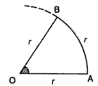
Radians, Arcs and Sectors

6

The practical unit of measurement for angles is the degree (°) which is $\frac{1}{360}$ th of a complete revolution. The number 360 comes from Babylonian times but it is an arbitrary choice. There is another system of measurement called **circular** or **radian** measure which is more suitable for further mathematics, particularly in Calculus. This system does not depend on the choice of any particular number.

RADIAN MEASURE

Fig. 6.1a



In a circle of radius r, centre O, we take an arc AB also of length r (Fig. 6.1a). Then the angle AOB is the *unit of radian measurement*, one **radian**.
In Fig. 6.1b, for example, arc PQ = $\frac{1}{2}r$ so ∠POQ = $\frac{1}{2}$ radian. If arc PR = $2r$, then ∠POR is 2 radians, and so on. If the arc is kr then the angle subtended is k radians. Note that the size of 1 radian does not depend on the length of r or on any arbitrary number.
So we define 1 radian thus:

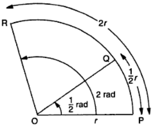


Fig. 6.1b

One radian is the angle made by an arc of length equal to the radius.

θ radians is sometimes written θ rad or θ^r or θ^c but normally just as θ. So we write sin θ meaning sin (θ radians). If degree measure is used, the degree symbol $\circ$ **must** be written.

Now the circumference of a circle of radius r has length $2\pi r$ (Fig.6.2).

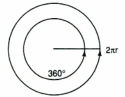


Fig. 6.2

So a complete revolution is 2π radians = 360°.

Therefore π rad = 180°

Hence $\frac{\pi}{2}$ rad = 90°, $\frac{\pi}{4}$ rad = 45°, etc.

x rad $= \left(\frac{180x}{\pi}\right)^\circ$ and $x^\circ = \frac{\pi x}{180}$ rad.

As π rad = 180°, then 1 rad $= \left(\frac{180}{\pi}\right)^\circ \approx 57.3°$.

This value cannot be found exactly as π is an irrational number. Usually radian measures are left as multiples of π, for example $\frac{3}{4}\pi$.

Tables or calculators may be used if necessary for conversion. When we use a calculator to work with trigonometrical ratios (sine, cosine and tangent) involving radians, it is convenient to put the calculator in the 'radian' mode. The input and output of angles will then be in radians.

Example 1

Convert (a) 36° to radian measure and (b) $\frac{5\pi}{6}$ *to degree measure.*

(a) 180° = π rad so 36° = $\frac{\pi}{180} \times 36 = \frac{\pi}{5}$ rad.

(b) π rad = 180° so $\frac{5\pi}{6}$ rad = $\frac{5\pi}{6} \times \frac{180°}{\pi}$ = 150°.

Example 2

Find the value of (a) sin 0.4 (b) tan 1.5.

Put the calculator in 'radian' mode and key in the appropriate function.

(a) $\sin 0.4 = 0.389$

(b) $\tan 1.5 = 14.1$

Example 3

Find the value of θ *(in radians) for* $0 \le \theta \le \frac{\pi}{2}$ *if*
(a) $\cos\theta = 0.5$ *(b)* $\tan\theta = 0.5$

Again put the calculator in the 'radian' mode.

(a) $\cos\theta = 0.5$

$\theta = \cos^{-1} 0.5 = 1.05$ rad

(b) $\tan\theta = 0.5$

$\theta = \tan^{-1} 0.5 = 0.46$ rad

Exercise 6.1 *(Answers on page 619.)*

1 Convert the following radians to degree measure:

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{10}$ (c) $\frac{2\pi}{3}$ (d) 4π (e) $\frac{\pi}{6}$ (f) $\frac{\pi}{9}$

(g) $\frac{3\pi}{4}$ (h) $\frac{11\pi}{6}$ (i) $\frac{5\pi}{4}$ (j) $\frac{\pi}{8}$ (k) 2 (l) 1.5

2 Convert the following to radian measure as a multiple of π:

(a) 30° (b) 135° (c) 270° (d) 540° (e) 105°

(f) 40° (g) 200° (h) $22\frac{1}{2}$° (i) 400° (j) 75°

3 Find the value of

(a) $\sin\frac{\pi}{6}$ (b) $\cos\frac{\pi}{3}$ (c) $\tan\frac{\pi}{4}$ (d) $\cos\frac{3\pi}{4}$

