# Vectors

# 8

## SCALARS AND VECTORS

A **scalar** is a purely numerical quantity with a unit, such as $20 or a mass of 2 kg. No idea of *direction* is involved. A **vector** quantity, however, has a direction which must be stated, such as a velocity of 20 m $s^{-1}$ northeast (NE). A velocity of 20 m $s^{-1}$ southeast (SE) would be quite different.

To specify a vector, its **magnitude** (e.g. 20 m $s^{-1}$) *and* its direction (e.g. NE) must **both** be given.

Scalars are added and subtracted by the usual rules of arithmetic but to 'add' or 'subtract' vectors, we use a special rule – the **parallelogram law.**

## REPRESENTATION OF VECTORS

A simple example of a vector is a **displacement**. Suppose a piece of board is moved, without rotation, across a flat surface (Fig.8.1).

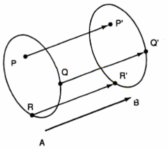



*Fig. 8.1*

Points on the board such as P, Q, R are displaced through the same distance and in the same direction to points P', Q', R'. So we can represent this vector by *any* line segment

AB where AB = PP' = QQ' = RR' and AB // PP' // QQ' // RR'. The arrow head shows the sense of the direction. AB is drawn to scale to give the correct magnitude of the displacement. We write such a vector as $\overrightarrow{AB}$.

## EQUALITY OF VECTORS

In Fig.8.2, the line segments AB, CD and EF are parallel (in the same direction) and equal in length. Then these lines can each represent the same vector and $\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{EF}$.

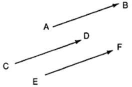



*Fig. 8.2*

Conversely, if $\overrightarrow{AB} = \overrightarrow{CD}$ (Fig.8.3), then

(a) the line segments AB and CD are equal in length **and**

(b) AB // CD.

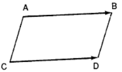



*Fig. 8.3*

It is important to remember that both parts are implied by the statement $\overrightarrow{AB} = \overrightarrow{CD}$. The figure ABDC is therefore a parallelogram.

## NOTATION FOR VECTORS

We state the endpoints of a vector by writing it as $\overrightarrow{AB}$, as above or we can use a single letter (Fig.8.4). A vector could then be given as **a** (printed in **bold**). We write this as $\vec{a}$ or $\underset{\sim}{a}$. Always distinguish a vector **a** in this way from an algebraic quantity *a*.

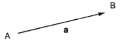



*Fig. 8.4*

## MAGNITUDE OF A VECTOR

The magnitude or **modulus** of a vector $\overrightarrow{AB}$ is the length of the line segment representing the vector to the scale used. We denote this as $|\overrightarrow{AB}|$.
If $\overrightarrow{AB}$ in Fig.8.4 is drawn to a scale of 1 cm ≡ 10 m $s^{-1}$ for example, then $|\overrightarrow{AB}| = 30$ m $s^{-1}$.
The magnitude of the vector **a** is written as | **a** | or as *a*.
Note this carefully: **a** is the vector but | **a** | or *a* is its magnitude.

### Zero Vector

The vector which has no magnitude (and of course no direction) is the **zero** vector, written $\underset{\sim}{0}$ or $\vec{0}$.

### Scalar Multiple of a Vector

Given a vector **a** (Fig.8.5), we can make multiples of this vector.
For example, $\overrightarrow{PQ} = 2\mathbf{a}$. $\overrightarrow{PQ}$ has the same direction as **a** but twice its magnitude.
$|\overrightarrow{PQ}| = 2|\mathbf{a}| = 2a$.

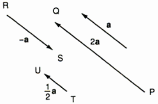



*Fig. 8.5*

$\overrightarrow{RS} = -\mathbf{a}$, i.e. it has the same magnitude as **a** but is in the reverse direction.
Note that $\overrightarrow{RS} = -\overrightarrow{SR}$.
$\overrightarrow{TU} = \frac{1}{2}\mathbf{a}$.

If $\mathbf{a} = k\mathbf{b}$, where $k$ is a scalar (a number) $\neq 0$, then the vectors **a** and **b** are parallel and in the same direction if $k > 0$ but in opposite directions if $k < 0$.
$|\mathbf{a}| = |k| \times |\mathbf{b}|$

Conversely, if **a** and **b** are parallel, then $\mathbf{a} = k\mathbf{b}$. ($k$ will be positive if **a** and **b** are in the same direction, negative if they are in opposite directions.)
Scalar multiples of a vector can be combined arithmetically. For example $2\mathbf{a} + 3\mathbf{a} = 5\mathbf{a}$ and $4(2\mathbf{a}) = 8\mathbf{a}$.
So $m\mathbf{a} + n\mathbf{a} = (m + n)\mathbf{a}$ and $m(n\mathbf{a}) = mn\mathbf{a}$ for all values of $m$ and $n$.

### *Example 1*

*Given the vector* **a** *(Fig.8.6(a)), draw the vectors (i) 3***a***, (ii)* $-\frac{1}{3}$**a**.

The vectors are shown in Fig.8.6(b). They are all parallel but (ii) is in the opposite direction to **a**.

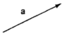



Fig. 8.6(a)

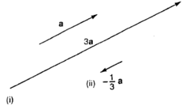



Fig. 8.6(b)

### *Example 2*

*What type of quadrilateral is ABCD if (a)* $\overrightarrow{AB} = \overrightarrow{DC}$, *(b)* $\overrightarrow{AB} = 3\overrightarrow{DC}$?

(a) AB = DC and AB // DC. Then ABCD is a parallelogram (Fig.8.7(a)).
It follows therefore that $\overrightarrow{AD} = \overrightarrow{BC}$.

(b) AB = 3DC and AB // DC. Then ABCD is a trapezium (Fig. 8.7(b)).

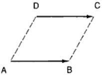



Fig. 8.7(a)

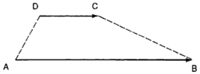
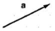



Fig. 8.7(b)

## Exercise 8.1 *(Answers on page 624.)*

1 Copy Fig.8.8 and draw the vectors (a) 2**a**, (b) –**a**, (c) $\frac{3}{4}$**a**.

Fig. 8.8



**2** In Fig.8.9, state each of the vectors **p**, **q** and **r** in the form $k$**a**.

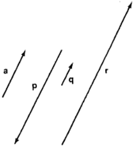



*Fig. 8.9*

**3** The line AB is divided into three equal parts at C and D. If $\overrightarrow{AD} = \mathbf{a}$, state as scalar multiples of **a**, (a) $\overrightarrow{AB}$, (b) $\overrightarrow{CB}$, (c) $\overrightarrow{BD}$.

**4** In Fig.8.10, ABCDEF is a regular hexagon. Given that $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{BC} = \mathbf{b}$ and $\overrightarrow{CD} = \mathbf{c}$, state the following vectors as scalar multiples of **a**, **b** or **c**:
(a) $\overrightarrow{DE}$, (b) $\overrightarrow{EF}$, (c) $\overrightarrow{FA}$, (d) $\overrightarrow{BE}$, (e) $\overrightarrow{AD}$.

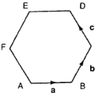



*Fig. 8.10*

**5** If $\overrightarrow{AB} = k\overrightarrow{BC}$ ($k \neq 0$), what can be said about the points A, B and C?

**6** A is the point (4,0) and B the point (0,3). State the value of $|\overrightarrow{AB}|$.

**7** If P is (–2,–5) and Q is (3,7), find $|\overrightarrow{PQ}|$.

**8** O is the origin, $|\overrightarrow{OR}| = 3$ and the line OR makes an angle $\theta$ with the $x$-axis where $\sin\theta = \frac{2}{3}$. Find the possible coordinates of R.

# ADDITION OF VECTORS

To 'add' two vectors **a** and **b**, i.e. to combine them into one vector, we place them so as to start from the same point O (Fig.8.11).

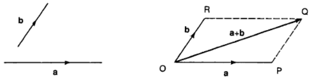



*Fig. 8.11*

Now complete the parallelogram OPQR.

We define $\mathbf{a} + \mathbf{b} = \overrightarrow{OQ}$ i.e. **the diagonal starting from O.**

$\overrightarrow{OQ}$ is called the **resultant** of **a** and **b**.

This is the **parallelogram law** for the addition of vectors. Note that we use the symbol '+' though here it means 'combined with' and not arithmetical addition.

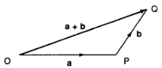
As RQ is parallel and equal to OP, $\overrightarrow{RQ} = \mathbf{a}$.

Then $\overrightarrow{OR} + \overrightarrow{RQ} = \mathbf{b} + \mathbf{a} = \overrightarrow{OQ} = \mathbf{a} + \mathbf{b}$.

Hence $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$.

In practice, it is not necessary to draw the parallelogram. The vectors can be placed 'end-on'. PQ is equal and parallel to OR so $\overrightarrow{PQ} = \mathbf{b}$. We draw **a** and then **b** starting from the end of **a** (Fig.8.12).

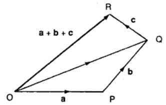
*Fig.8.12*




The third side OQ of the triangle gives **a** + **b**.

More than 2 vectors can be combined in this way. For example, in Fig.8.13, $\mathbf{a} + \mathbf{b} = \overrightarrow{OQ}$ and $\overrightarrow{OQ} + \mathbf{c} = \mathbf{a} + \mathbf{b} + \mathbf{c} = \overrightarrow{OR}$.

*Fig. 8.13*



# DIAGONALS OF A PARALLELOGRAM

In Fig.8.14, $\overrightarrow{OP} = \mathbf{a}$, $\overrightarrow{OR} = \mathbf{b}$.
Then $\overrightarrow{OQ} = \mathbf{a} + \mathbf{b}$.

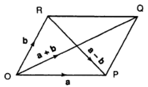



*Fig.8.14*

$\overrightarrow{RP} = \overrightarrow{RQ} + \overrightarrow{QP} = \mathbf{a} + (-\mathbf{b}) = \mathbf{a} - \mathbf{b}$.
Also $\overrightarrow{PR} = -\overrightarrow{RP} = -(\mathbf{a} - \mathbf{b}) = \mathbf{b} - \mathbf{a}$.

These last two results are important and can be remembered as follows:

$\mathbf{a} - \mathbf{b}$ is the vector from the endpoint of **b** to the endpoint of **a**;
$\mathbf{b} - \mathbf{a}$ is the vector from the endpoint of **a** to the endpoint of **b**

where **a** and **b** start from the same point (Fig.8.15).

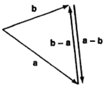



*Fig.8.15*

### *Example 3*

*The vectors* **a** *and* **b** *are given (Fig.8.16(a)). Draw the vectors*
*(i)* $2\mathbf{a} + \mathbf{b}$, *(ii)* $\mathbf{a} - \mathbf{b}$, *(iii)* $\mathbf{a} - 2\mathbf{b}$.

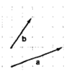



*Fig. 8.16(a)*

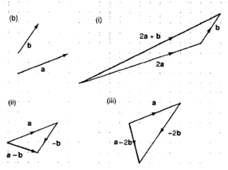



*Fig. 8.16(b)*

(The vectors are shown in Fig. 8.16(b).)

(i) Draw 2**a** followed by **b**.

