gradient of } AC$$
$$= -\left(\frac{11 - 1}{5 - 0}\right)-$$
$$= -\left(\frac{10}{5}\right)$$
$$= -2$$

Hence, equation of *PQ*:

$$y - 5 = -2(x - 2)$$
$$y - 5 = -2x + 4$$
$$y = -2x + 9 \text{ (Ans)}$$

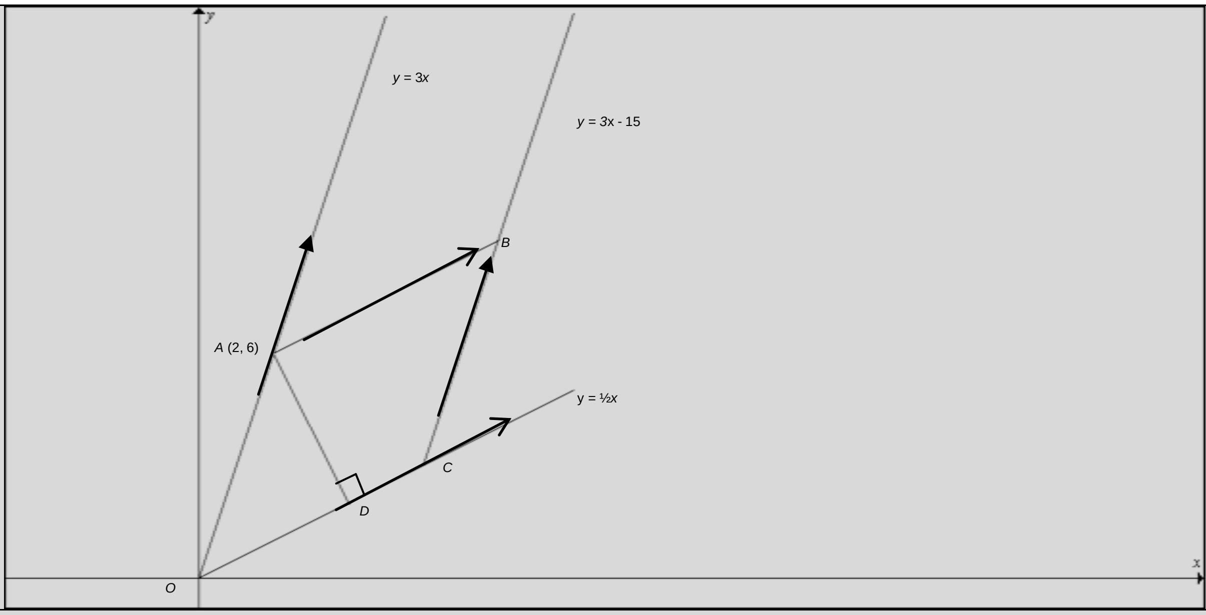
(ii) The coordinates of *Q*

Since *Q* lies on *x*-axis, then y = 0. Thus, let $Q = (Q_x = 0)$ and sub into equation of *PQ*:

$$0 = -2(Q_x) + 9$$
$$-9 = -2(Q_x)$$
$$2Q_x = 9$$
$$Q_x = 4.5$$

Hence, $Q = (4.5, 0)$

[N2001/I/3]



The diagram, which is not drawn to scale, shows a parallelogram *OABC* where *O* is the origin and *A* is the point (2, 6). The equations of *OA*, *OC* and *CB* are $y = 3x$, $y = \frac{1}{2}x$ and $y = 3x - 15$ respectively. The perpendicular from *A* to *OC* meets *OC* at the point *D*. Find

(i) The coordinates of *C*, *B* and *D*. [8]

To find coordinates of *C*, equate $y = \frac{1}{2}x$ and $y = 3x - 15$:

$$\frac{1}{2}x = 3x - 15$$
$$15 = 3x - \frac{1}{2}x$$
$$= 2.5x$$
$$x = 6 \text{ (Ans)}$$

