The Quadratic Function

4

You have solved quadratic equations such as $x^2 - 4x - 5 = 0$ in previous work. In this Chapter we study the **quadratic function**

$$\mathbf{f}: x \longmapsto ax^2 + bx + c \quad (a \neq 0)$$

First we review some essential techniques for solving quadratic equations. These will always give two solutions or **roots**, though sometimes they may be equal.

SOLVING QUADRATIC EQUATIONS

I By factorization

This is the simplest method if it is possible. For example $x^2 - 4x - 5 = 0$ gives $(x - 5)(x + 1) = 0$ so $x = 5$ or -1. However, certain quadratic equations, like $x^2 - 4x - 4 = 0$ for example, cannot be factorized.

It is useful to remember that the **equation with roots** α and β is

$$(x - \alpha)(x - \beta) = 0.$$

II By completing the square

To solve $x^2 - 4x - 4 = 0$, we can complete the square i.e. we make the $x^2 - 4x$ part into a square.

Rewrite $x^2 - 4x$ as $(x - 2)^2 - 4$. (Check by expanding this.)

Then $x^2 - 4x - 4 = 0$ becomes $(x - 2)^2 - 8 = 0$ or $(x - 2)^2 = 8$.

Now take the square root of each side: $x - 2 = \pm\sqrt{8}$ and $x = 2 \pm \sqrt{8}$ giving $x = 4.83$ or -0.83 (correct to 2 decimal places).

III By formula

We can derive a formula for the roots of any equation as follows.

$$ax^2 + bx + c = 0$$

so $\quad x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

Then $\quad x^2 + \frac{b}{a}x = -\frac{c}{a}$

Completing the square: $\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} = -\frac{c}{a}$

So $\quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$

and then $\quad x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$

giving

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{D}}{2a}$$

where $D = b^2 - 4ac$. D is called the *discriminant.* You will find out why later.

When using the formula note carefully that it begins with $-b$ and that the denominator is $2a$.

Note: The formula is the preferred method but it is essential to know the technique of completing the square for later use.

Example 1

Solve $2x^2 - 3x - 1 = 0$.

Check that the left hand side does not factorize.
Using the formula, $a = 2$, $b = -3$, $c = -1$.

Then $x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-1)}}{2(2)} = \frac{3 \pm \sqrt{17}}{4}$

giving $x = 1.78$ or -0.28 (2 decimal places).

Example 2

Solve $2x^2 - 3x + 4 = 0$.

$$x = \frac{3 \pm \sqrt{9 - 4(2)(4)}}{4} = \frac{3 \pm \sqrt{-23}}{4}$$

But -23 has no (real) square root. Hence the equation has no real roots. We shall see the significance of this later. Such an equation is said to have **complex** roots. We shall not however use complex numbers in our work.

GRAPH OF THE QUADRATIC FUNCTION $f(x) = ax^2 + bx + c$

As you will have noticed in drawing such graphs, the graph of a quadratic function, $y = ax^2 + bx + c$, has a characteristic shape. It is a curve called a **parabola** (Fig.4.1).

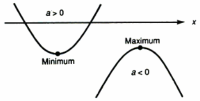


Fig.4.1

When $a > 0$, as in $2x^2 - 3x - 1$, the parabola has a **minimum** value at the bottom of the curve.

When $a < 0$, as in $1 - x - 2x^2$, the graph has a **maximum** value at the top of the curve.

The position of the curve relative to the x-axis depends on the type of the roots of the equation $f(x) = 0$. These roots are the values of x where the curve meets the x-axis.

TYPES OF ROOTS OF $ax^2 + bx + c = 0$

The roots are given by $x = \frac{-b \pm \sqrt{D}}{2a}$ where $D = b^2 - 4ac$.

I If D is negative ($D < 0$ i.e. $b^2 < 4ac$), then there is no value of $\sqrt{D}$. The equation has **no real roots** and the curve does not meet the x-axis (Fig.4.2).

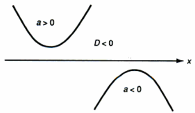


Fig.4.2

Example 3

What type of roots does the equation $5x^2 - 3x + 1 = 0$ have?

Using the formula, $a = 5$, $b = -3$, $c = 1$.

Then $D = (-3)^2 - 4(5)(1) = -11$.

As $D < 0$, the equation has no real roots.

II If D is positive ($D > 0$ i.e. $b^2 > 4ac$), then $\sqrt{D}$ has two values. The equation has **two different real roots** and the curve meets the x-axis at two points (Fig.4.3).

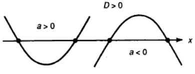


Fig.4.3

Example 4

For what values of p will the equation $x^2 + px + 9 = 0$ have two real roots?

Using the formula, $a = 1$, $b = p$, $c = 9$.

Then $D = p^2 - 36$.

For real roots, D must be > 0.

So $p^2 - 36 > 0$ i.e. $p^2 > 36$.

It follows that p must be numerically greater than 6, i.e. $p > 6$ or $p < -6$. (We could also write this as $|p| > 6$).

