Functions

3

RELATIONS AND FUNCTIONS

A **relation** links the members of two sets together. Relations can be of many kinds, e.g. "is the father of", "is a divisor of", "is the same age as", "is the square of" etc. Fig. 3.1 illustrates the relation "is the father of" linking the set of men {A, B, C, D} and the set of children {p, q, r, s, t, u, v}. An arrow identifies the relation between a father and child. The diagram shows that A has two children (p and q), B has 1 child (t), C has 3 children (r, s and u) and D 1 child (v). So 2 arrows leave from A, 1 from B, 3 from C and 1 from D.

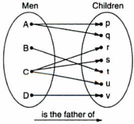


Fig. 3.1

In our work the relation will usually be some mathematical operation. Fig. 3.2 shows the relation "$y = 1 + x^2$" where the starting values (the **inputs**) are chosen values of x. These are linked to the values of y produced by the relation (the **outputs**), i.e. the set {1, 2, 5, 26}.

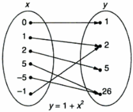


Fig. 3.2

Note that only ONE arrow leaves each input, unlike the relation in Fig. 3.1. In Fig. 3.2 each input produces a **unique** output. This is a special type of relation called a **function**, one of the most important concepts in Mathematics. The relation in Fig. 3.1 is NOT a function.

In Fig. 3.3, each member of set A is *squared* to produce the set of outputs B. As each input

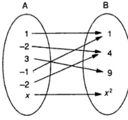


Fig. 3.3

has a unique square, Fig. 3.3 illustrates a function f. f is "square the input". So, if x is the input, the output is x^2.

A function is also called a **mapping** and we say that x is **mapped onto** x^2 *by the function* f. We symbolize this as

$$f: x \longmapsto x^2$$

Read this as 'f is the function which maps x onto x^2'.

f operates on the input x to produce x^2 so we write $f(x) = x^2$. Hence the image of 2 is $f(2) = 2^2 = 4$. The image of -3 is $f(-3) = (-3)^2 = 9$. The image of a is $f(a) = a^2$ and so on. What is the image of 5? What is $f(6)$, $f(-x)$ and $f(2x)$? If $f(x) = 49$, what is the value of x?

Now look at the relation illustrated in Fig. 3.4(a).

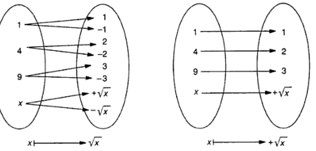


Fig. 3.4 (a) (b)

Is this a function? As you can see, each input has *two* outputs (2 arrows from each input). So this operation (taking the square root, $x \longmapsto \sqrt{x}$) is NOT a function. It does not produce a unique image as x has 2 square roots $+\sqrt{x}$ and $-\sqrt{x}$.

However, if we defined $\sqrt{\ }$ to mean the **positive** root only, then $f(x) = +\sqrt{x}$ would be a function (Fig. 3.4(b)).

Summarizing,

- a function f is a process or operation which takes an input x and maps it onto a **unique** output $f(x)$, the image of x;
- $f: x \longmapsto f(x)$;
- to define f, we write, for example, $f(x) = x^2$ or $f(x) = +\sqrt{x}$ or $f(x) = \sin x$ etc.

f and x are the usual letters for the function and the input respectively, but other letters can be used e.g. $F(x)$, $g(x)$ or $A(r)$, etc.

A function need not be defined algebraically. It may be stated in words, such as the function 'Y is the father of x', or given in the form of a table such as a table of sines.

Example 1

A function f *is given by* $f : x \longmapsto x^2 - x + 1$. *Find*
(a) $f(2)$, *(b)* $f(-3)$, *(c) the image of* -2, *(d)* $f(r)$, *(e)* $f(\frac{x}{2})$.

$f(x) = x^2 - x + 1$

(a) $f(2) = 2^2 - 2 + 1 = 3$

(b) $f(-3) = (-3)^2 - (-3) + 1 = 13$

(c) The image of -2 is $f(-2) = (-2)^2 - (-2) + 1 = 7$.

(d) $f(r) = r^2 - r + 1$

(e) $f(\frac{x}{2}) = (\frac{x}{2})^2 - (\frac{x}{2}) + 1 = \frac{x^2 - 2x + 4}{4}$

Example 2

The function h *is given by* $h(x) = \frac{x+1}{x-1}, x \neq 1$. *Find*
(a) $h(2)$, *(b)* $h(\frac{1}{2})$, *(c)* $h(x + 1)$

(a) $h(2) = \frac{2+1}{2-1} = 3$

(b) $h(\frac{1}{2}) = \frac{\frac{1}{2}+1}{\frac{1}{2}-1} = -3$

(c) $h(x + 1) = \frac{x+1+1}{x+1-1} = \frac{x+2}{x}, x \neq 0$

Note: A function may not produce an image for certain values of the input. In this example, $x \neq 1$. If $x = 1$, $h(1) = \frac{1+1}{1-1}$ which is impossible as division by zero is undefined. Hence 1 has no image under this function.

Example 3

$F(x) = x^2 + x - 1$. If $F(x) = 5$, find the values of x.

$F(x) = 5$ is the equation $x^2 + x - 1 = 5$ i.e. $x^2 + x - 6 = 0$ which we can solve for the values of x.

$x^2 + x - 6 = 0$
$(x + 3)(x - 2) = 0$

Hence $x = -3$ or $x = 2$.
These are the two values of x which have an image of 5.
Check by finding $F(-3)$ and $F(2)$.

DOMAIN AND RANGE

There are special names for the sets of inputs and outputs. The set of inputs is called the **domain** and the set of outputs the **range**.

Fig. 3.5 shows the domain and range for the function $f(x) = (1 - x)^2$. The domain is the set $\{-1, 0, 2, 4\}$ and the range is the set $\{1, 4, 9\}$.

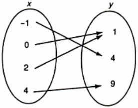


Fig. 3.5

The domain can be any set of numbers which have images. It could be just a few selected numbers or all positive numbers or all real numbers, etc. If it is not specified it is taken to be all real numbers. However, as we saw in *Example 2*, some numbers may have no image and these must be stated. They are excluded from the domain.

Example 4

State the domain for the function $f(x) = \frac{1}{x}$.

Every real number will have an image under this function **except** $x = 0$. So the domain will be $\{$all real values of x, $x \neq 0\}$.

This is often briefly stated as $f(x) = \frac{1}{x}$, $x \neq 0$.

Example 5

State the domain for $f(x) = \sqrt{x}$ *(positive root).*

Every positive number and 0 will have a square root but negative numbers will not. These must be excluded. So the domain is {all positive numbers and 0} or just $x \geq 0$.

Example 6

What values of x must be excluded from the domain of the function

$$f(x) = \frac{x+1}{x^2+x-2}\ ?$$

This function will always produce an image except when $x^2 + x - 2 = 0$ or $(x + 2)(x - 1) = 0$ i.e. when $x = -2$ or $x = 1$.

These values must be excluded from the domain.
Hence the domain is {all real values of x, $x \neq -2$ or 1}

Exercise 3.1 *(Answers on page 608.)*

1 For each of the following functions, find the images of –3, –1, 0, 1, 2, 4:
(a) $f(x) = x^2 - x - 5$
(b) $g(x) = (x + 1)^2$
(c) $h(x) = \frac{x-1}{x+1}$
(d) $F(x) = (x + 1)(x - 2)$

2 What value of x must be excluded from the domain of the function in Question 1 part (c)?

3 State the values of x which must be excluded from the domains of the following functions:
(a) $f(x) = \frac{2}{x-2}$
(b) $g(x) = \frac{x-2}{2x-3}$
(c) $h(x) = \frac{5}{x^2-x-2}$
(d) $F(x) = 3 - \frac{2}{x+3}$

4 f is the function 'square x and add 2'.
(a) Write f in the form $f(x) = ...$
(b) Find f(1), f(–1), f(0).
(c) If $f(x) = 27$, find the values of x.

5 F is the function 'add 2 to x and then square'.
(a) Write F in the form $F(x) = ...$
(b) Find F(1), F(–1), F(0).
(c) If $F(x) = 25$, find the values of x.
(d) Is this the same function as f in Question 4?

6 If $f(x) = 3x + 2$, what is the value of x which is mapped onto 8?

7 A function such as $f(x) = 5$ is a *constant* function. State the values of f(0), f(–1) and f(5).

8 The function E, where $E(x) = 2^x$, is an exponential function.
(a) Find the values of E(1), E(2) and E(5).
(b) If $E(x) = 16$, state the value of x.

