Coordinate Geometry

1

CARTESIAN COORDINATES

The position of a point in a plane can be given by an *ordered pair* of numbers, written as (x,y). These are called the **Cartesian coordinates** of the point. (The name comes from the French mathematician Rene Descartes (1596 – 1650)). The coordinates measure the displacement (+ or –) of the point from two perpendicular **axes,** the y-axis (Oy) and the x-axis (Ox), where O is the **origin.**

For example, in Fig.1.1, the coordinates of point A are (4,3) and the coordinates of point B are (3,4). 4 is the x-coordinate of A and 3 is its y-coordinate. (The x-coordinate is sometimes called the *abscissa* and the y-coordinate the *ordinate*).

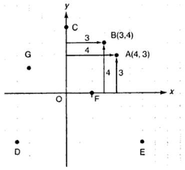


Fig.1.1

The x-coordinate is always stated first. As you can see, (4,3) is not the same point as (3,4). Now state the coordinates of the points C, D, E, F and G in Fig. 1.1.

MIDPOINT OF TWO POINTS

On graph paper, plot the points A(2,3) and B(8,7). Can you write down the coordinates of the midpoint of AB? Can you see how these are related to the coordinates of A and B? (Remember that the midpoint is *halfway* between A and B).

We can find a formula for the midpoint of AB. We could use different letters for coordinates such as (a,b), (c,d), etc. but it is neater to use *suffixes* attached to x and y for specific points. So we write the coordinates of A as (x_1,y_1) and B as (x_2,y_2).

Let the coordinates of the midpoint M be (x_M,y_M) (Fig. 1.2). AC and ME are parallel to the x-axis. MD and BC are parallel to the y-axis.

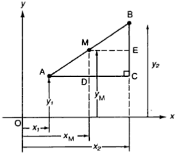


Fig. 1.2

Then AD = DC so $x_M - x_1 = x_2 - x_M$ (i)
and EC = BE so $y_M - y_1 = y_2 - y_M$ (ii)

From (i), $2x_M = x_1 + x_2$, and
from (ii), $2y_M = y_1 + y_2$.

Therefore, $x_M = \frac{x_1 + x_2}{2}$ and $y_M = \frac{y_1 + y_2}{2}$.

Midpoint of (x_1,y_1) and (x_2,y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

The coordinates of the midpoint are the *averages* of the two x-coordinates and of the two y-coordinates of the points.

Example 1

(a) Find the midpoint of (i) (3,4) and (5,2) (ii) (–2,–1) and (4,–4).
(b) If (–2,1) is the midpoint of AB, where A is (–3,2), find the coordinates of B.

(a) (i) The midpoint is $\left(\frac{3+5}{2}, \frac{4+2}{2}\right) = (4,3)$.

(ii) The midpoint is $\left(\frac{-2+4}{2}, \frac{-1-4}{2}\right) = (1,-2\frac{1}{2})$.

(b) If (x_B, y_B) are the coordinates of B, then –2 is the average of –3 and x_B,

so $(-2) = \frac{-3 + x_B}{2}$.

Hence $x_B = -1$.
Similarly $y_B = 0$.
Therefore the coordinates of B are (–1,0).

Exercise 1.1 *(Answers on page 606.)*

1 State the coordinates of the midpoints of:

(a) (0,4) and (3,–2)
(b) (–4,–2) and (–2,6)
(c) (4,–2) and (–6,9)
(d) (0,4) and (4,0)
(e) (–4,–1) and (–5,–2)
(f) (5,–3) and (–5,3)
(g) $(p,2p)$ and $(3p,-4p)$
(h) $(a + 2b, b - a)$ and $(a - 2b, 3a + b)$
(i) $(a, a - 4)$ and $(a + 2, 6 + a)$
(j) $\left(\frac{a+b}{2}, \frac{b-a}{2}\right)$ and $\left(\frac{a-b}{2}, \frac{a+b}{2}\right)$

2 A(1,5) and B(7,–9) are two points. AB is divided into four equal parts at C, D and E. Find the coordinates of C, D and E.

3 A$(3, 1\frac{1}{2})$, B(–5,–3) and C(7,–2) are the vertices of triangle ABC. What are the coordinates of M, the midpoint of BC and of Q, the midpoint of AM?

4 The midpoint of PQ is (2,3). If the coordinates of P are (–1,4), find the coordinates of Q.

5 A is $(a,3)$ and B is $(4,b)$. If the midpoint of AB is (3,5), find the values of a and b.

6 The points A and B are $(a,-4)$ and $(-3,b)$ respectively. If the midpoint of AB is (–2,3), find the values of a and b.

7 L is the point (–3,–2) and M is the point (5,4). N is the midpoint of LM. State the coordinates of N. P is the midpoint of NQ and the coordinates of P are $(2\frac{1}{2},4)$. Find the coòrdinates of Q.

8 ABCD is a parallelogram. A is the point (2,5), B is the point (8,8) and the diagonals intersect at $(3\frac{1}{2},2\frac{1}{2})$. What are the coordinates of C and D?

9 The coordinates of A and B are (–9,3) and (–3,4) respectively. B is the midpoint of AC and C is the midpoint of AD. Find the coordinates of C and of D.

10 A is the point (–1,4), B is the point (5, –2) and C is the point (–4,–5). If D is the midpoint of AB and E the midpoint of DC, find the coordinates of D and E and show that AE is parallel to the *y*-axis.

11 The coordinates of A and C are (–6,–3) and (–1,1) respectively.
(a) If C is the midpoint of AB, find the coordinates of B.
(b) BF is divided into three equal parts at D and E. If the coordinates of E are (6,–1), find the coordinates of D and F.

12 The points (x_1,y_1), (x_2,y_2), (x_3,y_3) and (x_4,y_4), in that order form a parallelogram ABCD. Show that $x_1 + x_3 = x_2 + x_4$ and $y_1 + y_3 = y_2 + y_4$.

13 ABCD is a quadrilateral where A is (1,7), B is (4,3), C is (–1,–3) and D is (–4,5). Is ABCD a parallelogram? If not, state new coordinates for B so that ABCD will be a parallelogram.

14 The points A(–1,4), B(4,10), C(6,–5) and D(–2,–8) form a quadrilateral ABCD. P, Q, R and S are the midpoints of the sides AB, BC, CD and DA respectively. Prove that PQRS is a parallelogram.

DISTANCE BETWEEN TWO POINTS

What is the length of AB in Fig.1.3?

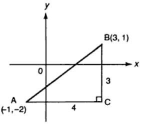


Fig. 1.3

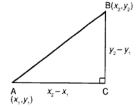
If we draw AC parallel to the *x*-axis and CB parallel to the *y*-axis, then AC = 3 – (–1) = 4 units and BC = 1 – (–2) = 3 units.
By Pythagoras' Theorem, $AB^2 = AC^2 + BC^2 = 16 + 9 = 25$ units^2.
Hence the length of AB = $\sqrt{25}$ = 5 units.
We can generalize this to find a formula for the distance between any two given points.

Take $A(x_1,y_1)$ and $B(x_2,y_2)$ to be the two points (Fig.1.4).
Now $AC = x_2 - x_1$ and $BC = y_2 - y_1$.
Then $AB^2 = AC^2 + BC^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$.
So the formula for the distance between the points (x_1,y_1) and (x_2,y_2) is:

$$\text{Distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Note: Take care with the subtractions if either or both of the coordinates are negative.