Sub $x = 6$ into $y = \frac{1}{2}x$:

$$y = \frac{1}{2}(6)$$
$$= 3 \text{ (Ans) Hence, } C = (6, 3)$$

Since *AB* is parallel to $y = \frac{1}{2}x$, then gradient of *AB* is also ½. Equation of *AB*:

$$y - 6 = \frac{1}{2}(x - 2)$$
$$= \frac{1}{2}x - 1$$
$$y = \frac{1}{2}x - 1 + 6$$
$$y = \frac{1}{2}x + 5 \text{ (Ans)}$$

Since *B* is intersection point of $y = \frac{1}{2}x + 5$ and $y = 3x - 15$, equate both to get *B*:

$$\frac{1}{2}x + 5 = 3x - 15$$
$$x + 10 = 6x - 30$$
$$10 + 30 = 6x - x$$
$$40 = 5x$$
$$x = 8 \text{ (Ans)}$$

Sub $x = 8$ into $y = \frac{1}{2}x + 5$:

$$y = \frac{1}{2}(8) + 5$$
$$= 9 \text{ (Ans) Hence } B = (8, 9)$$

$$\text{Gradient of } AD = -\left(\frac{1}{\text{Gradient of } OC}\right)$$
$$= -\left(\frac{1}{\frac{1}{2}}\right)$$
$$= -2$$

Equation of *AD*:

$$\begin{aligned} y - 6 &= -2(x - 2) \\ y - 6 &= -2x + 4 \\ y &= -2x + 4 + 6 \\ y &= -2x + 10 \end{aligned}$$

To find coordinates of D, equate $y = -2x + 10$ and $y = ½x$:

$$\begin{aligned} ½x &= -2x + 10 \\ x &= -4x + 20 \\ x + 4x &= 20 \\ 5x &= 20 \\ x &= 4 \text{ (Ans)} \end{aligned}$$

Sub $x = 4$ into $y = ½x$:

$$\begin{aligned} y &= ½(4) \\ &= 2 \text{ (Ans) Hence } D = (4, 2) \end{aligned}$$

(ii) The perimeter of the parallelogram *OABC*, correct to 1 decimal place [3]

$$\begin{aligned} \text{Length of } OA &= \sqrt{(2-0)^2 + (6-0)^2} \\ &= \sqrt{(2)^2 + (6)^2} \\ &= \sqrt{4 + 36} \\ &= \sqrt{40} \\ \text{Length of } OC &= \sqrt{(6-0)^2 + (3-0)^2} \\ &= \sqrt{(6)^2 + (3)^2} \\ &= \sqrt{36 + 9} \\ &= \sqrt{45} \\ \text{Perimeter of } OABC &= 2(\text{Length of } OA + \text{Length of } OC) \\ &= 2(\sqrt{40} + \sqrt{45}) \\ &= 26.06552 \text{ units}^2 \\ &= 26.1 \text{ units}^2 \text{ (Ans)} \end{aligned}$$

[N2003/I/11]

A parallelogram *ABCD* passes through vertices *A*(-3, 1), *B*(4, 9) and *C*(11, -3). Find:

(a) The midpoint of the diagonal *AC*

$$\begin{aligned} \text{Midpoint of } AC &= \left(\frac{-3 + 11}{2}, \frac{1 + (-3)}{2}\right) \\ &= \left(\frac{8}{2}, \frac{-2}{2}\right) \\ &= (4, -1) \text{ (Ans)} \end{aligned}$$

(b) The fourth vertex D

Since *ABCD* is a parallelogram, the midpoint of its diagonals would bisect each other (i.e. they would both be the same). Hence, let D be (h, k).

$$\begin{aligned} \text{Midpoint of } AC &= \text{Midpoint of } BD \\ (4, -1) &= \left(\frac{4 + h}{2}, \frac{9 + k}{2}\right) \end{aligned}$$

Focusing on x coordinate only:

$$\begin{aligned} 4 &= \frac{4 + h}{2} \\ 8 &= 4 + h \\ h &= 8 - 4 \\ &= 4 \end{aligned}$$

Focusing on y coordinate only:

$$\begin{aligned} -1 &= \frac{9 + k}{2} \\ -2 &= 9 + k \\ k &= 9 + 2 \\ &= 11 \end{aligned}$$

