Trigonometry

7

TRIGONOMETRIC FUNCTIONS FOR A GENERAL ANGLE

The trigonometric functions **sine, cosine** and **tangent** of an angle θ were originally defined as ratios of the sides of a right-angled triangle, i.e. for a domain $0° \leq \theta \leq 90°$. We now extend the definition to deal with any angle (the **general** angle).

The actual *values* of sin θ, cos θ and tan θ for any given angle can be found directly using a calculator. To solve equations, however, we must know how to use these definitions inversely.

Suppose the arm OR (of unit length) in Fig.7.1 can rotate about O in an anticlockwise direction and makes an angle θ with the positive x–axis. We divide the complete revolution into 4 **quadrants** and take the positive y–axis at 90°. Let (x,y) be the coordinates of R. x and y will be positive or negative depending on which quadrant R lies in.

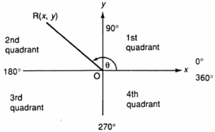


Fig.7.1

We define

$$\sin\theta = \frac{y\text{-coordinate of R}}{1}$$

$$\cos\theta = \frac{x\text{-coordinate of R}}{1}$$

$$\tan\theta = \frac{y\text{-coordinate of R}}{x\text{-coordinate of R}}$$

Note that both | sin θ | and | cos θ | are less than or equal to 1 as both | x | and | y | are less than or equal to 1, but that tan θ can have any value. In the first quadrant, where 0° ≤ θ ≤ 90°, each of these functions will be positive (Fig.7.2).

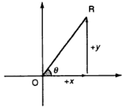


Fig. 7.2

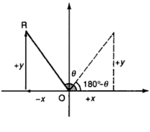
In the second quadrant (Fig.7.3), where 90° < θ ≤ 180°, the angle θ is linked to the corresponding angle 180° – θ in the first quadrant.

Fig.7.3

$\sin\theta = +y = \sin(180° - \theta)$

$\cos\theta = -x = -\cos(180° - \theta)$

$\tan\theta = \frac{+y}{-x} = -\tan(180° - \theta)$



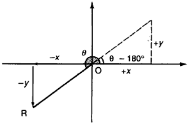
For the third quadrant (Fig.7.4), where 180° < θ ≤ 270°, the corresponding angle in the first quadrant is θ – 180°.

$\sin\theta = -y = -\sin(\theta - 180°)$

$\cos\theta = -x = -\cos(\theta - 180°)$

$\tan\theta = \frac{-y}{-x} = \tan(\theta - 180°)$

Fig.7.4



For the fourth quadrant (Fig.7.5), where $270° < \theta \le 360°$, the corresponding angle in the first quadrant is $360° - \theta$.

$\sin\theta = -y = -\sin(360° - \theta)$

$\cos\theta = +x = \cos(360° - \theta)$

$\tan\theta = \frac{-y}{x} = -\tan(360° - \theta)$

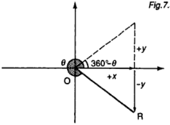


Fig.7.5

Summarizing:

SIN +	All +
$\sin\theta = \sin(180° - \theta)$ $\cos\theta = -\cos(180° - \theta)$ $\tan\theta = -\tan(180° - \theta)$	$\sin\theta$ $\cos\theta$ $\tan\theta$
2nd	1st
3rd	4th
$\sin\theta = -\sin(\theta - 180°)$ $\cos\theta = -\cos(\theta - 180°)$ $\tan\theta = \tan(\theta - 180°)$	$\sin\theta = -\sin(360° - \theta)$ $\cos\theta = \cos(360° - \theta)$ $\tan\theta = -\tan(360° - \theta)$
TAN +	COS +

Each function is positive (+) in the first quadrant and one other.
Each function is negative (–) in two quadrants.

Note on Special Angles 30°, 45°, 60°

As these angles are often used, it will be useful for future work to have their trigonometrical ratios in fractional form.

45°

In Fig.7.6(a), ABC is an isosceles right-angled triangle with AB = BC = 1. Hence $AC = \sqrt{2}$ and $\angle A = \angle C = 45°$.

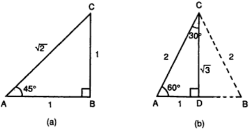


Fig.7.6

Then $$\sin 45° = \frac{1}{\sqrt{2}} = \cos 45°$$
$$\tan 45° = 1$$

30°, 60°

In Fig.7.6(b), ABC is an equilateral triangle with side 2.
CD is the perpendicular bisector of AB so AD = 1 and CD = $\sqrt{3}$.
$\angle A = 60°$ and $\angle ACD = 30°$.

Then $$\sin 60° = \cos 30° = \frac{\sqrt{3}}{2}$$
$$\sin 30° = \cos 60° = \frac{1}{2}$$
$$\tan 60° = \sqrt{3},\ \tan 30° = \frac{1}{\sqrt{3}}$$

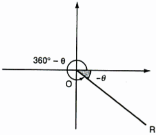
Using the special ratios above, the ratios for other angles related to 30°, 45° and 60° can be found in a similar form if required.
For example, $\cos 210° = -\cos(210° - 180°) = -\cos 30° = -\frac{\sqrt{3}}{2}$
Copy and complete this table:

θ	120°	135°	150°	210°	240°	300°	315°	330°
sin θ								
cos θ								
tan θ								

NEGATIVE ANGLES

If the arm OR rotates in a clockwise direction (Fig.7.7), it will describe a **negative** angle, $-\theta$. To find the value of a function of a negative angle, convert the angle to $360° - \theta$ or $2\pi - \theta$, if working in radians.

Fig.7.7



Thus $\sin(-30°) = \sin 330°$, $\tan(-\frac{\pi}{3}) = \tan(2\pi - \frac{\pi}{3}) = \tan(\frac{5\pi}{3})$ and so on.

BASIC TRIGONOMETRIC EQUATIONS

We apply the above trigonometric functions to the solution of basic trigonometric equations, i.e. equations in **one** function such as $\sin\theta = 0.44$, $\cos\theta = -0.78$ or $\tan\theta = 1.25$. As we shall see later, all other equations are reduced to one (or more) of these. **A basic equation will usually have two solutions for $0° \le \theta \le 360°$.**

To solve a basic equation, such as $\sin\theta = k$,

step 1 find the 1st quadrant angle α for which $\sin\alpha = |k|$;
step 2 find the quadrants in which θ will lie;
step 3 determine the corresponding angles for those quadrants.