Fig. 1.4



Example 2

Find the distance between
(a) (7,13) and (2,1),
(b) (2,–3) and (–3,4).

(a) Distance = $\sqrt{(7-2)^2 + (13-1)^2} = \sqrt{25 + 144} = \sqrt{169} = 13$ units.

(b) Distance = $\sqrt{[2-(-3)]^2 + [-3-4]^2} = \sqrt{25 + 49} = \sqrt{74} \approx 8.6$ units.

Note that this could also be done as:

distance = $\sqrt{[-3-2]^2 + [4-(-3)]^2} = \sqrt{25 + 49}$ units as before.

The coordinates can be subtracted in *either* order as the results are the same after squaring. Verify this for part (a).

Example 3

The vertices of a triangle ABC are A(–2,5), B(4,4) and C(5,–2).
(a) Which is the longest side?
(b) Is the triangle right-angled?
(c) What type of triangle is ABC?

We need only find the squares of the lengths of the sides.

$AB^2 = [-2-4]^2 + [5-4]^2 = 37$ units^2
$BC^2 = [4-5]^2 + [4-(-2)]^2 = 37$ units^2
$CA^2 = [5-(-2)]^2 + [-2-5]^2 = 98$ units^2

(a) AC is the longest side.
(b) $AC^2 \neq AB^2 + BC^2$ so the triangle is not right-angled.
(c) $AB^2 = BC^2$ so the triangle is isosceles.

Example 4

The vertices of a triangle ABC are A(1,3), B(5,11) and C(9,5). Find the lengths of the medians.

You will recall that a median is a line from a vertex to the midpoint of the opposite side.

The midpoint of BC is (7,8).
Hence the length of the median from A is $\sqrt{6^2 + 5^2} = \sqrt{61} \approx 7.8$ units.
Now find the lengths of the other two medians.
You should find that they are 7 units and $\sqrt{40}$ units.

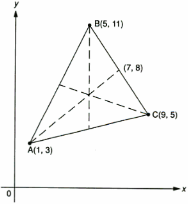


Fig 1.5

Example 5

The vertices of a triangle are A(–2,3), B(3,5) and C(0,–6) (Fig.1.5). D is the midpoint of AB and E is the midpoint of BC. Show that DE = ½ AC.

It is simpler to work with squares of distances, so we find DE^2 and AC^2.
D is $(\frac{1}{2}, 4)$ and E is $(1\frac{1}{2}, -\frac{1}{2})$.

Then $DE^2 = (\frac{1}{2} - 1\frac{1}{2})^2 + (4 + \frac{1}{2})^2 = 1 + \frac{81}{4} = \frac{85}{4}$

$AC^2 = (-2 - 0)^2 + (3 + 6)^2 = 85$

Hence $DE^2 = \frac{1}{4} AC^2$ which means that $DE = \frac{1}{2} AC$.

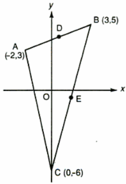


Fig 1.6

Exercise 1.2 *(Answers on page 606.)*

1 Find the distance between the following pairs of points. [Where necessary give your answer correct to 2 significant figures.]

(a) (1,2), (4,6)
(b) (–1,–3), (2,1)
(c) (–4,–5), (1,7)
(d) (0,–3), (4,0)
(e) (–1,–3), (–2,–5)
(f) (–2,1), (4,2)
(g) (–5,0), (–7,–4)
(h) (–5,–2), (0,–3)
(i) $(a,0)$, $(0,a)$
(j) $(a,a+b)$, $(a-b,b)$

2 A circle has centre at (1,2). One point on its circumference is (–3,–1). What is the radius of the circle?

3 The vertices of a triangle are A(–4,–2), B(4,2) and C(2,6).
(a) Is the triangle right-angled?
(b) If a circle is drawn round this triangle, what are the coordinates of its centre?
(c) Hence find the radius of this circle.

4 The vertices of triangle ABC are A(–1,3), B(2,7) and C(6,4).
(a) Find the squares of the lengths of the sides.
(b) Hence state completely what type of triangle ABC is.
(c) Find the area of the triangle.

5 The vertices of triangle PQR are P(3,4), Q(5,8) and R(7,4).
(a) What kind of triangle is PQR?
(b) State the coordinates of the midpoint S of side PR.
(c) Find the length of QS and deduce the area of the triangle PQR.

6 The vertices of triangle ABC are A(–4,4), B(2,6) and C(0,–6). Find the lengths of the three medians of the triangle.

7 A(–6,3), B(2,5) and C(0,–5) form a triangle. D is the midpoint of BC.
(a) State the coordinates of D.
(b) Find the values of AC^2, AB^2, AD^2 and DC^2.
(c) Hence show that $AC^2 + AB^2 = 2(AD^2 + DC^2)$.

8 The vertices of triangle ABC are A(2,3), B(4,5) and C(8,–2). P and Q are the midpoints of AB and BC respectively.
(a) State the coordinates of P and Q.
(b) Find the values of PQ^2 and AC^2.
(c) What fraction of AC is PQ?

9 Circle C_1 has centre (–3,4) and radius 2 units. Circle C_2 has centre (1,7) and radius 3 units. Find the distance between the two centres and hence show that the circles touch each other.

10 The centre of a circle is (–1,3) and its radius is 10 units. The centre of a second circle is (2,7) and its radius is 5 units. Show that the two circles touch each other and make a sketch showing the positions of the circles.

11 The vertices of triangle PQR are P(2,5), Q(4,3) and R(–2,–3). If S is the midpoint of PR, show that triangle PSQ is isosceles.

12 A circle has its centre at the origin and its radius is 3 units. P(*x*,*y*) is *any* point on the circumference. State an equation in *x* and *y* which is true for all possible positions of P.

13 A(–3,2) and B(4,3) are two fixed points. The point P(*x*,*y*) moves so that it is always equidistant from A and B (i.e. AP = PB).
(a) Describe the locus of P .
(b) Show that $(x + 3)^2 + (y - 2)^2 = (x - 4)^2 + (y - 3)^2$.
(c) Simplify this equation. (The result is called the equation of the locus of P).

AREAS OF RECTILINEAR FIGURES (Optional)

A rectilinear figure has straight line sides. The following method will be found useful but it is not essential in this Syllabus. It gives a quick way of finding the area of such a figure using the coordinates of the vertices, written in a certain way. We will start with a triangle with one vertex at the origin O (Fig. 1.7). The other vertices are $A(x_1,y_1)$ and $B(x_2,y_2)$. Then the area of $\triangle$OAB = area of $\triangle$OBC + area of trapezium CDAB – area of $\triangle$ODA. Verify that this is

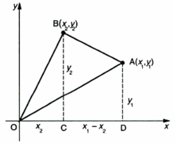


Fig 1.7

$= \frac{1}{2}(x_2y_2 + x_1y_1 + x_1y_2 - x_2y_1 - x_2y_2 - x_1y_1)$

$= \frac{1}{2}(x_1y_2 - x_2y_1)$

So, for example, the area of $\triangle$OAB, where A is (6,3) and B is (4,5) will be $\frac{1}{2}(6 \times 5 - 3 \times 4) = 9$. The vertices were taken in the order O – A – B, i.e. anticlockwise. If we take them in the order O – B – A, i.e. clockwise, the result would be –9 (check this). We now extend this to $\triangle$ABC (Fig. 1.8). Then the area of $\triangle$ABC = $\triangle$OAB – $\triangle$OAC – $\triangle$OCB (taking each triangle anticlockwise)

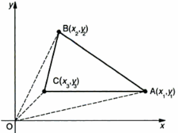
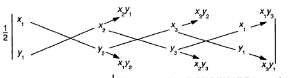


Fig. 1.8

$= \frac{1}{2}(x_1y_2 - x_2y_1) - \frac{1}{2}(x_1y_3 - x_3y_1) - \frac{1}{2}(x_3y_2 - x_2y_3)$

$= \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3)$

This result can be easily calculated by arranging the coordinate pairs as columns of a matrix, repeating the first pair at the end:

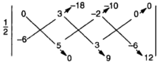


Find the products shown. The area = $\frac{1}{2}$[The sum of the DOWNWARD ↘ products – the sum of the UPWARD ↗ products].