9 $f(x) = \frac{x+3}{x-1}$
(a) What value of x must be excluded from the domain of this function?
(b) Find the positive value of x for which $f(x) = x$.

10 If $f(x) = \frac{x+3}{x+2}$, find the values of x for which $f(x) = 2x$ to 2 decimal places.

11 Given that $g(x) = x^2 - 4x - 6$ solve the equation $g(x) = x$.

12 Given that $f(x) = x^2 - 4x + 1$ solve the equations
(a) $f(x) = x - 3$, (b) $f(2x) = 13$.

13 For the linear function $f(x) = ax + b$, where a and b are constants, $f(-2) = 7$ and $f(2) = -1$. Find the values of a and b.

14 $f(x) = ax^2 + bx + c$, where a, b and c are constants. If $f(0) = 7$, what is the value of c? Given also that $f(1) = 6$ and $f(-1) = 12$, find the value of a and of b.

15 For the function $f(x) = px^2 + qx + r$, where p, q and r are constants, $f(0) = 4$, $f(-1) = 8$ and $f(-2) = 18$. Find the values of p, q and r.

16 $F(x) = x^2 - 2x$. What values of x have an image of 15?

17 The function h is given by $h(t) = 7t - 2t^2$. Find the values of t whose image is 5.

18 $f(x) = \frac{x-1}{2x^2 - x - 3}$
(a) Find f(2) and $f(\frac{1}{2})$.
(b) Find x if $f(x) = 0$.
(c) What values of x must be excluded from the domain?

19 The number of diagonals in a polygon with n sides is given by the function $D(n) = \frac{n(n-3)}{2}$.
(a) State the domain of this function.
(b) Find the number of diagonals in polygons with 4, 5 and 10 sides.
(c) If $D(n) = 20$, find the value of n.

20 The domain for the function $f(x) = 2x^2 + 1$ is $\{-2, -1, 0, 1, 2\}$.
Find the range.

21 The domain of the function $f(x) = \frac{2x+1}{x-1}$ is $\{0, 2, 4\}$.
Find the range of the function.

22 If the range for the function $g(x) = x^2 - 2$ is $\{-2, -1, 7\}$, find the domain.

23 The range of the function $f(x) = 1 - \frac{2}{x}$ is $\{-1, 2, 4\}$. Find the domain.

24 S is the function $S: x \longmapsto \sin x°, 0 \le x \le 180$.
(a) Find (correct to 2 decimal places) S(30), S(50), S(120).
(b) If $S(x°) = 1$, what is the value of x?
(c) State the range of S.

25 Functions f and g are given as $f(x) = x^2 - x$ and $g(x) = 2x - 3$.
(a) Find f(0), f(–1), g(0) and g(–1).
(b) If $f(x) + g(x) = 3$, find x.
(c) If $f(p) + g(-p) = 1$, find p.
(d) If $f(z) = g(z) + 1$, find z.

26 Given that $f(x) = x^2 - 3x + 6$ and that $g(x) = x + 6$, solve the equations
(a) $f(x) = 2g(x)$, (b) $f(x) = g(2x)$, (c) $f(2x) = g(x) - 3$.

27 If $f(x) = \frac{x+1}{x^2 - x + 1}$, find the value of k (other than $k = 1$) such that $f(k) = f(1)$.

28 Given the function $f(x) = x^2 - 3x - 2$, express $f(2a) - f(a)$ in its simplest form in terms of a.

29 $f{:}x \longmapsto x^2 - x + 3$
Find $f(p)$, $f(-2p)$ and $f(p - 1)$ in their simplest forms.

30 If $f(x) = 3x + 1$, find $f(a)$, $f(b)$ and $f(a + b)$.
Is $f(a + b) = f(a) + f(b)$?

31 If $f(x) = x^2 + x - 3$, find $f(x + h)$ where h is a constant.
Hence express $\frac{f(x+h) - f(x)}{h}$ in its simplest form.

32 If $f(-x) = f(x)$, f is called an *even* function, but if $f(-x) = -f(x)$, f is called an *odd* function. Which of the following functions are even, which are odd and which are neither?
(a) $2x$ (b) $3x^2$ (c) x^3
(d) $1 - x$ (e) $\frac{1}{x}$ $(x \neq 0)$ (f) $x - \frac{1}{x}$ $(x \neq 0)$

GRAPHICAL REPRESENTATION OF FUNCTIONS

A simple way of illustrating a function graphically is to use two parallel number lines, one for values of the domain, the other for the range. Fig. 3.6 shows the function $f(x) = x - 2$, $x = -1, 0, 1, 2, 3, 4$. An arrowed line joins x in the domain to $f(x)$ in the range.

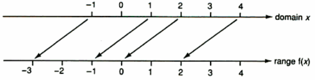


Fig. 3.6

Example 7

Illustrate the function $f(x) = x^2 - x + 2$ *on two number lines for the domain* $\{-2, -1, 0, 1, 2, 3\}$.

Verify that the range is $\{2, 4, 8\}$. Fig. 3.7 shows the result.

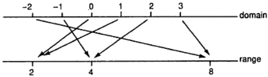


Fig. 3.7

This method is only suitable if the domain consists of a few values. If the domain was all real numbers for example, it would be impossible to show all the arrowed lines. Furthermore, the pattern of the arrowed lines gives no idea of the type of function.

A far better method is to use a **Cartesian graph**, with which you are already familiar. Here we use two perpendicular lines, the x-axis and the y-axis (Fig. 3.8). Values of the domain are placed on the x-axis and the range on the y-axis. Then x and its image $f(x)$ give the coordinates (x,y) of a point. If sufficient points are plotted and joined up, we have the **graph** of the function. $y = f(x)$ is the Cartesian equation of the curve.

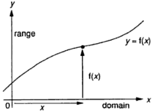


Fig. 3.8

Using this method of representing a function, we find that the graphs of various kinds of functions have characteristic shapes. Hence functions can be recognized from their graphs.

Common Functions And Their Graphs

Fig. 3.9 shows the graphs of some common functions.

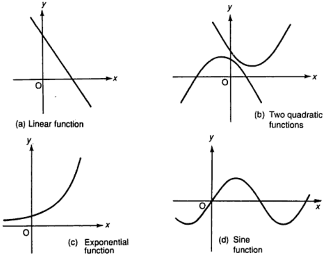
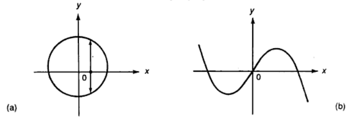


Fig. 3.9

(a) is a linear function such as $y = -3x + 4$.
(b) shows two quadratic functions such as $y = x^2 - x + 4$ (upper graph) and $y = 2 - x - x^2$.
(c) is an exponential function such as 2^x.
(d) is the graph of $y = \sin x$ (see Chapter 7).

Example 8

Which of the graphs in Fig. 3.10 is the graph of a function?



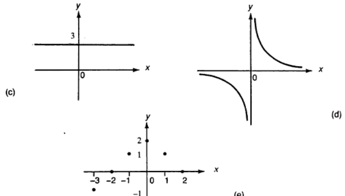


Fig. 3.10

For a function, each value of x in the domain must give just one and only one value of y. If there is more than one value of y for the same value of x in the domain, the graph does not represent a function.

(a) is not the graph of a function, as there are 2 values of y for each value of x.
(b) is the graph of a function.
(c) is the graph of a constant function $y = 3$. The domain is the set of all real numbers but the range is just 3.
(d) is the graph of a function provided $x = 0$ is excluded from the domain.
(e) is the graph of a function for the domain $\{-3,-2,-1,0,1,2\}$. The graph consists only of the points marked and these must not be joined up. The range is $\{2,1,0,-1\}$.

Example 9

Fig. 3.11 shows part of the graph of a function $y = f(x)$. *Sketch the corresponding* ***parts of the functions (a)*** $y_1 = -f(x)$, ***(b)*** $y_2 = f(-x)$, ***(c)*** $y_3 = 2 + f(x)$, ***(d)*** $y_4 = 3 - f(x)$, ***(e)*** $y_5 = f(x + 1)$, ***(f)*** $y_6 = f(x - 2)$.