(ii) Draw **a** followed by –**b**. Alternatively draw **a** and **b** from the same point and use the rule above.

(iii) Draw **a** followed by –2**b**.

## Example 4

*In △ABC,* $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{AC} = \mathbf{b}$ *and M is the midpoint of AB (Fig.8.17). State in terms of* **a** *and* **b**, *(a)* $\overrightarrow{AM}$, *(b)* $\overrightarrow{MC}$, *(c)* $\overrightarrow{CM}$.

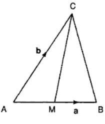
(a) $\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}\mathbf{a}$

(b) $\overrightarrow{MC} = \overrightarrow{AC} - \overrightarrow{AM}$ (from the end of AM to end of AC)

$= \mathbf{b} - \frac{1}{2}\mathbf{a}$

(c) $\overrightarrow{CM} = -\overrightarrow{MC} = \frac{1}{2}\mathbf{a} - \mathbf{b}$.

Fig. 8.17




## Example 5

$\overrightarrow{OA}$, $\overrightarrow{OB}$ *and* $\overrightarrow{OC}$ *are the vectors* **a**, **b** *and* **c** *respectively. D is the midpoint of AB and E lies on BC where BE = 2BC (Fig.8.18). State in terms of* **a**, **b** *and* **c**,

*(a)* $\overrightarrow{AB}$, *(b)* $\overrightarrow{AD}$, *(c)* $\overrightarrow{OD}$, *(d)* $\overrightarrow{BC}$, *(e)* $\overrightarrow{BE}$, *(f)* $\overrightarrow{OE}$, *(g)* $\overrightarrow{DE}$.

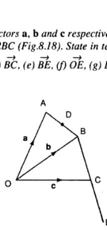



Fig. 8.18

(a) $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

(b) $\overrightarrow{AD} = \frac{1}{2}(\mathbf{b} - \mathbf{a})$

(c) $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD} = \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a}) = \frac{1}{2}(\mathbf{a} + \mathbf{b})$

(d) $\overrightarrow{BC} = \mathbf{c} - \mathbf{b}$

(e) $\overrightarrow{BE} = 2(\mathbf{c} - \mathbf{b})$

(f) $\overrightarrow{OE} = \overrightarrow{OB} + \overrightarrow{BE} = \mathbf{b} + 2(\mathbf{c} - \mathbf{b}) = 2\mathbf{c} - \mathbf{b}$

(g) $\overrightarrow{DE} = \overrightarrow{OE} - \overrightarrow{OD} = 2\mathbf{c} - \mathbf{b} - \frac{1}{2}(\mathbf{a} + \mathbf{b}) = 2\mathbf{c} - \frac{\mathbf{a}}{2} - \frac{3\mathbf{b}}{2}$

## Exercise 8.2 *(Answers on page 624.)*

**1** Given the vectors **a** and **b** in Fig.8.19, draw the vectors
(a) $\mathbf{a} + 2\mathbf{b}$, (b) $2\mathbf{a} - \mathbf{b}$, (c) $3\mathbf{a} - 2\mathbf{b}$.

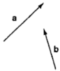



*Fig. 8.19*

**2** In △ABC, $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{BC} = \mathbf{b}$. State in terms of **a** and **b**, (a) $\overrightarrow{AC}$ and (b) $\overrightarrow{CA}$.

**3** Given the vectors **a**, **b** and **c** in Fig.8.20, draw
(a) $\mathbf{a} + 2\mathbf{b}$, (b) $\mathbf{a} + 2\mathbf{b} + \mathbf{c}$, (c) $\mathbf{a} - \mathbf{b} + \mathbf{c}$, (d) $\frac{1}{2}\mathbf{a} + \mathbf{b} - 2\mathbf{c}$.

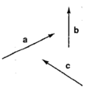



*Fig.8.20*

**4** If $|\mathbf{a}| = |\mathbf{b}|$ but $\mathbf{a} \neq \mathbf{b}$, explain why $\mathbf{a} + \mathbf{b}$ bisects the angle between **a** and **b** and is perpendicular to $\mathbf{a} - \mathbf{b}$.

**5** In △OAB, $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and M is the midpoint of AB. State in terms of **a** and **b**, (a) $\overrightarrow{AB}$, (b) $\overrightarrow{AM}$, (c) $\overrightarrow{OM}$.

**6** In $\triangle OPQ$, $\overrightarrow{OP} = \mathbf{p}$, $\overrightarrow{OQ} = \mathbf{q}$. R is the midpoint of OP and S lies on OQ such that OS = 3SQ. State in terms of **p** and **q**, (a) $\overrightarrow{OR}$, (b) $\overrightarrow{PQ}$, (c) $\overrightarrow{OS}$, (d) $\overrightarrow{RS}$.

**7** In $\triangle OAB$, $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$. BC is drawn parallel to OA (in the same direction) and BC = 2OA. State in terms of **a** and **b**, (a) $\overrightarrow{AB}$, (b) $\overrightarrow{BC}$, (c) $\overrightarrow{OC}$, (d) $\overrightarrow{AC}$

**8** OACB is a parallelogram with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. AC is extended to D where AC = 2CD. Find, in terms of **a** and **b**, (a) $\overrightarrow{AD}$, (b) $\overrightarrow{OD}$, (c) $\overrightarrow{BD}$.

**9** OAB is a triangle with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. M is the midpoint of OA and G lies on MB such that MG = $\frac{1}{2}$GB. State in terms of **a** and **b** (a) $\overrightarrow{OM}$, (b) $\overrightarrow{MB}$, (c) $\overrightarrow{MG}$, (d) $\overrightarrow{OG}$.

**10** $\overrightarrow{OA} = \mathbf{p} + \mathbf{q}$, $\overrightarrow{OB} = 2\mathbf{p} - \mathbf{q}$, where **p** and **q** are two vectors and M is the midpoint of AB. Find in terms of **p** and **q**, (a) $\overrightarrow{AB}$, (b) $\overrightarrow{AM}$, (c) $\overrightarrow{OM}$.

## POSITION VECTORS

If O is the origin, then the vector $\overrightarrow{OA}$ is called the **position vector** of A. For example, if the position vector of A is $2\mathbf{a} - 3\mathbf{b}$, then $\overrightarrow{OA}$ is $2\mathbf{a} - 3\mathbf{b}$.

### Using Vectors

The following principles should be carefully noted:

(1) If $m\mathbf{a} + n\mathbf{b} = p\mathbf{a} + q\mathbf{b}$ then $m = p$ and $n = q$. (See Examples 6 and 9).

(2) If the points P, Q and R are collinear, then $\overrightarrow{PQ} = k\overrightarrow{QR}$ (and conversely) because $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are parallel but meet at Q. (See Examples 7 and 8). We could also use $\overrightarrow{PQ} = k\overrightarrow{PR}$.

(3) If the vectors $m\mathbf{a} + n\mathbf{b}$ and $p\mathbf{a} + q\mathbf{b}$ are parallel, then $\frac{m}{p} = \frac{n}{q}$. (See Example 10.)

### Example 6

*If* $\mathbf{p} = 2\mathbf{a} - 3\mathbf{b}$ *and* $\mathbf{q} = \mathbf{a} + 2\mathbf{b}$, *find numbers x and y such that* $x\mathbf{p} + y\mathbf{q} = \mathbf{a} - 12\mathbf{b}$.

$x\mathbf{p} + y\mathbf{q} = x(2\mathbf{a} - 3\mathbf{b}) + y(\mathbf{a} + 2\mathbf{b}) = (2x + y)\mathbf{a} + (-3x + 2y)\mathbf{b}$

By (1) above, if this vector is to equal $\mathbf{a} - 12\mathbf{b}$, then the multiples of **a** and the multiples of **b** on each side must be separately equal.
Hence $2x + y = 1$ and $-3x + 2y = -12$.
Solving these equations, $x = 2$, $y = -3$.
Checking this, $2(2\mathbf{a} - 3\mathbf{b}) - 3(\mathbf{a} + 2\mathbf{b}) = \mathbf{a} - 12\mathbf{b}$ as required.

## Example 7

*The position vectors of P, Q and R are* $\mathbf{a} - 2\mathbf{b}$, $2\mathbf{a} - 3\mathbf{b}$ *and* $\mu\mathbf{a} - 6\mathbf{b}$, *where* $\mu$ *is a scalar constant. If the points P, Q and R are collinear, find*
*(a) the value of* $\mu$ *and (b) the ratio PQ:QR.*

First we find $\overrightarrow{PQ}$ and $\overrightarrow{QR}$.

(a) $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = 2\mathbf{a} - 3\mathbf{b} - (\mathbf{a} - 2\mathbf{b}) = \mathbf{a} - \mathbf{b}$
$\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ} = \mu\mathbf{a} - 6\mathbf{b} - (2\mathbf{a} - 3\mathbf{b}) = (\mu - 2)\mathbf{a} - 3\mathbf{b}$

Now if P, Q and R are to be collinear, $\overrightarrow{PQ} = k\overrightarrow{QR}$.
$\overrightarrow{PQ} = \mathbf{a} - \mathbf{b}$ and $\overrightarrow{QR} = (\mu - 2)\mathbf{a} - 3\mathbf{b}$
Comparing these, the multiple of $-\mathbf{b}$ in $\overrightarrow{QR}$ is 3 so the multiple of $\mathbf{a}$ must also be 3. Hence $\mu - 2 = 3$ or $\mu = 5$.

(b) When $\mu = 5$, $|\overrightarrow{PQ}| = |\mathbf{a} - \mathbf{b}|$ and $|\overrightarrow{QR}| = |3(\mathbf{a} - \mathbf{b})|$ which gives the ratio PQ:QR as 1:3.

## Example 8

$\overrightarrow{OP} = 3\mathbf{a} + \mathbf{b}$, $\overrightarrow{OQ} = \mu(\mathbf{a} - \mathbf{b})$ *and* $\overrightarrow{OR} = 4\mathbf{a} + 4\mathbf{b}$.
*Given that P, Q and R are collinear, find the value of* $\mu$ *and the ratio PQ:QR.*

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \mu\mathbf{a} - \mu\mathbf{b} - 3\mathbf{a} - \mathbf{b} = (\mu - 3)\mathbf{a} - (\mu + 1)\mathbf{b}$
$\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ} = 4\mathbf{a} + 4\mathbf{b} - \mu\mathbf{a} + \mu\mathbf{b} = (4 - \mu)\mathbf{a} + (\mu + 4)\mathbf{b}$

The relation between these vectors is not as straightforward as it was in Example 7. We shall have to find an equation for $\mu$. If P, Q and R are collinear, $\overrightarrow{PQ} = k\overrightarrow{QR}$ so the multiples of $\mathbf{a}$ and of $\mathbf{b}$ in the two vectors must be in the same ratio.

Then $\frac{\mu - 3}{4 - \mu} = \frac{-\mu - 1}{\mu + 4}$ which leads to $\mu^2 + \mu - 12 = \mu^2 - 3\mu - 4$ giving $\mu = 2$.

Hence $\overrightarrow{PQ} = -\mathbf{a} - 3\mathbf{b}$ and $\overrightarrow{QR} = 2\mathbf{a} + 6\mathbf{b} = -2(-\mathbf{a} - 3\mathbf{b})$.