This gives $\frac{1}{2}[(x_1y_2 + x_2y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)]$.

Check that this is the formula given above.

For example, the area of the triangle shown in Fig. 1.6 will be

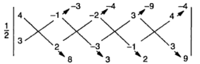


Area = $\frac{1}{2}[(0 + 9 + 12) - (-18 - 10 + 0)]$

= $24\frac{1}{2}$ units^2

This method can be extended to give the area of a polygon, provided the vertices are taken in order anticlockwise.

For example, the area of the quadrilateral whose vertices are (4,3), (–2,–3), (–1,2) and (3,–1) is given by



Draw a sketch to make sure the vertices are taken in order.

Write the pairs as before repeating the first one at the end. Then the area = $\frac{1}{2}[(8 + 3 + 2 + 9) - (-3 -4 -9 -4)] = 21$ units^2

Optional Exercise

Find the areas of the figures whose vertices are

(a) (0,0), (3,7), (5,1)

(b) (–1,–2), (–2,3), (4,–4)

(c) (–4,2), (0,–8), (5,11)

(d) (5,3), (2,5), (10,–1), (–6,3)

(e) (–2,–4), (3,1), (–1,5), (6,–3)

GRADIENT OR SLOPE OF A STRAIGHT LINE

The rest of this Chapter deals with the coordinate geometry of straight lines. An important concept is the **gradient** or **slope** of a line. This is a measure of the *steepness* of the line relative to the *x*-axis. It corresponds to the slope of a path or road which we measure relative to the horizontal. Mathematically, if A and B are *any* two points on a line (Fig. 1.9) then the gradient is the value of the ratio

$$\frac{\text{vertical rise (or fall)}}{\text{horizontal distance}} \text{ i.e. } \frac{y\text{-step}}{x\text{-step}} \text{ in going from A to B.}$$

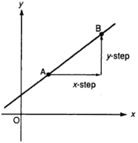


Fig. 1.9

The x-step and the y-step must be taken parallel to the x-axis and the y-axis respectively and either may be positive, negative or zero.
Then, as we shall see, a gradient can be zero, or a positive or negative number. In a special case, it may have no value.

Example 6

Find the gradient of the line through (–2,–3) and (3,5) as shown in Fig. 1.10.

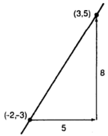


Fig. 1.10

Gradient = $\frac{5-(-3)}{3-(-2)} = \frac{8}{5}$, a **positive** gradient.
If the coordinates are taken in the reverse order, then the gradient is $\frac{-3-5}{-2-3} = \frac{-8}{-5}$ $= \frac{8}{5}$, giving the same value. The gradient is usually left as a fraction.

Example 7

What is the gradient of the line through the points (–2,5) and (4,–2) (Fig. 1.11)?

Gradient = $\frac{5-(-2)}{-2-4} = \frac{7}{-6} = -\frac{7}{6}$, a **negative** gradient.

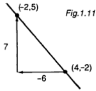


Fig. 1.11

Hence, if the coordinates of A and B are (x_1,y_1) and (x_2,y_2) respectively, as in Fig. 1.12, then the gradient of AB is $\frac{y_2 - y_1}{x_2 - x_1}$, or alternatively $\frac{y_1 - y_2}{x_1 - x_2}$.
(The coordinates must be subtracted in the same order).

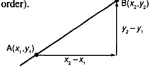


Fig. 1.12

Gradient of line through (x_1,y_1) and $(x_2,y_2) = \frac{y_2 - y_1}{x_2 - x_1}$

As the steepness of a straight line is clearly the same at all points on the line, we can take **any** two points on it to calculate its gradient.

Example 8

State the gradients of the following lines:
(a) through (–2,3) and (5,3),
(b) through (3,–4) and (3,2).

(a) Gradient $= \frac{3-3}{5-(-2)} = \frac{0}{7} = 0$

As we see in Fig. 1.13, the line is parallel to the x-axis.

The gradient of any line parallel to the x-axis is zero.

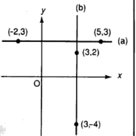


Fig. 1.13

(b) Gradient $= \frac{2-(-4)}{3-3} = \frac{6}{0}$ which is undefined as division by zero is not possible.

From Fig. 1.11, we see that the line is parallel to the y-axis.

The gradient of any line parallel to the y-axis is undefined.

Angle of Slope

In Fig. 1.14, the slope or gradient of the line AB is
$\frac{y\text{-step}}{x\text{-step}} = \frac{BC}{AC} = \tan \angle BAC$.
But $\angle BAC = \theta$ where θ is the angle between the line and the *positive* x-axis. So the gradient = $\tan \theta$.

θ is called the **angle of slope** and $0° \le \theta \le 180°$ (Fig. 1.15).

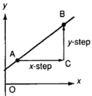


Fig. 1.14

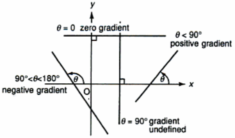


Fig. 1.15

If $\theta = 0°$, $\tan\theta = 0$; gradient = 0. The line is parallel to the *x*-axis.

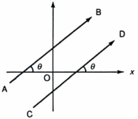
If $0° < \theta < 90°$, θ is an acute angle; $\tan\theta$ is positive and the gradient is positive. The line slopes upwards from left to right. ↗

If $\theta = 90°$, $\tan\theta$ and the gradient are undefined. The line is parallel to the *y*-axis.

If $90° < \theta < 180°$, θ is an obtuse angle; $\tan\theta$ is negative and the gradient is negative. The line slopes downwards from left to right. ↘

PARALLEL LINES

Fig. 1.16



In Fig. 1.16, the lines AB and CD are parallel. Then the angle of slope of each line is θ. Hence they have the same gradient.

> Parallel lines have equal gradients.
>
> Lines with equal gradients are parallel.

Example 9

A(2,3), B(5,7), C(0,–1) and D(–3,–5) are four points.

(a) Which of the lines AB, BC, CA and DA are parallel?

(b) What type of quadrilateral is ABCD?

(a) The gradients of AB, BC, CA and DA are $\frac{4}{3}$, $\frac{8}{5}$, 2 and $\frac{8}{5}$ respectively. Hence BC is parallel to DA.

(b) As it has 2 parallel sides, ABCD is a trapezium.

