Conic Sections

11

In Chapter 11, we present several new types of graphs, called conic sections. These include circles, parabolas, ellipses, and hyperbolas. These shapes are found in a variety of applications. For example, a reflecting telescope has a mirror whose cross section is in the shape of a parabola, and planetary orbits are modeled by ellipses.

As you work through the chapter, you will encounter a variety of equations associated with the conic sections. Match the equation with its description, and then use the letter next to each answer to complete the puzzle.

_____ 1. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

_____ 2. $x^2 + y^2 = r^2$

_____ 3. $x = a(y - k)^2 + h$

_____ 4. $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

_____ 5. $y = a(x - h)^2 + k$

_____ 6. $(x - h)^2 + (y - k)^2 = r^2$

_____ 7. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

a. standard form of a parabola with horizontal axis of symmetry

o. standard form of an ellipse centered at the origin

f. standard form of a circle centered at (h, k)

t. standard form of a hyperbola with horizontal transverse axis

s. standard form of a circle centered at the origin

c. standard form of a parabola with vertical axis of symmetry

r. distance formula

He became a math teacher due to some prime $\underset{6}{\underline{\quad}}\ \underset{3}{\underline{\quad}}\ \underset{5}{\underline{\quad}}\ \underset{7}{\underline{\quad}}\ \underset{1}{\underline{\quad}}\ \underset{4}{\underline{\quad}}\ \underset{2}{\underline{\quad}}$

Section 11.1 Distance Formula and Circles

Concepts

1. Distance Formula
2. Circles
3. Writing an Equation of a Circle

1. Distance Formula

Suppose we are given two points (x_1, y_1) and (x_2, y_2) in a rectangular coordinate system. The distance between the two points can be found by using the Pythagorean theorem (Figure 11-1).

First draw a right triangle with the distance d as the hypotenuse. The length of the horizontal leg a is $|x_2 - x_1|$, and the length of the vertical leg b is $|y_2 - y_1|$. From the Pythagorean theorem we have

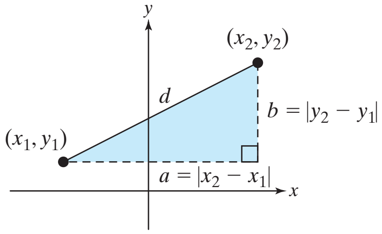


Figure 11-1

$$d^2 = a^2 + b^2 \qquad \text{Pythagorean theorem}$$
$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$d = \pm\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Because distance is positive, reject the negative value.

The Distance Formula

The distance d between the points (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 1 Finding the Distance Between Two Points

Find the distance between the points $(-2, 3)$ and $(4, -1)$ (Figure 11-2).

Solution:

$(-2, 3)$ and $(4, -1)$

(x_1, y_1) and (x_2, y_2) Label the points.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{[(4) - (-2)]^2 + [(-1) - (3)]^2} \qquad \text{Apply the distance formula.}$$
$$= \sqrt{(6)^2 + (-4)^2}$$
$$= \sqrt{36 + 16}$$
$$= \sqrt{52}$$
$$= \sqrt{2^2 \cdot 13}$$
$$= 2\sqrt{13}$$

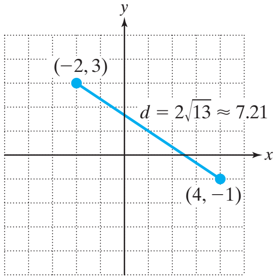


Figure 11-2

Skill Practice

1. Find the distance between the points $(-4, -2)$ and $(2, -5)$.

Skill Practice Answers

1. $3\sqrt{5}$

TIP: The order in which the points are labeled does not affect the result of the distance formula. For example, if the points in Example 1 had been labeled in reverse, the distance formula would still yield the same result:

$(-2, 3)$ and $(4, -1)$

(x_2, y_2) and (x_1, y_1)

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{[(-2) - (4)]^2 + [3 - (-1)]^2} \\ &= \sqrt{(-6)^2 + (4)^2} \\ &= \sqrt{36 + 16} \\ &= \sqrt{52} \\ &= 2\sqrt{13} \end{aligned}$$

2. Circles

A **circle** is defined as the set of all points in a plane that are equidistant from a fixed point called the **center**. The fixed distance from the center is called the **radius** and is denoted by r, where $r > 0$.