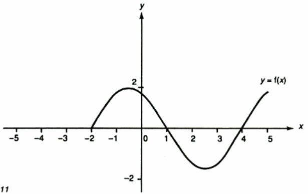


Fig. 3.11

(a) For each value of x, $y_1 = -y$. **So the graph of of** y_1 **is the reflection of** $y = f(x)$ **in** the x-axis (Fig. 3.12(a)).
Points where $y = f(x)$ **meets the** x**-axis are unchanged.**

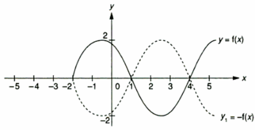


Fig. 3.12(a)

(b) When $x = a$, $y = f(a)$ and $y_2 = f(-a)$.
Now $f(-a)$ is the value of y when $x = -a$. For example when $x = 2$, the value of y_2 is the same as the value of y when $x = -2$.
So the graph of y is reflected in the y-axis to produce the graph of y_2 (Fig. 3.12(b)).
Points where $y = f(x)$ meets the y-axis will be unchanged.

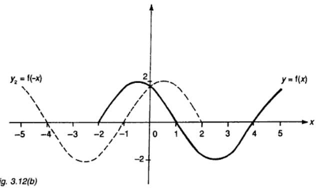


Fig. 3.12(b)

(c) Here 2 is added to each value of y.
So the original graph is shifted upwards through 2 units (Fig. 3.12(c)).

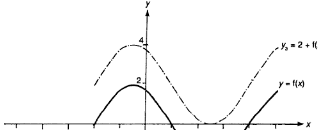


Fig. 3.12(c)

(d) $y_4 = 3 - f(x) = 3 + (-y) = 3 + y_1$.
So the graph of y_1 is shifted upwards through 3 units to obtain the graph of y_4 (Fig. 3.12(d)).

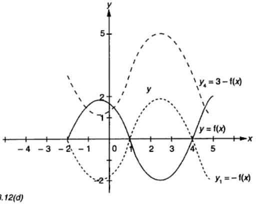


Fig. 3.12(d)

(e) Suppose $x = 1$. Then $y_5 = f(1 + 1) = f(2)$, which is the value of y when $x = 2$.
Again when $x = 3$, $y_5 = f(3 + 1) = f(4)$, which is the value of y when $x = 4$.
All the values of y have been shifted 1 unit **to the left** to obtain y_5 (Fig. 3.12(e)).

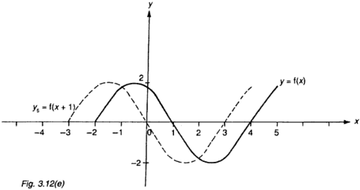


Fig. 3.12(e)

(f) You will be able to work out that y_6 is the original curve shifted 2 units **to the right** (Fig. 3.12(f)).

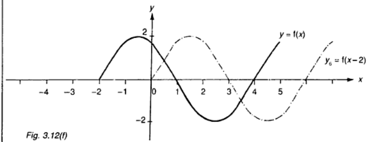


Fig. 3.12(f)

It would be useful to summarize such transformations of the graph of a function $y = f(x)$.

- $y = -f(x)$ is the reflection in the x-axis.
- $y = f(-x)$ is the reflection in the y-axis.
- $y = a + f(x)$ shifts the graph through a units **upwards** if a is **positive**, and **downwards** if a is **negative**.
- $y = f(x + a)$ shifts the graph through a units to the **left** if a is **positive**, but to the **right** if a is **negative**.

Exercise 3.2 *(Answers on page 609.)*

1 Which of the following are graphs of functions?

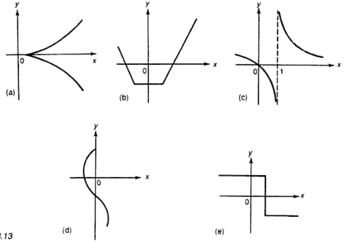


Fig. 3.13

2 Each of the diagrams in Fig. 3.14 shows part of the graph of a function f(x). Copy each diagram and sketch the corresponding parts of

(i) $y_1 = f(-x)$ (ii) $y_2 = f(x - 1)$

(iii) $y_3 = f(x + 1)$ (iv) $y_4 = 1 + f(x + 1)$

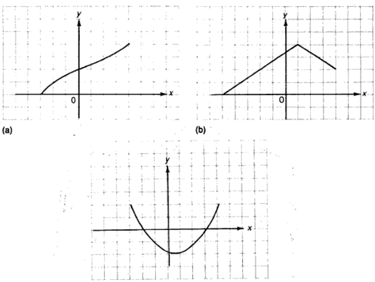


Fig. 3.14

3 On another copy of the diagrams in Fig. 3.14, sketch the corresponding parts of

(i) $y_5 = f(x - 2)$

(ii) $y_6 = 2 - f(x - 2)$

(iii) $y_7 = f(1 - x)$

4 Fig. 3.15 shows part of the graph of $y = f(x)$ with three graphs derived from it. State y_1, y_2 and y_3 in terms of f(x).

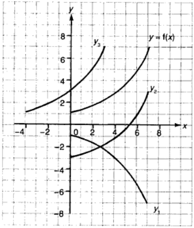


Fig. 3.15

5 The domain of a function f(x) is –1 to 4 inclusive. What would be the corresponding domain for the following?

(a) $y_1 = f(-x)$ (b) $y_2 = f(x - 2)$

(c) $y_3 = f(x + 1)$

6 The range of the function $y = f(x)$ is 0 to 5 inclusive. What is the corresponding range for the following?

(a) $y_1 = -f(x)$ (b) $y_2 = 1 + f(x)$

(c) $y_3 = f(x - 3)$ (d) $y_4 = f(x) - 3$

THE MODULUS OF A FUNCTION

If $y = x$, the values of y are negative when x is negative. They can be converted to positive values by using the **modulus** $y = |x|$, read as 'y = mod x'. $|x|$ gives the **numerical** or **absolute** value of x. For example $|-3.5| = 3.5$. It does not alter 0 or any positive number: $|0| = 0$, $|2| = 2$ etc. $|x|$ is *never negative*.

So we define the modulus of x as

$$|x| = \begin{cases} -x \text{ if } x < 0 \\ x \text{ if } x \geq 0 \end{cases}$$

Similarly the modulus of a function f(x) written $|f(x)|$ is the numerical value of f(x).

Example 10

State the values of $|1 - x|$ for $x = -3, 2, 4$.

When $x = -3$, $|1 - x| = |1 + 3| = 4$.

When $x = 2$, $|1 - x| = |1 - 2| = 1$.

When $x = 4$, $|1 - x| = |1 - 4| = 3$.

Example 11

$f(x) = x^2 - x - 6$. *Find the values of $|f(x)|$ for $x = -1, 0, 2, 4$.*

x	–1	0	2	4
f(x)	–4	–6	–4	6
$\lvert f(x) \rvert$	4	6	4	6

Example 12

What is the least value of x if $|2x - 3| = 2x - 3$?

$|2x - 3|$ will be equal to $2x - 3$ if $2x - 3$ is 0 or greater than 0. Hence the least value of x will be when $2x - 3 = 0$, i.e. when $x = 1\frac{1}{2}$.

Example 13

Draw the graph of $y = |x - 1|$ for the domain $-2 \leq x \leq 3$ and state the range of y.

$-2 \leq x \leq 3$ means that x can take any value between -2 and 3 (inclusive).

We make a table for the integer values of x:

x	-2	-1	0	1	2	3
$x - 1$	-3	-2	-1	0	1	2
$y = \lvert x - 1 \rvert$	3	2	1	0	1	2

Plotting the points given by x and y, the graph is seen to consist of the two lines AB and AC (Fig. 3.16). The range is $0 \leq y \leq 3$.

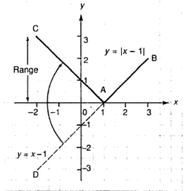


Fig. 3.16

However, if we extend BA to D (shown dotted) where D is $(-2, -3)$ we see that the part AC is the reflection of AD in the x-axis. So a quicker method of drawing the graph is to draw $y = x - 1$ for the given domain first and then reflect any negative part in the x-axis.

To draw a graph of the type $y = |\,f(x)\,|$, draw $y = f(x)$ first and then reflect any negative part in the x-axis.

Example 14

Draw the graph of $y = |2 - x|$ *for the domain* $-1 \le x \le 3$ *and state the range of* y.

Draw the line $y = 2 - x$ first (Fig. 3.17). (The negative part is dotted).