Example 10

Two lines are drawn from A(–1,–3), one to B(4,2) and the other to C(–4,2). What are their angles of slope?

Gradient of AB = $\frac{5}{5}$ = 1 = tan θ, so θ = 45°.

Gradient of AC = $\frac{5}{-3}$ = tan θ, so θ ≈ 121°.

COLLINEAR POINTS

Do the points A(–3,–5), B(0,–1) and C(3,3) lie in a straight line, i.e. are they **collinear**?

If they are, then the gradient of AB must be the same as that of BC or AC, as these will be segments of the same line.

Gradient of AB = $\frac{4}{3}$ and gradient of BC = $\frac{4}{3}$. (Check gradient of AC).

Hence the three points are collinear.

Example 11

If C(p,q) is a point on the line AB, where A is (–2,1) and B is (3,2), find a relationship between p and q.

The three points are collinear.

Hence the gradient of AC = the gradient of AB.

Then $\frac{q-1}{p+2} = \frac{1}{5}$.

Now verify that this gives $5q - p = 7$, which is the relationship required.

Exercise 1.3 *(Answers on page 606.)*

1 State the gradient of the line through the following pairs of points:

(a) (2,3), (1,5) (b) (0,3), (3,0) (c) (2,2), (5,5)
(d) (–3,–9), (1,–1) (e) (1,4), (–3,4) (f) (3,–4), (3,–1)
(g) (–1,–2), (–2,–4) (h) (–4,0), (3,–2) (i) $(a,0)$, $(0,-a)$
(j) (a,b), (b,a) (k) (p,p^2), (q,q^2)

2 A(–4,–2), B(5,–2), C(0,3) and D(1,0) are four points. State the gradients of (a) AB, (b) CD, (c) AC and (d) BD.

3 Which of the lines through the following pairs of points are parallel?

(a) (–1,3), (4,5) (b) (3,–2), (5,1) (c) (–4,–3), (1,–1)
(d) (–7,4), (2,4) (e) (0,–4), (2,–1) (f) $(a,b-1)$, $(a+5,b+1)$

4 Find the angle of slope of the line through the following pairs of points:
(a) (–2,–1), (3,4) (b) (–2,–1), (2,–5) (c) (1,3), (3,7)

5 Are the points (–7,5), (–5,8) and (1,17) collinear?

6 A(–6,–3), B(–2,8), C(0,5) and D(2,2) are four points.
(a) Show that B, C and D are collinear.
(b) P, Q and R are the midpoints of AB, AC and AD respectively. Show that P, Q and R are also collinear.

7 If the point (a,b) lies on the line joining (–2,3) and (2,1), find a relationship between a and b.

8 If the points (–2,–3), (3,5) and $(13,p)$ are collinear, find the value of p.

9 The coordinates of a point are given as $(t - 1, 2t + 1)$. Show that the points where $t = 0$, 1 and 2 are collinear.

10 (a) If the line joining the points (2,4) and (5,–2) is parallel to the line joining (–1,–2) and $(p,6)$ find the value of p.
(b) The line joining (–1,–4) and $(a,0)$ is parallel to the line joining $(a,1)$ to (11,3). Find the value of a.

11 (a) Show that the points (2,–4), (5,0) and (8,4) are collinear.
(b) The point $(d,d - 2)$ also lies on this line. Find the value of d.

12 If the points (–3,–2), $(-1, a - 2)$ and $(a, 7)$ are collinear, find the two possible values of a.

PERPENDICULAR LINES

The vertices of triangle ABC are A(–4, –2), B(4, 2) and C(2, 6). Verify that this triangle is right-angled. Which two sides are perpendicular? Now state the gradients of these sides. If you multiply the two gradients, what result do you obtain?

The result is surprising so we investigate it further. Given the points A(–5, –4), B(–2, 3) and C(–16, 9) show by using Pythagoras' theorem that AB and BC are perpendicular. Now find the product of their gradients. We can show that this result is true in general excluding undefined or zero gradients.

In Fig. 1.17, AB is a line with gradient m_1 and CD a line with gradient m_2. The lines intersect at right angles at T. The small triangle PQR shows that $m_1 = \frac{-a}{b}$.

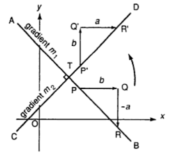


Fig. 1.17

Now imagine that line AB is rotated through 90° about T to lie along CD. Then triangle PQR takes a new position P'Q'R'.

This shows that $m_2 = \frac{b}{a}$, as a and b are now both positive.

Then $m_1 m_2 = -\frac{a}{b} \times \frac{b}{a} = -1$ and this will be true for any pair of perpendicular lines (except for lines parallel to the x- or the y-axis).

If m_1, m_2 are the gradients of two perpendicular lines,

then $m_1 m_2 = -1$ or $m_1 = -\frac{1}{m_2}$ ($m_1 \neq 0$, $m_2 \neq 0$).

Conversely, if m_1 and m_2 are the gradients of two lines ($m_1 \neq 0$, $m_2 \neq 0$) and $m_1 m_2 = -1$, then the lines are perpendicular.

Example 12

The vertices of triangle ABC are A(–2,–4), B(2,–1) and C(5,–5).

(a) Show that the triangle is right-angled.

(b) State the gradient of the altitude through B.

(a) This could be done using the Pythagoras' Theorem but here we use gradients.

The gradients of AB, BC and CA are $\frac{3}{4}$, $-\frac{4}{3}$ and $-\frac{1}{7}$ respectively.

As $\frac{3}{4} \times (-\frac{4}{3}) = -1$, AB is perpendicular to BC. Hence the triangle is right-angled ($\angle B = 90°$).

(b) The altitude through B will be perpendicular to AC. Hence its gradient

$= -\frac{1}{-\frac{1}{7}} = 7.$

Exercise 1.4 *(Answers on page 607.)*

1 Which of the lines through these pairs of points are perpendicular?

(a) (–4,–2), (–1,0)
(b) (0,–5), (4,–2)
(c) (–2,1), (1,5)
(d) (–1,–4), (2,–8)
(e) (1,2), (5,–4)
(f) (–2,3), (–2,7)

2 State the gradient of a line which is (a) parallel, (b) perpendicular, to AB where
(i) A is (3,–2), B is (0,4) (ii) A is (0,–1), B is (2,1)
(iii) A is (–3,–3), B is (2,4) (iv) A is (–4,1), B is (3,0)

3 Is the triangle formed by the points (–3,2), (0,4) and (4,–2) right-angled?

4 Find the gradient of a line perpendicular to the longest side of the triangle formed by A(–3,4), B(5,2) and C(0,–3).

5 (a) Show that the triangle formed by A(–2,–3), B(2,5) and C(10,1) is right-angled and isosceles.
(b) State the gradients of the three altitudes.

6 Find the angle of slope of a line with gradient $\frac{1}{2}$ and that of another line perpendicular to it.

7 Find the gradient of a line perpendicular to the line joining the points $(a,3a)$ and $(2a,-a)$.

8 CD is the perpendicular bisector of the line joining A(2,3) and B(5,7).
(a) State (i) the coordinates of the point where CD intersects AB and (ii) the gradient of CD.
(b) If the point (p,q) lies on CD, find a relationship between p and q.