The ratio PQ:QR = 1:–2 which means that QR is twice as long as PQ but in the *opposite direction* as shown in Fig.8.21.

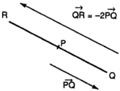



*Fig.8.21*

### *Example 9*

*In Fig.8.22,* $\vec{OA} = \mathbf{a}$ *and* $\vec{OB} = \mathbf{b}$. *C lies on OA where* $OC = \frac{2}{3}$ *OA, D is the midpoint of AB and BC and OD intersect at M.*

*(a) By taking* $\vec{OM} = p\vec{OD}$ *and* $\vec{BM} = q\vec{BC}$, *where p and q are numbers, find two vector expressions for* $\vec{OM}$.

*Hence find (b) the values of p and q, (c) the ratios OM:MD and BM:MC.*

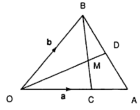



*Fig.8.22*

(a) First find $\vec{OM}$ as part of $\vec{OD}$. To obtain $\vec{OD}$ we find $\vec{AD}$.

$\vec{AB} = \mathbf{b} - \mathbf{a}$ so $\vec{AD} = \frac{1}{2}(\mathbf{b} - \mathbf{a})$

Then $\vec{OD} = \vec{OA} + \vec{AD} = \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a}) = \frac{1}{2}(\mathbf{a} + \mathbf{b})$

So $\vec{OM} = p\vec{OD} = \frac{p}{2}(\mathbf{a} + \mathbf{b}) = \frac{p}{2}\mathbf{a} + \frac{p}{2}\mathbf{b}$. (i)

Now to find another expression for $\vec{OM}$, we use $\vec{OB} + \vec{BM}$.

First find $\vec{BM}$ as part of $\vec{BC}$.

$\vec{BC} = \vec{OC} - \vec{OB} = \frac{2}{3}\mathbf{a} - \mathbf{b}$ so $\vec{BM} = q(\frac{2}{3}\mathbf{a} - \mathbf{b})$

Then $\vec{OM} = \vec{OB} + \vec{BM} = \mathbf{b} + q(\frac{2}{3}\mathbf{a} - \mathbf{b}) = \frac{2q}{3}\mathbf{a} + (1 - q)\mathbf{b}$ (ii)

(b) As (i) and (ii) are expressions for the *same* vector, then by (1) above, the multiples of **a** and **b** are equal.

So $\frac{p}{2} = \frac{2q}{3}$ and $\frac{p}{2} = 1 - q$. We solve these equations.

$\frac{2q}{3} = 1 - q$ giving $q = \frac{3}{5}$ and hence $p = \frac{4}{5}$.

(c) OM = $\frac{4}{5}$OD so OM:MD = 4:1 and BM = $\frac{3}{5}$BC so BM:MC = 3:2.

## Ratio Theorem (Optional)

This theorem is not necessary for this syllabus but may be found useful. It gives a direct way of finding the position vector of a point dividing a line in a given ratio. In Fig. 8.23, $\vec{OA} = \mathbf{a}$, $\vec{OB} = \mathbf{b}$ and P divides AB in the ratio $p$:$q$. We wish to find $\vec{OP}$.

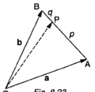



Fig. 8.23

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \mathbf{a} + \frac{p}{p+q}\ \overrightarrow{AB} = \mathbf{a} + \frac{p}{p+q}(\mathbf{b} - \mathbf{a}) = \frac{q\mathbf{a} + p\mathbf{b}}{q + p}$

This is known as the **ratio theorem** for vectors. Note carefully that $q$ multiplies **a** (on the other side of P) and $p$ multiplies **b**.

For example, if P was the midpoint of AB, then $p = q = 1$.

So $\overrightarrow{OP} = \frac{\mathbf{a} + \mathbf{b}}{2}$. If $\overrightarrow{AP} = \frac{1}{3}\overrightarrow{AB}$ then $p = 1$, $q = 2$ and $\overrightarrow{OP} = \frac{2\mathbf{a} + \mathbf{b}}{3}$. Similarly, if $\overrightarrow{AP} = \frac{3}{5}\overrightarrow{AB}$ where the position vectors of A and B are $3\mathbf{a} - 2\mathbf{b}$ and $-\mathbf{a} + 5\mathbf{b}$, then $p = 3$, $q = 2$ and the position vector of P will be $\frac{2(3\mathbf{a} - 2\mathbf{b}) + 3(-\mathbf{a} + 5\mathbf{b})}{5} = \frac{3\mathbf{a} + 11\mathbf{b}}{5}$.

*Note:* Care must be taken when P divides $\overrightarrow{AB}$ **externally** i.e. P lies outside AB. One of $p$ or $q$ must then be taken as negative (Fig. 8.24).

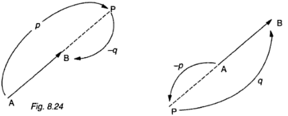



Fig. 8.24

$\overrightarrow{OP} = \frac{-q\mathbf{a} + p\mathbf{b}}{-q + p}$ $\qquad$ $\overrightarrow{OP} = \frac{q\mathbf{a} - p\mathbf{b}}{q - p}$

### *Example 10*

*In Fig. 8.25, the position vectors of the points A, B and C are **2a – b**, **4a + 5b** and **–a + 4b** respectively. L and N are the midpoints of AB and AC respectively. M is a point such that* $\overrightarrow{LM} = \frac{1}{3}\overrightarrow{LC}$.

*(a) Find the position vectors of L, M and N and (b) show that B, M and N are collinear and state the ratio BM:MN. (c) P is a point on BN produced such that BP = pBN. If PC is parallel to AM, find the value of p.*

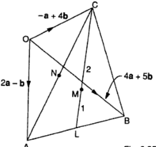


*Fig. 8.25*

(a) The position vector of L is $\frac{(\mathbf{2a - b}) + (\mathbf{4a + 5b})}{2} = \mathbf{3a + 2b}$.

As LM = $\frac{1}{3}$LC, LM:MC = 1:2.

The position vector of M is $\frac{2(\mathbf{3a + 2b}) + 1(\mathbf{-a + 4b})}{2 + 1} = \frac{\mathbf{5a + 8b}}{3}$

The position vector of N is $\frac{(\mathbf{2a - b}) + (\mathbf{-a + 4b})}{2} = \frac{\mathbf{a + 3b}}{2}$

(b) We find $\overrightarrow{BM}$ and $\overrightarrow{MN}$.

$\overrightarrow{BM} = \overrightarrow{OM} - \overrightarrow{OB} = \frac{\mathbf{5a + 8b}}{3} - (\mathbf{4a + 5b}) = \frac{\mathbf{-7a - 7b}}{3} = -\frac{7}{3}(\mathbf{a + b})$

$\overrightarrow{MN} = \overrightarrow{ON} - \overrightarrow{OM} = \frac{\mathbf{a + 3b}}{2} - \frac{\mathbf{5a + 8b}}{3} = \frac{\mathbf{-7a - 7b}}{6} = -\frac{7}{6}(\mathbf{a + b})$

Then B, M and N are collinear as $\overrightarrow{BM}$ is a multiple of $\overrightarrow{MN}$.

BM:MN = $-\frac{7}{3}:-\frac{7}{6}$ = 2:1.

(c) $\overrightarrow{BP} + \overrightarrow{PC} = \overrightarrow{BC}$

so $p\overrightarrow{BN} + \overrightarrow{PC} = (\mathbf{-a + 4b}) - (\mathbf{4a + 5b})$

i.e. $p[\frac{\mathbf{a + 3b}}{2} - (\mathbf{4a + 5b})] + \overrightarrow{PC} = \mathbf{-5a - b}$

so $p(\frac{\mathbf{-7a - 7b}}{2}) + \overrightarrow{PC} = \mathbf{-5a - b}$

giving $\overrightarrow{PC} = (\frac{7p - 10}{2})\mathbf{a} + (\frac{7p - 2}{2})\mathbf{b}$

$\overrightarrow{AM} = \frac{\mathbf{5a + 8b}}{3} - (\mathbf{2a - b}) = \frac{\mathbf{-a + 11b}}{3} = -\frac{1}{3}\mathbf{a} + \frac{11}{3}\mathbf{b}$

If these are parallel, the multiples of **a** and of **b** must be in the same ratio.

Hence $\frac{\frac{7p - 10}{2}}{-\frac{1}{3}} = \frac{\frac{7p - 2}{2}}{\frac{11}{3}}$ i.e. $\frac{7p - 10}{-1} = \frac{7p - 2}{11}$

which simplifies to $77p - 110 = -7p + 2$ or $p = \frac{4}{3}$.

## Exercise 8.3 *(Answers on page 625.)*

**1** Given that $\mathbf{p} = 3\mathbf{a} - \mathbf{b}$ and $\mathbf{q} = 2\mathbf{a} - 3\mathbf{b}$, find numbers $x$ and $y$ such that $x\mathbf{p} + y\mathbf{q} = \mathbf{a} + 9\mathbf{b}$.

**2** If $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j}$, $\mathbf{b} = -2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{c} = 12\mathbf{i} - 5\mathbf{j}$, find numbers $p$ and $q$ such that $p\mathbf{a} + q\mathbf{b} = \mathbf{c}$.

**3** Given $\mathbf{p} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{q} = \mathbf{i} - \mathbf{j}$, find numbers $x$ and $y$ such that $x\mathbf{p} + y\mathbf{q} = -4\mathbf{i} - 11\mathbf{j}$.

**4** If $\mathbf{p} = 2\mathbf{a} - 5\mathbf{b}$, $\mathbf{q} = \mathbf{a} + 2\mathbf{b}$ and $\mathbf{r} = \mathbf{a} - 16\mathbf{b}$, find numbers $x$ and $y$ such that $x\mathbf{p} + y\mathbf{q} = \mathbf{r}$.

**5 If $\overrightarrow{OP} = 2\mathbf{a} - 5\mathbf{b}$, $\overrightarrow{OQ} = 5\mathbf{a} - \mathbf{b}$ and $\overrightarrow{OR} = 11\mathbf{a} + 7\mathbf{b}$, show that P, Q and R are collinear and state the ratio PQ:QR.**

**6** The position vectors of P, Q and R are $\mathbf{a} - 2\mathbf{b}$, $2\mathbf{b}$ and $-4\mathbf{a} + k\mathbf{b}$ respectively. If P, Q and R are collinear, find the value of $k$. What is the ratio PQ:QR?

**7 Given that $\overrightarrow{OP} = \mathbf{a} + \mathbf{b}$, $\overrightarrow{OQ} = k\mathbf{a}$ and $\overrightarrow{OR} = 7\mathbf{a} - 2\mathbf{b}$, find the value of $k$ if Q lies on PR.**

**8 The position vectors of P, Q and R are $2\mathbf{a} - \mathbf{b}$, $\mu(\mathbf{a} - \mathbf{b})$ and $\mathbf{a} + \mathbf{b}$ respectively. Find the value of $\mu$ if PQR is a straight line. State the ratio PQ:QR.**

**9** (a) The position vectors of L, M and N are $\mathbf{p} + 2\mathbf{q}$, $m(\mathbf{p} + \mathbf{q})$ and $\mathbf{p} - \mathbf{q}$ respectively. Find the value of $m$ for which LMN is a straight line, and state the ratio LM:MN.