If D is a perfect square, $\sqrt{D}$ will be an integer. Then the roots will be *rational numbers*, i.e. fractions and whole numbers.

Example 5

What type of roots does the equation $2x^2 + 3x - 5 = 0$ have?

$D = 3^2 - 4(2)(-5) = 49$

As D is positive, the equation has two different real roots.

The roots are $\frac{-3 \pm 7}{4} = 1$ or $-\frac{5}{2}$.

The equation could have been solved by factorization.

III If $D = 0$ (i.e. $b^2 = 4ac$), then $x = \frac{-b}{2a}$. This means that the **roots are equal** (also called repeated or coincident roots). The curve *touches* the x-axis with the two roots merging into one (Fig.4.4).

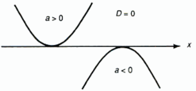


Fig.4.4

Example 6

(a) *For what values of k will the x-axis be a tangent to the curve* $y = kx^2 + (1 + k)x + k$?

(b) *With these values, find the equations of the curves.*

(a) On the x-axis, $y = 0$. So the roots of $kx^2 + (1 + k)x + k = 0$ must be equal if the x-axis is to be a tangent. Then $b^2 = 4ac$ where $a = k$, $b = 1 + k$ and $c = k$.

Therefore $(1 + k)^2 = 4kk = 4k^2$.

So $1 + 2k + k^2 = 4k^2$ or $3k^2 - 2k - 1 = 0$.

Solving this we get $(3k + 1)(k - 1) = 0$ giving $k = 1$ or $-\frac{1}{3}$.

(b) If $k = 1$, the equation is $y = x^2 + 2x + 1$.

If $k = -\frac{1}{3}$, the equation is $y = -\frac{x^2 - 2x + 1}{3}$.

Summarizing the conditions for the various types of roots of the equation $ax^2 + bx + c = 0$:

$D < 0$	$D \geq 0$	
$b^2 < 4ac$	$b^2 \geq 4ac$	
No real roots	Real roots	
	Different	Equal
	if $b^2 > 4ac$	if $b^2 = 4ac$

As we have learnt, D is called the **discriminant**: it discriminates between the types of roots.

Example 7

The equation $px^2 - 2(p+3)x + p - 1 = 0$ has real roots. What is the range of values of p?

For real roots, $b^2 \geq 4ac$.

Here $a = p$, $b = -2(p+3)$ and $c = p - 1$.

Then $[-2(p+3)]^2 \geq 4p(p-1)$.

Simplifying, $4(p^2 + 6p + 9) \geq 4p^2 - 4p$

or $6p + 9 \geq -p$.

Hence $7p \geq -9$ and $p \geq -\frac{9}{7}$.

Example 8

Find the range of values of p for which the line $2x - y = p$ (i)

meets the curve $x(x - y) = 4$. (ii)

The line may meet the curve at two points or touch the curve. The coordinates of these points will be the solutions of the simultaneous equations (i) and (ii).

From (i), $y = 2x - p$.

Substituting in (ii), $x(x - 2x + p) = 4$

which simplifies to $x^2 - px + 4 = 0$.

The roots of this equation are the x-coordinates of the point(s) where the line meets the curve. These must be real. So $b^2 \geq 4ac$ where $a = 1$, $b = -p$ and $c = 4$.

Then $(-p)^2 \geq 4(1)(4)$ or $p^2 \geq 16$ which gives $p \geq 4$ or $p \leq -4$.

Example 9

(a) Find the relation between m and k if the line $y = mx + k$ is a tangent to the curve $y^2 = 8x$.

(b) If $m = \frac{1}{2}$, find the equation of the tangent and the coordinates of its point of contact.

(c) Find the equations of the two tangents to this curve which pass through the point (–3,–5).

(a) As in Example 8, we solve the simultaneous equations.
Substituting $y = mx + k$ in the equation of the curve:

$$(mx + k)^2 = 8x$$

Then $m^2x^2 + 2mkx + k^2 - 8x = 0$ i.e. $m^2x^2 + (2mk - 8)x + k^2 = 0$

Now this equation must have equal roots as the line is a tangent.

Then $b^2 = 4ac$ where $a = m^2$, $b = (2mk - 8)$ and $c = k^2$, so $(2mk - 8)^2 = 4m^2k^2$ or $4m^2k^2 - 32mk + 64 = 4m^2k^2$ which gives $mk = 2$, the relation required.

(b) If $m = \frac{1}{2}$, then $k = 4$. The equation of the tangent is therefore $y = \frac{x}{2} + 4$.

To find the coordinates of the point of contact, we solve this equation with that of the curve.

Then $(\frac{x}{2} + 4)^2 = 8x$ i.e. $\frac{x^2}{4} + 4x + 16 = 8x$

or $x^2 - 16x + 64 = 0$.

Hence $(x - 8)^2 = 0$ giving $x = 8$.

The corresponding value of y is $\frac{8}{2} + 4 = 8$.

Hence the coordinates of the point of contact are (8,8).