(e) $\sin\frac{\pi}{2}$ (f) $\sin 2$ (g) $\cos 0.5$

4 Find the value of θ (in radians) for $0 \le \theta \le \frac{\pi}{2}$ if

(a) $\sin\theta = 0.5$ (b) $\cos\theta = 0.6$

(c) $\tan\theta = 1.5$ (d) $\cos\theta = 0.25$

5 Find the value of $\theta - \sin\theta$ if $\theta = 0.75$ rad.

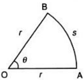
6 Using a calculator, investigate the value of $\frac{\sin\theta}{\theta}$ when θ is small.
(Take $\theta = 0.5, 0.3, 0.1, 0.05, 0.01$ for example).

LENGTH OF AN ARC

In Fig.6.3, the arc AB is of length s in a circle with centre O and radius r. The arc subtends an angle of θ radians ($\angle AOB$) at the centre.

As we saw above, an arc of length kr subtends an angle of k radians. Here the arc length is s so $s = kr$ and $k = \frac{s}{r} = \theta$.

Fig. 6.3



So $$s = r\theta; \quad \theta = \frac{s}{r}$$

This formula is only valid if θ is in radians.

If $\angle AOB = \theta°$, then $s = \pi r \times \frac{\theta}{180}$. The formula is simpler in radian measure.

Example 4

In a circle of radius 8 cm, find
(a) the length of the arc which subtends an angle of $\frac{3\pi}{4}$ radians at the centre,
(b) the angle subtended by an arc of length 6 cm.

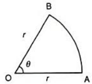
(a) $s = r\theta = 8 \times \frac{3\pi}{4} = 6\pi$ cm ($\approx$ 18.8 cm)

(b) From the formula, $\theta = \frac{s}{r} = \frac{6}{8} = 0.75$ rad ($= 0.75 \times \frac{180}{\pi} \approx 43.0°$)

AREA OF A SECTOR OF A CIRCLE

In Fig.6.4, AOB is a sector of angle θ rad in a circle with centre O and radius r. $\angle AOB = \theta$ rad.

Fig. 6.4



The area of the sector will be proportional to θ.

Hence $\frac{\text{area of sector AOB}}{\text{area of circle}} = \frac{\theta}{2\pi}$ and

area of sector $= \frac{\theta}{2\pi} \times \pi r^2 = \frac{1}{2}r^2\theta$.

Area of sector of angle $\theta = \frac{1}{2}r^2\theta$

Once again, this formula is only valid if θ is in radians.

Example 5

In Fig.6.5, O is the centre of a circle of radius r and $\angle AOB = \theta$. State
(a) the area of sector AOB
(b) the area of $\triangle AOB$.
Hence deduce the area of the segment which is shaded.
(c) Find the difference in length between the arc AB and the chord AB.

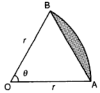


Fig. 6.5

(a) Area of sector = $\frac{1}{2}r^2\theta$

(b) Area of $\triangle$AOB = $\frac{1}{2}r^2 \sin\theta$ (using the formula: area of $\triangle = \frac{1}{2}bc \sin A$)

Hence the area of the segment = $\frac{1}{2}r^2\theta - \frac{1}{2}r^2 \sin\theta = \frac{1}{2}r^2(\theta - \sin\theta)$.

(c) Lehgth of arc AB = $r\theta$

The length of the chord AB is found from the isosceles $\triangle$AOB (Fig.6.6), where AD is the perpendicular bisector of AB.

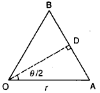
$\angle$DOA = $\frac{\theta}{2}$

Then AD = $r \sin\frac{\theta}{2}$ and AB = $2r \sin\frac{\theta}{2}$.

Hence the difference in length between arc AB and chord AB

$= r\theta - 2r \sin\frac{\theta}{2} = r(\theta - 2\sin\frac{\theta}{2})$.