Hence, $D = (4, 11)$ (Ans)

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SUB-TOPIC 2.2: COORDINATE GEOMETRY IN TWO DIMENSIONS

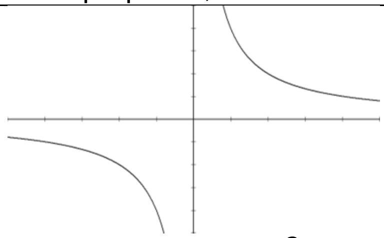
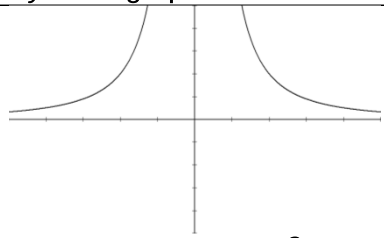
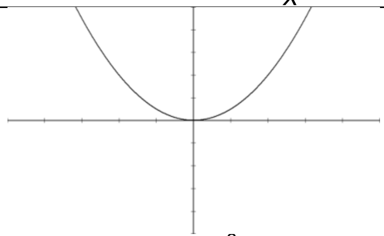
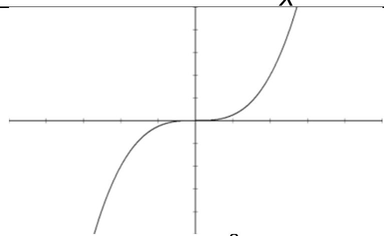
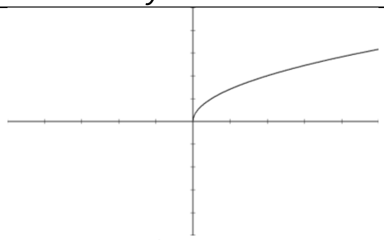
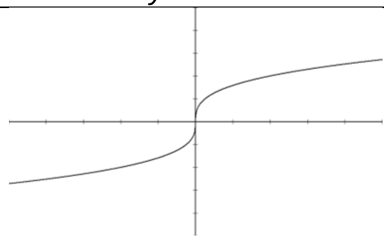
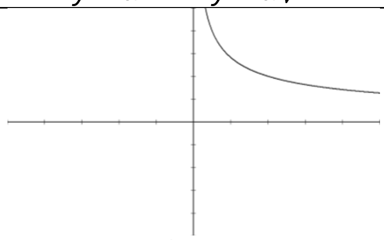
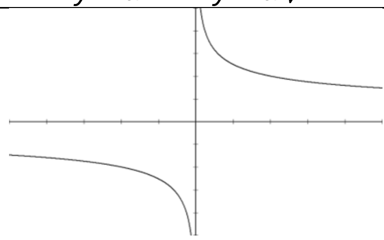
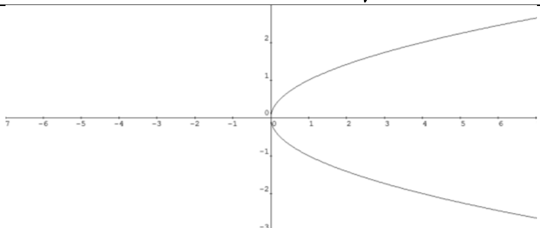
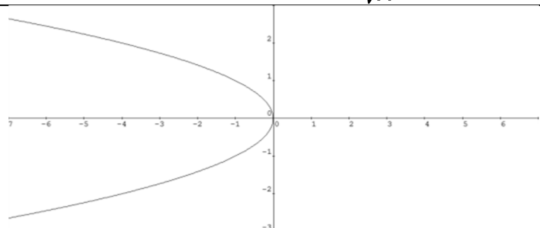
G Graphs of Equations $y = ax^n$ and $y^2 = kx$

For such questions, you would only be required to:

- Draw the graphs given the values
- Find the solution set to a particular equation by adding a suitable straight line
- Find the corresponding *x* or *y* coordinate, given one.

As such, refer to your notes from Mathematics on Functions and Graphs for how to answer these questions.

For reference purposes, here are the various $y = ax^n$ and $y^2 = kx$ graphs.