9 (a) Show that the point (7,1) lies on the perpendicular bisector of the line joining (2,4) and (4,6).
(b) The point $(a,4)$ also lies on this bisector. Find the value of a.

10 A semicircle with centre O (the origin) and radius 5 units, meets Ox at A and B and the positive y-axis at C.
(a) State the coordinates of A, B and C.
(b) If a point (x,y) lies on the semicircle, show that $x^2 + y^2 = 25$.
(c) Verify that the point P(–3,4) lies on the semicircle and show by using gradients that $\angle APB = 90°$.

11 A(–1,–2), B(b,1) and C(6,–3) are three points and AB is perpendicular to BC.
(a) State, in terms of b, the gradients of AB and BC.
(b) Hence show that $(b + 1)(b - 6) = -12$.
(c) Now find the two possible values of b.

EQUATION OF A STRAIGHT LINE

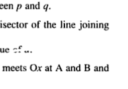


Fig. 1.18

The point P(x,y) lies on the line through A(–2,3) and B(4,–1) (Fig. 1.18). Can we find a relationship between x and y? (Note that we use the coordinates (x,y) as P is *any* point on the line).

Since the three points are collinear, the gradient of AP = gradient of AB.

Then $\frac{y-3}{x+2} = \frac{-4}{6} = -\frac{2}{3}$

i.e. $3(y - 3) = -2(x + 2)$ or $3y + 2x = 5$.

This relationship is called the **equation of the line** through A and B.

If the coordinates (x,y) of a point are substituted in the equation and both sides are equal, **then the point lies on the line.** We say the coordinates **satisfy** the equation. Conversely, if the point lies on the line, its coordinates must satisfy the equation.

For example, the point (3,–5) lies on the line $2x + 3y = -9$ because $2 \times \mathbf{3} + 3 \times (\mathbf{-5}) = -9$. The coordinates (3,–5) satisfy the equation.

The point (2,3) does **not** lie on the line because $2 \times \mathbf{2} + 3 \times \mathbf{3} \neq -9$. The coordinates (2,3) do not satisfy the equation.

Such an equation is called a **linear** equation, as it is the equation of a straight line. Its general form is $ax + by = c$ where a, b and c are constants. For example, $2x - 3y = 1$, $y = 3x - 5$ are linear equations. Note that $y = 2$ (no x term) or $2x + 1 = 0$ (no y term) are also linear equations.

We now look at various forms of a linear equation and how to find them. The position of a line can be fixed in two ways.

1 Given one point $A(x_1,y_1)$ on the line and its gradient m.

If P(x,y) is any point on the line (Fig. 1.19), then its gradient is $\frac{y - y_1}{x - x_1} = m$.

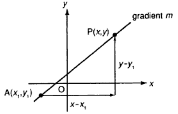


Fig. 1.19

So the equation of the line is

$$y - y_1 = m(x - x_1) \qquad \text{one-point, gradient form}$$

Example 13

(a) *What is the equation of the line through (3,–1) with gradient $\frac{2}{3}$?*
(b) *Does the point (2,3) lie on this line?*
(c) *Find the coordinates of the points where this line cuts the axes.*

(a) Using the one-point, gradient form, the equation of the line is $y - (-1) = \frac{2}{3}(x - 3)$ i.e. $3(y + 1) = 2(x - 3)$ which simplifies to $3y = 2x - 9$ or $3y - 2x = -9$.

(b) Substituting in the equation, $3 \times 3 - 2 \times 2 = 5$. But $5 \neq -9$ so the coordinates do not satisfy the equation and hence the point (2,3) does not lie on the line.

(c) The y-coordinate of any point on the x-axis is 0.
·Substitute $y = 0$ in the equation of the line.

Then $0 = 2x - 9$ giving $x = 4\frac{1}{2}$. The line cuts the x-axis at $(4\frac{1}{2},0)$.

Similarly, to find where the line cuts the y-axis, put $x = 0$ in the equation. Verify that this gives the point (0,–3).

II Given two points $A(x_1,y_1)$ and $B(x_2,y_2)$

Let $P(x,y)$ be any point on the line (Fig. 1.20).

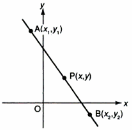


Fig. 1.20

Then by gradients, $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$.

Rewriting this in a more symmetrical form, the equation of the line is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

two-point form
(note the order of the terms)

Example 14

Find the equation of the line through (2,–3) and (–1,4).

It does not matter which point is taken as (x_1,y_1). Take (2,–3).

Using the two-point form, the equation is $\frac{y-(-3)}{4-(-3)} = \frac{x-2}{-1-2}$

i.e. $\frac{y+3}{7} = \frac{x-2}{-3}$.

Now remove the fractions to get $-3(y + 3) = 7(x - 2)$, which simplifies to $3y + 7x = 5$.

Lines Parallel to the *x*- or *y*-axis

Equations for these lines are special cases.

Example 15

Find the equation of the line through
(a) (–3,2) and (5,2),
(b) (3,–1) and (3,5).

(a) If we use the two-point form, we get $\frac{y-2}{2-2} = \frac{x+3}{5+3}$ which is not defined. We can see however that the line is parallel to the x-axis (Fig. 1.21). Every point of the line will have coordinates of the form $(x,2)$ so its equation will be $y = 2$ as y is always $= 2$, whatever the value of x.

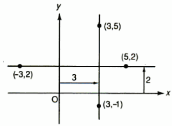


Fig. 1.21

(b) Similarly this line is parallel to the y-axis. Every point will have coordinates of the form $(3,y)$. So the equation is $x = 3$. Hence, if k is a constant, then

$y = k$ is the equation of a line parallel to the x-axis
$x = k$ is the equation of a line parallel to the y-axis

Exercise 1.5 *(Answers on page 607.)*

1 Find, in its simplest form, the equation of the line
(a) through (2,3) with gradient 1
(b) through (–1,–1) with gradient $\frac{3}{4}$
(c) through (1,3) with gradient $-\frac{1}{5}$
(d) through (1,0) and (–2,3)
(e) through (0,1) and (–1,3)
(f) through (3,–2) and (7,–2)
(g) through (–2,4) parallel to the *y*-axis
(h) through (1,2) and parallel to a line with gradient 2
(i) through (–3,–1) and perpendicular to a line with gradient $-\frac{1}{3}$
(j) through (–1,2) and (–1,7)
(k) through (0,–3) and (0,5)

2 Find the coordinates of the points where each of the lines in Question 1 cut the axes.

3 A line cuts the *x*-axis at (3,0) and the *y*-axis at (0,–2). Find the equation of the line.

4 P(0,9) and Q(6,0) are two points. A line is drawn from the origin perpendicular to PQ. Find the equation of this line.

5 Find the equations of the lines through (–1,–4) which are (a) parallel and (b) perpendicular to another line with gradient $-\frac{2}{3}$.

6 The gradient of a line is 2 and it cuts the *y*-axis at (0,3). Find its equation and the coordinates of the point where it cuts the *x*-axis.

7 Find the equations of the sides of triangle ABC where A is (–2,3), B is (0,5) and C is (3,–1).

8 The points A(4,4), B(–2,0) and C(6,–2) form a triangle.
(a) Find the equations of the medians of this triangle.
(b) If AD is an altitude of the triangle, find the equation of AD.