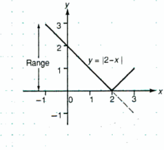


Fig. 3.17

Then reflect the negative part in the x-axis.
The graph consists of two lines meeting on the x-axis where $x = 2$.
The range is $0 \le y \le 3$.

MODULAR INEQUALITIES

Suppose we know that $|\,x\,| > 3$. Suggest some values that x could take to satisfy the inequality.

From the definition of a modulus, $|\,x\,| > 3$ means that either $x > 3$ or $-x > 3$. $-x > 3$ means that $x < -3$ (dividing by -1 and reversing the inequality sign). So the range of x is $x < -3$ **or** $x > 3$. We can show the range on a number line:

x must lie on the thick lined parts. o means this value is excluded.

So if $|x| > k$ then $x < -k$ or $x > k$.

Next suppose $|x| < 3$. Then $x < 3$ or $-x < 3$ i.e. $x > -3$. Hence x lies between -3 and 3 (not inclusive) and we write $-3 < x < 3$.

On a number line we have

So if $|x| < k$ then $-k < x < k$.

These rules apply also to linear and quadratic functions.

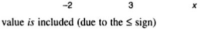
Example 15

Find and show on a number line, the range of values of x if (a) $|x + 1| > 4$, (b) $|1 - 2x| \leq 5$.

(a) From the above, if $|x + 1| > 4$ then $x + 1 < -4$ or $x + 1 > 4$. Hence $x < -5$ or $x > 3$.

(b) If $|1 - 2x| \leq 5$ then $-5 \leq 1 - 2x \leq 5$. Taking each part, $-5 \leq 1 - 2x$ gives $-6 \leq -2x$ or $3 \geq x$ i.e. $x \leq 3$; $1 - 2x \leq 5$ gives $-2x \leq 4$ i.e. $x \geq -2$.

So $-2 \leq x \leq 3$



In this case the ● means the value *is* included (due to the $\leq$ sign)

Exercise 3.3 *(Answers on page 610.)*

1 State the values of
(a) $|-6|$ (b) $|-\frac{1}{2}|$ (c) $|\cos 120°|$ (d) $|3^2 - 6^2|$

2 By testing with $x = -3, 0, 2$ verify that $|1 - x| = |x - 1|$.

3 What is the least value of x for which $|2x - 1| = 2x - 1$?

4 For what domain will the graph of $y = |3 - x|$ be the same as the graph of $y = x - 3$?

5 Find and show on a number line the range of values of x which satisfy the inequalities:
(a) $|2x - 3| > 5$ (b) $\left|\frac{x-3}{2}\right| \leq 4$
(c) $\left|\frac{2x-1}{3}\right| \geq 2$ (d) $\left|1 - \frac{x}{3}\right| < 3$

6 For the domain $-3 \le x \le 4$, draw the graphs of

(a) $y = |x|$ (b) $y = |x + 1|$ (c) $y = -|x - 2|$

(d) $y = |2x - 1|$ (e) $y = |3 - x|$

7 State the range for each of the functions in Question 6.

8 Using the graph you have drawn for part (a) in Question 6, add the graph of $y = -|x|$.

9 On the same piece of graph paper, draw the graphs of $y = |3x|$ and $y = |x - 3|$ for the domain $-2 \le x \le 3$. Hence solve the equation $|3x| = |x - 3|$.

10 By drawing two graphs, solve the equation $|x - 1| = |2x - 5|$. (Take $0 \le x \le 7$ as domain).

11 The range of the function $y = |x - 1|$ is $0 \le y \le 3$. Find a possible domain. What is the widest possible domain?

12 The domain of the function $y = |2x - 3|$ ends where $x = 2$. If the upper limit of the range is 7, what is the least value of the domain?

13 Draw the graph of $y = |x - 1|$ for the domain $-1 \le x \le 2$. Now add the graph of $y = 2 - |x - 1|$ for the same domain. State the range of this function.

THE INVERSE OF A FUNCTION

Fig. 3.18 shows the mapping of the domain $\{-3, 0, 1, 2\}$ by the function $f : x \longmapsto 3x - 2$. Verify that the range is $\{-11, -2, 1, 4\}$.

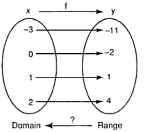


Fig. 3.18

Is there a function that will map the range back to the domain?

The function f in Fig.3.18 mapped x onto y where $y = 3x - 2$. Now we wish to start with y and return to x. If $3x - 2 = y$, then $x = \frac{y+2}{3}$.

So this new function will map y onto $\frac{y+2}{3}$.

Testing this with $y = -11$, we get $\frac{-11+2}{3} = -3$ which is the original value of x. Check the other values.

Such a function, *if it exists*, is called the **inverse** function of f and is written as f^{-1}. (Read this as 'inverse f'.) It is usual to take x as the 'starting' letter so we have

$f^{-1} : x \longmapsto \frac{x+2}{3}$.

Summarizing,

function $f: x \longmapsto 3x - 2$ and inverse $f^{-1}: x \longmapsto \frac{x+2}{3}$.

It then follows that the inverse of f^{-1} is f.

Example 16

Find the inverse function to $f: x \longmapsto \frac{x-3}{2}$.

f maps x onto y where $y = \frac{x-3}{2}$.

Make x the subject of this equation.

$\frac{x-3}{2} = y$ so $x - 3 = 2y$ and $x = 2y + 3$.

Hence $f^{-1} : y \longmapsto 2y + 3$.

Changing to the usual letter x, $f^{-1} : x \longmapsto 2x + 3$.

Suppose -4 was a value in the original domain. Then f will map this onto $3\frac{1}{2}$. f^{-1} will now map this value onto $2(-3\frac{1}{2}) + 3 = -4$, which is the original value. Repeat this check with other values of x, say 0, 1 and 5.

Example 17

Given the function $f: x \longmapsto \frac{x+p}{x-3}$ $(x \neq 3)$, *where p is a constant,*

(a) find the value of p if $f(5) = 1\frac{1}{2}$,

(b) find f^{-1} *in a similar form,*

(c) state the value of x for which f^{-1} *is undefined.*

(a) $f(5) = \frac{5+p}{5-3} = 1\frac{1}{2}$

Then $5 + p = 3$ and $p = -2$.

(b) From (a), $f(x) = \frac{x-2}{x-3}$ i.e. $y = \frac{x-2}{x-3}$

or $yx - 3y = x - 2$, and $x(y - 1) = 3y - 2$.

Hence $x = \frac{3y-2}{y-1}$.

Therefore $f^{-1} : x \longmapsto \frac{3x-2}{x-1}$.

(c) f^{-1} is undefined for $x = 1$. (This means that there is no value of x in the original domain which had an image of 1. So 1 does not exist in the range and therefore cannot be used).

Example 18

Find the inverse of f : $x \longmapsto 3 - x$.

f maps x onto y where $y = 3 - x$.
So $x = 3 - y$ and the inverse function will be $f^{-1} : x \longmapsto 3 - x$, which is the same function as f.
Check this by taking $x = 3, -1$ and 5.
Such a function f is called **self-inverse**, i.e. it is its own inverse.

Functions With No Inverse

Some functions do not have an inverse. Take the function f : $x \longmapsto x^2$ (Fig. 3.19).

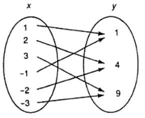


Fig. 3.19

Two arrows arrive at 1 in the range. An inverse would have two paths to return from 1 to the domain and so could not be a function. There is no inverse function.

An inverse function can only exist if the original function is a **one-to-one** function (Fig. 3.20(a)), i.e. there is only one arrow reaching each member of the range. There will be no inverse if the function is a **many-to-one** function (Fig. 3.20(b)), i.e. more than one arrow reaches some members of the domain.

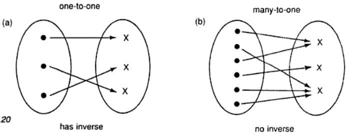


Fig. 3.20

The inverse function f^{-1} exists only if f is one-to-one.

Graphical Illustration of an Inverse Function

Verify that the inverse of $f : x \longmapsto 2x - 3$ is $f^{-1} : x \longmapsto \frac{x+3}{2}$.

Now draw the lines

$$y = 2x - 3 \qquad \text{(i)}$$

and

$$y = \frac{x+3}{2} \qquad \text{(ii)}$$

on graph paper (Fig. 3.21). Add the line $y = x$ (shown dotted).