 $y = ax^{-1} \leftrightarrow y = \dfrac{a}{x}$	 $y = ax^{-2} \leftrightarrow y = \dfrac{a}{x^2}$
 $y = ax^2$	 $y = ax^3$
 $y = ax^{\frac{1}{2}} \leftrightarrow y = a\sqrt{x}$	 $y = ax^{\frac{1}{3}} \leftrightarrow y = a\sqrt[3]{x}$
 $y = ax^{-\frac{1}{2}} \leftrightarrow y = \dfrac{a}{\sqrt{x}}$	 $y = ax^{-\frac{1}{3}} \leftrightarrow y = \dfrac{a}{\sqrt[3]{x}}$
 $y^2 = kx$ (where $k > 0$)	 $y^2 = kx$ (where $k < 0$)

H Equation of a Circle

We know that the length of a line is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. We now have a circle with a radius CF of length r, where C (a, b) is the centre of the circle and F (x, y) is a point on the circumference of the circle.

$$\sqrt{(x-a)^2 + (y-b)^2} = r$$

Taking square on both sides, $(x - a)^2 + (y - b)^2 = r^2$

We thus get the equation that a circle of centre C (a, b) would have an equation of $(x - a)^2 + (y - b)^2 = r^2$ where r is the radius.

Now consider a circle with the equation: $x^2 + y^2 + 2gx + 2fy + c = 0$. How can we determine the centre and the radius? Let us try manipulating the equation.

$$\begin{aligned}
x^2 + y^2 + 2gx + 2fy + c &= 0\\
x^2 + 2gx + y^2 + 2fy &= -c\\
x^2 + 2(g)(x) + y^2 + 2(f)(y) &= -c\\
\text{Complete the square: } x^2 + 2(g)(x) + g^2 - g^2 + y^2 + 2(f)(y) + f^2 - f^2 &= -c\\
(x+g)^2 - g^2 + (y+f)^2 - f^2 &= -c\\
(x+g)^2 + (y+f)^2 &= -c + g^2 + f^2\\
(x+g)^2 + (y+f)^2 &= \left(\sqrt{-c + g^2 + f^2}\right)^2
\end{aligned}$$

We thus gather that the centre of the circle is $(-g, -f)$ and the radius is $\sqrt{g^2 + f^2 - c}$.

In summary,

FORM 1	Property	FORM 2
(a, b)	**Centre of a Circle, C**	$(-g, -f)$
r	**Radius of a Circle, r**	$\sqrt{g^2 + f^2 - c}$
$(x - a)^2 + (y - b)^2 = r^2$	**Equation of a Circle**	$x^2 + y^2 + 2gx + 2fy + c = 0$

WORKED EXAMPLES

The equation of a circle, C, is $x^2 + y^2 - 6x - 8y + 16 = 0$

(i) Find the coordinates of the centre of C and find the radius of C. **[3]**

$$\begin{aligned}
x^2 + y^2 - 6x - 8y + 16 &= \\
x^2 - 6x + y^2 - 8y &= -16\\
x^2 - 2(3)(x) + y^2 - (2)(4)(y) &= -16\\
x^2 - 2(3)(x) + 3^2 - 3^2 + y^2 - (2)(4)(y) + 4^2 - 4^2 &= -16\\
(x-3)^2 - 3^2 + (y-4)^2 - 4^2 &= -16\\
(x-3)^2 - 9 + (y-4)^2 - 16 &= -16\\
(x-3)^2 + (y-4)^2 - 25 &= -16\\
(x-3)^2 + (y-4)^2 &= -16 + 25\\
&= 9\\
&= 3^2
\end{aligned}$$

Hence, coordinates of centre of C is (3, 4) and radius of C is 3 units.