9 From the point (2,5), a perpendicular is drawn to the line joining (–1,–4) and (5,2). Find the equation of this perpendicular.

10 ABCD is a parallelogram where A is (2,–1), B is (6,2) and C is (11,–2).
(a) State the coordinates of the midpoint of AC.
(b) Hence find the coordinates of D.
(c) Find the equations of the diagonals of the parallelogram.

11 A(–1,2) and C(3,4) are opposite vertices of a rhombus ABCD. Find
(a) the coordinates of the point where the diagonals intersect,
(b) the gradient of AC,
(c) the equation of the diagonal BD.

GRADIENT-INTERCEPT FORM

Suppose the equation of a line is $2x - 3y = 5$. How can we find its gradient?

To do this we convert the equation to a special form – the **gradient-intercept** form. Fig. 1.22 shows a line with gradient m which cuts the y-axis at C(0,c). c is called the y-intercept of the line. Let P(x,y) be any point on this line. Then the gradient of the line = $\frac{y-c}{x} = m$ so $y - c = mx$.

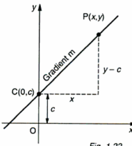


Fig. 1.22

i.e. $y = mx + c$ (gradient ↑ m, y-intercept ↑ c) gradient-intercept form

Hence, if an equation is written in this form, the gradient is given by the **coefficient** of x and the y-intercept by the **constant term**.

To verify this, suppose the equation of line is $y = 2x - 3$ (gradient-intercept form). This line cuts the y-axis where $x = 0$, so $y = -3$ (the constant term). The points (2,1) and (5,7) lie on the line (check this). The gradient is $\frac{6}{3} = 2$ which is the coefficient of x.

Example 16

Find the gradients of the lines (a) $2x - 3y = 5$, (b) $2y + x = -4$.

(a) Convert to the gradient-intercept form, $y = mx + c$:

$-3y = -2x + 5$

Then $y = \frac{2}{3}x - \frac{5}{3}$ (dividing by −3)

↑ gradient ↑ y-intercept

So the gradient is $\frac{2}{3}$ (and the y-intercept is $-\frac{5}{3}$).

(b) $2y + x = -4$ i.e. $2y = -x - 4$ so $y = -\frac{1}{2}x - 2$

The gradient is $-\frac{1}{2}$.

It is useful to practise this conversion, i.e. making y the subject of the equation. The gradient is then obtained quickly.

Equations of Parallel and Perpendicular Lines

Example 17

Find the equations of the lines through the point (1,2) which are (a) parallel, (b) perpendicular, to the line $2x - 3y = 4$.

(a) The gradient of the line $2x - 3y = 4$ is $\frac{2}{3}$. So the gradient of any parallel line is also $\frac{2}{3}$. Hence its equation will be $y = \frac{2}{3}x + c$.

To find c, we substitute the coordinates (1,2) in the equation as (1,2) lies on the line. Then $2 = \frac{2}{3} + c$ giving $c = \frac{4}{3}$.

The equation is $y = \frac{2}{3}x + \frac{4}{3}$ i.e. $3y = 2x + 4$.

(b) The gradient of any perpendicular line will be $-\frac{3}{2}$ so its equation is $y = -\frac{3}{2}x + c$. Substitute (1,2) to find c and verify that the required equation is $2y = -3x + 7$.

Exercise 1.6 *(Answers on page 607.)*

1 State the gradients of the following lines:

(a) $x + y = 2$	(b) $x - y = -1$	(c) $y - 2x = 3$
(d) $2x + y = 1$	(e) $3x + 2y = 6$	(f) $5x - 2y = 5$
(g) $y = 4$	(h) $x - 2y = 0$	(i) $2x + 3y = 1$
(j) $2x - 3y = 4$	(k) $4x = 3y - 2$	(l) $5x - 2y = 10$
(m) $tx - y = t$	(n) $py + x = 2p$	(o) $ax + by = 1$

2 Find the equation of the line which is
- (a) parallel to $x - y = 1$ and passes through (2,3)
- (b) parallel to $2x + y = 3$ and passes through (0,1)
- (c) perpendicular to $2x + y = 0$ and passes through (–1,–2)
- (d) perpendicular to $3x + y = 5$ and passes through (–2,–1)
- (e) parallel to $y = 4$ and passes through (0,1)
- (f) perpendicular to $x - 3y = 1$ and passes through (–3,0)
- (g) perpendicular to $x = 2$ and passes through (–2,3)

3 Find the equations of the lines parallel and perpendicular to
- (a) $x + y = 3$ passing through (–1,2)
- (b) $2x - y = 4$ passing through (0,3)
- (c) $4x + 3y = 1$ passing through (0,–2)
- (d) $x - 3y = 1$ passing through (–1,–1)

4 A line is drawn through the point (–1,2) parallel to the line $y + 5x = 2$. Find its equation and that of the perpendicular line through the same point.

5 The side BC of a triangle ABC lies on the line $2x - 3y = 4$. A is the point (2,3). Find the equation of the altitude through A.

INTERSECTION OF LINES

At what point do the lines $2x - 3y = -7$ and $3x + 8y = 2$ intersect? This point lies on **both** these lines so its coordinates must satisfy **both** equations. Hence its coordinates will be the solution of the simultaneous equations

$$2x - 3y = -7 \quad \text{(i)}$$

and $$3x + 8y = 2 \quad \text{(ii)}$$

These can be solved by any of the methods you have learnt previously. We use the elimination method here.

Multiply (i) by 3: $6x - 9y = -21$

Multiply (ii) by 2: $6x + 16y = 4$

Subtract: $-25y = -25$ so $y = 1$

Substitute in (i): $2x - 3 = -7$ so $x = -2$

The point is $(-2,1)$.

Suppose the lines were $2x - 3y = -7$ and $4x - 6y = 3$. What happens in the solution? Explain this.

Example 18

From the point P(–1,3), a perpendicular PQ is drawn to the line joining A(–4,–8) and B(4,4). Find

(a) the equations of AB and the perpendicular,
(b) the coordinates of the point where they intersect,
(c) the distance of P from the line AB.

A sketch diagram should always be drawn to help in such questions (Fig. 1.23).

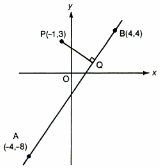


Fig. 1.23

(a) The equation of AB is $\frac{y+8}{12} = \frac{x+4}{8}$ i.e. $3x - 2y = 4$.

Now check that the equation of PQ is $2x + 3y = 7$.

(b) Solving the equations $3x - 2y = 4$ and $2x + 3y = 7$, we get (2,1) as the coordinates of Q.

(c) The distance of P from AB is PQ.

$PQ^2 = (-1 - 2)^2 + (3 - 1)^2 = (-3)^2 + 2^2 = 13$ so $PQ = \sqrt{13}$

Example 19

ABCD is a rectangle where A is (–3,2), D is (2,5) and B lies on the y-axis. Find

(a) the equation of AD,

(b) the equation of AB,

(c) the coordinates of B,

(d) the coordinates of C.

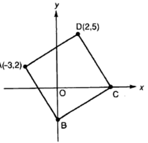


Fig. 1.24

Fig. 1.24 shows the facts given

(a) Equation of AD is $\frac{y-2}{3} = \frac{x+3}{5}$ i.e. $3x - 5y = -19$.