(b) The position vectors of A, B and C are $\mathbf{a} + 2\mu\mathbf{b}$, $\mu\mathbf{a} - \mathbf{b}$ and $2\mathbf{a} - 3\mathbf{b}$ respectively. If AB is parallel to OC, where O is the origin, find the value of $\mu$.

**10** The position vectors of A and B are $-\mathbf{a} - 2\mathbf{b}$ and $3\mathbf{a} + 4\mathbf{b}$ respectively. Using the ratio theorem or otherwise, find the position vector of P where (a) $\overrightarrow{AP} = 2\overrightarrow{PB}$, (b) $\overrightarrow{AP} = \frac{1}{3}\overrightarrow{AB}$, (c) $4\overrightarrow{AP} = 3\overrightarrow{AB}$, (d) P lies on AB extended and AP = 3BP, (e) P lies on BA extended and AP = 2BA.

**11** $\overrightarrow{OA} = 2\mathbf{a} - 4\mathbf{b}$ and $\overrightarrow{OB} = 4\mathbf{a} + 6\mathbf{b}$, where O is the origin. P and Q are the midpoints of OA and AB respectively. (a) State the position vectors of P and Q. (b) G lies on BP such that BG = 2GP. Find the position vector of G. (c) Show that O, G and Q are collinear and state the ratio OG:GQ. (d) R lies on OA where $\overrightarrow{OR} = p\overrightarrow{OA}$. If BR is parallel to GA find the value of $p$.

**12 P and Q divide the sides BC and AC respectively of $\triangle$ABC in the ratio 2:1. If $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AC} = \mathbf{b}$, find (a) $\overrightarrow{QP}$ and (b) show that QP is parallel to AB and one-third its length.**

**13 OABC is a parallelogram with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$. D lies on OB where OD:DB = 1:4. AD meets OC at E. By taking $\overrightarrow{OE} = p\overrightarrow{OC}$ and $\overrightarrow{AD} = q\overrightarrow{AE}$, show that $\frac{1}{5}(\mathbf{a} + \mathbf{c}) = (1 - q)\mathbf{a} + pq\mathbf{c}$. Hence find the values of $p$ and $q$ and the ratios OE:EC and AD:DE.**

**14** OABC is a parallelogram in which $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{b}$. M is the midpoint of AB and MC meets OB at X.

(a) By taking $\overrightarrow{MX} = p\overrightarrow{MC}$ and $\overrightarrow{OX} = q\overrightarrow{OB}$, express $\overrightarrow{OX}$ in terms of (i) $p$, $\mathbf{a}$ and $\mathbf{b}$, (ii) $q$, $\mathbf{a}$ and $\mathbf{b}$.

(b) Hence evaluate $p$ and $q$ and state the ratios OX:XB and CX:XM.

**15** C lies on the side OA of $\triangle$OAB where OC:CA = 2:1. D lies on the side OB where OD:DB = 1:2. AD meets BC at T.

(a) Taking $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{AT} = p\overrightarrow{AD}$ and $\overrightarrow{CT} = q\overrightarrow{CB}$, find two expressions for $\overrightarrow{OT}$. Hence find (b) the values of $p$ and $q$ and (c) the ratios CT:TB and AT:TD.

**16** C and D divide OA and OB respectively in the ratio 1:3. E divides CB in the ratio 1:4. Taking $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$, use vector methods to prove that DEA is a straight line and find the ratio DE:EA.

**17** In $\triangle$OAB, C divides OA in the ratio 2:3 and D divides AB in the ratio 1:2. OD meets CB at E.

(a) Taking $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OE} = p\overrightarrow{OD}$ and $\overrightarrow{CE} = q\overrightarrow{CB}$, obtain two expressions for $\overrightarrow{OE}$.

(b) Hence find the values of $p$ and $q$.

(c) State the ratios OE:ED and CE:EB.

**18** The position vectors of A and B are $\mathbf{a}$ and $\mathbf{b}$ respectively relative to an origin O. C is the midpoint of AB and D divides OB in the ratio 2:1. AD and OC meet at P.

(a) Taking $\overrightarrow{OP} = p\overrightarrow{OC}$ and $\overrightarrow{AP} = q\overrightarrow{AD}$, express $\overrightarrow{OP}$ in two different forms.

Hence find (b) the values of $p$ and $q$ and (c) the ratio OP:PC. (d) Q lies on BA produced where $\overrightarrow{AQ} = k\overrightarrow{BA}$. State the position vector of Q. If OQ is parallel to DC, find the value of $k$.

**19** OABC is a parallelogram with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$. $\overrightarrow{OB}$ is extended to D where $\overrightarrow{OB} = \overrightarrow{BD}$ and $\overrightarrow{OA}$ is extended to E where $\overrightarrow{AE} = \frac{1}{2}\overrightarrow{OA}$. CE and AD meet at X.

(a) Taking $\overrightarrow{AX} = p\overrightarrow{AD}$ and $\overrightarrow{CX} = q\overrightarrow{CE}$, find two expressions for $\overrightarrow{OX}$.

(b) Hence find the values of $p$ and $q$ and the ratios AX:XD and CX:XE.

(c) F lies on AD and BF is parallel to CE. Taking $\overrightarrow{AF} = r\overrightarrow{AD}$, find the value of $r$.

(d) Hence state the ratio BF:CE.

**20** $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. OB is produced to C where $\overrightarrow{OB} = 2\overrightarrow{BC}$. D is the midpoint of AB. OD produced meets AC at E. Taking $\overrightarrow{OD} = p\overrightarrow{OE}$ and $\overrightarrow{AE} = q\overrightarrow{AC}$, derive two expressions for $\overrightarrow{OD}$ and hence find the values of $p$ and $q$ and the ratios OD:DE and AE:EC.

**21** OABC is a parallelogram with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$. D lies on OC where OD:DC = 1:2 and E is the midpoint of CB. DB meets AE at T. Taking $\overrightarrow{DT} = p\overrightarrow{DB}$ and $\overrightarrow{AT} = q\overrightarrow{AE}$, form two vector expressions for $\overrightarrow{OT}$ and hence find the values of $p$ and $q$.

**22** The position vectors of A and B are $\mathbf{a}$ and $\mathbf{b}$ respectively, relative to O. C lies on OB where OC:CB = 1:3. AC is produced to D where $\overrightarrow{AD} = p\overrightarrow{AC}$. If DB is parallel to OA, find the value of $p$.

**23** The position vectors of the points A, B and C are $7\mathbf{a} - 2\mathbf{b}$, $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - 2\mathbf{b}$ respectively. L is the point where $\overrightarrow{AL} = \frac{1}{3}\overrightarrow{AB}$. M is the midpoint of BC and N is the point such that $\overrightarrow{CN} = 2\overrightarrow{CA}$. Find the position vectors of L, M and N and show that these points are collinear. State the ratio ML:LN.

**24** A, B and C have position vectors $\mathbf{a} - \mathbf{b}$, $3\mathbf{a} + 2\mathbf{b}$ and $4\mathbf{a} - 3\mathbf{b}$ respectively. P lies on AB where AP:AB = 2:3, Q lies on BC where BQ:BC = 3:4 and R lies on AC extended so that AC = CR. Find the position vectors of P, Q and R and show that P, Q and R are collinear. State the ratio PQ:QR.

## COMPONENTS OF A VECTOR: UNIT COORDINATE VECTORS

Suppose $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{BC} = \mathbf{b}$ (Fig.8.24).

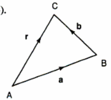



*Fig. 8.26*

The resultant of **a** and **b** is $\overrightarrow{AC} = \mathbf{r} = \mathbf{a} + \mathbf{b}$. The vectors **a** and **b** are called the **components** of **r**. The components of a vector **r** are *any* two vectors whose resultant is **r**. A vector can therefore be resolved into two components in an infinite number of ways. However if we take the components parallel to the $x$- and $y$-axes (Fig. 8.27), they will be unique and perpendicular.

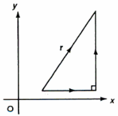



*Fig. 8.27*

We now define two standard unit vectors **i** and **j** called the **unit coordinate** (or **base**) **vectors** (Fig. 8.28).

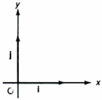



*Fig. 8.28*

**i** is a vector in the direction of the positive $x$-axis and $|\mathbf{i}| = 1$;
**j** is a vector in the direction of the positive $y$-axis and $|\mathbf{j}| = 1$.

The perpendicular components of any vector can now be expressed in terms of **i** and **j** in a standard form. For example, suppose the vector $\overrightarrow{AC} = \mathbf{r}$ has components of magnitude 3 and 4 parallel to the axes (Fig. 8.28). The horizontal component $\overrightarrow{AD} = 3\mathbf{i}$ and the vertical component $\overrightarrow{DC} = 4\mathbf{j}$.
Hence the vector $\mathbf{r} = \overrightarrow{AD} + \overrightarrow{DC} = 3\mathbf{i} + 4\mathbf{j}$.
**r** is now expressed in terms of the base vectors **i** and **j**.

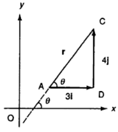



*Fig. 8.29*

**r** can also be written as $\binom{3}{4}$, i.e. in **column vector** form. [Do not confuse with coordinates (3,4)]. For example, $2\mathbf{i} - 3\mathbf{j}$ can be written as $\binom{2}{-3}$. $\mathbf{i} = \binom{1}{0}$ and $\mathbf{j} = \binom{0}{1}$.
Given **r** in terms of **i** and **j**, we can find $|\mathbf{r}|$ and the angle θ it makes with the positive $x$-axis.
From △ADC (Fig. 8.28), $AC^2 = AD^2 + DC^2$, so $|\mathbf{r}^2| = 3^2 + 4^2 = 25$ and $|\mathbf{r}| = \sqrt{25} = 5$.
$\tan\theta = \frac{4}{3}$ giving $\theta = 53.13°$.

*Note:* To find θ for a given vector, draw a diagram to locate the correct quadrant as $\tan\theta = \frac{b}{a}$ will give two values for $0° \le \theta \le 360°$.

In general for a vector **r** with perpendicular components of magnitude $a$ and $b$ (Fig. 8.30):

$\mathbf{r} = a\mathbf{i} + b\mathbf{j} = \binom{a}{b}$

$|\mathbf{r}| = |a\mathbf{i} + b\mathbf{j}| = \sqrt{a^2 + b^2}$ and $\tan\theta = \frac{b}{a}$

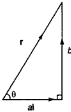



*Fig. 8.30*

## Example 11

*The position vector of A is* $-2\mathbf{i} + 3\mathbf{j}$.
*(a) State the coordinates of A.*
*(b) Find* $|\overrightarrow{OA}|$ *and the angle the vector OA makes with the x-axis.*

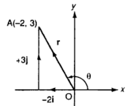



*Fig. 8.31*

(a) The coordinates of A are (–2,3) (Fig.8.29).

(b) $|\overrightarrow{OA}| = |-2\mathbf{i} + 3\mathbf{j}| = \sqrt{(-2)^2 + 3^2} = \sqrt{13} \approx 3.6$

$\tan\theta = \frac{3}{-2}$ giving $\theta = 123.7°$ (2nd quadrant, Fig.8.29).