Fig. 6.6

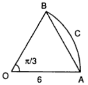


Example 6

O is the centre of a circle of radius 6 cm. AOB is a sector of angle $\frac{\pi}{3}$ (Fig.6.7). Find

(a) the area of sector AOB,
(b) the area of segment ABC,
(c) the difference in length between arc AB and chord AB.

Fig. 6.7



(a) Area of sector = $\frac{1}{2}r^2\theta = \frac{1}{2} \times 36 \times \frac{\pi}{3}$
$= 6\pi$ ($\approx$ 18.8 cm²)

(b) Area of segment = area of sector − area of $\triangle$
$= 6\pi - \frac{1}{2} \times 6^2 \times \sin\frac{\pi}{3}$
$= 6\pi - 18 \times 0.866 = 3.26$ cm²

(c) Difference in lengths = $r\theta - 2r\sin\frac{\theta}{2}$
$= 6 \times \frac{\pi}{3} - 12 \sin\frac{\pi}{6}$
$= 2\pi - 6$ (as $\sin\frac{\pi}{6} = 0.5$) $\approx$ 0.28 cm

Example 7

In Fig.6.8, OACB is a sector of a circle centre O and radius 5 cm. AB = 8 cm. Find
(a) θ in radians,
(b) the length of the arc ACB.

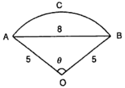


Fig. 6.8

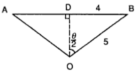


Fig. 6.9

(a) If OD bisects ∠AOB (Fig.6.9)
then D is the midpoint of AB and BD = 4 cm.
Then $\sin \frac{\theta}{2} = \frac{4}{5}$ and $\frac{\theta}{2} \approx 0.927$ rad.
$\theta \approx 1.85$ rad

(b) Length of the arc = $5 \times 1.85 = 9.25$ cm

Example 8

A circular disc, centre O and radius 30 cm, rests on two vertical supports AB, CD, each 20 cm tall and 45 cm apart (Fig. 6.10). Calculate, correct to 3 significant figures
(a) ∠AOC in radians,
(b) the height of the lowest point of arc AC above BD,
(c) the fraction of the area of the disc that lies above the level of AC.

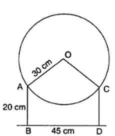


Fig. 6.10

Take ∠AOC = θ.

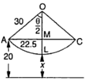


Fig. 6.11

(a) In Fig. 6.11, OM bisects ∠AOC, AM = 22.5, OA = 30 and ∠AOM = $\frac{\theta}{2}$.
$OM^2 = 30^2 - 22.5^2$ giving OM = 19.8 cm.
$\sin\frac{\theta}{2} = \frac{22.5}{30}$ so $\frac{\theta}{2} = 0.848$ rad and θ = 1.70 rad.

(b) L is the lowest point of AC. OL = 30 and OM = 19.8 so ML = 10.2.
Then ML + x = 20 so x = 9.8 cm.

(c) Area of the disc = $\pi r^2 = 900\pi$ cm².
Area of segment ALC = $\frac{1}{2} \times 30^2 \times \theta - \frac{1}{2} \times 30^2 \times \sin\theta$
$= 450(1.70 - \sin 1.70) = 318.8$ cm²
Then area above AC = $900\pi - 318.8$ and fraction of area above AC
$= \frac{900\pi - 318.8}{900\pi} = 0.887$.

Example 9

The area of the sector OAB (Fig.6.12) is 150 cm². Calculate
(a) θ (in radians),
(b) the length of arc AB.
(c) If this sector is folded up to form a cone, what is the radius of the cone?

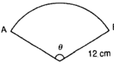


Fig.6.12

(a) Area of sector = $\frac{1}{2}r^2\theta = 150$
Then $\frac{1}{2} \times 144\theta = 150$ and θ = 2.083 rad.

(b) Length of arc = $r\theta = 12 \times 2.083 = 25.0$ cm

(c) When folded up, the arc AB becomes a circle of radius, say R (Fig.6.13).
Then $2\pi R = 24.96$ and $R = 3.97$ cm.

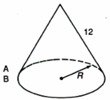


Fig.6.13

Example 10

A sector of a circle has radius r and angle θ. Find the value of θ correct to 3 significant figures if the perimeter of the sector equals half the circumference of the circle.