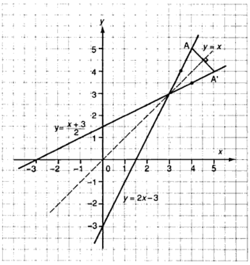


Fig. 3.21

How do the two lines (i) and (ii) appear in relation to the line $y = x$?

Consider the point where $x = 4$ (point A) on (i). The image of 4 from f is 5, so the coordinates of A are (4,5).

Now if we take $x = 5$, its image in f^{-1} will be 4. This gives point A'(5,4) which lies on line (ii).

The gradient of AA' is −1 so AA' is perpendicular to the line $y = x$ and the midpoint of AA' $(4\frac{1}{2}, 4\frac{1}{2})$ lies on the line $y = x$. Hence A and A' are reflections of each other in the line $y = x$.

We can repeat this for any other point. The coordinates will be interchanged by the inverse function, so the two points are reflections of each other. Hence lines (i) and (ii) are reflections of each other in the line $y = x$. You can also test this by folding the graph paper along the line $y = x$.

The graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of each other in the line $y = x$.

Exercise 3.4 *(Answers on page 611.)*

1 Find the inverses of the following functions in the same form:

(a) $f: x \longmapsto x$
(b) $f: x \longmapsto x + 2$
(c) $f: x \longmapsto 2x - 1$
(d) $g: x \longmapsto 3x + 4$
(e) $f: x \longmapsto \frac{x+1}{3}$
(f) $f: x \longmapsto 9 - x$
(g) $f: x \longmapsto 2x - 5$
(h) $f: x \longmapsto 8 - 2x$
(i) $f: x \longmapsto \frac{x}{3} - 1$
(j) $h: x \longmapsto \frac{8}{x}\ (x \neq 0)$
(k) $f: x \longmapsto \frac{5}{x+1}\ (x \neq -1)$
(l) $F: x \longmapsto \frac{3}{x} + 2\ (x \neq 0)$
(m) $f: x \longmapsto \frac{x+1}{x-2}\ (x \neq -2)$
(n) $h: x \longmapsto \frac{2x-1}{x-3}\ (x \neq 3)$

2 Which of the functions in Question 1 are self-inverse?

3 Given $f^{-1}: x \longmapsto 2x - 3$, find f in the same form.

4 If $f^{-1}: x \longmapsto \frac{x+3}{2}$, find f in the same form.

5 $f: x \longmapsto a - x$, where a is a constant, is a self-inverse function. Given that $f^{-1}(4) = 3$, find the value of a.

6 Given the function $h: x \longmapsto \frac{1-x}{x-4}\ (x \neq 4)$, find the value of $h^{-1}(-3)$.

7 Given the function $g: x \longmapsto \frac{x+3}{x+2}\ (x \neq -2)$, find $g^{-1}(-1)$.

8 Given the function $f: x \longmapsto \frac{x+d}{x-1} (x \neq 1)$ and that $f(2) = 5$,
find (a) the value of d, (b) f^{-1}.
What can be said about this function?

9 $f: x \longmapsto \frac{x+r}{x+s}$, where r and s are constants and $f(4) = 6$, $f(-1) = -\frac{1}{4}$. Find
(a) the values of r and s,
(b) the value of x for which f is undefined,
(c) f^{-1} in the same form,
(d) the value of x for which f^{-1} is undefined.

10 Fig. 3.22 shows part of the graph of a function $y = f(x)$. Copy the diagram and sketch the graph of f^{-1}.

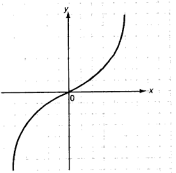


Fig. 3.22

11 On graph paper, draw the graph of $f : x \longmapsto 3 - x$. Construct the reflection of this graph in $y = x$. Explain your result.

12 (a) Find the inverse of $f : x \longmapsto \frac{6-3x}{2}$.

(b) On graph paper,draw the graph of $y = \frac{6-3x}{2}$.

(c) Construct the reflection of the graph in part (b) in $y = x$. Show that this is the graph of f^{-1}.

13 The function f is defined as $f : x \longmapsto \begin{cases} x + 3 \text{ for } x \geq 0 \\ 2x + 3 \text{ for } x < 0 \end{cases}$

Sketch the graphs of f and f^{-1}.

14 Copy Fig. 3.23 and sketch the inverse of the function $y = f(x)$.

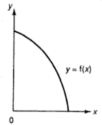


Fig. 3.23

15 (a) If $f(x) = 3 - \frac{2}{x}$, solve the equation $f(x) = x$.

(b) Draw the graph of $f(x)$ for $\frac{1}{2} \leq x \leq 2$.

(c) Add a sketch of the graph of $f^{-1}(x)$ for $-1 \leq x \leq 2$.

Composite Functions

Consider the function $f: x \longmapsto 2x - 3$ (Fig. 3.24). 4 is mapped onto 5.

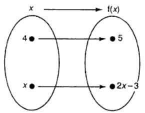


Fig. 3.24

Let g be another function such that g : $x \longmapsto x + 1$.
We now use g on f(4) to obtain g[f(4)] = 6. So 4 has been mapped onto 6 by f followed by g (Fig. 3.25).

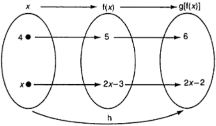


Fig. 3.25

Can we find a single function h which combines f and g?

x is mapped onto $2x - 3$ by f and this is the starting value for g. So g maps $2x - 3$ onto $(\mathbf{2x - 3}) + 1 = 2x - 2$. Hence h : $x \longmapsto 2x - 2$. If $x = 4$, the final result is 6 as we have seen. h is called the **composite** (or **combined**) function g[f(x)] which we write briefly as **h = gf.**

second first

Note carefully that the *first* function is written on the *right*.

Now suppose we do g first, followed by f, i.e. fg.

$$x \xmapsto{\text{g}} x + 1 \xmapsto{\text{f}} 2(\boldsymbol{x} + \mathbf{1}) - 3 = 2x - 1$$

fg

The result is different. **fg is not the same function as gf**. We say the combination of functions is **not commutative**, i.e. the order in which they are done is important and cannot (in general) be interchanged. However for some values of x, fg may be equal to gf.

NB: Take care! fg does **NOT** mean f × g when dealing with functions.

Example 19

If f : $x \longmapsto 2x - 3$, *find (a)* f^{-1} *and (b)* f^{-1}f.

(a) $y = 2x - 3$ so $x = \frac{y + 3}{2}$

Therefore $\text{f}^{-1} : x \longmapsto \frac{x + 3}{2}$

(b) f^{-1}f means that we do f first, f^{-1} second,

$$x \xmapsto{\text{f}} 2x - 3 \xmapsto{\text{f}^{-1}} \frac{2x - 3 + 3}{2} = x$$

So $\text{f}^{-1}\text{f} : x \longmapsto x$

i.e. $\text{f}^{-1}\text{f}(x) = x$

Verify that ff^{-1} gives the same result.

Part (b) above is an example of an important result:

$$f^{-1}f(x) = ff^{-1}(x) = x$$

This follows from the definition of the inverse function. f maps x onto the range giving $f(x)$. f^{-1} operates on $f(x)$ to return to the original element x. So $f^{-1}f(x) = x$. Similarly, if we start from the range, $ff^{-1}(x) = x$.

Example 20

If $f : x \longmapsto x^2$ *and* $g : x \longmapsto x - 1$, *find in a similar form, (a)* fg *and (b)* gf.

(a) fg is g first, f second.

$x \overset{g}{\longmapsto} x - 1 \overset{f}{\longmapsto} (x - 1)^2$

So the combined function fg is $fg : x \longmapsto (x - 1)^2$.

(b) gf is f first, g second.

$x \overset{f}{\longmapsto} x^2 \overset{g}{\longmapsto} x^2 - 1$

The combined function gf is $x^2 - 1$.

Note that $fg \neq gf$.

Example 21

Functions f *and* g *are defined as*

$$f : x \longmapsto \frac{2}{x+1} \text{ and } g : x \longmapsto 3x - 2.$$

Find (a) fg, *(b)* gf, *(c)* $(fg)^{-1}$, *(d)* $(gf)^{-1}$.
(e) For what value(s) of x *is* gf = fg?

(a) $x \overset{g}{\longmapsto} 3x - 2 \overset{f}{\longmapsto} = \dfrac{2}{(3x-2)+1} = \dfrac{2}{3x-1}$

Hence $fg : x \longmapsto \dfrac{2}{3x-1}, x \neq \frac{1}{3}$.