## Example 12

*(a)* $\overrightarrow{OC}$ *has the same direction as* $\binom{-3}{4}$ *and* $|\overrightarrow{OC}| = 30$. *Express* $\overrightarrow{OC}$ *as a column vector.*

*(b) The position vectors of A and B are* $\binom{-2}{3}$ *and* $\binom{1}{-2}$ *respectively. Find*

*(i)* $\overrightarrow{AB}$, *(ii) the equation of AB.*

(a) $\overrightarrow{OC}$ must be a scalar multiple of $\binom{-3}{4}$ so $\overrightarrow{OC} = \binom{-3k}{4k}$ where $k > 0$.
$|\overrightarrow{OC}|^2 = 9k^2 + 16k^2 = 25k^2$ so $|\overrightarrow{OC}| = 5k = 30$ and $k = 6$. Hence $\overrightarrow{OC} = \binom{-18}{24}$.

(b) (i) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{i} - 2\mathbf{j} - (-2\mathbf{i} + 3\mathbf{j}) = 3\mathbf{i} - 5\mathbf{j}$.

(ii) The coordinates of A and B are (–2,3) and (1,–2).
Hence the equation of AB is $\frac{y-3}{-2-3} = \frac{x+2}{1+2}$ i.e. $5x + 3y = -1$.

### *Example 13*

*The position vectors of A, B and C are* $2\mathbf{i} - \mathbf{j}$, $3\mathbf{i} + 2\mathbf{j}$ *and* $-3\mathbf{i} + 4\mathbf{j}$ *respectively (Fig. 8.32).*

*(a) Find* $|\overrightarrow{AB}|$, $|\overrightarrow{BC}|$ *and* $|\overrightarrow{AC}|$ *and show that* $\triangle ABC$ *is right angled.*

*(b) If the position vector of D is* $-4\mathbf{i} - \mathbf{j}$, *find the angle BD makes with the x-axis.*

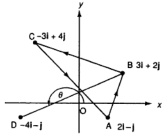



*Fig. 8.32*

(a) $\overrightarrow{AB} = 3\mathbf{i} + 2\mathbf{j} - (2\mathbf{i} - \mathbf{j}) = \mathbf{i} + 3\mathbf{j}$ ; $|\overrightarrow{AB}| = \sqrt{10}$

$\overrightarrow{BC} = -3\mathbf{i} + 4\mathbf{j} - (3\mathbf{i} + 2\mathbf{j}) = -6\mathbf{i} + 2\mathbf{j}$ ; $|\overrightarrow{BC}| = \sqrt{40}$

$\overrightarrow{CA} = 2\mathbf{i} - \mathbf{j} - (-3\mathbf{i} + 4\mathbf{j}) = 5\mathbf{i} - 5\mathbf{j}$ ; $|\overrightarrow{CA}| = \sqrt{50}$

As $|\overrightarrow{CA}|^2 = |\overrightarrow{AB}|^2 + |\overrightarrow{BC}|^2$, the triangle is right angled.

(b) $\overrightarrow{BD} = -4\mathbf{i} - \mathbf{j} - (3\mathbf{i} + 2\mathbf{j}) = -7\mathbf{i} - 3\mathbf{j}$

$\tan\theta = \frac{-3}{-7}$ giving $\theta = 203.2°$ (3rd quadrant).

## Unit Vectors

The magnitude of the vector $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$ is $|3\mathbf{i} + 4\mathbf{j}| = 5$ so the vector is 5 units long. Hence the vector $\frac{3\mathbf{i} + 4\mathbf{j}}{5}$ is one unit long and is in the **same** direction as **a**. This is the **unit vector** in the direction of **a** (Fig. 8.33). It is written as $\hat{\mathbf{a}}$ (read 'vector **a** cap').

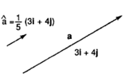



*Fig. 8.33*

If $\mathbf{r} = a\mathbf{i} + b\mathbf{j}$, then $\hat{\mathbf{r}} = \frac{a\mathbf{i} + b\mathbf{j}}{\sqrt{a^2 + b^2}}$

For example, if $\mathbf{a} = 2\mathbf{i} - \mathbf{j}$ then $\hat{\mathbf{a}} = \frac{2\mathbf{i} - \mathbf{j}}{\sqrt{5}}$

Again, if $\overrightarrow{OA} = \binom{2}{3}$, the unit vector parallel to $\overrightarrow{AO}$ would be $\frac{-2\mathbf{i} - 3\mathbf{i}}{\sqrt{13}}$.

## Exercise 8.4 *(Answers on page 625.)*

**1** On graph paper, mark the positions of the points with position vectors $\mathbf{i} + \mathbf{j}$, $-2\mathbf{i} - \mathbf{j}$, $3\mathbf{i} + 2\mathbf{j}$, $-3\mathbf{j}$.

**2** A, B and C are points with position vectors $2\mathbf{i} - 3\mathbf{j}$, $\mathbf{i} + 2\mathbf{j}$ and $4\mathbf{i} - \mathbf{j}$ respectively. Find in terms of $\mathbf{i}$ and $\mathbf{j}$, the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$.

**3** The position vectors of A and B are $3\mathbf{i} + \mathbf{j}$ and $2\mathbf{i} + 3\mathbf{j}$ respectively. Points C and D have position vectors given by $\overrightarrow{OC} = \overrightarrow{AO}$ and $\overrightarrow{CD} = \overrightarrow{AB}$.
(a) Find the position vectors of C and D in terms of $\mathbf{i}$ and $\mathbf{j}$ and show the positions of the four points on a diagram.
(b) Express $\overrightarrow{DB}$ in terms of $\mathbf{i}$ and $\mathbf{j}$.
(c) Find $|\overrightarrow{DB}|$ and the angle $\overrightarrow{DB}$ makes with the $x$-axis.

**4** Find the magnitude and the angle made with the $x$-axis of the vectors
(a) $\binom{-1}{2}$ (b) $\binom{3}{-3}$ (c) $2\mathbf{i} + 3\mathbf{j}$ (d) $-4\mathbf{i} - 2\mathbf{j}$

**5** (a) The coordinates of A are $(-3,2)$ and the position vector of B is $2\mathbf{i} + 4\mathbf{j}$. Find the vector $\overrightarrow{BA}$.
(b) The vector $\overrightarrow{OA}$ has magnitude 25 units and is in the same direction as $\binom{3}{-4}$. The vector $\overrightarrow{OB}$ has magnitude 6.5 units and is opposite in direction to $\binom{5}{12}$. State the vectors $\overrightarrow{AO}$ and $\overrightarrow{AB}$ in column vector form.

**6** The point with position vector $3\mathbf{i} - 2\mathbf{j}$ is displaced by a vector $\binom{4}{2}$. Find its new position vector.

**7** If the coordinates of A are (2, 4) and $\overrightarrow{AB} = \mathbf{i} + 2\mathbf{j}$, find the position vector of B.

**8** (a) If $\mathbf{a} = 6\mathbf{i} - 8\mathbf{j}$ and $\mathbf{b} = \mathbf{i} + 3\mathbf{j}$, find $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$.
(b) $\overrightarrow{OA} = \binom{3}{-4}$ and $\overrightarrow{OB} = \binom{7}{-1}$. Find the unit vectors parallel to $\overrightarrow{AO}$ and $\overrightarrow{BA}$.

**9** In this question, take $\mathbf{a} = 3\mathbf{i} + \mathbf{j}$, $\mathbf{b} = -\mathbf{i} + 2\mathbf{j}$, $\mathbf{c} = 3\mathbf{i}$. Find (i) $\hat{\mathbf{a}}$ and (ii) $\hat{\mathbf{b}}$.
Express in terms of $\mathbf{i}$ and $\mathbf{j}$, (iii) $\mathbf{a} + 2\mathbf{b}$, (iv) $2\mathbf{c} - 3\mathbf{b}$, (v) $\mathbf{a} + \mathbf{b} - 2\mathbf{c}$.
Find (vi) $|\mathbf{a} + 2\mathbf{b}|$, (vii) $|2\mathbf{c} - 3\mathbf{b}|$, (viii) $|\mathbf{a} + \mathbf{b} - 2\mathbf{c}|$.

**10** The position vectors of A, B, C and D are $\mathbf{i} + 3\mathbf{j}$, $2\mathbf{i} - \mathbf{j}$, $-\mathbf{i} - 4\mathbf{j}$ and $3\mathbf{i} + 2\mathbf{j}$ respectively. Find in terms of $\mathbf{i}$ and $\mathbf{j}$ the vectors (a) $\overrightarrow{AB}$, (b) $\overrightarrow{BD}$, (c) $\overrightarrow{CA}$, (d) $\overrightarrow{AD}$.

**11** The position vectors of A and B are $2\mathbf{i} + 3\mathbf{j}$ and $3\mathbf{i} - 8\mathbf{j}$ respectively. D is the midpoint of AB and E divides OD in the ratio 2:3. Find the coordinates of E.

**12** P and Q have position vectors $5\mathbf{i} + 2\mathbf{j}$ and $\mathbf{i} - 4\mathbf{j}$ respectively. If $\overrightarrow{OP} = 3\overrightarrow{OQ} + 2\overrightarrow{OR}$, find the position vector of R.

**13** A, B and C have coordinates (1,2), (2,5) and (0,–4) respectively. If $\overrightarrow{AB} = \overrightarrow{CD}$, find the position vector of D.

**14** The position vectors of A and B are $3\mathbf{i} + \mathbf{j}$ and $-4\mathbf{i} + 2\mathbf{j}$ respectively. Find the position vector of C if $\overrightarrow{AB} = \overrightarrow{BC}$.

**15** Points A and B have position vectors $2\mathbf{i} - \mathbf{j}$ and $\mathbf{i} + 3\mathbf{j}$ respectively.

(a) Given that $\overrightarrow{OC} = \overrightarrow{AB}$ and $\overrightarrow{AD} = \overrightarrow{CB}$, find the position vectors of C and D.

(b) Show the positions of the four points on a diagram.

(c) Find $|\overrightarrow{CD}|$ and the angle $\overrightarrow{CD}$ makes with the $x$-axis.

**16** The position vectors of A and B are $4\mathbf{i} + 5\mathbf{j}$ and $\mathbf{i} - 2\mathbf{j}$ respectively. Find the position vector of C if $3\overrightarrow{OA} = 2\overrightarrow{OB} + \overrightarrow{OC}$.

**17** The coordinates of A and B are (2,3) and (–2,5) respectively. Find the position vector of C if $2\overrightarrow{OA} = 2\overrightarrow{OB} + \overrightarrow{BC}$.

**18** Show that the points with position vectors $4\mathbf{i} + 5\mathbf{j}$, $3\mathbf{i} + 3\mathbf{j}$ and $-3\mathbf{j}$ are collinear.

**19** What is the gradient of the line joining the points with position vectors $2\mathbf{i} + \mathbf{j}$ and $\mathbf{i} + 3\mathbf{j}$?

**20** Show that the triangle whose vertices have position vectors $2\mathbf{i} + 4\mathbf{j}$, $5\mathbf{i} + 2\mathbf{j}$ and $3\mathbf{i} + 5\mathbf{j}$ is isosceles.

**21** (a) The velocity $\mathbf{v}$ m s$^{-1}$ of a body is given by the vector $\mathbf{v} = \mathbf{i} + 3\mathbf{j}$.
Find the speed of the body and the angle its path makes with the $x$-axis.