Suppose a circle is centered at the point (h, k) and has radius, r (Figure 11-3). The distance formula can be used to derive an equation of the circle.

Let (x, y) be any arbitrary point on the circle. Then, by definition, the distance between (h, k) and (x, y) must be r.

$$\sqrt{(x - h)^2 + (y - k)^2} = r$$

$$(x - h)^2 + (y - k)^2 = r^2 \qquad \text{Square both sides.}$$

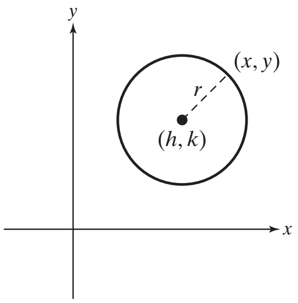


Figure 11-3

Standard Equation of a Circle

The **standard equation of a circle**, centered at (h, k) with radius r, is given by

$$(x - h)^2 + (y - k)^2 = r^2 \qquad \text{where } r > 0$$

Note: If a circle is centered at the origin $(0, 0)$, then $h = 0$ and $k = 0$, and the equation simplifies to $x^2 + y^2 = r^2$.

Example 2 Graphing a Circle

Find the center and radius of each circle. Then graph the circle.

a. $(x - 3)^2 + (y + 4)^2 = 36$

b. $x^2 + \left(y - \frac{10}{3}\right)^2 = \frac{25}{9}$

c. $x^2 + y^2 = 10$

Solution:

a. $(x - 3)^2 + (y + 4)^2 = 36$

$(x - 3)^2 + [y - (-4)]^2 = (6)^2$

$h = 3, k = -4, \text{ and } r = 6$

The center is $(3, -4)$ and the radius is 6 (Figure 11-4).

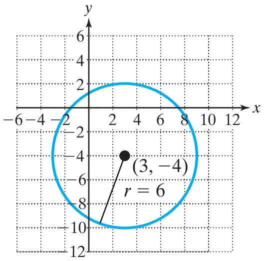


Figure 11-4

Calculator Connections

Graphing calculators are designed to graph *functions,* in which y is written in terms of x. A circle is not a function. However, it can be graphed as the union of two functions—one representing the top semicircle and the other representing the bottom semicircle.

Solving for y in Example 2(a), we have

$$(x-3)^2+(y+4)^2=36$$

$$(y+4)^2=36-(x-3)^2$$

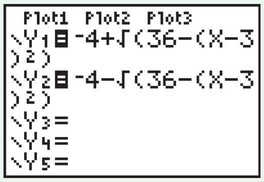
$$y+4=\pm\sqrt{36-(x-3)^2}$$

$$y=-4\pm\sqrt{36-(x-3)^2}$$

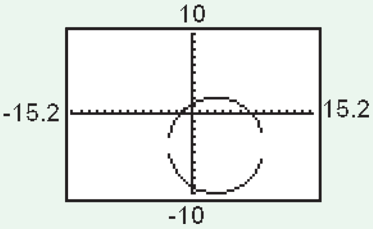
Graph these functions as Y_1 and Y_2, using a square viewing window.

$$Y_1=-4+\sqrt{36-(x-3)^2}$$

$$Y_2=-4-\sqrt{36-(x-3)^2}$$



```
Plot1 Plot2 Plot3
\Y1■-4+√(36-(X-3
)2)
\Y2■-4-√(36-(X-3
)2)
\Y3=
\Y4=
\Y5=
```



Notice that the image from the calculator does not show the upper and lower semicircles connecting at their endpoints, when in fact the semicircles should "hook up." This is due to the calculator's limited resolution.

b. $x^2+\left(y-\frac{10}{3}\right)^2=\frac{25}{9}$

$$(x-0)^2+\left(y-\frac{10}{3}\right)^2=\left(\frac{5}{3}\right)^2$$

The center is $(0, \frac{10}{3})$ and the radius is $\frac{5}{3}$ (Figure 11-5).