(c) As $mk = 2$, the equation of any tangent is $y = mx + \frac{2}{m}$.

If (–3,–5) lies on the tangent, then $-5 = -3m + \frac{2}{m}$

which simplifies to $3m^2 - 5m - 2 = 0$.

Solving this, $(3m + 1)(m - 2) = 0$ giving $m = 2$ or $-\frac{1}{3}$.

Hence the equations are $y = 2x + 1$ and $y = -\frac{x}{3} - 6$ i.e. $x + 3y = -18$.

Exercise 4.1 *(Answers on page 616.)*

1 Without solving these equations, state the type of roots they have i.e., real, real and equal or not real:

(a) $x^2 - 10x + 25 = 0$
(b) $x^2 - 6x + 10 = 0$
(c) $x^2 = 4x + 7$
(d) $2x^2 - x + 2 = 0$
(e) $3x^2 + x = 1$
(f) $4x^2 - 20x + 25 = 0$
(g) $\frac{1}{x} + \frac{1}{x-1} = 2$
(h) $\frac{2}{x} + 1 = \frac{2}{x+1} - 1$
(i) $2x^2 = px + p^2$
(j) $ax^2 - x = a$ $(a > 0)$

2 Find the values of k if the equation $x^2 + (k - 2)x + 10 - k = 0$ has equal roots.

3 What is the largest value m can have if the roots of $3x^2 - 4x + m = 0$ are real?

4 For what values of p does the equation $x^2 - 2px + (p + 2) = 0$ have equal roots?

5 The equation $x^2 - 2x + 1 = p(x - 3)$ has equal roots. Find the possible values of p.

6 Show that the equation $a^2x^2 + ax + 1 = 0$ can never have real roots.

7 Find the values of k if the line $x + y = k$ is a tangent to the circle $x^2 + y^2 = 8$.

8 The equation $kx^2 + 2(k + a)x + k = a$ has equal roots. Express k in terms of a. Show that the line $y = k(x-3)$ is a tangent to the curve $y = k(x^2-3x + 1)$ for any value of k except 0.

9 (a) Find the range of values of m for which the line $y = mx + 5$ meets the curve $y = x^2 + 9$ in two distinct points.

(b) If this line is to be a tangent, find the two possible equations of the tangent and the coordinates of the points of contact.

10 The line $y = mx + 1$ is a tangent to the curve $y^2 = 2x - 3$. Find the values of m.

11 The line $y = 2x + p$ is a tangent to the curve $x(x + y) + 12 = 0$. Find the possible values of p.

12 (a) Find the relation between m and c if the line $y = mx + c$ is a tangent to the curve $y^2 = 2x$.

(b) Hence find the equations of the two tangents to this curve which pass through the point $(2,2\frac{1}{2})$.

13 What is the range of values of c if the line $y = 2x + c$ is to meet the curve $x^2 + 2y^2 = 8$ in two distinct points?

14 The equation $(p + 3)x^2 + 2px + p = 1$ has real roots. Find the range of values of p.

15 If the equation $x^2 - (p - 2)x + 1 = p(x - 2)$ is satisfied by only one value of x, what are the possible values of p?

16 If the equation $x^2 - 2kx + 3k + 4 = 0$ has equal roots, find the possible values of k and solve the two equations.

17 Find the values of k for which the line $x + y = k$ is a tangent to the curve $x(x - y) + 2 = 0$.

MAXIMUM AND MINIMUM VALUES OF A QUADRATIC FUNCTION

The maximum or minimum values of the function $f(x) = ax^2 + bx + c$ are the values of $f(x)$ at the top or bottom of the curve. These are also called the **turning points** of the curve.

By completing the square, we find that $ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$, where $a > 0$. Now the least or **minimum** value of this expression will be when the squared term is 0 (it cannot be negative) as the other terms are fixed. This occurs when $x = -\frac{b}{2a}$.

Hence the minimum value of $f(x) = ax^2 + bx + c$ $(a > 0)$, i.e. at the bottom of the curve, is $\mathbf{f(-\frac{b}{2a})}$.

If $a < 0$, the turning point will be a **maximum** (the top of the curve) where $x = -\frac{b}{2a}$. This can be proved in a similar way and is illustrated in Example 11.

Example 10

What is the minimum value of $3x^2 - 4x + 1$ and for what value of x does it occur?

$f(x) = 3x^2 - 4x + 1$, $a = 3$, $b = -4$.

As $a > 0$, the minimum value of f(x) occurs when $x = -\frac{b}{2a} = -\frac{-4}{6} = \frac{2}{3}$.

The minimum value $= f(\frac{2}{3}) = 3 \times \frac{4}{9} - \frac{8}{3} + 1 = -\frac{1}{3}$.

This is illustrated in Fig.4.5. The line $x = \frac{2}{3}$ through the turning point is called the **axis** of the curve and the curve is symmetrical about this line.