(ii) Show that C touches the y-axis **[2]**

When C touches y-axis, x = 0. Hence, sub $x = 0$ into equation.

$$\begin{aligned}
(0-3)^2 + (y-4)^2 &= 9\\
9 + (y-4)^2 &= 9\\
(y-4)^2 &= 0\\
y - 4 &= 0\\
y &= 4
\end{aligned}$$

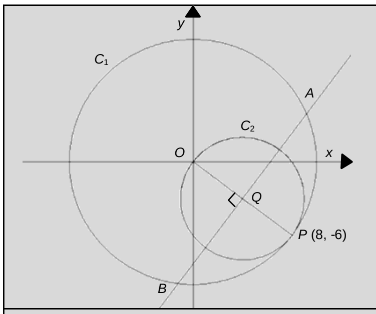
Therefore, as there is a corresponding *y* coordinate for the value of $x = 0$, *C* does touch the *y*-axis.

(iii) Find the equation of the circle which is a reflection of *C* in the *y*-axis. [3]

Centre of $C = (3, 4)$. Hence, a circle which has a reflection of *C* in the *y*-axis has a centre of (-3, 4). The radius is the same as it the same size as *C*.

$$(x - (-3))^2 + (y - 4)^2 = 3^2$$
$$(x + 3)^2 + (y - 4)^2 = 9$$
$$X^2 + 6x + 9 + y^2 - 8y + 16 = 9$$
$$X^2 + 6x + y^2 - 8y + 16 = 0$$
$$X^2 + y^2 + 6x - 8y + 16 = 0 \text{ (Ans)}$$

[SP2008/I/7]



The diagram shows two circles, C_1 and C_2. Circle C_1 has its centre at the origin *O*. Circle C_2 passes through *O* and has its centre at *Q*. The point *P* (8, -6) lies on both circles and *OP* is a diameter of C_2.

(i) Find the equation of C_1. [2]

Since centre of circle is origin, it is (0, 0)

$$\begin{aligned}
\text{Radius of circle } C_1 &= \text{Diameter of Circle } C_2\\
&= \text{Length of } OP\\
&= \sqrt{(0 + 8)^2 + (0 - 6)^2}\\
&= \sqrt{8^2 + (-6)^2}\\
&= \sqrt{64 + 36}\\
&= \sqrt{100}\\
&= 10 \text{ units}\\
\text{Hence, equation of } C_1\text{: } (x - 0)^2 + (y - 0)^2 &= 10^2\\
x^2 + y^2 &= 100\\
x^2 + y^2 - 100 &= 0 \text{ (Ans)}
\end{aligned}$$

(ii) Find the equation of C_2. [3]

$$\begin{aligned}
\text{Centre of } C_2 &= \text{Midpoint of } OP, Q\\
&= \left(\frac{0 + 8}{2}, \frac{0 + (-6)}{2}\right)\\
&= \left(\frac{8}{2}, \frac{-6}{2}\right)\\
&= (4, -3)\\
\text{Radius of } C_2 &= \tfrac{1}{2} (\text{Length of } OP)\\
&= \tfrac{1}{2} (10 \text{ units})\\
&= 5 \text{ units}\\
\text{Hence, equation of } C_2\text{: } (x - 4)^2 + (y - (-3))^2 &= 5^2\\
(x - 4)^2 + (y + 3)^2 &= 25\\
x^2 - 8x + 16 + y^2 + 6y + 9 &= 25\\
x^2 + y^2 - 8x + 6y + 25 &= 25\\
x^2 + y^2 - 8x + 6y &= 0 \text{ (Ans)}
\end{aligned}$$

(iii) The line through *Q* perpendicular to *OP* meets the circle C_1 at the points *A* and *B*. Show that the *x* coordinates of *A* and *B* are $a + b\sqrt{3}$ [7]

and $a - b\sqrt{3}$ respectively, where a and b are integers to be found.