(b) AB is perpendicular to AD. The gradient of AD is $\frac{3}{5}$ so the gradient of AB is $-\frac{5}{3}$. Knowing the gradient and the point A, verify that the equation of AB is $3y = -5x - 9$.

(c) AB meets the y-axis where $x = 0$. Hence $y = -3$. The coordinates of B are (0,–3).

(d) Let the diagonals meet at M. M is the midpoint of BD, so M is (1,1). As M is also the midpoint of AC, therefore C is (5,0).

Example 20

The line $2x + 3y = 6$ meets the y-axis at A and the x-axis at B. C is the point such that $AB = BC$. CD is drawn perpendicular to AC to meet the line through A parallel to $5x + y = 7$ at D.

(a) Find the coordinates of A, B and C.

(b) State the equations of CD and AD, and hence find the coordinates of D.

(c) Calculate the area of the triangle ACD.

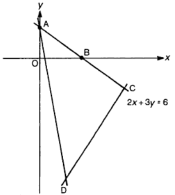


Fig. 1.25

(a) The line meets the y-axis where $x = 0$, $y = 2$, so A is (0,2).
It meets the x-axis where $y = 0$, $x = 3$, so B is (3,0).
Since B is the midpoint of AC, then C must be (6,–2).

(b) CD is perpendicular to AC. Therefore its gradient is $\frac{3}{2}$ and it passes through C. Verify that the equation of CD is $3x - 2y = 22$.

AD is parallel to $5x + y = 7$. Therefore its gradient is –5 and it passes through A. Verify that the equation of AD is $5x + y = 2$.

Solving these two equations gives $x = 2$ and $y = -8$. So the coordinates of D are (2,–8).

(c) As ACD is a right-angled triangle,

$$\text{its area} = \tfrac{1}{2} \times AC \times DC = \tfrac{1}{2} \times \sqrt{52} \times \sqrt{52} = 26 \text{ units}^2.$$

Exercise 1.7 *(Answers on page 607.)*

1 The line $4x - 3y = 12$ meets the axes at A and B. Find the length of AB.

2 Find the equation of the line through the point of intersection of $2x + 3y = 5$ and $3x - y = 2$, and which is parallel to $4y - x = 14$.

3 Through A(2,3) two lines are drawn with gradients –1 and 2. These lines meet the line $x - 2y = 5$ at B and C. Find
(a) the equations of AB and AC,
(b) the coordinates of B and C.

4 The lines $x + 3y = 1$ and $2x - 5y = -9$ intersect at A. Find the equation of the line through A and the point (–1,–2).

5 A line through A(5,2) meets the line $3x + 2y = 6$ at right angles at B. Find the coordinates of B and calculate the length of AB.

6 (a) Find the equation of the perpendicular bisector of the line joining A(–3,3) and B(1,–5).
(b) If this bisector meets the x-axis at C, find the coordinates of C.

7 The sides of a triangle lie on the lines $y = -1$, $2x + y = 1$ and $4x - 3y = -13$. Find the coordinates of the vertices and show that the triangle is isosceles.

8 The intersections of the lines $5x + 6y = 36$, $x - 2y = 4$ and $7x + 2y = 12$ are the vertices of a triangle.
(a) Find the coordinates of these vertices.
(b) Obtain the equation of the altitude drawn to the longest side.

9 OABC is a parallelogram where O is the origin and B is the point (5,7). C lies on the line $x - 2y = 0$ and A lies on the line $2x - y = 0$. Calculate the coordinates of A and C.

10 The sides of a triangle lie on the lines $y = 1$, $x + y = 6$ and $3x - y = 2$.
(a) Calculate the coordinates of the vertices of the triangle.
(b) Find the equations of the three altitudes.
(c) Show that these altitudes intersect at a point and find the coordinates of this point.

11 A(3,1) and B(0,6) are two points. BC is perpendicular to AB and meets the x-axis at C. Find
(a) the equation of BC,
(b) the coordinates of C,
(c) the area of triangle ABC.

12 The diagonals of a rhombus meet at the point (–1,5) and one of them is parallel to the line $2x - 5y = 3$.
(a) Find the equations of the diagonals.
(b) If two of the vertices of the rhombus are (–3,10) and (9,9), find the coordinates of the other two.

13 A is the point (–1,6). Lines are drawn through A with gradients 3 and –2, meeting the x-axis at B and C respectively. BD is perpendicular to AB and CD is perpendicular to AC.
(a) Find the coordinates of B and C.
(b) State the equations of BD and CD.
(c) Find the coordinates of D.
(d) Calculate the ratio BD:CD.

14 A(1,2) and C(5,4) are two vertices of the rectangle ABCD. AB and CD are parallel to the line $y - x = 5$.
(a) Find the equations of AB and BC.
(b) Find the coordinates of B and D.
(c) Hence find the area of the rectangle.

15 ABCD is a rectangle where A is (1,3) and D is (5,5). AC lies on the line $3y = 4x + 5$. Find
(a) the equation of DC,
(b) the coordinates of C,
(c) the coordinates of B,
(d) the area of ABCD.

16 The point B(a,b) is the reflection of A(5,–2) in the line $2x - 3y = 3$.
(a) Find the equation of AB and show that $3a + 2b = 11$.
(b) State the coordinates of the midpoint of AB in terms of a and b and show that $2a - 3b = -10$.
(c) Hence find the values of a and b.

SUMMARY

- Midpoint of (x_1,y_1) and (x_2,y_2) is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.
- Distance between (x_1,y_1) and (x_2,y_2) is $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$.
- Gradient of line through (x_1,y_1) and (x_2,y_2) is $\frac{y_2-y_1}{x_2-x_1}$.
- Parallel lines have equal gradients.
- Three points A, B and C are collinear if the gradient of AB equals the gradient of BC.
- If m_1 and m_2 are the gradients of perpendicular lines, $m_1m_2 = -1$. If m_1 and m_2 ($m_1 \neq 0$, $m_2 \neq 0$) are the gradients of two lines and $m_1m_2 = -1$, then the lines are perpendicular.
- Equation of line through (x_1,y_1) with gradient m is $y - y_1 = m(x - x_1)$.
- Equation of line through (x_1,y_1) and (x_2,y_2) is $\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$.
- The form $y = mx + c$ gives the gradient (m) and the y-intercept (c).

REVISION EXERCISE 1 *(Answers on page 607.)*

A

1 Find the equation of the line
(a) through (–2,3) with gradient $-\frac{1}{2}$,
(b) through the points (–3,2) and (–1,–5),
(c) through (–1,–1) perpendicular to the line $3x - 2y = 1$.

2 A and B are the points (–2,–1) and (4,1) respectively. BC is perpendicular to AB.
(a) Find the equation of BC.
(b) If the gradient of AC is 1, find the equation of AC and the coordinates of C.
(c) Hence find the area of triangle ABC.

3 A(–1,1) and B(3,4) are two vertices of triangle ABC. If the area of the triangle is 15 units^2, find the distance of C from AB.

4 The line $y = 2x + 3$ intersects the y-axis at A. The points B and C on this line are such that AB = BC. The line through B perpendicular to AC passes through the point D(–1,6). Find
(a) the equation of BD,
(b) the coordinates of B,
(c) the coordinates of C. (C)

5 (a) The line $\frac{x}{4} - \frac{y}{3} = 1$ meets the axes at A and B. Find the coordinates of the midpoint of AB and the length of AB.
(b) A circle is drawn with its centre at the origin. If the point P(4,3) lies on this circle, find the equation of the tangent to the circle at P.