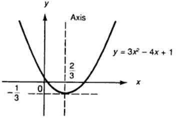


Fig. 4.5

Example 11

Express $5 - x - 2x^2$ in the form $a - b(x + c)^2$ and hence or otherwise find its maximum value and the value of x where this occurs.

$$5 - x - 2x^2 = 5 - 2\left(x^2 + \frac{x}{2}\right)$$
$$= 5 - 2\left[\left(x + \frac{1}{4}\right)^2 - \frac{1}{16}\right] \text{ by completing the square}$$
$$= 5 - 2\left(x + \frac{1}{4}\right)^2 + \frac{1}{8} = 5\frac{1}{8} - 2\left(x + \frac{1}{4}\right)^2.$$

Now the least value of $\left(x + \frac{1}{4}\right)^2$ is 0 when $x = -\frac{1}{4}$ so the **maximum** value of the expression is $5\frac{1}{8}$ when $x = -\frac{1}{4}$.

Alternatively as the question allows us to use another method (otherwise) we can use the rule stated above. Here $a = -2$, $b = -1$. Verify that the same result is obtained. This is illustrated in Fig. 4.6. The equation of the axis of summetry is $x = -\frac{1}{4}$.

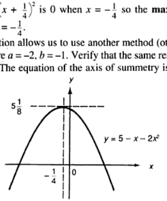


Fig. 4.6

SUMMARY

To find the maximum/minimum value of $f(x) = ax^2 + bx + c$, rewrite $f(x)$ as $a\left(x^2 + \frac{bx}{a} + \frac{c}{a}\right)$ and complete the square on $x^2 + \frac{bx}{a}$.

$f(x)$ is then converted to $a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right]$.

The turning point of $f(x)$ is a $\begin{matrix}\text{maximum if } a < 0\\ \text{minimum if } a > 0\end{matrix}$ and occurs where $x = -\frac{b}{2a}$.

The $\begin{matrix}\text{max}\\ \text{min}\end{matrix}$ value is $f(-\frac{b}{2a})$.

SKETCHING THE GRAPH OF A QUADRATIC FUNCTION

To *draw* the graph, we need a table of values. For a *sketch*, we need only know:

(1) the shape of the curve;

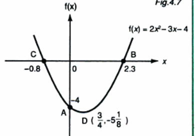
(2) where it cuts the y-axis. This is given by $f(0)$;

(3) the positions of the roots (if any). If $f(x)$ factorizes, the roots are easily found; otherwise, approximate values will be sufficient;

(4) the position of the turning point. Remember that the curve is symmetrical about the axis through this position.

Example 12

Fig.4.7



Sketch the graph of $f(x) = 2x^2 - 3x - 4$.

(1) As $a > 0$, the shape is $\bigcup$.

(2) $f(0) = -4$.
This is the point A (Fig.4.7).

(3) $f(x)$ does not factorize.
The roots of $f(x) = 0$ are given by $x = \frac{3 \pm \sqrt{41}}{4} \approx \frac{3 \pm 6}{4} \approx 2.3$ and -0.8 (points B and C respectively).

(4) $f(x) = 2(x^2 - \frac{3x}{2} - 2)$

$= 2\left[\left(x - \frac{3}{4}\right)^2 - \frac{9}{16} - 2\right]$

$= 2\left(x - \frac{3}{4}\right)^2 - \frac{41}{8}$.

So the minimum is at point $D\left(\frac{3}{4}, -\frac{41}{8}\right)$.

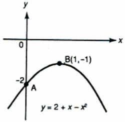
The curve can now be sketched through these points.

Example 13

Sketch the curve $y = -2 + 2x - x^2$.

(1) $a < 0$ so the shape is $\cap$.
(2) When $x = 0$, $y = -2$ (point A in Fig. 4.8).
(3) $b^2 < 4ac$ so the curve does not meet the x-axis.
(4) $f(x) = -(x^2 - 2x + 2)$
$= -[(x - 1)^2 - 1 + 2]$
$= -(x - 1)^2 - 1$
So the maximum is at (1,–1) (point B).

Fig. 4.8



Example 14

Sketch the graph of $f(x) = |x^2 - x - 2|$.

To sketch this graph, we use the same method as before.
First sketch $f(x) = x^2 - x - 2$ and then reflect the negative part in the x-axis.

$x^2 - x - 2 = (x - 2)(x + 1)$ so the graph meets the x-axis at –1 and 2 (Fig.4.9).

It meets the y-axis at –2 and the minimum is at $\left(\frac{1}{2}, -2\frac{1}{4}\right)$.

When reflected, these values become 2 and $\left(\frac{1}{2}, 2\frac{1}{4}\right)$ respectively.

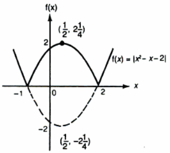


Fig. 4.9

RANGE OF A QUADRATIC FUNCTION

Example 15

Find the range of $f(x) = x^2 - 2x - 3$ *for the domain* $-2 \le x \le 5$.

At the end points, $f(-2) = (-2)^2 - 2(-2) - 3 = 5$ and $f(5) = 5^2 - 2(5) - 3 = 12$.