(b) $x \overset{f}{\longmapsto} \dfrac{2}{x+1} \overset{g}{\longmapsto} 3\left(\dfrac{2}{x+1}\right) - 2 = \dfrac{6-2x-2}{x+1} = \dfrac{4-2x}{x+1}$

Hence $gf : x \longmapsto \dfrac{4-2x}{x+1}, x \neq -1$.

(c) $(fg)^{-1}$ is the inverse of the combined function fg.

Now fg maps x onto $y = \dfrac{2}{3x-1}$ from (a).

So $3xy - y = 2$

i.e. $3xy = y + 2$ giving $x = \dfrac{y+2}{3y}$.

Hence $(fg)^{-1} : x \longmapsto \dfrac{x+2}{3x}, x \neq 0$.

(d) $(gf)^{-1}$ is the inverse of gf.

Verify that $(gf)^{-1} : x \longmapsto \frac{4-x}{x+2}, x \neq -2$.

(e) If gf = fg, then $\frac{4-2x}{x+1} = \frac{2}{3x-1}$, $x \neq -1, \frac{1}{3}$.

So $2x + 2 = (3x - 1)(4 - 2x) = -6x^2 + 14x - 4$ or $6x^2 - 12x + 6 = 0$.

Then $x^2 - 2x + 1 = 0$ or $(x - 1)(x - 1) = 0$ giving $x = 1$.

This is the only value of x for which gf = fg.

Example 22

Using the functions f *and* g *in Example 21, find* f^{-1} *and* g^{-1}. *Show that* $(fg)^{-1} = g^{-1}f^{-1}$. *Suggest and test a similar result for* $(gf)^{-1}$.

Verify that $f^{-1} : x \longmapsto \frac{2-x}{x}$ *and* $g^{-1} : x \longmapsto \frac{x+2}{3}$

From (c) in Example 21, $(fg)^{-1} : x \longmapsto \frac{x+2}{3x}$.

$g^{-1}f^{-1}$ is given by $x \xmapsto{f^{-1}} \frac{2-x}{x} \xmapsto{g^{-1}} \frac{\frac{2-x}{x}+2}{3} = \frac{2+x}{3x}$

Hence $(fg)^{-1} = g^{-1}f^{-1}$.

So the inverse of fg is the inverse of f followed by the inverse of g. This suggests that $(gf)^{-1} = f^{-1}g^{-1}$. Show that this is correct using $(gf)^{-1}$ from Example 21.

The results of Example 22 are true in general:

$$(fg)^{-1} = g^{-1}f^{-1}$$
$$(gf)^{-1} = f^{-1}g^{-1}$$

Example 23

Given that $f : x \longmapsto \frac{x}{x+2}$ $(x \neq -2)$, *find in a similar form (a)* f^2, *(b)* f^3, *(c)* f^4 *and deduce an expression for* f^n.

(a) f^2 means ff, i.e. f done twice in succession.

So $x \xmapsto{f} \frac{x}{x+2} \xmapsto{f} \frac{\frac{x}{x+2}}{\frac{x}{x+2}+2} = \frac{x}{x+2x+4} = \frac{x}{3x+4}$, $x \neq -2, -\frac{4}{3}$

(b) f^3 means f^2 followed by f.

So $f^3(x) = f[f^2(x)] = \frac{\frac{x}{3x+4}}{\frac{x}{3x+4}+2} = \frac{x}{7x+8}$, $x \neq -2, \frac{4}{3}, -\frac{8}{7}$

(c) $f^4(x) = f[f^3(x)] = \frac{\frac{x}{7x+8}}{\frac{x}{7x+8}+2} = \frac{x}{15x+16}$, $x \neq -2, -\frac{4}{3}, -\frac{8}{7}, -\frac{16}{15}$

Studying the pattern, the numerator is always x. The denominators are $3x + 4$, $7x + 8$, $15x + 16$ so the next denominator will be $31x + 32$.

f^5 is $x \longmapsto \frac{x}{31x + 32}$, $x \neq -2, -\frac{4}{3}, -\frac{8}{7}, -\frac{16}{15}, -\frac{32}{31}$.

Example 24

If $f : x \longmapsto ax + b$ $(a > 0)$ *and* $f^2 : x \longmapsto 9x - 8$, *find (a) the values of a and b, (b)* f^3, (c) f^4.
(d) Deduce f^5

(a) We first find f^2 in terms of a and b.

$$x \xmapsto{f} ax + b \xmapsto{f} a(ax + b) + b = a^2x + ab + b$$

But this is $9x - 8$.

Hence $a^2 = 9$ which gives $a = 3$ (since $a > 0$) and $ba + b = -8$ so $b = -2$.

(b) $f^3(x) = f[f^2(x)] = 3(9x - 8) - 2 = 27x - 26$

(c) $f^4(x) = f^2[f^2(x)] = 9(9x - 8) - 8 = 81x - 80$

(d) The pattern in these results is:

$f^2 : 9x - 8 = 3^2x - (3^2 - 1)$
$f^3 : 27x - 26 = 3^3x - (3^3 - 1)$
$f^4 : 81x - 80 = 3^4x - (3^4 - 1)$
so we can deduce that $f^5 = 3^5x - (3^5 - 1) = 243x - 242$.

Example 25

Given $f : x \longmapsto 2x - 5$, *find a function* g *such that* $fg : x \longmapsto 6x - 1$.

Clearly g must be a linear function as no squares appear in fg.
Take g as $x \longmapsto ax + b$.

Then $fg : x \longmapsto 2(ax + b) - 5 = 2ax + 2b - 5$.
But this must be identical to $6x - 1$.
Then $2a = 6$, giving $a = 3$ and $2b - 5 = -1$, giving $b = 2$.
Hence $g : x \longmapsto 3x + 2$.

Example 26

Express in terms of the functions $f : x \longmapsto x + 3$ *and* $g : x \longmapsto x^2$,
(a) $x^2 + 3$, *(b)* $x^2 + 6x + 9$, *(c)* $x + 6$, *(d)* $x^2 + 6x + 12$, *(e)* $x^2 - 6x + 9$.

(a) This is fg.

(b) Note that $x^2 + 6x + 9 = (x + 3)^2$. f gives $(x + 3)$. g gives the square. So this is gf.

(c) Here g is not involved as there is no square. Try ff.

(d) Note that $x^2 + 6x + 12 = (x + 3)^2 + 3$. We get $(x + 3)^2$ from gf.
If we now use f, we obtain the result. The answer is therefore fgf (first f, then g and lastly f again).

(e) $x^2 - 6x + 9 = (x - 3)^2$. Now f does not produce $(x - 3)$ but $f^{-1} : x \longmapsto x - 3$.
Hence the answer is gf^{-1}.

Exercise 3.5 *(Answers on page 613.)*

1 Using the functions $f : x \longmapsto x + 2$ and $g : x \longmapsto x - 3$, find in the same form (a) fg, (b) gf, (c) ff, (d) gg.

2 $S : x \longmapsto \sin x°$ and $T : x \longmapsto 2x$ are two functions.
Find (a) ST(20), (b) TS(20).

3 Taking $f : x \longmapsto x + 2$ and $g : x \longmapsto 3x - 1$, find (a) fg, (b) gf, (c) f^{-1}, (d) g^{-1}, (e) f^{-1}, (f) $g^{-1}f$.

4 If $f : x \longmapsto x + 1$, find (a) f^2, (b) f^3 and deduce (c) f^4, (d) f^5 (e) f^n.

5 Taking the function f as $f : x \longmapsto \frac{x-3}{x+2}$, $x \neq -2$, find (a) f^{-1}, (b) f^2, (c) $(f^2)^{-1}$. In each case, state the values of x which must be excluded from the domain.

6 If $f : x \longmapsto x^2 - 2$ and $g : x \longmapsto x + 3$, find (a) fg, (b) gf.
For what value of x is fg = gf?

7 Given that $g : x \longmapsto x + 2$ and $h : x \longmapsto x^2 - 3$, find the value of x for which gh = hg.

8 For the functions $f : x \longmapsto x - 4$ and $g : x \longmapsto 3x - 2$, find similarly (a) f^{-1}, (b) g^{-1}, (c) fg^{-1}, (d) $(fg)^{-1}$.