6 Fig. 1.26 shows a triangle ABC with A(1,1) and B(–1,4). The gradients of AB, AC and BC are $-3m$, $3m$ and m respectively.
(a) Find the value of m.
(b) Find the coordinates of C.
(c) Show that AC = 2AB. (C)

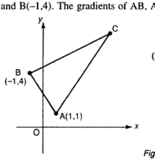


Fig. 1.26

7 A(–3,4) and C(4,–10) are opposite vertices of the parallelogram ABCD. The gradients of the sides AB and BC are $-\frac{1}{2}$ and 3 respectively. Find
(a) the equations of AB and BC,
(b) the coordinates of B and D.

8 Three points have coordinates A(1,–3), B(5,5) and C(5,9). Find the equation of the perpendicular bisector of (a) AB, (b) BC. Hence find the coordinates of the point equidistant from A, B and C. (C)

9 (a) Find the equation of the perpendicular bisector of AB, given that A is (2,7) and B is (6,–1).

(b) The bisector meets the y-axis at C. Find the coordinates of C and the area of triangle ABC.

10 A(0,6), B(1,3) and C(4,6) are three points. D is the foot of the perpendicular from A to BC. Find

(a) the coordinates of D,

(b) the length of AD.

11

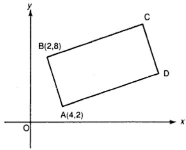


Fig. 1.27

In Fig. 1.27, ABCD is a rectangle, and A and B are the points (4,2) and (2,8) respectively. Given that the equation of AC is $y = x - 2$, find

(a) the equation of BC,

(b) the coordinates of C,

(c) the coordinates of D,

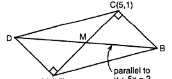
(d) the area of the rectangle ABCD. (C)

12 Two points have coordinates A(1,3) and C(7,7). Find the equation of the perpendicular bisector of AC.

B is the point on the y-axis equidistant from A and C and ABCD is a rhombus. Find the coordinates of B and D.

Show the area of the rhombus is 52 units^2 and hence calculate the perpendicular distance of A from BC. (C)

13



ABCD is a parallelogram, lettered anticlockwise, such that A and C are the points (–1,5) and (5,1) respectively. Find the coordinates of the midpoint of AC.

Given that BD is parallel to the line whose equation is $y + 5x = 2$, find the equation of BD.

Given that BC is perpendicular to AC, find the equation of BC. Calculate (i) the coordinates of B, (ii) the coordinates of D, (iii) the area of ABCD. (C)

14 A(–2,2) and C(4,–1) are opposite vertices of a parallelogram ABCD whose sides are parallel to the lines $x = 0$ and $3y = x$.

(a) Find the coordinates of B and D.

(b) If P and Q are the feet of the perpendiculars from D and B respectively to AC, find the coordinates of P and Q and show that PQ = $\frac{1}{3}$ AC.

15 Fig. 1.28 shows a quadrilateral ABCD in which A is (2,8) and B is (8,6). The point C lies on the perpendicular bisector of AB and the point D lies on the y-axis. The equation of BC is $3y = 4x - 14$ and angle DAB = 90°. Find

(a) the equation of AD,

(b) the coordinates of D,

(c) the equation of the perpendicular bisector of AB,

(d) the coordinates of C.

Show that the area of triangle ADC is 10 units² and find the area of the quadrilateral ABCD. (C)

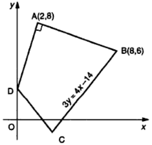


Fig. 1.28

16 The line $x + y = 3$ meets the y-axis at A and the x-axis at B. AC is perpendicular to AB and the equation of BC is $y = 3x - 9$.

(a) Find the equation of AC and the coordinates of C. AD is parallel to CB where D lies on the x-axis.

(b) Find the coordinates of D.

(c) Hence find the area of the trapezium ACBD.

17 Fig. 1.29 shows the quadrilateral OABC. The coordinates of A are $(k,2k)$ where $k > 0$, and the length of OA is $\sqrt{80}$ units.

(a) Calculate the value of k.

AB is perpendicular to OA and B lies on the y-axis.

(b) Find the equation of AB and the coordinates of B.

The point C lies on the line through O parallel to $y + 3x = 5$ and also on the perpendicular bisector of AB.

(c) Calculate the coordinates of C.

Calculate the area of the quadrilateral OABC. (C)

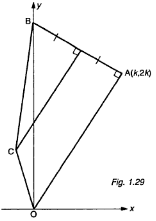


Fig. 1.29

18 The vertices of a triangle are (–3,5), (4,–2) and (6,2).
(a) Find the equations of the perpendicular bisectors of the sides.
(b) Show that they meet at the same point and find the coordinates of this point.
(c) Find the radius of the circle passing through the vertices.

19 A and B are the points (2,4) and (4,0) respectively.
(a) Find the equation of the perpendicular bisector of AB.
(b) The bisector meets the line through B parallel to the y-axis at C. Find the coordinates of C.
(c) Calculate the radius of the circle which passes through A and touches the x-axis at B.

20 The sides AB, BC and CA lie on the lines $2y = x - 4$, $x + y = 5$, and $y = mx$ respectively. If the origin O is the midpoint of AC, find the value of m.

B

21 A(h,k) lies on the line $y + 3x = -10$. B lies on the line $x + y = 4$. If the origin is the midpoint of AB, find the value of h and of k.

22 A(1,5) lies on the line $y = 2x + 3$. P lies on the perpendicular to that line through A.
(a) Show that the coordinates of P can be written as $(11 - 2a, a)$.
(b) If OP = $\sqrt{34}$, where O is the origin, find the possible values of a.

23 A line with gradient m passes through the point P(3,2) and meets the y-axis at A. A line perpendicular to the first also passes through P and meets the x-axis at B.
(a) Express the coordinates of A and B in terms of m.
(b) If AB = $\sqrt{65}$, find the possible values of m.

24 P and Q are the points of intersection of the line $\frac{y}{2} + \frac{x}{3} = 1$ with the x– and y-axes respectively. The gradient of QR is $\frac{1}{2}$ and R is the point whose x-coordinate is $2t$, where t is positive. Express the y-coordinate of R in terms of t and evaluate t given that the area of triangle PQR is 21 units^2. **(C)**

25 A line through (3,1) has gradient m ($> \frac{1}{3}$). It meets the x-axis at A and the y-axis at B. From A and B, perpendiculars to the line are drawn to meet the y-axis at C and the x-axis at D respectively. Show that the gradient of CD is $\frac{1}{m^3}$.

26 $A(x_1,y_1)$, $B(x_2,y_2)$, $C(x_3,y_3)$ and $D(x_4,y_4)$ are the vertices of a parallelogram ABCD.

(a) Show that $x_1 + x_3 = x_2 + x_4$ and $y_1 + y_3 = y_2 + y_4$.

(b) If ABCD is a rhombus show that $(x_1 - x_3)(x_2 - x_4) + (y_1 - y_3)(y_2 - y_4) = 0$.

(c) If however ABCD is a rectangle show that $x_1x_3 + y_1y_3 = x_2x_4 + y_2y_4$.