We might be tempted to say that the range is 5 to 12, but does the curve rise continuously from 5 to 12? It may go down to the minimum and then rise.
Verify that the minimum is -4 at $x = 1$ and sketch the curve (Fig.4.10).
The minimum lies within the domain.
So the actual range is $-4 \le f(x) \le 12$.
Hence for such questions it would be wise to make a sketch.

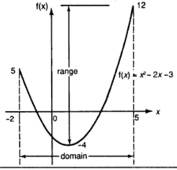


Fig. 4.10

Example 16

Sketch the graph of $3 - 5x - 2x^2$ *and state the range of the function* $f: x \longmapsto 3 - 5x - 2x^2$ *for real x.*

As $a < 0$, the curve has a maximum where $x = -\frac{b}{2a} = -\frac{(-5)}{2(-2)} = -\frac{5}{4}$.

The maximum value is $3 + \frac{25}{4} - \frac{50}{16} = \frac{49}{8} = 6\frac{1}{8}$.

Fig. 4.11 shows the sketch. As x can take any real value, the function takes values $\le 6\frac{1}{8}$ so the range is $f(x) \le 6\frac{1}{8}$.

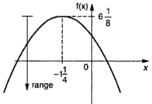


Fig. 4.11

Example 17

Find the range of the function $f(x) = |x(x-2)|$ *for the domain* $\frac{1}{2} \le x \le 2\frac{1}{2}$.

First sketch the graph (Fig.4.12).

The minimum of $x(x-2)$ is -1 at $x = 1$ which becomes a value of 1 when reflected.

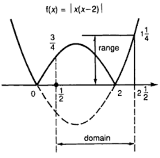


Fig.4.12

At the end points, $f(\frac{1}{2}) = \left|\frac{1}{2}(-\frac{3}{2})\right| = \frac{3}{4}$ and $f(2\frac{1}{2}) = \left|-\frac{5}{2}(\frac{1}{2})\right| = \frac{5}{4}$.

At $x = 2$ however, $f(2) = 0$.

Hence the range is $0 \le f(x) \le 1\frac{1}{4}$.

Exercise 4.2 *(Answers on page 616.)*

1 Find the maximum or minimum values of the following functions and the values of x where this occurs:

(a) $x^2 - 6x - 1$
(b) $x^2 + 2x - 3$
(c) $1 - 4x - 2x^2$
(d) $3 - x - 2x^2$
(e) $2x^2 - x - 4$
(f) $x^2 + 3$
(g) $4x^2 - 3x - 1$
(h) $5 - 2x - 4x^2$
(i) $(1 - x)(x + 2)$
(j) $x^2 + 2bx + c$

2 Sketch the graphs of the functions in Question 1 (except part (j)).

3 The graph of a quadratic function meets the x-axis where $x = 3$ and $x = k$. If the turning point of the function occurs where $x = \frac{1}{2}$, find the value of k.

4 Sketch the graph of $f(x) = |x^2 - 4x + 3|$ and find the range for the domain $0 \le x \le 2$.

5 Sketch the graph of $f(x) = |x(2-x)|$. State the range if the domain is $-2 \le x \le 3$.

6 Sketch the graph of the function $f(x) = |x^2 + x - 2|$ and find its range for $0 \le x \le 2$.

7 Find the range of the function $y = |3 + 2x - x^2|$ for the domain $0 \le x \le 2$.

8 **(a) A function V is given by $V(t) = 2t^2 - 8t + 30$. Find the minimum value of V and the value of t where this occurs.**

(b) What is the range of this function for $0 \le t \le 3$?

9 Find the range of the function $f : x \longmapsto 2x^2 - 6x - 1$ for real values of x.

10 **Find the range of the functions (a) $1 - 3x - x^2$ and (b) $2x^2 - x - 3$ for the domain $-1 \le x \le 2$.**

11 Find the range of (a) $2x^2 - x - 3$ and of (b) $1 - 2x - x^2$ for the domain $-2 \le x \le 2$.

12 **Convert $y = \frac{1}{2}[(x + 4)^2 + (x - 2)^2]$ to the form $y = (x + p)^2 + q$ and hence find the minimum value of y and the value of x where this occurs.**

13 (a) Express $7 - x - 3x^2$ in the form $a - b(x + c)^2$, showing the values of a, b and c. Hence state the range of the function $f : x \longmapsto 7 - x - 3x^2$ for all real values of x.

(b) If the minimum value of $x^2 + 4x + k$ is -7 find the value of k.

14 **The height (h m) of a ball above the ground is given by the function $h(t) = 15t - 5t^2$ where t is the time in seconds since the ball left ground-level. Find the range of the height for $1 \le t \le 3$.**

15 **A spot of light is made to travel across a computer screen in a straight line so that, at t seconds after starting, its distance from the left hand edge (d cm) is given by the function $d(t) = 7t - t^2 + 2$. Find the furthest distance the spot travels and how long it takes to travel this distance .**

16 The function $f(x) = 1 + bx + ax^2$ has a maximum value of 4 where $x = -1$. Find the value of a and of b.

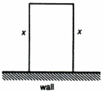
17 The function $f(x) = ax^2 + bx + c$ has a minimum value of $-5\frac{1}{4}$ where $x = \frac{1}{4}$ and $f(0) = -5$. Find the value of a, of b and of c.