9 Functions f and g are defined by $x \longmapsto 2x + 1$ and $x \longmapsto 1 - 3x$ respectively.
For what value of x is $fg^{-1} = f^{-1}g$?

10 Functions f and g are defined as $f : x \longmapsto \frac{x-1}{2}$ and $g : x \longmapsto \frac{1}{x}$ $(x \neq 0)$ respectively. Find similarly (a) fg, (b) $g^{-1}f$, (c) $f^{-1}g^{-1}$. In each case, state the values of x which must be excluded from the domain. (d) For what values of x is $g^{-1}f = f^{-1}g^{-1}$?

11 The functions f and g are defined as $f : x \longmapsto 3x + 2$ and $g : x \longmapsto \frac{1}{x}$ $(x \neq 0)$.
Find similar expressions for (a) fg, (b) gf, (c) $f^{-1}g$, (d) gf^{-1}. In each case, state the values of x which must be excluded from the domain.
Find the value(s) of x for which (e) fg = gf, (f) $f^{-1}g = gf^{-1}$.

12 Given $f : x \longmapsto 1 - \frac{1}{x}$, $x \neq 0$, and $g : x \longmapsto \frac{1}{1-x}$, $x \neq 1$, find fg(x) and gf(x). Hence state the inverses of f and g.

13 f is the function that maps x onto $\frac{x+1}{x-1}$ $(x \neq 1)$.
(a) Show that f is self-inverse.
(b) Find f^2.
(c) Show that $f^3 = f$.

14 $f : x \longmapsto ax + b$ (a, b constants) and $g : x \longmapsto 2x + 3$ are two functions.
(a) If fg = gf, find a relation between a and b.
(b) Given that $f^{-1}(7) = -1$, find the values of a and b.

15 If f maps x onto $5 - \frac{x}{2}$ and g maps x onto $2x + 1$, show that fg and gf are both self-inverse.

16 $f : x \longmapsto \frac{x-2}{x+1}$, $x \neq -1$
(a) Find f^2. State the value of x which must be excluded from the domain.
(b) If $f^2(x) = -1$, find the value of x.

17 If $f(x) = \frac{x-1}{x+2}$, $x \neq -2$, find f^2 and f^3. In each case, state the values of x which must be excluded from the domain. Solve the equation $f^3(x) = 1$.

18 $f : x \longmapsto 3x + 1$. Find a function g so that $gf : x \longmapsto 3x + 2$.

19 If $f : x \longmapsto 2x + 3$, find a function g so that $fg : x \longmapsto 2x - 1$.

20 Express the following in terms of the functions $g : x \longmapsto x + 2$ and $h : x \longmapsto 3x$.
(a) $x \longmapsto 3x + 2$ (b) $x \longmapsto 3x + 6$
(c) $x \longmapsto x + 4$ (d) $x \longmapsto 3x + 12$
(e) $x \longmapsto 9x$ (f) $x \longmapsto 9x + 2$
(g) $x \longmapsto x - 2$ (h) $x \longmapsto 3x - 6$

21 Given $f : x \longmapsto x + 3$ and $g : x \longmapsto x^2 - 1$, state the following in terms of f and g.
(a) $x \longmapsto x^2 + 2$ (b) $x \longmapsto x^2 + 6x + 8$
(c) $x \longmapsto x + 6$ (d) $x \longmapsto x^2 + 12x + 35$
(e) $x \longmapsto x^2 - 6x + 8$ (f) $x \longmapsto x^2 - 4$

22 Given $f : x \longmapsto \sqrt{x}$ (positive root) and $g : x \longmapsto x + 2$, express the following in terms of f and g:
(a) $x \longmapsto \sqrt{x+2}$ (b) $x \longmapsto \sqrt{x} + 2$
(c) $x \longmapsto x + 4$ (d) $x \longmapsto \sqrt{x+4}$
(e) $x \longmapsto \sqrt{x-2}$ (f) $x \longmapsto x^2 + 4x + 4$
(g) $x \longmapsto x^2 - 4x + 4$ (h) $x \longmapsto x^2 + 8x + 16$

23 If $f : x \longmapsto x - 3$, what is the function g which makes $gf : x \longmapsto x^2 - 6x + 10$?

24 $f : x \longmapsto 2 + \frac{3}{x-1}$, $x \neq 1$, and $g : x \longmapsto x + 4$. Find the inverse of fg in a similar form.

25 f is given by $f : x \longmapsto \frac{x}{x-3}$ $(x \neq 3)$. Find (a) f^2, (b) f^3, (c) f^4. Deduce f^5.
In each case, state the values of x that must be excluded from the domain.

SUMMARY

- A function f maps an input x (domain) onto a unique image y (range).
  $\mathrm{f}: x \longmapsto y = \mathrm{f}(x)$
- $y = \mathrm{f}(x)$ is the equation of the graph of the function.
  $y = -\mathrm{f}(x)$ is the reflection of $y = \mathrm{f}(x)$ in the x-axis.
  $y = \mathrm{f}(-x)$ is the reflection of $y = \mathrm{f}(x)$ in the y-axis.
  $y = a + \mathrm{f}(x)$ shifts the graph upwards through a units if $a > 0$, and downwards if $a < 0$.
  $y = \mathrm{f}(x + a)$ shifts the graph through a units to the left if $a > 0$, and to the right if $a < 0$.
- Modulus of x : $|x| = x$ for $x \geq 0$,
  $= -x$ for $x < 0$.
- If $|x| > k$, then $x < -k$ or $x > k$;
  if $|x| < k$, then $-k < x < k$
- To draw the graph of $y = |\mathrm{f}(x)|$, first draw the graph of $y = \mathrm{f}(x)$ and then reflect any negative part in the x-axis.
- If f is one-to-one, the inverse function f^{-1} exists.
  $\mathrm{ff}^{-1}(x) = \mathrm{f}^{-1}\mathrm{f}(x) = x$
- If $\mathrm{f} = \mathrm{f}^{-1}$, f is self-inverse.
- The graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ are reflections of each other in the line $y = x$.
- Functions may be combined, but the order is important.

$$\underset{\text{second}}{\mathrm{g}}\underset{\text{first}}{\mathrm{f}} : x \xmapsto{\ \mathrm{f}\ } \mathrm{f}(x) \xmapsto{\ \mathrm{g}\ } \mathrm{g}[\mathrm{f}(x)]$$

$$\mathrm{fg} : x \xmapsto{\ \mathrm{g}\ } \mathrm{g}(x) \xmapsto{\ \mathrm{f}\ } \mathrm{f}[\mathrm{g}(x)]$$

- f^2 means ff, and so on.
- $(\mathrm{fg})^{-1} = \mathrm{g}^{-1}\mathrm{f}^{-1}$; $(\mathrm{gf})^{-1} = \mathrm{f}^{-1}\mathrm{g}^{-1}$

REVISION EXERCISE 3 *(Answers on page 614.)*

A

1 $\mathrm{f}: x \longmapsto 2x - 3$. Find the domain of x if $-5 \leq \mathrm{f}(x) \leq 3$.

2 f is a function given by $\mathrm{f}: x \longmapsto \frac{2x+1}{x-3}$, $(x \neq 3)$.
(a) Find f^{-1}.
(b) State the value of x for which f^{-1} is undefined.

3 (a) Solve these inequalities and show the results on a number line for each one:

(i) $| 4x - 3 | \geq 2$ (ii) $\left| 1 - \frac{3x}{4} \right| < 4$

(b) Given that $| ax + b | \leq 5$ where a and b are constants and that $-4 \leq x \leq 1$, find the value of a and of b.

4 On the same diagram, sketch the graphs of

(a) $y = | x - 2 |$,

(b) $y = 2 | x - 2 |$,

(c) $y = 2 - | x - 2 |$ for the domain $-2 \leq x \leq 4$.

5 On graph paper, sketch the graphs of

(a) $y = | x + 1 |$,

(b) $y = | 3 - x |$.

Hence solve the equation $| x + 1 | = | 3 - x |$.

6 Fig. 3.26 shows part of the graph of $y = f(x)$. Copy the diagram and add the graphs of

(a) $y_1 = f(-x)$,

(b) $y_2 = f(x - 1)$,

(c) $y_3 = f(1 - x)$.

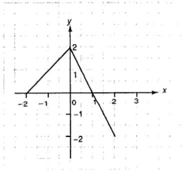


Fig. 3.26

7 Given the function $f : x \longmapsto 3x - \frac{10}{x}, x \neq 0$, find the value of f(2) and the values of x whose image under f is 1.