Unless exact, angles in degrees are to be given to one decimal place.

Example 1

Solve (a) $\sin\theta = 0.57$, (b) $\sin\theta = -0.38$ for $0° \le \theta \le 360°$.

(a) If $\sin\alpha = 0.57$, then $\alpha = 34.75°$.
θ will lie in the 1st and 2nd quadrants (θ and $180° - \theta$)
Then $\theta = 34.75°$ or $180° - \theta = 34.75°$ i.e. $\theta = 145.25°$.
The solutions are $34.8°$ and $145.3°$.

(b) From $\sin\alpha = +0.38$, $\alpha = 22.33°$.
θ will lie in the 3rd and 4th quadrants.
Then $\theta - 180° = 22.3°$ or $360° - \theta = 22.3°$ giving $\theta = 202.3°$ and $337.7°$.

Solutions for the equations $\cos\theta = k$ and $\tan\theta = k$ are found in the same way.

Example 2

Solve (a) $\cos\theta = -0.3814$, (b) $\tan\theta = 1.25$ for $0° \le \theta \le 360°$.

(a) The 1st quadrant angle for $\cos\alpha = +0.3814$ is $67.58°$.
θ lies in the 2nd and 3rd quadrants.
Then $180° - \theta = 67.58°$ or $\theta - 180° = 67.58°$ giving $\theta = 112.4°$ and $247.6°$.

(b) The 1st quadrant angle for $\tan\theta = 1.25$ is $51.34°$.
θ lies in the 2nd and 3rd quadrants.
Then $\theta = 51.34°$ and $\theta - 180° = 51.34°$ i.e. $\theta = 231.34°$.
Hence the solutions are $\theta = 51.3°$ and $231.3°$.

Example 3

Solve the equation $3\cos^2\theta + 2\cos\theta = 0$ for $0° \le \theta \le 360°$.

The left hand side factorizes giving $\cos\theta(3\cos\theta + 2) = 0$ which separates into 2 basic equations:

$\cos\theta = 0$

and $3\cos\theta + 2 = 0$ which gives $\cos\theta = -\frac{2}{3} = -0.6667$.

Note: Do not divide through by the factor $\cos\theta$. This would lose the equation $\cos\theta = 0$. Never divide by a factor containing the unknown angle.

For $\cos\theta = 0$, $\theta = 90°$ or $270°$.

For $\cos\theta = -0.6667$, θ lies in the 2nd and 3rd quadrants.

The 1st quadrant angle is 48.19°.

Then $180° - \theta = 48.19°$ and $\theta - 180° = 48.19°$ giving $\theta = 131.8°$ and $\theta = 228.2°$.

Hence the solutions are 90°, 131.8°, 228.2° and 270°.

Example 4

For $0° \le \theta \le 360°$, solve $6\cos^2\theta + \cos\theta = 1$.

This is a quadratic equation in $\cos\theta$:

$$6\cos^2\theta + \cos\theta - 1 = 0$$

and so

$$(3\cos\theta - 1)(2\cos\theta + 1) = 0$$

which separates into $\cos\theta = 0.3333$ and $\cos\theta = -0.5$.

Verify that the solutions are $\theta = 70.5°$, 120°, 240° and 289.5°.

Example 5

Solve the equation $\sin(\theta - 30°) = 0.4$ for $0° \le \theta \le 360°$.

Write $\phi = \theta - 30°$.
Then $\sin\phi = 0.4$.
Solve for ϕ.
Verify that $\phi = 23.6°$ and 156.4°.
Then $\theta = 53.6°$ and 186.4°.

OTHER TRIGONOMETRIC FUNCTIONS

There are three other functions which are the reciprocals of the sine, cosine and tangent. They are

cosecant: $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$

secant: $\sec\theta = \dfrac{1}{\cos\theta}$

cotangent: $\cot\theta = \dfrac{1}{\tan\theta}$

Example 6

Solve (a) cosec $\theta = -1.58$, (b) $4 \cot \theta = \tan \theta$, for $0° \le \theta \le 360°$.

(a) Replace cosec θ by $\frac{1}{\sin \theta}$.

$\frac{1}{\sin \theta} = -1.58$ so $\sin \theta = -\frac{1}{1.58} = -0.6329$

Now verify that $\theta = 219.3°$ or $320.7°$.

(b) Replace $\cot \theta$ by $\frac{1}{\tan \theta}$.

Then $\frac{4}{\tan \theta} = \tan \theta$ i.e. $\tan^2 \theta = 4$.

So $\tan \theta = \pm 2$ (NB: don't forget the negative root)

Verify that the solutions of these equations are 63.4°, 116.6°, 243.4° and 296.6°.

Exercise 7.1 *(Answers on page 620.)*

1 Solve the following equations for $0° \le \theta \le 360°$:

(a) $\sin \theta = \frac{1}{3}$ (b) $\cos \theta = 0.762$ (c) $\tan \theta = 1.15$
(d) $\cos \theta = -0.35$ (e) $\sin \theta = -0.25$ (f) $\tan \theta = -0.81$
(g) $\sin \theta = -0.1178$ (h) $\sin \theta = -0.65$ (i) $\cos \theta = 0.23$
(j) $\tan \theta = -1.5$ (k) cosec $\theta = 1.75$ (l) $\cos \theta = -0.14$
(m) $\sec \theta = -1.15$ (n) $\cot \theta = 0.54$ (o) $\sec \theta = 2.07$

2 Solve the following equations for $0° \le \theta \le 360°$:

(a) $5 \sin^2 \theta = 2 \sin \theta$ (b) $9 \tan \theta = \cot \theta$
(c) $3 \tan^2 \theta + 5 \tan \theta = 2$ (d) $4 \cos^2 \theta + 3 \cos \theta = 0$
(e) $5 \sin^2 \theta = 2$ (f) $6 \sin^2 \theta + 7 \sin \theta + 2 = 0$
(g) $\cos(\theta + 20°) = -0.74$ (h) $\tan(\theta - 50°) = -1.7$
(i) $3 \sin^2 \theta = \sin \theta$ (j) $4 \sec^2 \theta = 5$
(k) $\cos^2 \theta = 0.6$ (l) $6 \sin^2 \theta = 2 + \sin \theta$
(m) $2 \sec^2 \theta = 3 - 5 \sec \theta$ (n) $\sec(\theta - 50°) = 2.15$
(o) $\sin(\theta + 60°) = -0.75$