The perimeter of the sector $= r + r + r\theta = r(2 + \theta)$.

Then $r(2 + \theta) = \pi r$ and $2 + \theta = \pi$ giving $\theta = \pi - 2 \approx 1.14$ rad.

Example 11

A sector of angle θ in a circle of radius r cm has an area of 5 cm² and its perimeter is 9 cm. Find the values of r and θ.

Area $= \frac{1}{2}r^2\theta = 5$ so $r^2\theta = 10$ (i)

Perimeter $= r + r + r\theta = 2r + r\theta = 9$ (ii)

We solve these two equations.

From (i), $\theta = \frac{10}{r^2}$.

Substitute in (ii), $2r + r\frac{10}{r^2} = 9$ which gives $2r^2 + 10 = 9r$ or $2r^2 - 9r + 10 = 0$.

Hence $(2r - 5)(r - 2) = 0$ and $r = 2\frac{1}{2}$ or 2 cm.

From (i), the corresponding values of θ are 1.6 or 2.5 rad.

Exercise 6.2 *(Answers on page 619.)*

Where the answer is not exact, 3-figure accuracy is sufficient.

1 The length of an arc in a circle of radius 5 cm is 6 cm. Find the angle subtended at the centre.

2 An arc of length 5 cm is drawn with radius 3 cm. What angle does it subtend at the centre?

3 The area of a sector of a circle is 9 cm^2. If its radius is 6 cm find the angle of the sector and the length of its arc.

4 The area of a sector of a circle is 15 cm^2 and the length of its arc is 3 cm. Calculate (a) the radius of the sector and (b) its angle.

5 Find the missing values in the following table for sectors:

	Radius (cm)	$\angle$ of sector (radians)	Length of arc (cm)	Area of sector (cm^2)
(a)	4	1.3		
(b)		0.8	8	
(c)	8			16
(d)			3	18
(e)		0.4		7.2

6 A sector of a circle of radius 4 cm has an angle of 1.2 radians. Calculate
(a) the area of the sector,
(b) its perimeter.

7 In Fig. 6.14, DOBC is a semicircle, centre O and radius 6 cm. AC is perpendicular to DOB where AB is 2 cm.
Calculate
(a) the length of AC,
(b) $\angle COA$ in radians and
(c) the perimeter of the shaded region.
(d) Express the area of the shaded region as a percentage of the area of the semicircle.

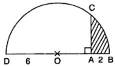


Fig. 6.14

8 If the area of a sector is 6.4 cm^2 and its angle is 0.8 radians, calculate the radius of the sector.

9 A chord AB is 8 cm long in a circle of radius 5 cm. Calculate
(a) the angle it subtends at the centre of the circle,
(b) the length of the shorter arc AB.

10 The perimeter of a sector is 128 cm and its area is 960 cm^2.
Find the possible values of the radius of the sector and its angle.

11 A wheel of radius 0.6 m rotates on its axis at a rate of 4.5 radians per second. Calculate the speed with which a point on its rim is moving.

12 A disc is rotating at $33\frac{1}{3}$ revolutions per minute.
(a) At what rate, in radians per second, is it rotating?
(b) At what speed, in metres per second, is a point on the rim moving, if the radius of the disc is 15 cm?

13 Fig.6.15 shows a cross–section through a tunnel, which is part of a circle of radius 5 m. The width AB of the floor is 8 m. Calculate
(a) the angle subtended at the centre of the circle by the chord AB,
(b) the length of the arc ACB.

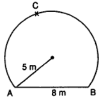


Fig. 6.15

14 Fig.6.16 represents the action of a windscreen wiper of a car. It rotates about O and travels from AB to A'B'and back. Calculate
(a) the area AA'B'B swept clear,
(b) the perimeter of this area.

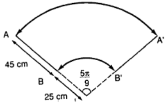


Fig. 6.16

15 A cylindrical barrel floats in water (Fig. 6.17). The diameter of the barrel is 120 cm and its highest point P is 80 cm above the water level AB.
(a) Calculate $\angle AOB$ in radians, where O is the centre of the circular face.
(b) What fraction of the volume of the barrel is below the water line?