8 $g : x \longmapsto \frac{2x + 3}{x - 2}, x \neq 2$. Show that $gg(x) = x$ for all values of x except $x = 2$.

9 For the functions $f : x \longmapsto x^2 - 4$ and $g : x \longmapsto 2x + 3$, find in a similar form

(a) fg, (b) gf.

(c) Find the values of x for which fg = gf.

10 The function R maps x onto the remainder when 16 is divided by x. If the domain is $\{2, 3, 5, 7\}$, state the range. Does R^{-1} exist?

11 A function f is defined as $f : x \longmapsto \frac{x}{x + 1}, x \neq -1$. Prove that

$f^2 : x \longmapsto \frac{x}{2x + 1}, x \neq -1, \frac{1}{2}$. Obtain a similar expression for f^3 and hence suggest a possible expression for f^n. (C)

12 Given that $f : x \longmapsto x + 2$ and $gf : x \longmapsto x^2 + 4x + 2$, find the function g. Hence express $x \longmapsto x^2 - 4x + 2$ in terms of f and g.

13 (a) The function $f : x \longmapsto 3x + a$ is such that $ff(6) = 10$. Find the value of a and of $f^{-1}(4)$.

(b) Functions f and g are defined by

$f : x \longmapsto \frac{8}{x-3}, x \neq 3, g : x \longmapsto 2x - 3$

(i) Find expressions for f^{-1}, fg and gf.

(ii) Find the value of x for which $fg(x) = gf(x)$.

(c) The function $f : x \longmapsto 2x - 5$ is defined for the domain $x \geq 1$. State the range of f and the corresponding range of ff (C)

14 Fig. 3.27 illustrates part of the function $f : x \longmapsto y$, where $y = ax + b$. Calculate the value of a and of b.

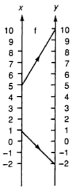


Fig. 3.27

Find the end-points of the shortest arrow that can be drawn for this function.

15 (a) Functions f and g are defined by $f : x \longmapsto 3x - 2$ and $g : x \longmapsto \frac{12}{x} - 4$ $(x \neq 0)$. Find an expression for the function (i) ff, (ii) fg, (iii) g^{-1}.

(b) The function $h : x \longmapsto x^3 + ax + b$ is such that the equation $h(x) = x$ has solutions of $x = 2$ and $x = 3$. Find the value of a and of b. (C)

16 The functions f and g are defined over the positive integers by $f : x \longmapsto 6 - 2x$ and $g : x \longmapsto \frac{2}{x}, x \neq 0$.

Express in similar form (a) fg, (b) gf, (c) f^{-1}, (d) g^{-1}, (e) $(fg)^{-1}$.
Find the value of x for which $ff(x) = gg(x)$.

17 Express in terms of the functions $f : x \longmapsto \sqrt{x}$, $x \geq 0$ and $g : x \longmapsto x + 5$

(a) $x \longmapsto \sqrt{x+5}, x \geq -5$ **(b)** $x \longmapsto x - 5$

(c) $x \longmapsto x + 10$ **(d)** $x \longmapsto \sqrt{x} + 10, x \geq 0$

(e) $x \longmapsto x^2 + 5$ **(C)**

18 Fig. 3.28 shows part of the mapping of x to y by the function $\mathrm{f}: x \longmapsto 9x - a$ and part of the mapping of y to z by the function $\mathrm{g}: y \longmapsto \frac{b}{12-y}$, $y \neq 12$.

(a) Find the values of a and b.

(b) Express in similar form the function which maps an element x to an element z.

(c) Find the element x which is unchanged when mapped to z.

(C)

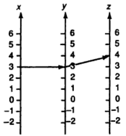


Fig. 3.28

19 Given that $\mathrm{f}: x \longmapsto \frac{x+p}{x-3}$ $(x \neq 3)$ and that $\mathrm{f}(4) = 9$, find

(a) the value of p,

(b) $\mathrm{f}^{-1}(-3)$.

(c) Obtain a similar expression for f^2.

(d) Find the value(s) of x which have the same image under f^2 and f^{-1}.

20 The function P maps x onto $\frac{18}{ax+b}$, $x \neq -\frac{b}{a}$.

(a) Given that $\mathrm{P}(3) = 2$ and $\mathrm{P}(-3) = -6$, find the values of a and b.

(b) Find the value of x whose image under P is $\frac{6}{5}$.

(c) Obtain a similar expression for P^{-1}.

21 (a) Given the functions $\mathrm{f}: x \longmapsto 2x - 5$ and $\mathrm{g}: x \longmapsto \frac{3}{x}$ (for $x \neq 0$), find in a similar form (i) fg, (ii) gf.
Hence solve the equation $\mathrm{fg}(x) = \mathrm{g}(x)$.

(b) Functions p and q are defined as $\mathrm{p}: x \longmapsto \frac{1}{x+3}$, $x \neq -3$, and $\mathrm{q}: x \longmapsto \frac{x}{4}$.
Find in a similar form (i) $\mathrm{p}^{-1}\mathrm{q}$ and (ii) pq^{-1}.

(c) The function h is defined by $\mathrm{h}: x \longmapsto \frac{tx+9}{1-x}$ $(x \neq 1)$.

Find the value of t for which the equation $\mathrm{h}(x) = x$ has the solution $x = 3$.

22 The function f is defined as $\mathrm{f}: x \longmapsto \begin{cases} 2 \text{ for } x \geq 0 \\ x + 2 \text{ for } x < 0 \end{cases}$

Sketch the graphs of f and f^{-1}.

23 If $\mathrm{f}(x) = 3 + \frac{2}{x}$, $x \neq 0$, sketch the graph of $\mathrm{f}(x)$ for $1 \leq x \leq 4$.
Now add a sketch of the graph of $\mathrm{f}^{-1}(x)$ for $3\frac{1}{2} \leq x \leq 5$.

24 (a) Given that $f : x \longmapsto \frac{1}{x-2}$ $(x \neq 2)$ find $f^{-1}(x)$ and $f^2(x)$. Hence solve the equation $f^2(x) + 2f^{-1}(x) = 5$.

(b) If $g : x \longmapsto \frac{a}{x-2}$ $(x \neq 2)$, find the values of a if $g^2(-1) + 2g^{-1}(-1) = -3$.

B

25 For the domain $-3 \leq x \leq 3$, sketch the graph of $y = |\,[x]\,|$, where $[x]$ means the greatest integer less than or equal to x (for example, $[3.4] = 3$, $[-3.4] = -4$ etc). State the range of this function for this domain.

26 Draw the graph of $y = |\,1 - |\,2 - x\,|\,|$ for the domain $-3 \leq x \leq 5$.

27 Fig. 3.29 illustrates the function $y = f(x)$ over the domains $-1 \leq x \leq 0$ and $0 < x \leq 3$. The function is undefined for all other values of x. Sketch the functions given by

(a) $y_1 = f(x) + 1$,

(b) $y_2 = f(x + 1)$.

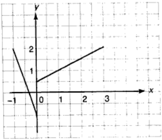


Fig. 3.29

28 f, g and h are functions defined by $f : x \longmapsto \sqrt{x}$, $g : x \longmapsto \frac{x}{2}$ and $h : x \longmapsto x + 1$. Express in terms of f, g and h:

(a) $x \longmapsto \sqrt{\frac{x+1}{2}}$ (b) $x \longmapsto \frac{(x+1)^2}{2}$

(c) $x \longmapsto 2(x + 1)^2$ (d) $x \longmapsto 2x^2 + 1$

29 The functions f and g are defined by

$f : x \longmapsto$ remainder when x^2 is divided by 7,
$g : x \longmapsto$ remainder when x^2 is divided by 5.

(a) Show that $f(5) = g(3)$.

(b) If n is an integer, prove that $f(7n + x) = f(x)$ and state the corresponding result for g. (C)

30 The function T maps (x,y) onto $(x + y, x - 2y)$.

(a) A is the point (2,1). T maps A onto B and B onto C. Find the coordinates of B and C.

(b) The point D is mapped onto E(1,7) by T. Find the coordinates of D.

(c) Another point F is mapped onto G(0,9) by T^2. Find the coordinates of F.

(d) Express T^{-1} in the same form as T.

31 Given that the range of $y = f(1 - x) - 1$ is $-2 \leq y \leq 3$, find the range of (a) $f(x)$, (b) $f(x + 1) + 1$.