3 Find θ for $0° \le \theta \le 360°$ if $3 \cos^2 \theta - 2 = 0$.

4 If $5 \tan \theta + 2 = 0$, find θ in the range $0° \le \theta \le 360°$.

5 Solve the equation $5 \cos \theta - 3 \sec \theta = 0$ for $0° \le \theta \le 360°$.

6 Find all the angles between 0° and 180° which satisfy the equations

(a) $\sin x = 0.45$ (b) $\cos y = -0.63$ (c) $\tan \theta = 2.15$

7 Find the values of

(a) $\sin(-30°)$ (b) $\cos(-\frac{\pi}{4})$ (c) $\tan(-200°)$ (d) $\sec(-150°)$
(e) $\cot(-300°)$ (f) $\sin(-\frac{4\pi}{3})$ (g) $\text{cosec}(-\frac{2\pi}{5})$

8 Show that

(a) $\sin(-\theta) = -\sin \theta$, (b) $\cos(-\theta) = \cos \theta$, (c) $\tan(-\theta) = -\tan \theta$.

9 Solve the equations

(a) $\sin(-\theta) = 0.35$,
(b) $\sin(-\theta) = -0.27$
(c) $\cos(-\theta) = -0.64$
(d) $\tan(-\theta) = 1.34$,

for $0° \leq \theta \leq 360°$.

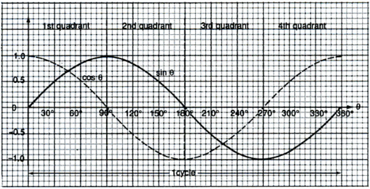
GRAPHS OF TRIGONOMETRIC FUNCTIONS

sin θ and cos θ

Complete the following table of values of sin θ and cos θ, taking a domain of 0° to 360° at 30° steps:

θ	0°	30°	60°	90°	120°	150°	180°	...	270°	...	360°
sin θ	0	0.5		1			0		−1		0
cos θ	1		0.5	0			−1				1

Plot these values on graph paper using scales of say 1 cm ≡ 30° on the θ-axis and 4 cm ≡ 1 unit on the function axis (Fig.7.8).



The graph shows **one cycle** of each function.

Fig.7.8

The sine curve has a maximum of 1 when θ = 90° and a minimum of −1 when θ = 270°. The cosine curve is identical to the sine curve but is shifted 90° to the left. This difference is called the *phase difference* between the two functions.

For angles greater than 360° or less than 0° the curves repeat themselves in successive cycles (Fig.7.9). Functions which repeat themselves like this are called **periodic** functions. The sine and cosine functions each have a **period** of 360° (or 2π). Hence

$$\sin(\theta + n360°) = \sin\theta \text{ or } \cos(\theta + 2n\pi) = \cos\theta$$

where n is any integer. This means that we can add or subtract 360° from any solution of $\sin\theta = k$ or $\cos\theta = k$ and obtain other solutions outside the domain $0° \leq \theta \leq 360°$.

For example, if the solutions of $\sin\theta = 0.5$ for $0° \leq \theta \leq 360°$ are 30° and 150°, then 30° + 360° = 390° and 150° − 360° = −210° are also solutions of the equation. These solutions are marked by dots on the graph of sin θ in Fig.7.9.

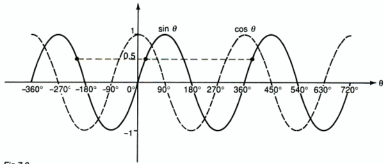


Fig.7.9

tan θ

Values of tan θ begin at 0 for θ = 0°, increase to 1 when θ = 45° and then increase rapidly as θ approaches 90°. tan 90° is undefined. Between 90° and 270° the function increases from large negative values through 0 to large positive values. The curve approaches the 90° and 270° axes but never reaches them. Hence the curve consists of 3 separate branches between 0° and 360° (Fig.7.10).

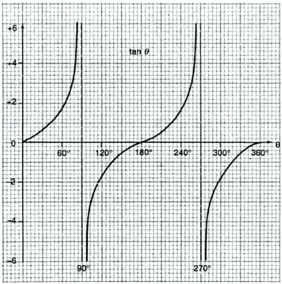


Fig.7.10

tan θ is also a periodic function but with a period of 180°. Hence tan(θ + $n\pi$) or tan(θ + n180°) = tan θ where n is an integer.

MULTIPLE ANGLE FUNCTIONS

Functions such as sin 2θ, cos $\frac{\theta}{2}$, etc. are multiple angle functions as 2θ, $\frac{\theta}{2}$ are multiples of θ.

Example 7

(a) Sketch the graph of y = *sin 2θ.*
(b) Solve the equation sin 2θ = 0.55 for 0° ≤ θ ≤ 360° and show the solutions on the graph.

(a) If the domain of θ is 0° to 360°, 2θ will take values from 0° to 720°. Hence the curve completes **two** cycles as θ increases from 0° to 360° (Fig.7.11).

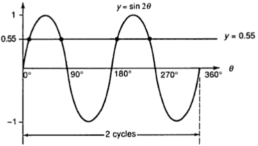


Fig.7.11

(b) For convenience, write 2θ = ø so sin ø = 0.55.
ø lies in the 1st and 2nd quadrants so ø = 33.37° or 180° − ø = 33.37°.
Hence ø = 33.37° or 146.63°.

But ø takes values from 0° to 720°, so we add 360° to each of these to obtain further solutions.

Then ø = 2θ = 33.37° or 146.63° or 393.37° or 506.63°.
Hence θ = 16.7° or 73.3° or 196.7° or 253.3°.

So we obtain 4 solutions, 2 for each cycle. These solutions are marked on the graph.

Note that all the solutions for 2θ must be obtained first before dividing by 2 to obtain the values of θ, which are then corrected to 1 decimal place.

Example 8

(a) Sketch the graph of y = cos $\frac{\theta}{2}$ for 0° ≤ θ ≤ 360°.
(b) Solve the equation cos $\frac{\theta}{2}$ = −0.16 for this domain.

(a) If the domain of θ is 0° to 360°, then $\frac{\theta}{2}$ will take values from 0° to 180° only. So the graph will be a half-cycle of the cosine curve (Fig. 7.12).