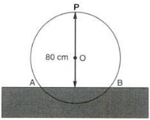


Fig. 6.17

16 In a circle centre O, AOB and COD are two concentric sectors as shown in Fig.6.18. The lengths of the arcs AB and DC are 2.8 cm and 2 cm respectively and AD = 2 cm. Calculate
(a) the length of OC,
(b) ∠AOB in radians,
(c) the area of ABCD.

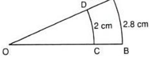
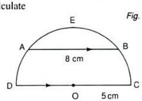


Fig. 6.18

17 In Fig.6.19, the chord AB, of length 8 cm, is parallel to the diameter DOC of the semicircle with centre O and radius 5 cm. Calculate
(a) ∠AOB in radians,
(b) the area of the segment ABE,
(c) what fraction the area of the segment ABE is of the area of the semicircle.

Fig. 6.19



18 A wheel of radius 20 cm rolls without slipping on level ground. A point P on the rim is in the position P_1 at the start (Fig.6.20). When the centre of the wheel has moved through 50 cm, P is now in the position P_2. Calculate the angle (in radians) through which the wheel has turned.

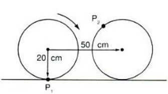


Fig. 6.20

19 A piece of wire, 10 cm long, is formed into the shape of a sector of a circle of radius r cm and angle θ radians.

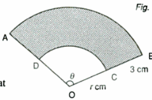
(a) Show that $\theta = \frac{10 - 2r}{r}$.

(b) Show also that the area A cm^2 of the sector is given by $A = 5r - r^2$.

(c) If $4 \le A \le 6$ and $\theta \le 3$, find the limits within which r must lie.

20 In Fig.6.21, O is the centre of the circle containing the sector OAB. DC is a parallel arc and BC = 3 cm.

Fig. 6.21



If OC = r cm and $\angle$AOB = θ rad, show that

(a) the shaded area = $\frac{\theta}{2}(6r + 9)$ cm^2,

(b) the perimeter of the shaded area equals $6 + \theta(2r + 3)$ cm.

(c) Given that the shaded area is three-quarters of the area of the sector OAB, calculate the value of r.

(d) If, however, the total perimeter of the shaded area equals the total perimeter of the sector OAB, find the value of θ.

SUMMARY

- 1 radian is the angle subtended by an arc of length equal to the radius.
- π rad = 180°
- Length s of an arc of radius r subtending an angle θ is $s = r\theta$ (θ in radians).
- Area A of a sector of radius r and angle θ is $A = \frac{1}{2}r^2\theta$ (θ in radians)

REVISION EXERCISE 6 *(Answers on page 620.)*

Where the answer is not exact, 3-figure accuracy is sufficient.

A

1 In Fig.6.22, ADB is a semicircle with centre O and radius 20 cm. DC is perpendicular to AB where C is the midpoint of OB. Calculate

Fig. 6.22

(a) $\angle$DOC in radians,

(b) the area of the shaded region.

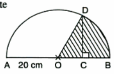


Fig. 6.22

2 Fig.6.23 shows a circle, centre O, radius 10 cm. The tangent to the circle at A meets OB produced at T. Given that the area of the triangle OAT is 60 cm^2 calculate the area of the sector OAB. (C)

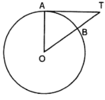


Fig. 6.23

3 In Fig.6.24, O is the centre of the sector OAB. CD is another arc, with centre O and radius r cm. DB = 2 cm. If the area of ABDC is one-third the area of the sector OCD, find the value of r.

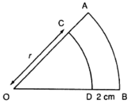


Fig. 6.24

4 In Fig.6.25, ADC is an arc of a circle, centre O, radius r and $\angle AOC = 2\theta$ radians. ABC is a semicircle on AC as diameter. Show that $AC = 2r \sin \theta$.

Find expressions, in terms of r and θ, for the areas of
(i) the sector OADC,
(ii) the segment ADC,
(iii) the shaded region. (C)

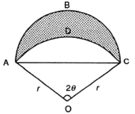


Fig. 6.25

5 OBD is a sector of a circle with centre O and radius 6 cm. $\angle BOD = \frac{2\pi}{5}$
A is a point on OB where OA = 2 cm and C is a point on OD such that OC = 4 cm. Find the area of the region bounded by BA, AC, CD and the arc BD.