18 **A rectangular enclosure is made against a straight wall using three lengths of fencing, two of length x m (Fig.4.13). The total length of fencing available is 50 m.**

(a) Show that the area enclosed is given by $50x - 2x^2$.

(b) Hence find the maximum possible area which can be enclosed and the value of x for this area.

Fig. 4.13



QUADRATIC INEQUALITIES

For $D > 0$ and $a > 0$, the equation $f(x) = ax^2 + bx + c = 0$ will have unequal roots. Call these α and β (where $\alpha < \beta$). Then we see from the graph of such a function (Fig.4.14) that

for $x < \alpha$,	**$f(x) > 0$**
for $\alpha < x < \beta$,	**$f(x) < 0$**
for $x > \beta$,	**$f(x) > 0$**

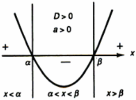


Fig. 4.14

For $D > 0$ and $a < 0$, the signs of f(x) will be reversed (Fig. 4.15),

for $x < \alpha$, $f(x) < 0$
for $\alpha < x < \beta$, $f(x) > 0$
for $x > \beta$, $f(x) < 0$

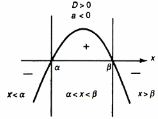


Fig.4.15

Keep the graphical illustrations in mind when dealing with such inequalities.

If, however, $D < 0$ (Fig.4.16) then

$D < 0$	f(x) is always positive if $a > 0$ f(x) is always negative if $a < 0$

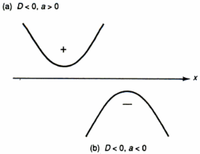


Fig. 4.16

Example 18

Show that $3x^2 - 2x + 4$ is always greater than 1.

This means we have to show that $3x^2 - 2x + 3$ is always greater than 0.
Now for this function, $D = b^2 - 4ac = 4 - 36 < 0$. Therefore the function is always positive. (Similar to Fig.4.16 (a)).

Example 19

For what domain of values of x is $3x^2 - 2x \le 1$?

This is equivalent to $3x^2 - 2x - 1 \le 0$ i.e. $(3x + 1)(x - 1) \le 0$.
The roots of the function are $\alpha = -\frac{1}{3}$ and $\beta = 1$.
Then $3x^2 - 2x - 1$ is 0 or negative if $-\frac{1}{3} \le x \le 1$ (Fig.4.17).

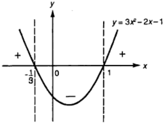
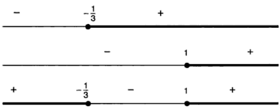


Fig. 4.17

Alternative method

This method uses the **signs** of the two factors $(3x + 1)$ and $(x - 1)$ to find the sign of the product $(3x + 1)(x - 1)$.



On the first number line, $3x + 1$ will be negative for $x < -\frac{1}{3}$, 0 at $x = -\frac{1}{3}$ and positive thereafter.
On the second number line, $(x - 1)$ will be negative if $x < 1$, 0 at $x = 1$ and positive thereafter.
The signs of the product are shown on the third number line. Hence we see that $(3x + 1)(x - 1) \le 0$ for $-\frac{1}{3} \le x \le 1$ as before.

Example 20

For what range of values of p will the equation $x^2 - (p + 2)x + p^2 + 3p = 3$ have real roots?

The equation is $x^2 - (p + 2)x + p^2 + 3p - 3 = 0$.

For real roots, $b^2 \ge 4ac$ where $a = 1$, $b = -(p + 2)$ and $c = p^2 + 3p - 3$.

Then $[-(p + 2)]^2 \ge 4(p^2 + 3p - 3)$
i.e. $p^2 + 4p + 4 \ge 4p^2 + 12p - 12$

which simplifies to $3p^2 + 8p - 16 \le 0$
i.e. $(3p - 4)(p + 4) \le 0$.

Hence, as in Fig.4.14, $-4 \le p \le \frac{4}{3}$.

Example 21

Find the domain of x for which $|x^2 - 3x - 7| \le 3$.

Extending the result for $|x| < k$ found in Chapter 3, if $|x^2 - 3x - 7| \le 3$, then $-3 \le x^2 - 3x - 7 \le 3$.

Take these separately:

(1) $-3 \le x^2 - 3x - 7$
gives $x^2 - 3x - 4 \ge 0$
i.e. $(x - 4)(x + 1) \ge 0$
This is true if $x \le -1$ or $x \ge 4$.

(2) $x^2 - 3x - 7 \le 3$
gives $x^2 - 3x - 10 \le 0$
i.e. $(x - 5)(x + 2) \le 0$
This is true if $-2 \le x \le 5$.

These inequalities are shown on number lines.

Now x must satisfy **both** sets of conditions. Hence x must lie in the shaded regions i.e. between -2 and -1 (inclusive) and between 4 and 5 (inclusive). Hence $-2 \le x \le -1$ and $4 \le x \le 5$.