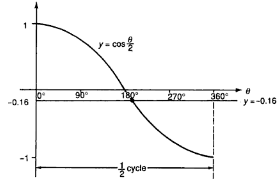


Fig.7.12

(b) Write $\frac{\theta}{2} = \phi$. Then $\cos \phi = -0.16$.
ϕ lies in the 2nd and 3rd quadrants.
Then $180° - \phi = 80.79°$ or $\phi - 180° = 80.79°$.
Hence $\phi = 99.21°$ or $260.79°$ and therefore $\theta = 198.4°$ or $521.6°$.

The second solution is outside the domain and is therefore discarded.
The only solution to the equation is $\theta = 198.4°$. This is to be expected as there is only a half cycle of the function.

Example 9

Solve the equation $5 \sin \frac{3\theta}{4} + 4 = 0$ for the domain $0° \le \theta \le 360°$.

Let $\frac{3\theta}{4} = \phi$. Then $\sin \phi = -\frac{4}{5} = -0.8$.
ϕ lies in the 3rd and 4th quadrants.
Then $\phi - 180° = 53.13°$ and $360° - \phi = 53.13°$.
Hence $\phi = 233.13°$ or $306.87°$.

If the domain of θ is 0° to 360°, then $\phi = \frac{3\theta}{4}$ takes values from 0° to 270°.

Hence the only solution is $\phi = 233.13°$ i.e. $\theta = \frac{4}{3} \times 233.13° = 310.8°$.
($\phi = 306.87°$ would give $\theta = 409.2°$).

Example 10

Solve $\cos(2\theta + 60°) = -0.15$ for $0° \le \theta \le 360°$.

Put $\phi = 2\theta + 60°$. Then $\cos \phi = -0.15$ giving $\phi = 98.63°$ and $261.37°$.
However if the domain of θ is 0° to 360°, then the domain of ϕ is 60° to 780°.
So we must add 360° to each of the above values.
Therefore $\phi = 2\theta + 60° = 98.63°$ or $261.37°$ or $458.63°$ or $621.37°$
and hence $\theta = 19.3°$ or $100.7°$ or $199.3°$ or $280.7°$.

Exercise 7.2 *(Answers on page 620.)*

1 Sketch the graphs of (a) $y = \sin 3\theta$, (b) $y = \cos 3\theta$ for $0° \le \theta \le 360°$. What is the period of each of these functions?

2 Sketch the graphs of (a) $y = \tan 2\theta$, (b) $y = \tan \frac{\theta}{2}$ for $0° \le \theta \le 360°$.

3 On the same diagram, sketch the graphs of $y = \sin 2\theta$ and $y = \cos \theta$ for $0° \le \theta \le 360°$. How many solutions of the equation $\sin 2\theta = \cos \theta$ are there in this domain?

4 Sketch on the same diagram, the graphs of $y = \sin \frac{\theta}{2}$ and $y = \cos \theta$ for $0° \le \theta \le 360°$. State the number of solutions which the equation $\sin \frac{\theta}{2} = \cos \theta$ will have in this domain.

5 On the same diagram, sketch the graphs of $y = \cos 3\theta$ and $y = \sin \frac{\theta}{2}$ for $0° \le \theta \le 360°$. State the number of solutions of the equation $\cos 3\theta = \sin \frac{\theta}{2}$ you would expect to obtain in this domain.

6 Solve, for $0° \le \theta \le 360°$, the following equations:

(a) $\sin 2\theta = 0.67$	(b) $\cos 3\theta = 0.58$
(c) $\tan \frac{\theta}{2} = 1.5$	(d) $\sin \frac{\theta}{3} = 0.17$
(e) $3 \cos 2\theta = 2$	(f) $\sec \frac{\theta}{2} = -1.7$
(g) $\sin \frac{\theta}{3} = -0.28$	(h) $3 \tan 2\theta + 1 = 0$
(i) $3 \sin \frac{2\theta}{3} = 2$	(j) $4 \cos \frac{3\theta}{2} + 3 = 0$
(k) $2 \operatorname{cosec} 2\theta + 3 = 0$	(l) $\cot \frac{\theta}{2} = 1.35$
(m) $\cos \frac{3\theta}{4} = \frac{3}{4}$	(n) $\tan 2\theta = -1$
(o) $3 \sin^2 2\theta + 2 \sin 2\theta = 1$	(p) $2 \cos^2 \frac{\theta}{2} = \cos \frac{\theta}{2}$
(q) $\sin 2\theta = -0.76$	(r) $\sec \frac{\theta}{2} = 1.88$
(s) $\cos 2\theta = -0.65$	(t) $\tan \frac{2\theta}{3} + 2 = 0$
(u) $5 \sin \frac{4\theta}{5} + 3 = 0$	(v) $2 \operatorname{cosec} \frac{\theta}{2} = 3$

7 For $0° \le \theta \le 360°$, solve the following

(a) $\sin(\frac{\theta}{3} + 20°) = 0.47$	(b) $\tan(2\theta - 60°) = 1.55$
(c) $\cos(\frac{\theta}{2}) = 0.75$	(d) $\sin(2\theta + 80°) = -0.54$
(e) $\sec^2(\frac{\theta}{3} - 50°) = 1.2$	

8 State the values of (a) $\sin(30° + n360°)$, (b) $\cos(n360° - 50°)$, (c) $\tan(45° + n180°)$ where n is an integer.

9 State the values of (a) $\sin(2n + 1)\pi$, (b) $\cos(6n - 1)\frac{\pi}{3}$, (c) $\tan(3n + 1)\frac{\pi}{3}$, where n is an integer.

10 Solve the equation $4 \cos^2 \frac{2\theta}{3} = 1$ for $0° \le \theta \le 360°$.

MODULUS OF TRIGONOMETRIC FUNCTIONS

$|\sin \theta|$ has the same meaning as $|x|$, i.e. it is the numerical value of $\sin \theta$. For example, $|\sin 300°| = |-0.866| = 0.866$, and so on.

Example 11

For $0° \leq \theta \leq 360°$, sketch separate graphs of (a) $y = 2 \sin \theta$, (b) $y = |2 \sin \theta|$, (c) $y = 1 + |\cos 2\theta|$, (d) $y = 1 - |\cos 2\theta|$.

(a) First sketch $y = \sin \theta$ (Fig.7.13)
For $y = 2 \sin \theta$, each value of $y = \sin \theta$ is doubled to give the graph of $y = 2 \sin \theta$.