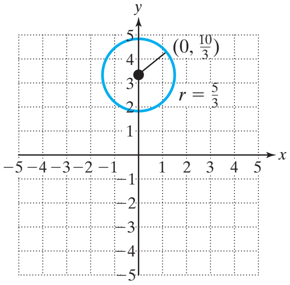


Figure 11-5

c. $x^2+y^2=10$

$$(x-0)^2+(y-0)^2=(\sqrt{10})^2$$

The center is (0, 0) and the radius is $\sqrt{10}\approx 3.16$ (Figure 11-6).

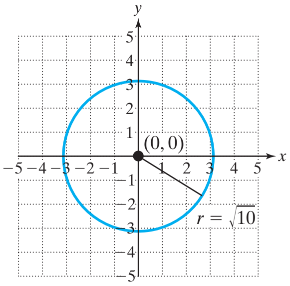


Figure 11-6

Skill Practice Find the center and radius of each circle. Then graph the circle.

2. $(x + 1)^2 + (y - 2)^2 = 9$

3. $\left(x + \frac{7}{2}\right)^2 + y^2 = \frac{9}{4}$

4. $x^2 + y^2 = 15$

Sometimes it is necessary to complete the square to write an equation of a circle in standard form.

Example 3 Writing the Equation of a Circle in the Form $(x - h)^2 + (y - k)^2 = r^2$

Identify the center and radius of the circle given by the equation $x^2 + y^2 + 2x - 16y + 61 = 0$.

Solution:

$$x^2 + y^2 + 2x - 16y + 61 = 0$$

To identify the center and radius, write the equation in the form $(x - h)^2 + (y - k)^2 = r^2$.

$$(x^2 + 2x \quad) + (y^2 - 16y \quad) = -61$$

Group the x-terms and group the y-terms. Move the constant to the right-hand side.

$$(x^2 + 2x + 1) + (y^2 - 16y + 64) = -61 + 1 + 64$$

- Complete the square on x. Add $[\frac{1}{2}(2)]^2 = 1$ to both sides of the equation.
- Complete the square on y. Add $[\frac{1}{2}(-16)]^2 = 64$ to both sides of the equation.

$$(x + 1)^2 + (y - 8)^2 = 4$$

Factor and simplify.

$$[x - (-1)]^2 + (y - 8)^2 = 2^2$$

The center is $(-1, 8)$ and the radius is 2.

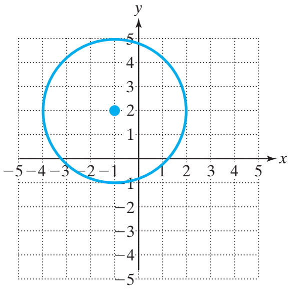
Skill Practice

5. Identify the center and radius of the circle given by the equation

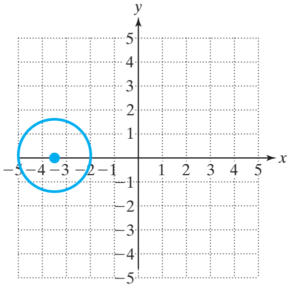
$$x^2 + y^2 - 10x + 4y - 7 = 0$$

Skill Practice Answers

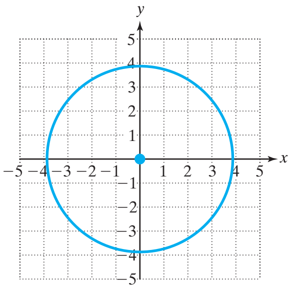
2. Center; $(-1, 2)$; radius: 3



3. Center: $(-\frac{7}{2}, 0)$; radius: $\frac{3}{2}$



4. Center: $(0, 0)$; radius: $\sqrt{15}$



5. Center: $(5, -2)$; radius: 6

3. Writing an Equation of a Circle

Example 4 Writing an Equation of a Circle

Write an equation of the circle shown in Figure 11-7.