$$
\begin{aligned}
x^2 + y^2 - 100 &= 0 \text{ --- (1)} \\
\text{Gradient of } AB &= -\left(\frac{1}{\text{Gradient of } OP}\right) \\
&= -\left(\frac{1}{\frac{0-(-6)}{0-8}}\right) \\
&= -\left(\frac{0-8}{0+6}\right) \\
&= \frac{8}{6} \\
&= \frac{4}{3} \\
\text{Equation of } AB\text{: } y-(-3) &= \frac{4}{3}(x-4) \\
y + 3 &= \frac{4}{3}x - \frac{16}{3} \\
y &= \frac{4}{3}x - \frac{16}{3} - 3 \\
&= \frac{4}{3}x - \frac{16}{3} - \frac{9}{3} \\
&= \frac{4}{3}x - \frac{25}{3} \text{ ---(2)} \\
\text{Sub (1)into (2): } x^2 + \left(\frac{4}{3}x - \frac{25}{3}\right)^2 - 100 &= 0 \\
x^2 + \left(\frac{16}{9}x^2 - \frac{2(4)(25)}{9}x + \frac{625}{9}\right) - 100 &= 0 \\
\frac{9}{9}x^2 + \frac{16}{9}x^2 - \frac{200}{9}x + \frac{625}{9} - \frac{900}{9} &= 0 \\
9x^2 + 16x^2 - 200x + 625 - 900 &= 0 \\
25x^2 - 200x - 275 &= 0 \\
x^2 - 8x - 11 &= 0 \\
x &= \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(-11)}}{2(1)} \\
&= \frac{8 \pm \sqrt{64+44}}{2} \\
&= \frac{8 \pm \sqrt{108}}{2} \\
&= 4 \pm \sqrt{27} \\
&= 4 \pm 3\sqrt{3} \\
&= 4 + 3\sqrt{3} \text{ or } 4 - 3\sqrt{3} \text{ (proven)}
\end{aligned}
$$

[N2008/II/11]

4038 ADDITIONAL MATHEMATICS TOPIC 2: GEOMETRY AND TRIGONOMETRY

SUB-TOPIC 2.2: COORDINATE GEOMETRY IN TWO DIMENSIONS

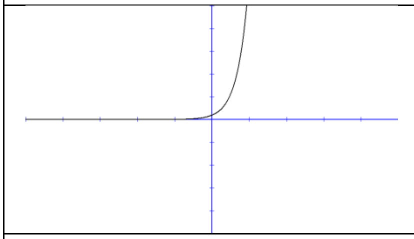
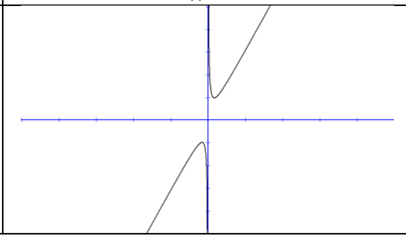
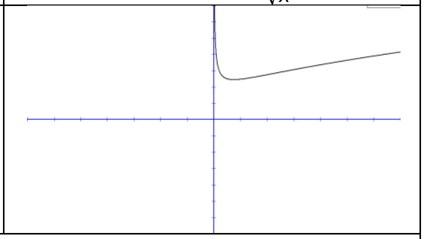
I Linear Law

(Source: New Syllabus Additional Mathematics 8th Edition (2007), Dr Yeap Ban Har/ Teh Keng Seng/ Loh Cheng Yee)

In research work, when two variables are believed to be related, a set of corresponding values are obtained. Any equation must be in the form of $Y = mX + c$, whereby:

- Y is a function of y (i.e. must contain y in any form such as ln y, sin y, y^2, $\frac{x}{y}$)
- m is a constant term which is the gradient (i.e. cannot contain x or y terms)
- X is a function of x (i.e. must contain y in any form such as lg x, cos x, x^2y, $\frac{y}{x^2}$)
- c is a constant term which is the Y-intercept (i.e. cannot contain x or y terms)

However, not all experimental results obey a linear relationship. Examples of such graphs include:

$y = ax^b$	$y = \frac{a}{x} + bx$	$y = a\sqrt{x} + \frac{b}{\sqrt{x}}$
		
Where a and b are constants		

With a little algebraic manipulation, each of the above relationships can be converted to a linear relationship. The table below shows the manipulation of some common graph types.