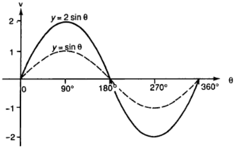


Fig. 7.13

(b) As we did earlier, we reflect the negative part of $y = 2 \sin \theta$ in the θ-axis to obtain $y = |2 \sin \theta|$ (Fig.7.14).

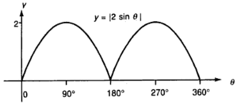


Fig. 7.14

(c) First sketch $y = \cos 2\theta$ (Fig.7.15) which has two cycles. Now reflect the negative part in the θ-axis to obtain $y = |\cos 2\theta|$.
This curve is now shifted up through 1 unit to obtain $y = 1 + |\cos 2\theta|$.

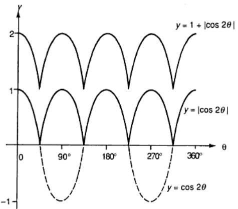


Fig.7.15

(d) Start by sketching $y = |\cos 2\theta|$ as in part (c) (Fig.7.16).
Then obtain $y = -|\cos 2\theta|$ by reflection of the whole curve in the θ-axis.
This is now shifted up through 1 unit to obtain $y = 1 - |\cos 2\theta|$ (Fig.7.17).

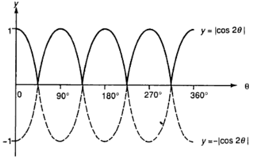


Fig.7.16

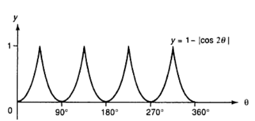


Fig.7.17

Example 12

Sketch on the same diagram, the graphs of $y = |2 \sin x|$ and $y = \frac{x}{\pi}$ for $0 \le x \le 2\pi$. Hence state the number of solutions of the equations $|2\pi \sin x| = x$ and $2\pi \sin x = x$ for $0 \le x \le 2\pi$.

We have to work in radians here as $y = \frac{x}{\pi}$ is a linear equation.
($y = \frac{x}{180°}$ is not meaningful.)

The graph of $y = 2 \sin x$ is drawn and then $y = |2 \sin x|$ (Fig.7.18).

To draw the line $y = \frac{x}{\pi}$ we take the points $x = 0$, $y = 0$ and $x = 2\pi$, $y = 2$.

The equation $|2\pi \sin x| = x$ is the same as $|2 \sin x| = \frac{x}{\pi}$ as π is positive. The solutions will occur at the intersections of the curve and the line, giving 4 solutions at the points marked O, A, B and C.

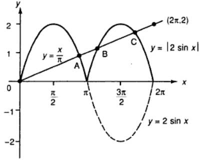


Fig.7.18

The equation $2\pi \sin x = x$ is the same as $2 \sin x = \frac{x}{\pi}$. So we look for the intersections of the original curve $y = 2 \sin x$ with the line, which reduces the number of solutions to 2 (points O and A).

Example 13

Sketch on the same diagram the graphs of $y = |2 \cos x|$ and $3y = x$ for the domain $0 \le x \le 2\pi$. Hence state the number of solutions in this domain of the equation $6|\cos x| = x$.

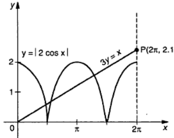


Fig. 7.19

Fig. 7.19 shows the graphs. The graph of $3y = x$ i.e. $y = \frac{x}{3}$ is the line OP, where O is the origin and P is the point $(2\pi,\ \frac{2\pi}{3} \approx 2.1)$. There are 3 solutions to the equation $|\,2\cos x\,| = \frac{x}{3}$ i.e. $6|\cos x\,| = x$.

Exercise 7.3 *(Answers on page 621.)*

1 State the values of (a) $|\sin 200°|$, (b) $|\cos \frac{2\pi}{3}|$, (c) $\sin |-200°|$, (d) $|\tan \frac{5\pi}{6}|$.

2 By sketching the graph of $y = \sin 2\theta$ for $0° \le \theta \le 360°$, find how many solutions the equation $\sin 2\theta = k$ will have in this interval, where $0 < k < 1$. How many solutions will the equation $|\sin 2\theta| = k$ have in the same interval?

3 Sketch the graphs of $y = |\cos\theta|$ and $y = |\cos\theta| - 1$ for $0° \le \theta \le 360°$.

4 On the same diagram, sketch the graphs of $y = |\sin\theta|$ and $y = |\cos\theta|$ for $0° \le \theta \le 360°$. How many solutions will the equation $|\sin\theta| = |\cos\theta|$ have in this interval?

5 Sketch the graphs of $y = 1 + 2\sin\theta$ and $y = |1 + 2\sin\theta|$ for $0° \le \theta \le 360°$. On another diagram, sketch the graph of $y = 1 + |2\sin\theta|$.

6 On the same diagram, for $0° \le \theta \le 360°$, sketch the graphs of $y = 2\cos\theta$ and $y = |2\cos\theta|$. Now add the graph of $y = 1 - |2\cos\theta|$.

7 On the same diagram, sketch the graphs of $y = |2\cos x|$ and $y = \frac{x}{2\pi}$ for $0 \le x \le 2\pi$. Hence state the number of solutions of the equations $|4\pi\cos x| = x$ and $4\pi\cos x = x$ for $0 \le x \le 2\pi$.

8 Sketch the graph of $y = |\tan\theta|$ for $0° \le \theta \le 360°$.

9 Sketch on the same diagram the graphs of $y = |\cos 2x|$ and $2y = x$ for the domain $0 \le x \le \pi$. Hence state the number of solutions in this domain of the equation $|2\cos 2x| = x$.

10 For $0 \le x \le 2\pi$, sketch the graphs of $y = |\cos x|$ and $y = \sin 2x$ on the same axes. State the number of solutions of the equation $\sin 2x = |\cos x|$ in this interval.

11 Sketch the graphs of $y = |\sin 3x|$ and $2\pi y = x$ for $0 < x \leq 2\pi$. How many solutions do the equations $2\pi \sin 3x = x$ and $|2\pi \sin 3x| = x$ have in this interval?

12 On the same diagram, sketch the graphs of $y = |\sin x - 1|$ and $y = 2 \cos x$ for $0 \leq x \leq 2\pi$. Hence find the number of solutions of the equation $2 \cos x = |\sin x - 1|$ in this interval.

IDENTITIES

We have defined earlier, for an angle θ, $\sin \theta = y$, $\cos \theta = x$ and $\tan \theta = \frac{y}{x}$ where (x,y) were the coordinates of R and OR = 1 unit (*Fig. 7.19*).