(b) If its position vector at the start was $\mathbf{i} + \mathbf{j}$, what is its position vector (i) after 1 sec, (ii) after 3 secs, (iii) after $t$ secs?

(c) After what time will it reach the position given by $7\mathbf{i} + 19\mathbf{j}$?

**22** A body is moving with velocity $\mathbf{v}$ m s$^{-1}$ where $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$. If it started from the position $\mathbf{i} + 4\mathbf{j}$, what is its position after 3 seconds? How long will it take to reach the position $11(\mathbf{i} - \mathbf{j})$?

**23** The position vector $\mathbf{r}$ of a point on a straight line is given by $\mathbf{r} = \mathbf{i} + \mathbf{j} + t(2\mathbf{i} - \mathbf{j})$ where $t$ is a number.

(a) What is its position vector when $t = 2$?

(b) Find the position vector of another point on the line by taking any other value of $t$.

(c) Hence find the gradient and the equation of the line .

**24** Find the gradient and equation of the line given by $\mathbf{r} = \mathbf{i} - \mathbf{j} + k(\mathbf{i} - \mathbf{j})$ where $k$ is a number.

**25** The position vector $\mathbf{r}$ of a point is given by $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + t(\mathbf{i} + 2\mathbf{j})$, where $t$ is a number. What is its position vector when (a) $t = -1$, (b) $t = 3$?

(c) What is the value of $t$ when its position vector is $7\mathbf{i} + 9\mathbf{j}$?

**26** If the vectors $m\mathbf{i} - 2\mathbf{j}$ and $4\mathbf{i} - 6\mathbf{j}$ are parallel, state the value of $m$.

**27** The position vectors of A and B are $3\mathbf{i} - 2\mathbf{j}$ and $t\mathbf{i} + \mathbf{j}$ respectively. Find the value of $t$ if OAB is a straight line.

**28** OABC is a parallelogram where O is the origin. The position vectors of A and B are $4\mathbf{i} + 6\mathbf{j}$ and $6\mathbf{i} + 8\mathbf{j}$ respectively. D is the midpoint of CB and E is the midpoint of AB. OD meets CE at F.

(a) State the position vectors of C, D and E.

(b) By taking $\overrightarrow{OF} = m\overrightarrow{OD}$ and $\overrightarrow{CF} = n\overrightarrow{CE}$, find the values of $m$ and $n$ and the ratio OF:FD.

**29** (a) State the condition for the lines $y = m_1x + c_1$ and $y = m_2x + c_2$ to be perpendicular.

(b) The points A, B, C and D have position vectors $\mathbf{i} + \mathbf{j}$, $3\mathbf{i} - 2\mathbf{j}$, $-3\mathbf{i} - 3\mathbf{j}$ and $-\mathbf{j}$ respectively. Find the gradients of AB and CD and show that these lines are perpendicular.

**30** The points A, B, C and D have position vectors $\mathbf{i}$, $2\mathbf{i} + 3\mathbf{j}$, $2\mathbf{i} + \mathbf{j}$ and $5\mathbf{i}$ respectively. Show that AB and CD are perpendicular.

**31** P, Q, R and S have position vectors $\mathbf{i} + 2\mathbf{j}$, $3\mathbf{i} - \mathbf{j}$, $-\mathbf{i} - \mathbf{j}$ and $k\mathbf{i} + \mathbf{j}$ respectively, where $k$ is a number.

(a) Find the gradients of PQ and RS.

(b) For what value of $k$ will the lines PQ and RS be perpendicular?

**32** If $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$ and $|\mathbf{b}| = 2$, what are the greatest and smallest values of $|\mathbf{a} + \mathbf{b}|$?

## SCALAR PRODUCT OF TWO VECTORS

Vectors can be 'multiplied' in two ways. In one, the result is another vector, called the vector product but we shall not use this method. In the other, the result is a scalar so it is called the **scalar** (or **dot**) product.

We write the scalar product of **a** and **b** as **a.b** and define it as

$$\mathbf{a}.\mathbf{b} = |\mathbf{a}| \times |\mathbf{b}| \times \cos\theta = ab\cos\theta$$

where $\theta$ is the angle between the vectors (Fig. 8.34).

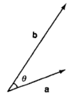


*Fig. 8.34*

For example, if $|\mathbf{a}| = 2$, $|\mathbf{b}| = 3$ and the angle between **a** and **b** = 60°, then $\mathbf{a}.\mathbf{b} = 2 \times 3 \times 0.5 = 3$.

**The scalar product is always a number.** This will be negative if $90° < \theta < 180°$ as $\cos\theta$ is then negative. $|\mathbf{a}|$ and $|\mathbf{b}|$ are always positive. From the definition we can derive the following important facts:

***1 The scalar product is commutative***

$\mathbf{a}.\mathbf{b} = |\mathbf{a}| \times |\mathbf{b}| \times \cos\theta$ and $\mathbf{b}.\mathbf{a} = |\mathbf{b}| \times |\mathbf{a}| \times \cos\theta$

Hence $\qquad \mathbf{a}.\mathbf{b} = \mathbf{b}.\mathbf{a}$

*II Parallel vectors*

If **a** and **b** are parallel but in the same direction, then $\theta = 0°$ (Fig. 8.35(a)).
$\mathbf{a}.\mathbf{b} = ab \cos 0° = ab = |\,\mathbf{a}\,| \times |\,\mathbf{b}\,|$.

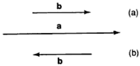



*Fig. 8.35*

If **a** and **b** are parallel but in opposite directions, then $\theta = 180°$ (Fig. 8.35(b)).
$\mathbf{a}.\mathbf{b} = ab \cos 180° = -ab$.

Hence **i.i** (written as $\mathbf{i}^2$) = 1 and $\mathbf{j}^2 = 1$.
Also $(3\mathbf{i} + 4\mathbf{j}).(3\mathbf{i} + 4\mathbf{j}) = |\,3\mathbf{i} + 4\mathbf{j}\,|^2 = 25$.

*III Perpendicular vectors*

If **a** and **b** are perpendicular, then $\theta = 90°$ and $\cos 90° = 0$.
Hence $\mathbf{a}.\mathbf{b} = ab \cos 90° = 0$.

**a.b** = 0 if **a** is perpendicular to **b**

Hence **i.j** = **j.i** = 0.

Conversely, if **a.b** = 0, then **a** and **b** are at right angles (unless either **a** or **b** or both are **0**).

*IV Distributive law for a scalar product*

In ordinary algebra, $a(b + c) = a \times (b + c) = ab + ac$, i.e. we can 'remove the brackets'. The '×' is distributed over the '$b + c$'. This is known as the **distributive law** for products. The same law is true for scalar products: **a.(b + c) = a.b + a.c**.

## Example 14

*If* $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$ *and* $\mathbf{b} = \mathbf{i} - 2\mathbf{j}$, *find* **a.b**.

$\mathbf{a}.\mathbf{b} = (2\mathbf{i} + 3\mathbf{j}).(\mathbf{i} - 2\mathbf{j})$.
$= 2\mathbf{i}.(\mathbf{i} - 2\mathbf{j}) + 3\mathbf{j}(\mathbf{i} - 2\mathbf{j})$ by the distributive law
$= 2\mathbf{i}.\mathbf{i} - 2\mathbf{i}.2\mathbf{j} + 3\mathbf{j}.\mathbf{i} - 3\mathbf{j}.2\mathbf{j}$ using the law again
$= \mathbf{2} - 0 + 0 - \mathbf{6}$ as $\mathbf{i}.\mathbf{i} = \mathbf{j}.\mathbf{j} = 1$ and $\mathbf{i}.\mathbf{j} = \mathbf{j}.\mathbf{i} = 0$
$= -4$

From this we see that we only need to multiply the two **i** and the two **j** terms and add the results. The **i.j** terms are ignored.

So $(\overset{\;2}{\overbrace{2\mathbf{i}} + \underbrace{3\mathbf{j}).(\mathbf{i}}_{-6} - 2\mathbf{j})} = 2 - 6 = -4$.

By the same method we can show that in general:

$$(a_1\mathbf{i} + b_1\mathbf{j}).(a_2\mathbf{i} + b_2\mathbf{j}) = \binom{a_1}{b_1}.\binom{a_2}{b_2} = a_1a_2 + b_1b_2$$

## Example 15

*The position vectors of P and Q are* **2i + j** *and* **–3i + 2j** *respectively. Find ∠POQ (Fig. 8.36).*

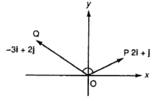



*Fig. 8.36*

From the definition, $\mathbf{a.b} = ab \cos \theta$

Then $\cos \theta = \frac{\mathbf{a.b}}{ab}$

i.e. **the cosine of the angle between two vectors is the scalar product divided by the product of their moduli.**

Hence $\cos \angle POQ = \frac{(2\mathbf{i} + \mathbf{j}).(-3\mathbf{i} + 2\mathbf{j})}{|2\mathbf{i} + \mathbf{j}| \times |-3\mathbf{i} + 2\mathbf{j}|} = \frac{-6+2}{\sqrt{5}\sqrt{13}} = -0.4961$

giving ∠POQ = 119.74°.

## Example 16

*The position vectors of A and B are* **2i – 3j** *and* **ti + 2j** *respectively.*

*(a) Find the value of t for which $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are perpendicular.*

*(b) If t = 4, find ∠AOB to the nearest degree.*

(a) If $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{OB}$, then their scalar product = 0.
So $(2\mathbf{i} - 3\mathbf{j}).(t\mathbf{i} + 2\mathbf{j}) = 2t - 6 = 0$ and $t = 3$.

(b) $\cos \angle AOB = \frac{(2\mathbf{i} - 3\mathbf{j}).(4\mathbf{i} + 2\mathbf{j})}{|2\mathbf{i} + 3\mathbf{j}| \times |4\mathbf{i} + 2\mathbf{j}|} = \frac{2}{\sqrt{13}\sqrt{20}} = 0.1240$
Hence ∠AOB = 83°.

### *Example 17*

*Find the relationship between p and q if the vectors* $\mathbf{a} = p\mathbf{i} + 3\mathbf{j}$ *and* $\mathbf{b} = 2\mathbf{i} + q\mathbf{j}$ *are at right angles. Given that* $q = -2$, *find* $|\mathbf{a} + \mathbf{b}|$.

To be at right angles, $(p\mathbf{i} + 3\mathbf{j}).(2\mathbf{i} + q\mathbf{j}) = 0$.
Then $2p + 3q = 0$.
If $q = -2$, then $p = 3$.
$|\mathbf{a} + \mathbf{b}| = |3\mathbf{i} + 3\mathbf{j} + 2\mathbf{i} - 2\mathbf{j}| = |5\mathbf{i} + \mathbf{j}| = \sqrt{26}$

### *Example 18*

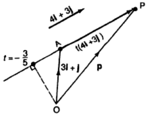
*A straight line passes through the point A whose position vector is* $3\mathbf{i} + \mathbf{j}$ *and is parallel to the vector* $4\mathbf{i} + 3\mathbf{j}$.

*(a) If the position vector of any point P on the line is* $\mathbf{p}$, *show that* $\mathbf{p} = 3\mathbf{i} + \mathbf{j} + t(4\mathbf{i} + 3\mathbf{j})$.