$y = ax + bx^2$	y	=	ax	+	bx^2
Divide throughout by x	$\frac{y}{x}$	=	a	+	bx
Hence, a straight line graph is obtained when $\frac{y}{x}$ (Y) is plotted against x (X). Here, a denotes the Y-intercept and b denotes the gradient.					
$y = \frac{a}{x} + b$	y	=	$\frac{a}{x}$	+	b
	y	=	$a(\frac{1}{x})$	+	b
Hence, a straight line graph is obtained when y (Y) is plotted against $\frac{1}{x}$ (X). Here, a denotes the gradient and b denotes the Y-intercept.					
$y = a\sqrt{x} + \frac{b}{\sqrt{x}}$	y	=	$a\sqrt{x}$	+	$\frac{b}{\sqrt{x}}$
Multiply throughout by $\sqrt{x}$	$y\sqrt{x}$	=	ax	+	b
Hence, a straight line graph is obtained when $y\sqrt{x}$ (Y) is plotted against x (X). Here, a denotes the gradient and b denotes the Y-intercept.					
$xy = \frac{a}{x} + bx$	xy	=	$\frac{a}{x}$	+	bx
Divide throughout by x	y	=	$\frac{a}{x^2}$	+	bx
	y	=	$a(\frac{1}{x^2})$	+	bx
Hence, a straight line graph is obtained when y (Y) is plotted against $\frac{1}{x^2}$ (X). Here, a denotes the gradient and b denotes the Y-intercept.					

$y = ae^{bx}$	y	=	ae^{bx}		
Taking ln on both sides	$\ln y$	=	$\ln (ae^{bx})$		
Using product law to split up $\ln (ae^{kx})$	$\ln y$	=	$\ln a$	+	$\ln e^{bx}$
Using power law to change $\ln e^{kx}$	$\ln y$	=	$\ln a$	+	$bx \ln e$
Since $\ln e = 1$	$\ln y$	=	$\ln a$	+	bx
Hence, a straight line graph is obtained when $\ln y$ (Y) is plotted against x (X). Here, $\ln a$ denotes the Y-intercept and b denotes the gradient.					
$y = ax^b$	y	=	ax^b		
Taking lg on both sides	$\lg y$	=	$\lg (ax^b)$		
Using product law to split up $\lg (ax^b)$	$\lg y$	=	$\lg a$	+	$\lg x^b$
Using power law to change $\lg x^b$	$\lg y$	=	$\lg a$	+	$b \lg x$
Hence, a straight line graph is obtained when $\lg y$ (Y) is plotted against $\lg x$ (X). Here, $\lg a$ denotes the Y-intercept and b denotes the gradient.					
$xy = a$	xy	=	a		
Divide throughout by x	y	=	$\frac{a}{x}$		
	y	=	$a(\frac{1}{x})$		
Hence, a straight line graph is obtained when y (Y) is plotted against $\frac{1}{x}$ (X). Here, a denotes the gradient and 0 denotes the Y-intercept (i.e. passes through origin O).					

<u>WORKED EXAMPLE</u>

(a) The table shows experimental values of two variables, x and y.