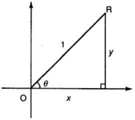


Fig. 7.20

Then $\tan \theta \equiv \dfrac{\sin \theta}{\cos \theta}$ (i)

This is an **identity** which is true for all values of θ. So we use the symbol ≡ meaning 'identical to' or 'equivalent to'. In any expression, tan θ could be replaced by $\dfrac{\sin \theta}{\cos \theta}$ or vice-versa.

As $\cot \theta = \dfrac{1}{\tan \theta}$, then $\cot \theta \equiv \dfrac{\cos \theta}{\sin \theta}$ (ii)

From Fig.7.19, $x^2 + y^2 = 1$ for all values of x and y.

Hence $\mathbf{\sin^2 \theta + \cos^2 \theta \equiv 1}$ (iii)

[*Note*: $\sin^2 \theta$ means $(\sin \theta)^2$]

and $\mathbf{\sin^2 \theta \equiv 1 - \cos^2 \theta}$ (iv)

and $\mathbf{\cos^2 \theta \equiv 1 - \sin^2 \theta}$ (v)

Taking identity (iii), divide both sides by $\cos^2 \theta$:

then $\dfrac{\sin^2 \theta}{\cos^2 \theta} + 1 \equiv \dfrac{1}{\cos^2 \theta}$

i.e. $\mathbf{\tan^2 \theta + 1 \equiv \sec^2 \theta}$ (vi)

Dividing both sides of identity (iii) by $\sin^2 \theta$:

then $1 + \dfrac{\cos^2 \theta}{\sin^2 \theta} \equiv \dfrac{1}{\sin^2 \theta}$

i.e. $\mathbf{1 + \cot^2 \theta \equiv \operatorname{cosec}^2 \theta}$ (vii)

Summarizing:

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta} \qquad \cot\theta \equiv \frac{\cos\theta}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta \equiv 1$$

$$\sin^2\theta \equiv 1 - \cos^2\theta \qquad \cos^2\theta \equiv 1 - \sin^2\theta$$

$$\tan^2\theta + 1 \equiv \sec^2\theta \qquad \cot^2\theta + 1 \equiv \operatorname{cosec}^2\theta$$

These identities are used to transform trigonometric expressions into another form.

Example 14

Prove that $\cot\theta + \tan\theta \equiv \operatorname{cosec}\theta \sec\theta$.

We take one side and convert it to the expression on the other side. It is usually easier to start with the side which is more complicated or which involves sums of functions. This gives more scope for manipulation.

Taking the left hand side (LHS):

$$\begin{aligned}\cot\theta + \tan\theta &\equiv \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}\\ &\equiv \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}\\ &\equiv \frac{1}{\sin\theta\cos\theta}\\ &\equiv \frac{1}{\sin\theta} \times \frac{1}{\cos\theta}\\ &\equiv \operatorname{cosec}\theta\sec\theta\end{aligned}$$

If we start with the RHS, then

$$\frac{1}{\sin\theta} \times \frac{1}{\cos\theta} \equiv \frac{1}{\sin\theta\cos\theta}$$

but it is not obvious that we should now replace 1 by $\sin^2\theta + \cos^2\theta$. Do this and then divide the numerator by $\sin\theta\cos\theta$ to complete the proof.

Example 15

Show that $\frac{1}{1+\sin\theta} + \frac{1}{1-\sin\theta} \equiv 2\sec^2\theta$

We take the more complicated LHS.

Then

$$\begin{aligned}\frac{1}{1+\sin\theta} + \frac{1}{1-\sin\theta} &\equiv \frac{1-\sin\theta+1+\sin\theta}{(1+\sin\theta)(1-\sin\theta)}\\ &\equiv \frac{2}{1-\sin^2\theta}\\ &\equiv \frac{2}{\cos^2\theta} \equiv 2\sec^2\theta\end{aligned}$$

Example 16

Prove that $\tan^2\theta \equiv \sin^2\theta(1 + \tan^2\theta)$

$$\text{RHS} \equiv \sin^2\theta\left(1 + \frac{\sin^2\theta}{\cos^2\theta}\right)$$

$$\equiv \sin^2\theta\left(\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}\right)$$

$$\equiv \sin^2\theta\left(\frac{1}{\cos^2\theta}\right) \equiv \tan^2\theta$$

Exercise 7.4

Prove the following identities:

1 $\sin\theta \cot\theta \equiv \cos\theta$

2 $(1 + \tan^2\theta)\cos^2\theta \equiv 1$

3 $(1 + \tan^2\theta)(1 - \sin^2\theta) \equiv 1$

4 $\cos^2\theta - \sin^2\theta \equiv 1 - 2\sin^2\theta$

5 $\sec\theta - \cos\theta \equiv \sin\theta\tan\theta$

6 $\cot^2\theta\,(1 - \cos^2\theta) \equiv \cos^2\theta$

7 $\dfrac{1}{\cos^2\theta} - \dfrac{1}{\cot^2\theta} \equiv 1$

8 $\dfrac{\cot\theta}{\tan\theta} + 1 \equiv \text{cosec}^2\,\theta$

9 $\tan^2\theta - \sin^2\theta \equiv \sin^4\theta\sec^2\theta$

10 $(\sin\theta + \cos\theta)(\tan\theta + \cot\theta) \equiv \sec\theta + \text{cosec}\,\theta$

11 $\sin^4\theta - \cos^4\theta \equiv 1 - 2\cos^2\theta$

12 $(\cos\theta + \sin\theta)^2 + (\cos\theta - \sin\theta)^2 \equiv 2$

13 $\dfrac{1 - \tan^2\theta}{\cot^2\theta - 1} \equiv \tan^2\theta$

14 $\sec\theta + \tan\theta \equiv \dfrac{1}{\sec\theta - \tan\theta}$

15 $\sec^4\theta - \sec^2\theta \equiv \tan^2\theta + \tan^4\theta$

16 $(\text{cosec}\,\theta - \cot\theta)^2 \equiv \dfrac{1 - \cos\theta}{1 + \cos\theta}$

EQUATIONS WITH MORE THAN ONE FUNCTION

Further types of trigonometrical equations can be solved using the identities we have just learnt. Some methods of solution are now shown. The object is to reduce the equation to one function.

Example 17

Solve the equation 3 cos θ + 2 sin θ = 0 for 0° ≤ θ ≤ 360°.