This solution is shown on the graph of $y = | x^2 - 3x - 7 |$ (Fig.4.18).

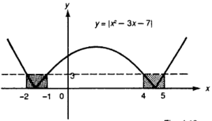


Fig. 4.18

Exercise 4.3 *(Answers on page 617.)*

1 Find the domain of x if

(a) $x^2 - x \ge 2$ (b) $x^2 + x \le 6$

(c) $x^2 + 5x > 6$ (d) $2x^2 \ge x + 1$

(e) $x(6x - 5) \le -1$ (f) $x^2 \ge 4x$

(g) $3x^2 \le x + 2$ (h) $(x + 3)(x + 1) > 24$

(i) $3x^2 < 4 - 11x$ (j) $2x^2 + 7x \ge 4$

2 If $8x^2 + 4x + k$ is never negative, find the least possible value of k.

3 Find the range of values of t if the equation $3x^2 - 3tx + (t^2 - t - 3) = 0$ has real roots.

4 If the roots of $p(x^2 + 2) = 1 - 2x$ are real, find the range of values of p.

5 If the equation $px(x - 1) + p + 3 = 0$ has real roots, find the range of values of p.

6 Find the domain of x if

(a) $| x^2 + x - 7 | < 5$ (b) $| x^2 - 5x - 10 | \ge 4$

(c) $| 4 + x - x^2 | \le 2$

7 Find the range of values of p if the roots of the equation $p^2x^2 - (p + 2)x + 1 = 0$ are real.

8 Show that the equation $(t + 3)x^2 + (2t + 5)x + (t + 2) = 0$ has real roots for all values of t.

9 Show that the equation $px^2 + (2p + 1)x + (p + 1) = 0$ has real roots for all values of p.

10 A rectangle has sides of length $(2x + 3)$ cm and $(x + 1)$ cm. What is the domain of x if the area of the rectangle lies between 10 cm^2 and 36 cm^2 inclusive?

11 If the equation $x^2 + 3 = t(x + 1)$ does not have real roots, find the range of values of t.

12 For what domain of x is $|2x^2 - x - 3| \leq 3$? Illustrate your result on a sketch of $y = |2x^2 - x - 3|$.

13 If $|(x + 3)(x - 2)| < 6$, find the domain of x and show your result on a sketch graph.

14 State the domain of x for which $3x - 2$ and $x + 3$ are
(a) both positive,
(b) both negative.
Hence find the domain of x for which $3x^2 + 7x \leq 6$.

15 The equation $px^2 + px + 2p = 3$ has real roots. Find the range of values of p.

16 The function $x^2 + 3x + k$ is never negative. Find the least whole number value of k. If $k = 4$, find the minimum value of the function.

SUMMARY

- The roots of $ax^2 + bx + c = 0$ are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
- Types of roots: if $b^2 > 4ac$, the roots are real and different
  if $b^2 = 4ac$, the roots are real and equal
  if $b^2 < 4ac$, the equation has no real roots.
- If $a > 0$, the function $ax^2 + bx + c$ has a minimum value;
  if $a < 0$, it has a maximum value.
- To find the maximum/minimum, write as $a\left(x^2 + \frac{bx}{a} + \frac{c}{a}\right)$ and complete the square on $x^2 + \frac{bx}{a}$.
- If the roots of $f(x) = ax^2 + bx + c = 0$ are α and β (where $\alpha < \beta$), then

  $f(x) > 0$ for $x < \alpha$
  $f(x) < 0$ for $\alpha < x < \beta$ } when $b^2 > 4ac$ and $a > 0$ (Fig.4.19)
  $f(x) > 0$ for $x > \beta$

  and

  $f(x) < 0$ for $x < \alpha$
  $f(x) > 0$ for $\alpha < x < \beta$ } when $b^2 > 4ac$ and $a < 0$ (Fig.4.20)
  $f(x) < 0$ for $x > \beta$

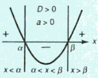


Fig. 4.19

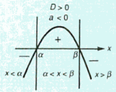


Fig. 4.20

If $b^2 < 4ac$ and $a > 0$, f(x) is always > 0.
If $b^2 < 4ac$ and $a < 0$, f(x) is always < 0.

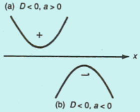


Fig.4.21

REVISION EXERCISE 4 *(Answers on page 617.)*

A

1 Without solving the following equations, state the nature of their roots:
(a) $3x^2 - x = 1$ (b) $(x + 1)(x - 2) = 5$
(c) $(1 - x) = \frac{4}{x + 2}$ (d) $\frac{1}{x} + 3 = \frac{1}{x - 1}$
(e) $(2x + 5)(2x + 3) = 2(6x + 7)$

2 Find the range of values of x for which $3x^2 < 10x - 3$.

3 Show that the equation $(t - 3)x^2 + (2t - 1)x + (t + 2) = 0$ has rational roots for all values of t.

4 Show that the equation $(p + 1)x^2 + (2p + 3)x + (p + 2) = 0$ has real roots for all values of p. **(C)**

5 The quadratic equation $x^2 + px + q = 0$ has roots -2 and 6. Find (i) the value of p and of q, (ii) the range of values of r for which the equation $x^2 + px + q = r$ has no real roots. **(C)**

6 Express $8 + 2x - x^2$ in the form $a - (x + b)^2$. Hence or otherwise find the range of $8 + 2x - x^2$ for $-1 \leq x \leq 5$.