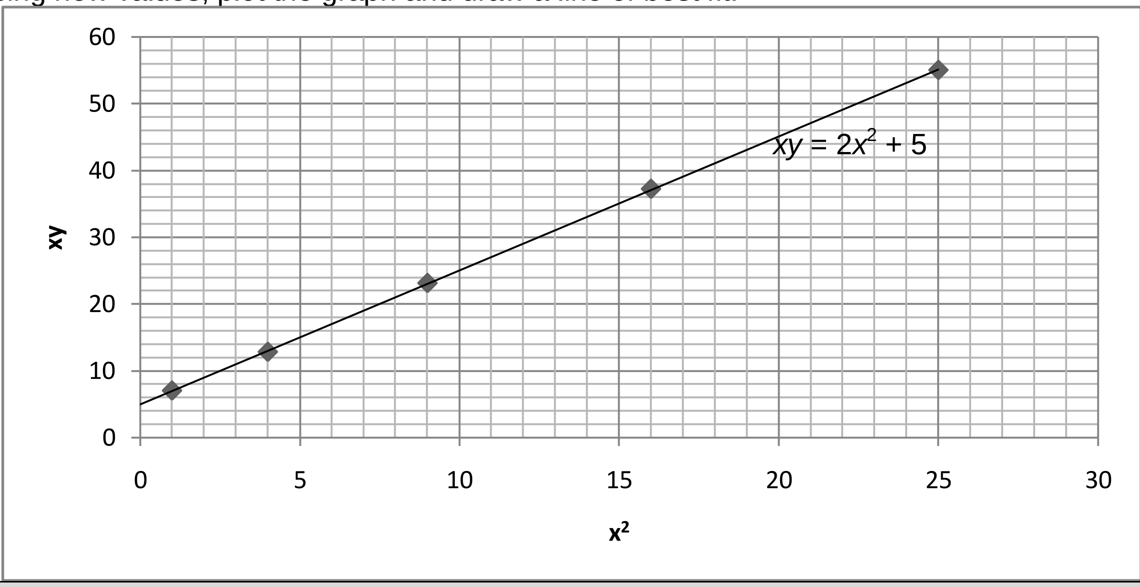
x	1	2	3	4	5
y	7.0	6.4	7.7	9.3	11.0

Using the vertical axis for xy and the horizontal axis for x^2, plot xy against x^2 and obtain a straight line graph. Use your graph to: [10]

Plot a new table of values:

x^2	1.0	4.0	9.0	16.0	25.0
xy	7.0	12.8	23.1	37.2	55.0

Using new values, plot the graph and draw a line of best fit.



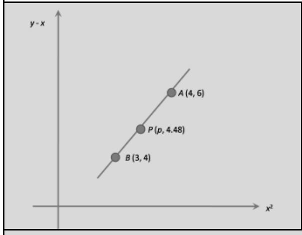
(i) Express y in terms of x

To find gradient, choose two points on the graph (e.g. (1.0, 7.0) and (25.0, 55.0)

$$\begin{aligned}
\text{Gradient} &= \frac{55.0 - 7.0}{25.0 - 1.0}\\
&= \frac{48.0}{24.0}\\
&= 2\\
\text{From the graph, } Y\text{-intercept} &\approx 5.0\\
\text{Hence, } Y &= mX + c\\
xy &= 2(x^2) + 5.0\\
y &= 2x + \frac{5}{x} \text{ (Ans)}
\end{aligned}$$

(ii) Estimate the value of x when $y = \frac{30}{x}$.

$$\begin{aligned}
y &= \frac{30}{x}\\
xy &= 30\\
\text{Hence, from the graph, when } xy = 30,\ x^2 &\approx 12.5\\
x &= 3.536\\
&= 3.54 \text{ (Ans to 3sf)}
\end{aligned}$$



Variables x and y are related in such a way that, when $y - x$ is plotted against x^2, a straight line is produced which passes through the points $A(4, 6)$, $B(3, 4)$ and $P(p, 4.48)$, as shown in the diagram. Find [6]

(i) y in terms of x

To find gradient, choose the two given points on the graph (i.e. (4, 6) and (3, 4))

$$\begin{aligned}
\text{Gradient} &= \frac{6 - 4}{4 - 3}\\
&= \frac{2}{1}\\
&= 2\\
\text{Hence, equation of line: } Y - Y_1 &= m(X - X_1)\\
(y - x) - (4) &= 2(x^2 - 3)\\
y - x - 4 &= 2x^2 - 6\\
y &= 2x^2 + x + 4 - 6\\
&= 2x^2 + x - 2 \text{ (Ans)}
\end{aligned}$$

(ii) The value of p

Find value of $x^2 = p$ when $(y - x) = 4.48$

$$\begin{aligned}
y &= 2x^2 + x - 2\\
y - x &= 2x^2 - 2\\
4.48 &= 2x^2 - 2\\
6.48 &= 2x^2\\
x^2 &= 3.24\\
p &= 3.24 \text{ (Ans)}
\end{aligned}$$

(iii) The value of x and of y at the point P.

$$\begin{aligned}
\text{At } P,\ x^2 &= 3.24\\
x &= 1.8 \text{ (Ans)}\\
\text{At } P,\ y - x &= 4.48\\
Y - 1.8 &= 4.48\\
y &= 4.48 + 1.8\\
&= 6.28 \text{ (Ans)}
\end{aligned}$$

[N2000/II/4]