*(b) Find the value of t for which* $\overrightarrow{OP}$ *is perpendicular to the line.*

*(c) Hence find the distance of the origin from the line.*

*Fig. 8.37*




(a) In Fig. 8.37, A is the point with position vector $3\mathbf{i} + \mathbf{j}$.
$\overrightarrow{AP}$ is parallel to $4\mathbf{i} + 3\mathbf{j}$ so $\overrightarrow{AP} = t(4\mathbf{i} + 3\mathbf{j})$ where $t$ is any number.
Then $\overrightarrow{OP} = \mathbf{p} = \overrightarrow{OA} + \overrightarrow{AP} = 3\mathbf{i} + \mathbf{j} + t(4\mathbf{i} + 3\mathbf{j})$.

(b) $\overrightarrow{OP} = (3 + 4t)\mathbf{i} + (1 + 3t)\mathbf{j}$
If OP is perpendicular to $4\mathbf{i} + 3\mathbf{j}$ then $(3 + 4t) \times 4 + (1 + 3t) \times 3 = 0$.
This gives $12 + 16t + 3 + 9t = 0$ or $t = -\frac{3}{5}$.

(c) When $t = -\frac{3}{5}$, the position vector of $P = 3\mathbf{i} + \mathbf{j} - \frac{3}{5}(4\mathbf{i} + 3\mathbf{j}) = \frac{3\mathbf{i}}{5} - \frac{4\mathbf{j}}{5}$.

The distance from O to the line will then be $|OP| = \sqrt{\frac{9}{25} + \frac{16}{25}} = 1$.

## Exercise 8.5 *(Answers on page 626.)*

*For questions 1 – 7, take* $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = -\mathbf{i} + 2\mathbf{j}$, $\mathbf{c} = 3\mathbf{i} - 2\mathbf{j}$, $\mathbf{d} = -4\mathbf{i}$.

**1** Find the scalar products (a) **a.c**, (b) **b.d**, (c) **b.c**.

**2** Which two of the vectors are perpendicular?

**3** Evaluate $\mathbf{b}^2$.

**4** Find the angles between (a) **b** and **c**, (b) **a** and **d**.

**5** The vector $t\mathbf{i} + \mathbf{j}$ is perpendicular to $\mathbf{b} - \mathbf{a}$. Find the value of $t$.

**6** Find the angle between $\mathbf{a} + \mathbf{b}$ and $\mathbf{c} + \mathbf{d}$.

**7** Find the relation between $m$ and $n$ if $m\mathbf{a} + n\mathbf{b}$ is perpendicular to $\mathbf{c} - \mathbf{b}$.

**8** Given $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j}$, $\mathbf{b} = 4\mathbf{i} + \mathbf{j}$ and $\mathbf{c} = 2\mathbf{i} + 7\mathbf{j}$, verify that $\mathbf{a}.(\mathbf{b} + \mathbf{c}) = \mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{c}$.

**9** If $\mathbf{p} = 3\mathbf{i} + 4\mathbf{j}$, $\mathbf{q} = 2\mathbf{i} - 3\mathbf{j}$ and $\mathbf{r} = 5\mathbf{i} + \mathbf{j}$, show that $\mathbf{p}.(\mathbf{q} - \mathbf{r}) = \mathbf{p}.\mathbf{q} - \mathbf{p}.\mathbf{r}$ and find the angle between **p** and **q**.

**10** The position vectors of A and B are $4\mathbf{i} + 3\mathbf{j}$ and $7\mathbf{i} - \mathbf{j}$ respectively. Show that OA is perpendicular to AB and find $\angle$AOB.

**11** The position vectors of A, B and C are $3\mathbf{i} + \mathbf{j}$, $4\mathbf{i} + 3\mathbf{j}$ and $6\mathbf{i} + 2\mathbf{j}$ respectively.
(a) Show the positions of the points on a diagram.
(b) Find $\angle$B.
(c) Hence find the area of $\triangle$ABC.

**12** Two bodies are moving in a plane, one parallel to the vector $3\mathbf{i} - \mathbf{j}$, the other parallel to $-4\mathbf{i} + 2\mathbf{j}$. Find the angle between their paths.

**13** If the vectors $t\mathbf{i} + 2\mathbf{j}$ and $t\mathbf{i} - 8\mathbf{j}$ are perpendicular, find the values of $t$.

**14** If the vectors $2p\mathbf{i} - 3\mathbf{j}$ and $p\mathbf{i} + 6\mathbf{j}$ are perpendicular, find the values of $p$.

**15** A(2,3), B(−1,4) and C(5,−2) are three points. Evaluate $\overrightarrow{BA}.\overrightarrow{BC}$ and hence find $\angle$ABC.

**16** The position vectors of A, B and C are $2\mathbf{j}$, $3\mathbf{i} + 4\mathbf{j}$ and $5\mathbf{i}$ respectively. Find $\overrightarrow{BA}.\overrightarrow{BC}$ and hence find $\angle$ABC.

**17** OABC is a parallelogram with $\overrightarrow{OA} = 4\mathbf{i} + 2\mathbf{j}$ and $\overrightarrow{OC} = -6\mathbf{i} + 4\mathbf{j}$. P and Q are the midpoints of BC and AB respectively.
(a) Find the position vectors of P and Q.
(b) Evaluate $\overrightarrow{OP}.\overrightarrow{OQ}$ and hence find $\angle$POQ.

**18** OABC is a parallelogram where the position vectors of A and C are $3\mathbf{i} + 6\mathbf{j}$ and $-2\mathbf{i} + 4\mathbf{j}$ respectively. (a) Find the position vector of B. D is the midpoint of OC and E divides OA in the ratio 2:1. Find (b) the position vectors of D and E and (c) $\angle$BDE.

**19** ABCD is a quadrilateral where the position vectors of A, B, C and D are **a**, **b**, **c** and **d** respectively.
(a) State in terms of **a**, **b**, **c**, **d** (i) $\overrightarrow{AB}$, (ii) $\overrightarrow{CD}$, (iii) $\overrightarrow{AC}$, (iv) $\overrightarrow{BD}$.
(b) If $(\mathbf{c} - \mathbf{d}) = k(\mathbf{b} - \mathbf{a})$ where $k > 1$, what type of quadrilateral is ABCD?

(c) State in terms of **a, b, c, d**, the condition for the diagonals to be at right angles.
(d) If $|\mathbf{d} - \mathbf{b}|\,|\mathbf{c} - \mathbf{a}| = 2(\mathbf{d} - \mathbf{b}).(\mathbf{c} - \mathbf{a})$, what is the angle between the diagonals?

**20** The position vectors of A, B and C are $3\mathbf{i} + \mathbf{j}$, $-\mathbf{i} - 3\mathbf{j}$ and $5\mathbf{i}$ respectively. P is a point such that $\overrightarrow{AP} = k\overrightarrow{AB}$ where $k$ is any number.
(a) Find the position vector of P in terms of $k$, **i** and **j**.
(b) Find the value of $k$ if PC is perpendicular to AC.

**21** If $\mathbf{r}_1 = 3\mathbf{i}$, $\mathbf{r}_2 = \mathbf{i} + \mathbf{j}$ and $\mathbf{r}_3 = -\mathbf{i} - 3\mathbf{j}$, find the values of $t$ so that $t\mathbf{r}_1 + \mathbf{r}_2$ will be perpendicular to $t\mathbf{r}_2 + \mathbf{r}_3$.

**22** A and B have position vectors $3\mathbf{i} + 6\mathbf{j}$ and $6\mathbf{i} + 3\mathbf{j}$ respectively. C lies on OA where OC:CA = 1:2 and D lies on OB where OD:DB = 2:1.
(a) Find the position vectors of C and D.
(b) Find $\overrightarrow{CD}$ and $\overrightarrow{AB}$ in terms of **i** and **j**.
(c) Hence find the angle between $\overrightarrow{CD}$ and $\overrightarrow{AB}$.

**23** The position vectors of A and B are $4\mathbf{i} + \mathbf{j}$ and $\mathbf{i} + 7\mathbf{j}$ respectively.
(a) Find ∠AOB.
(b) C lies on AB where AC:CB = 2:1. Find the position vector of C.
(c) Hence find ∠AOC.

**24** The points A, B and C have position vectors $3\mathbf{i} + 3\mathbf{j}$, $8\mathbf{i} + 2\mathbf{j}$ and $\mu\mathbf{i} + 11\mathbf{j}$ respectively, where $\mu$ is a positive number. D lies on BC where BD:DC = 1:2.
(a) Find the position vector of D in term of $\mu$, **i** and **j**.
(b) Express $\overrightarrow{AD}$ in terms of $\mu$, **i** and **j**.
(c) If AD is perpendicular to BC, find the value of $\mu$.

**25** The position vectors of A, B and C are $3\mathbf{i} + 4\mathbf{j}$, $8\mathbf{i} - 6\mathbf{j}$ and $m\mathbf{i} + n\mathbf{j}$ respectively, where $m$ and $n$ are numbers.
(a) Evaluate $(3\mathbf{i} + 4\mathbf{j}).(m\mathbf{i} + n\mathbf{j})$.
(b) Find $|\overrightarrow{OA}|$ and $|\overrightarrow{OC}|$.
(c) Hence express cos ∠AOC and cos ∠BOC in terms of $m$ and $n$.
(d) If ∠AOC = ∠BOC, find the relation between $m$ and $n$.
(e) Hence find the equation of the line OC.

## SUMMARY

- Magnitude of vector $\mathbf{a} = |\mathbf{a}| = a$.
- If $\mathbf{a} = k\mathbf{b}$, where $k$ is a scalar (a number) $\neq 0$, then the vectors **a** and **b** are parallel and in the same direction if $k > 0$ but in opposite directions if $k < 0$.
  $|\mathbf{a}| = |k| \times |\mathbf{b}|$
- Conversely if **a** and **b** are parallel ($\mathbf{a} \neq 0$, $\mathbf{b} \neq 0$), then $\mathbf{a} = k\mathbf{b}$. P, Q and R are collinear if $\overrightarrow{PQ} = k\overrightarrow{QR}$ (or $\overrightarrow{PQ} = k\overrightarrow{PR}$) and conversely.