7 (a) Find the range of values of x for which $6x^2 - 11x \geq 7$.
(b) Find the coordinates of the turning point of the curve $y = (2x - 3)^2 + 6$ and sketch the curve. **(C)**

8 Find the range of the function $2x^2 - 7x + 3$ for the domain $0 \leq x \leq 4$.

9 State the range of values of k for which $2k - 1$ and $k + 2$ are (i) both positive, (ii) both negative. Hence, or otherwise, find the range of values of k for which $2k^2 + 3k < 2$. **(C)**

10 (a) Find the value of p for which the line $y = 6$ is a tangent to the curve $y = x^2 + (1 - p)x + 2p$.

(b) Find the range of values of q for which the line $x + 2y = q$ meets the curve $x(x + y) + 8 = 0$. (C)

11 Find the domain of x for which $| 2x^2 - 4x - 3 | > 3$.

12 (a) The quadratic equation $kx^2 + 2(k + a)x + (k + b) = 0$ has equal roots. Express k in terms of a and b.

(b) The quadratic equation $(p + 1)x^2 + 2px + (p + 2) = 0$ has real roots. Find the range of values of p. (C)

13 The function $2ax^2 - 4x - a$ has a maximum value of 3. Find the values of a.

14 Sketch the graph of the function $| x^2 - x - 6 |$ and find its range for $0 \le x \le 3$.

15 The curve $y = ax^2 + bx + c$ has a maximum point at (2,18) and passes through the point (0,10). Evaluate a, b and c. (C)

16 The two shortest sides of a right-angled triangle have lengths $(x + 1)$ cm and $(x + 2)$ cm. If the area A cm² of the triangle is such that $15 \le A \le 28$, find the range of values of x.

17 The equation of a curve is $y = 4x^2 - 8x - 5$. Find

(i) the range of values of x for which $y \ge 0$,

(ii) the coordinates of the turning point of the curve.

State the coordinates of the maximum point of the curve $y = | 4x^2 - 8x - 5 |$ and sketch the curve $y = | 4x^2 - 8x - 5 |$. (C)

18 A square has side x cm and a rectangle has sides x cm and $2(x + 1)$ cm. For what range of values of x is the total area not less than 1 cm² and not more than 5 cm²?

19 (a) Find the value of p for which the equation $(1 - 2p)x^2 + 8px - (2 + 8p) = 0$ has two equal roots.

(b) Show that the line $x + y = q$ will intersect the curve $x^2 - 2x + 2y^2 = 3$ in two distinct points if $q^2 < 2q + 5$. (C)

20 (a) The function $f : x \longmapsto x^2 + px + q$ is negative only between the values $x = 2$ and $x = 5$.

(i) Find the value of p and of q.

(ii) If $f(x) = -2$, find the value of x.

(b) The function $ax^2 + bx + c$ is positive only when $-\frac{3}{2} < x < 2$ and meets the y-axis where $y = 6$. Find the value of a, of b and of c.

21 Find the domain of x if $\sqrt{5x - 2 - 2x^2}$ is real.

22 $f(x) = 0.3x^2 - 0.2x$. If $0.1 \le f(x) \le 0.5$, find the domain of x.

B

23 Find the domain of x if $2 < \sqrt{(2x^2 + x + 3)} < 3$.

24 If the equation $x^2 + 3 = k(x + 1)$ has real roots, find the range of values of k. Hence find the two values of x for which the function $\frac{x^2+3}{x+1}$ has (i) a maximum, (ii) a minimum value.

25 The function $f(x) = ax^2 + bx + c$ has a minimum value of 5 when $x = 1$ and $f(2) = 7$. Find the values of a, b and c.

26 The roots of the quadratic equation $x^2 + 2x + 3 = p(x^2 - 2x - 3)$ are real. Show that p cannot have a value between -1 and $\frac{1}{2}$.

27 The function $x^2 + px + q$ is negative for $2 < x < 4$. Find
(a) the values of p and q,
(b) the domain of x if $15 \le x^2 + px + q \le 48$.

28 (a) Solve the equation $x^2 + 2ax + 2 = 2a^2 + 5a$ to obtain x in terms of a.
(b) If these values of x are real, find the range of values of a.

29 If α and β are the roots of $ax^2 + bx + c = 0$, show that $\alpha + \beta = -\frac{b}{a}$ and that $\alpha\beta = \frac{c}{a}$. Hence show that $\alpha^2 + \beta^2 = \frac{b^2-2ac}{a^2}$ and that $\alpha - \beta = \frac{\sqrt{b^2 - 4ac}}{a}$.