The equation contains two functions but if we divide throughout by cos θ, this will be reduced to one function.

Then $3 + 2\frac{\sin\theta}{\cos\theta} = 0$ or $\tan\theta = -1.5$.

Now solve this basic equation.
Verify that the solutions are 123.7° and 303.7°.

Example 18

Solve the equation 2 sin θ = tan θ for 0° ≤ θ ≤ 360°.
Illustrate the solutions graphically.

Rewrite the equation as $2\sin\theta = \frac{\sin\theta}{\cos\theta}$

i.e. $2\sin\theta\cos\theta - \sin\theta = 0$
or $\sin\theta(2\cos\theta - 1) = 0$.

This can be separated into two basic equations $\sin\theta = 0$ and $2\cos\theta - 1 = 0$
i.e. $\cos\theta = 0.5$.

The solutions of $\sin\theta = 0$ are 0°, 180° and 360°.
The solutions of $\cos\theta = 0.5$ are 60° and 300°.
Hence the solutions are 0°, 60°, 180°, 300° and 360°.
The graphs of $y = 2\sin\theta$ and $y = \tan\theta$ are shown in Fig. 7.21, with the positions of the solutions marked.

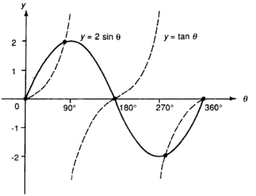


Fig. 7.21

Example 19

Solve $3 \sin \theta + 5 \cot \theta = \text{cosec } \theta$ for $0° \leq \theta \leq 360°$.

This involves three functions. Reduce this to two by replacing $\cot \theta$ and $\text{cosec } \theta$.

Then $3 \sin \theta + 5\frac{\cos \theta}{\sin \theta} = \frac{1}{\sin \theta}$.

Now remove the fractions: $3 \sin^2 \theta + 5 \cos \theta = 1$

We can now reduce to one function by replacing $\sin^2 \theta$ by $1 - \cos^2 \theta$.

Then $3(1 - \cos^2 \theta) + 5 \cos \theta = 1$ or $3 \cos^2 \theta - 5 \cos \theta - 2 = 0$.

This is a quadratic in $\cos \theta$ and gives $(3 \cos \theta + 1)(\cos \theta - 2) = 0$.

We now have two basic equations:

$$\cos \theta = -\tfrac{1}{3} \text{ which gives } \theta = 109.47° \text{ or } 250.53°,$$

and $\cos \theta = 2$ which has no solution.

Hence, the solutions are $\theta = 109.5°$ and $250.5°$.

Example 20

Solve the equation $4 \text{ cosec}^2 \theta - 7 = 4 \cot \theta$ for $0° \leq \theta \leq 180°$.

If we replace $\text{cosec}^2 \theta$ by $1 + \cot^2 \theta$, we shall have an equation in $\cot \theta$ only.

Then $4(1 + \cot^2 \theta) - 7 = 4 \cot \theta$ i.e. $4 \cot^2 \theta - 4 \cot \theta - 3 = 0$.

This is a quadratic in $\cot \theta$ and gives $(2 \cot \theta - 3)(2 \cot \theta + 1) = 0$ leading to the basic equations $\cot \theta = 1.5$ and $\cot \theta = -0.5$.

Hence $\tan \theta = 0.6667$ and $\tan \theta = -2$.

Now solve these but note that the domain is $0°$ to $180°$.

The only solutions are therefore $\theta = 33.7°$ and $116.6°$.

Exercise 7.5 *(Answers on page 622.)*

Solve the following equations for $0° \leq \theta \leq 360°$:

1 $8 \cot \theta = 3 \sin \theta$

2 $\sin \theta + 4 \cos^2 \theta = 1$

3 $8 \sin \theta = 3 \cos^2 \theta$

4 $2 \sec^2 \theta = 3 - \tan^2 \theta$

5 $\cot \theta + \tan \theta = 2 \sec \theta$

6 $\tan \theta + 3 \cot \theta = 4$

7 $\cot^2 \theta + 3 \text{ cosec}^2 \theta = 5$

8 $3(\sec \theta - \tan \theta) = 2 \cos \theta$

9 $2 \cot^2 \theta + 11 = 9 \text{ cosec } \theta$

10 $3 \sin^2 \theta = 1 + \cos \theta$

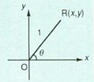
11 $5 \cos \theta - \sec \theta = 4$

12 $3 \cot 2\theta + 2 \sin 2\theta = 0$

SUMMARY

- If θ is any angle, $\sin\theta = y$, $\cos\theta = x$ and $\tan\theta = \frac{y}{x}$ where (x,y) are the coordinates of R and OR = 1 (Fig. 7.22).

Fig. 7.22



SIN +	All +
$\sin\theta = \sin(180° - \theta)$ $\cos\theta = -\cos(180° - \theta)$ $\tan\theta = -\tan(180° - \theta)$	$\sin\theta$ $\cos\theta$ $\tan\theta$
2nd	1st
3rd	4th
$\sin\theta = -\sin(\theta - 180°)$ $\cos\theta = -\cos(\theta - 180°)$ $\tan\theta = \tan(\theta - 180°)$	$\sin\theta = -\sin(360° - \theta)$ $\cos\theta = \cos(360° - \theta)$ $\tan\theta = -\tan(360° - \theta)$
TAN +	COS +

- To solve a basic equation such as $\sin\theta = k$:
  (1) find the angle α in the 1st quadrant such that $\sin\alpha = |k|$;
  (2) find the quadrants in which θ will lie;
  (3) determine the corresponding angles in these quadrants and solve for θ. A basic equation will usually have 2 solutions in the interval 0° to 360°.

- $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$ $\sec\theta = \frac{1}{\cos\theta}$ $\cot\theta = \frac{1}{\tan\theta}$

- Graphs of sin, cos, tan (Fig. 7.23).

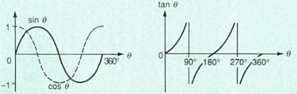


Fig. 7.23

sin and cos have a period of 360°:
$\sin(n360° + \theta) = \sin\theta$, $\cos(n360° + \theta) = \cos\theta$, where n is an integer.

tan has a period of 180°: $\tan(n180° + \theta) = \tan\theta$.

- For equations with a multiple angle $k\theta$, solve for $k\theta$ first and then derive the values of θ.