- If $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\mathbf{a} + \mathbf{b}$ is the diagonal OC of the parallelogram OACB.
- $\mathbf{a} - \mathbf{b}$ is the vector from the end of $\mathbf{b}$ to the end of $\mathbf{a}$,
  $\mathbf{b} - \mathbf{a}$ is the vector from the end of $\mathbf{a}$ to the end of $\mathbf{b}$,
  where $\mathbf{a}$, $\mathbf{b}$ start from the same point.
- If $m\mathbf{a} + n\mathbf{b} = p\mathbf{a} + q\mathbf{b}$ then $m = p$ and $n = q$.
- The position vector of A is the vector $\overrightarrow{OA}$ where O is the origin.
- $\mathbf{i}$, $\mathbf{j}$ are unit vectors in the directions of the positive coordinate axes.
  Column vector form: $\binom{a}{b} = a\mathbf{i} + b\mathbf{j}$
  If $\mathbf{r} = a\mathbf{i} + b\mathbf{j} = \binom{a}{b}$, then $|\mathbf{r}| = \sqrt{a^2 + b^2}$, $\tan\theta = \frac{b}{a}$ (check for the correct quadrant).
- The unit vector in the direction of $\mathbf{r}$ is $\hat{\mathbf{r}}$.
  If $\mathbf{r} = a\mathbf{i} + b\mathbf{j}$, then $\hat{\mathbf{r}} = \frac{a\mathbf{i} + b\mathbf{j}}{\sqrt{a^2 + b^2}}$.
- Scalar product of $\mathbf{a}$ and $\mathbf{b} = \mathbf{a}.\mathbf{b} = |\mathbf{a}| \times |\mathbf{b}| \cos\theta$ where $\theta$ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

  $\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$

  Commutative law: $\mathbf{a}.\mathbf{b} = \mathbf{b}.\mathbf{a}$
  Distributive law: $\mathbf{a}.(\mathbf{b} + \mathbf{c}) = \mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{c}$.
  If $\mathbf{a}$ and $\mathbf{b}$ are parallel, $\mathbf{a}.\mathbf{b} = ab$ (same direction), or $\mathbf{a}.\mathbf{b} = -ab$ (opposite directions).
  If $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, $\mathbf{a}.\mathbf{b} = 0$.

$$(a_1\mathbf{i} + b_1\mathbf{j}).(a_2\mathbf{i} + b_2\mathbf{j}) = \binom{a_1}{b_1}.\binom{a_2}{b_2} = a_1a_2 + b_1b_2.$$

## REVISION EXERCISE 8 *(Answers on page 626.)*

**A**

1 Given $\mathbf{a} = 2\mathbf{i} - 5\mathbf{j}$, $\mathbf{b} = -5\mathbf{i} - 12\mathbf{j}$ and $\mathbf{c} = m\mathbf{i} + n\mathbf{j}$, calculate
(a) $\mathbf{a}.\mathbf{b}$, (b) the angle between $\mathbf{a}$ and $\mathbf{b}$.
(c) If $\mathbf{a}.\mathbf{c} = \mathbf{b}.\mathbf{c}$, find the relation between $m$ and $n$.

2 The position vector of A relative to an origin O is $3\mathbf{i} + 5\mathbf{j}$. Given that $\overrightarrow{AB} = 8\mathbf{i} + 2\mathbf{j}$, evaluate $\overrightarrow{OA}.\overrightarrow{OB}$ and hence find angle AOB. (C)

3 The points A, B and C have position vectors $\mathbf{p} + \mathbf{q}$, $3\mathbf{p} - 2\mathbf{q}$ and $6\mathbf{p} + m\mathbf{q}$ relative to an origin O. Find the value of $m$ for which A, B and C are collinear. (C)

4 The position vectors of A and B relative to an origin O are $6\mathbf{i} + 4\mathbf{j}$ and $3\mathbf{i} + p\mathbf{j}$ respectively. Express $\overrightarrow{AO}.\overrightarrow{AB}$ in terms of $p$ and hence find (i) the value of $p$ for which AO is perpendicular to AB, (ii) the cosine of $\angle$OAB when $p = 6$. (C)

5 A, B and C are points with position vectors $4\mathbf{p} - \mathbf{q}$, $\mu(\mathbf{p} + \mathbf{q})$ and $\mathbf{p} + 2\mathbf{q}$ respectively, relative to an origin O. Obtain expressions for $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Given that B lies on AC, find the value of $\mu$. (C)

**6** (a) $\vec{OA}$ is perpendicular to $\binom{-4}{3}$ and $|\vec{OA}| = 15$. State $\vec{OA}$ in column vector form if A lies in the first quadrant.

(b) $|a\mathbf{i} + b\mathbf{j}| = 5$ and $a\mathbf{i} + b\mathbf{j}$ is perpendicular to $8\mathbf{i} - 6\mathbf{j}$. Find the value of $a$ and of $b$.

**7** Points A and B have position vectors **a** and **b** respectively relative to an origin O (Fig. 8.38). The point D is such that $\vec{OD} = p\vec{OA}$ and the point E is such that AE = $q$AB.

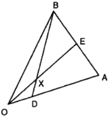



*Fig. 8.38*

The line segments BD and OE intersect at X. If OX = $\frac{2}{5}$OE and XB = $\frac{4}{5}$DB express $\vec{OX}$ and $\vec{XB}$ in terms of **a**, **b**, $p$ and $q$ and hence evaluate $p$ and $q$. (C)

**8** (a) Given that $\vec{OM} = \mathbf{i} + 3\mathbf{j}$ and $\vec{ON} = \mathbf{i} + \mathbf{j}$, evaluate $\vec{OM}.\vec{ON}$ and hence calculate $\angle$MON to the nearest degree.

(b) The position vectors, relative to an origin O, of two points S and T are 2**p** and 2**q** respectively. The point A lies on OS and is such that OA = AS. The point B lies on OT produced and is such that OT = 2TB. The lines ST and AB intersect at R. Given that $\vec{AR} = \lambda\,\vec{AB}$ and that $\vec{SR} = \mu\,\vec{ST}$, express $\vec{OR}$ (i) in terms of **p**, **q** and $\lambda$, (ii) in terms of **p**, **q**, $\mu$. Hence evaluate $\lambda$ and $\mu$ and express $\vec{OR}$ in terms of **p** and **q**. (C)

**9** The position vectors of A, B and C are **a**, **b** and **c** respectively relative to an origin O. Draw a diagram showing the positions of O, A, B and C given that (i) $\mathbf{a}.\mathbf{c} = 0$, (ii) $\mathbf{b} - \mathbf{a} = k(\mathbf{c} - \mathbf{a})$, (iii) $\mathbf{b}.(\mathbf{a} - \mathbf{c}) = 0$, (iv) $2|\mathbf{b}| = |\mathbf{a} - \mathbf{c}|$.
What is the position of B relative to A and C?

**10** The position vectors, relative to an origin O, of three points A, B and C are $2\mathbf{i} + 2\mathbf{j}$, $5\mathbf{i} + 11\mathbf{j}$ and $11\mathbf{i} + 9\mathbf{j}$ respectively.

(i) Given that $\vec{OB} = m\vec{OA} + n\vec{OC}$, where $m$ and $n$ are scalar constants, find the value of $m$ and of $n$.

(ii) Evaluate $\vec{AB}.\vec{BC}$ and state the deduction which can be made about $\angle$ABC.

(iii) Evaluate $\vec{AB}.\vec{AC}$ and **hence** find $\angle$BAC. (C)

**11** The position vectors of A and B are $\mathbf{i} + 3\mathbf{j}$ and $-2\mathbf{i} + \mathbf{j}$ respectively.

(a) Evaluate $\vec{OA}.\vec{OB}$ and hence find $\angle$AOB.
C is a point whose position vector is given by $\vec{OC} = \vec{OA} + t\vec{AB}$.

(b) Find the values of $t$ for which (i) OC is perpendicular to AB, (ii) $|\vec{OC}| = |\vec{AC}|$.

**12** The position vectors of points A and B relative to an origin O are **a** and **b** respectively. The point P is such that $\overrightarrow{OP} = 4\overrightarrow{OB}$. The midpoint of AB is the point Q. The point R is such that $\overrightarrow{OR} = \frac{8}{5}\overrightarrow{OQ}$.
Find, in terms of **a** and **b**, the vectors $\overrightarrow{OQ}$, $\overrightarrow{OR}$, $\overrightarrow{AR}$ and $\overrightarrow{RP}$. Hence show that R lies on AP and find the ratio AR:RP.
Given that the point S is such that $\overrightarrow{OS} = \mu\,\overrightarrow{OQ}$, find the value of $\mu$ such that PS is parallel to BA. (C)

**13** P and Q have position vectors $2t\mathbf{i} + (t+1)\mathbf{j}$ and $(t+1)\mathbf{i} - (t+2)\mathbf{j}$ respectively. If $|\overrightarrow{OP}| = |\overrightarrow{OQ}|$ show that $3t^2 - 4t - 4 = 0$ and hence find the possible values of $t$. For each one, calculate $\overrightarrow{OP}\,.\,\overrightarrow{OQ}$ and the angle POQ.

**14** OABC is a quadrilateral with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$.
If $(\mathbf{a} + \mathbf{c}).(\mathbf{a} - \mathbf{c}) = 0$, what type of quadrilateral is OABC?
If, in addition, $\mathbf{a}.\mathbf{c} = 0$ what is the quadrilateral?

**15** $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. C is the midpoint of AB and D divides OB in the ratio 2:1. AD and OC intersect at P.
(a) Taking $\overrightarrow{OP} = p\overrightarrow{OC}$ and $\overrightarrow{AP} = q\overrightarrow{AD}$, find two vector expressions for $\overrightarrow{OP}$ and hence find the ratio OP:PC.
(b) E divides OB in the ratio 1:2 and AE meets OC at Q. By a similar method, find the ratio OQ:QC.
(c) Hence find the ratio OQ:QP:PC.

**16** The position vectors of A, B and C are $-\mathbf{a} + 2\mathbf{b}$, $2\mathbf{a} + 3\mathbf{b}$ and $3\mathbf{a} + 5\mathbf{b}$ respectively. P divides BC in the ratio 3:4. Q lies on AB so that $AQ = \frac{2}{5}AB$. R lies on AC so that $\overrightarrow{CR} = 2\overrightarrow{AR}$. Find the position vectors of P, Q and R. Show that these points are collinear and state the ratio PQ:QR.

**B**

**17** The points A and B have position vectors $4\mathbf{i} + 3\mathbf{j}$ and $\mathbf{i} + t\mathbf{j}$ respectively.
If $\cos \angle AOB = \frac{2}{\sqrt{5}}$, find the values of $t$.

**18** The position vector **p** of a point P is given by $\mathbf{p} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$.
Find the equation of the curve on which P lies for all values of $\theta$.

**19** If $\mathbf{a}.\mathbf{b} = \mathbf{a}.\mathbf{c}$, show that **a** is perpendicular to $\mathbf{b} - \mathbf{c}$.

**20** In $\triangle$AOB, $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$. The altitudes BD and OE intersect at H and $\overrightarrow{OH} = \mathbf{h}$.
(a) State $\overrightarrow{BH}$ in terms of **h** and **b**.
(b) Show that $(\mathbf{h} - \mathbf{b}).\mathbf{a} = 0$ and that $\mathbf{h}.(\mathbf{b} - \mathbf{a}) = 0$ and hence deduce that $\mathbf{b}.(\mathbf{h} - \mathbf{a}) = 0$.
(c) Hence state a geometrical result about the altitudes of a triangle.

**21** If $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AC} = \mathbf{b}$, show that the area of the $\triangle$ABC is given by $\frac{1}{2}\sqrt{(ab)^2 - (\mathbf{a}.\mathbf{b})^2}$.
Hence find the area of $\triangle$ABC if the coordinates of A, B and C are (3,2), (–1,–1) and (5,–3) respectively.

**22** The points A, B and C have position vectors **a**, **b** and **c** respectively. L, M and N are the midpoints of AB, BC and CA respectively. G lies on CL so that CG = 2GL. Find the position vector of G. Show that A, G and M are collinear and state the ratio AG:GM. (Similarly B, G and N are collinear and BG:GN = 2:1. G is called the **centroid** of the triangle ABC.)