6 Fig.6.26 shows a circle, centre O, radius 5 cm and two tangents TA and TB, each of length 8 cm. Calculate
(i) ∠AOB,
(ii) the length of the arc APB,
(iii) the area of the shaded region. (C)

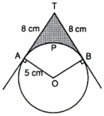


Fig. 6.26

7 Fig.6.27 shows part of a circle centre O of radius 6 cm.
(i) Calculate the area of sector BOC when θ = 0.8 radians.
(ii) Find the value of θ in radians for which the arc length BC is equal to the sum of the arc length CA and the diameter AB. (C)

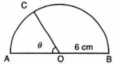


Fig. 6.27

8 In Fig.6.28, OAB is an equilateral triangle of side 10 cm. The arc ADB is drawn with centre O. A semicircle is drawn on AB as diameter. Find the area of the shaded region.

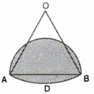


Fig.6.28

9 Fig.6.29 shows the circular cross-section of a uniform log of radius 40 cm floating in water. The points A and B are on the surface and the highest point X is 8 cm above the surface. Show that ∠AOB is approximately 1.29 radians. Calculate
(i) the length of the arc AXB,
(ii) the area of the cross-section below the surface,
(iii) the percentage of the volume of the log below the surface. (C)

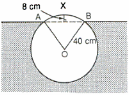


Fig.6.29

10 Fig.6.30 shows two arcs, AB and CD, of concentric circles, centre O. The radii OA and OC are 11 cm and 14 cm respectively and ∠AOB = θ radians.
Express in terms of θ the area of
(i) sector AOB,
(ii) the shaded region ABCD.

Given that the area of the shaded region ABCD is 30 cm², calculate
(iii) the value of θ,
(iv) the perimeter of the shaded region ABCD. (C)

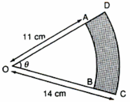


Fig. 6.30

11 Fig. 6.31 shows a semi-circle ABC, with centre O and radius 4 m, such that angle BOC = 90°.
Given that CD is an arc of a circle, centre B, calculate
(a) the length of the arc CD,
(b) the area of the shaded region. (C)

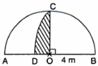


Fig. 6.31

B

12 In Fig.6.32, ABCD is a square of side 4 cm. Equal arcs AE and EB are drawn with radius 4 cm and centres B and A respectively. Calculate the area of the shaded region.

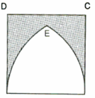


Fig. 6.32

13 The two circles in Fig.6.33 have centres A and B and radii 5 cm and 12 cm respectively. AB = 13 cm. Calculate the area of the shaded region.

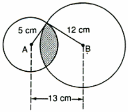


Fig. 6.33

14 A sector of a circle radius r has a total perimeter of 12 cm. If its area is A cm^2, show that $A = 6r - r^2$. Hence find the value of r for which A is a maximum and the corresponding value of the angle of the sector in radians.

15 In Fig.6.34, the sector OAB has centre O and radius 6 cm and $\angle AOB = \frac{\pi}{3}$ radians. OC is the bisector of $\angle AOB$ and P is the midpoint of OC. An arc DE of a circle is drawn with centre P to meet OA and OB at D and E respectively.

(a) Find the size of $\angle OPD$.

(b) Calculate the area of the shaded region.

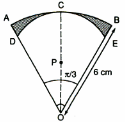


Fig. 6.34

16 In Fig.6.35, ABCD is a rhombus of side x and $\angle A = \theta$ radians. Arcs each of radius $\frac{x}{3}$ are drawn with centres A, B, C and D. If the shaded area is half the area of the rhombus, show that $\sin \theta = \frac{2\pi}{9}$ and find the two possible values of θ.

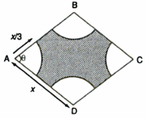


Fig. 6.35

17 A, B and C are three points in that order on the circumference of a circle radius 5 cm. The chords AB and BC have lengths 8 cm and 4 cm respectively. Find the ratio of the areas of the minor segments on AB and BC.