- Identities

$\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$ $\qquad$ $\cot\theta \equiv \dfrac{\cos\theta}{\sin\theta}$

$$\sin^2\theta + \cos^2\theta \equiv 1$$

$\sin^2\theta \equiv 1 - \cos^2\theta$ $\qquad$ $\cos^2\theta \equiv 1 - \sin^2\theta$

$\tan^2\theta + 1 \equiv \sec^2\theta$ $\qquad$ $\cot^2\theta + 1 \equiv \operatorname{cosec}^2\theta$

To solve equations with more than one function, use the above identities to reduce to one function.

REVISION EXERCISE 7 *(Answers on page 623.)*

A

1 Find all the angles between 0° and 360° which satisfy the equations
(a) $\cot 2x = -\frac{1}{2}$, (b) $2\sin y = 3\cos y$.

2 Sketch on the same diagram, for $0 \le x \le 2\pi$, the graph of $y = 2\cos x - 1$ and the graph of $y = \sin 2x$. Hence state the number of solutions in this interval of the equation $2\cos x - 1 = \sin 2x$. **(C)**

3 Sketch the graph of (a) $y = |\cos x|$, (b) $y = |\cos x| - 1$ and (c) $y = 1 - |\cos x|$ for values of x between 0 and 2π.

4 Prove the identity $\sec x - \cos x \equiv \sin x \tan x$.

5 Find all the angles between 0° and 180° which satisfy the equations
(a) $\cos \frac{2}{3}x = \frac{2}{3}$, (b) $3\cot y - 4\cos y = 0$,
(c) $3\sec^2 z = 7 + 4\tan z$. **(C)**

6 Solve for $0° \le \theta \le 360°$, the equations
(a) $\operatorname{cosec} 2\theta = 3$ (b) $4\cot\theta = 5\cos\theta$
(c) $10\sin^2\theta + 31\cos\theta = 13$.

7 Prove the identity $\dfrac{1+\sin x}{\cos x} + \dfrac{\cos x}{1+\sin x} \equiv 2\sec x$. **(C)**

8 On the same diagram, sketch the graphs of $y = 1 + \cos x$ and $y = |\sin x|$ for $0 \le x \le 2\pi$. Hence state the number of solutions of the equation $1 + \cos x = |\sin x|$ in this interval.

9 Find all the angles between 0° and 180° which satisfy the equations
(a) $\tan(x + 70°) = 1$,
(b) $8\sin y + 3\cos y = 0$,
(c) $3\sin^2\theta + 5\sin\theta\cos\theta - 2\cos^2\theta = 0$.

10 Sketch on the same diagram, the graphs of $y = |2\cos x|$ and $y = \frac{4x}{3\pi}$ for $0 \le x \le 2\pi$. State, for the range $0 \le x \le 2\pi$, the number of solutions of (i) $|3\pi\cos x| = 2x$, (ii) $3\pi\cos x = 2x$. **(C)**

11 State the range of $y = 2 - |\cos x|$ for the domain $0 \le x \le \frac{3\pi}{2}$.

12 On the same diagram, sketch the graphs of $y = \sin 2x$ and $y = \sin \frac{x}{2}$. for $0 \le x \le 2\pi$. Hence state the number of solutions of the equation $\sin 2x = \sin \frac{x}{2}$ in that interval. What would be the number of solutions of $| \sin 2x | = \sin \frac{x}{2}$?

13 For the domain $0° \le \theta \le 360°$, solve
(a) $\sin \theta + \cos \theta \cot \theta = 2$,
(b) $6 \cot^2 \theta = 1 + 4 \operatorname{cosec}^2 \theta$.

B

14 Solve the equation $\sin \theta = 4 \sin^3 \theta$ for $0° \le \theta \le 360°$.

15 Solve, for $0° \le \theta \le 360°$, the equations
(a) $8 \sin^2 \theta = \operatorname{cosec} \theta$,
(b) $4 \cos^2 \theta = 9 - 2 \sec^2 \theta$.

16 Sketch the graphs of $y = | 2 \sin x |$ and $y = | \frac{x}{\pi} - 1 |$ for $0 \le x \le 2\pi$. How many solutions are there of the equation $| 2\pi \sin x | = | x - \pi |$ in this interval?

17 A segment ACB in a circle is cut off by the chord AB where $\angle AOB = \theta$ radians (O is the centre). If the area of this segment is $\frac{1}{4}$ of the area of the circle, show that $\theta - \sin \theta = \frac{\pi}{2}$.
Draw the graphs of $y = \sin \theta$ and $y = \theta - \frac{\pi}{2}$ for $0 \le \theta \le \pi$, taking scales of 4 cm for $\frac{\pi}{2}$ on the x-axis and 4 cm per unit on the y-axis. (Take $\pi = 3.14$). Hence find an approximate solution of the equation $\theta - \sin \theta = \frac{\pi}{2}$.

18 In Fig. 7.24, ACB is a semicircle of radius r, centre O and $\angle ABC = \theta°$.
(a) Using the identity $2 \sin \theta \cos \theta = \sin 2\theta$, show that the area of the shaded region is $r^2 (\frac{\pi}{2} - \sin 2\theta)$.
(b) State in terms of r, the maximum and minimum possible values of this area and the corresponding values of θ.
(c) Find the values of θ for which the area of the shaded region equals $\frac{1}{2}$ the area of the semicircle.

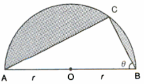


Fig. 7.24

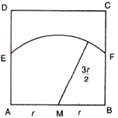


Fig. 7.25

19 A goat is tied to one end of a rope of length $\frac{3r}{2}$, the other end being fixed to the midpoint M of the side AB of a square field ABCD of side $2r$ (Fig. 7.25).
(a) Find, in radians, $\angle EMF$.
(b) Find in terms of r the area ABFE.
(c) Calculate what percentage of the area of the field the goat can cover.

20 In Fig. 7.26, OA and OB are two radii of a circle centre O where angle BOA = θ radians. The tangent to the circle at A meets OB produced at C. If the area of the sector OAB is twice the area of the shaded region, show that $2 \tan \theta = 3\theta$. By drawing the graphs of $y = \tan \theta$ and $y = \frac{3\theta}{2}$ for a suitable domain, or otherwise, find the approximate value of θ. (Otherwise, a solution could be found by trial and error using a calculator in radian mode. Test values of θ to make $\tan \theta - 1.5\theta$ reasonably small.)

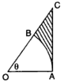


Fig. 7.26