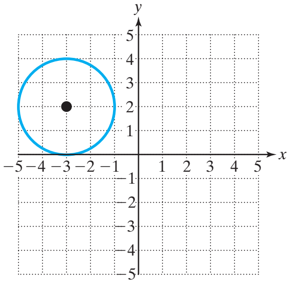


Figure 11-7

Solution:

The center is $(-3, 2)$; therefore, $h = -3$ and $k = 2$.
From the graph, $r = 2$.

$$(x - h)^2 + (y - k)^2 = r^2$$

$$[x - (-3)]^2 + (y - 2)^2 = (2)^2$$

$$(x + 3)^2 + (y - 2)^2 = 4$$

Skill Practice

6. Write an equation for a circle whose center is the point $(6, -1)$ and whose radius is 8.

Skill Practice Answers

6. $(x - 6)^2 + (y + 1)^2 = 64$

Section 11.1 Practice Exercises

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Study Skills Exercise

1. Define the key terms.

a. Distance formula **b. Circle** **c. Center** **d. Radius** **e. Standard equation of a circle**

Concept 1: Distance Formula

For Exercises 2–16, use the distance formula to find the distance between the two points.

2. $(-2, 7)$ and $(3, -9)$

3. $(1, 10)$ and $(-2, 4)$

4. $(0, 5)$ and $(-3, 8)$

5. $(6, 7)$ and $(3, 2)$

6. $\left(\frac{2}{3}, \frac{1}{5}\right)$ and $\left(-\frac{5}{6}, \frac{3}{10}\right)$

7. $\left(-\frac{1}{2}, \frac{5}{8}\right)$ and $\left(-\frac{3}{2}, \frac{1}{4}\right)$

8. $(4, 13)$ and $(4, -6)$

9. $(-2, 5)$ and $(-2, 9)$

10. $(8, -6)$ and $(-2, -6)$

11. $(7, 2)$ and $(15, 2)$

12. $(-6, -2)$ and $(-1, -4)$

13. $(-1, -5)$ and $(-3, -2)$

14. $(3\sqrt{5}, 2\sqrt{7})$ and $(-\sqrt{5}, -3\sqrt{7})$ **15.** $(4\sqrt{6}, -2\sqrt{2})$ and $(2\sqrt{6}, \sqrt{2})$ **16.** $(6, 0)$ and $(0, -1)$

17. Explain how to find the distance between 5 and −7 on the *y*-axis.

18. Explain how to find the distance between 15 and −37 on the *x*-axis.

19. Find the values of y such that the distance between the points $(4, 7)$ and $(-4, y)$ is 10 units.

20. Find the values of x such that the distance between the points $(-4, -2)$ and $(x, 3)$ is 13 units.

21. Find the values of x such that the distance between the points $(x, 2)$ and $(4, -1)$ is 5 units.

22. Find the values of y such that the distance between the points $(-5, 2)$ and $(3, y)$ is 10 units.

For Exercises 23–26, determine if the three points define the vertices of a right triangle.

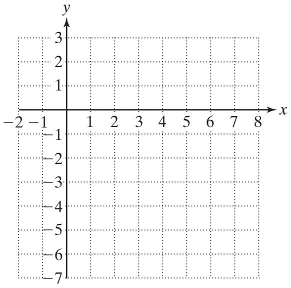
23. $(-3, 2)$, $(-2, -4)$, and $(3, 3)$ **24.** $(1, -2)$, $(-2, 4)$, and $(7, 1)$

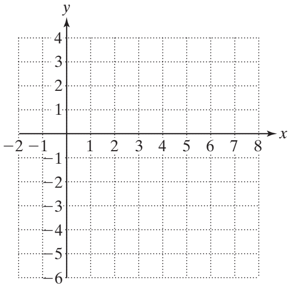
25. $(-3, -2)$, $(4, -3)$, and $(1, 5)$ **26.** $(1, 4)$, $(5, 3)$, and $(2, 0)$

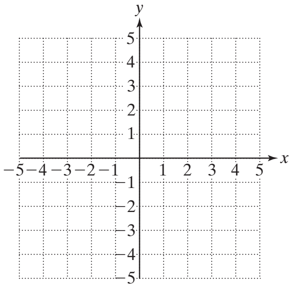
Concept 2: Circles

For Exercises 27–48, identify the center and radius of the circle and then graph the circle. Complete the square, if necessary.

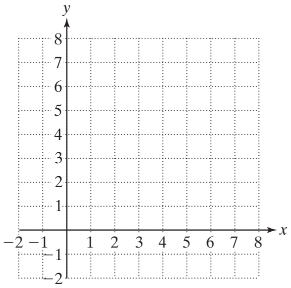
27. $(x - 4)^2 + (y + 2)^2 = 9$ **28.** $(x - 3)^2 + (y + 1)^2 = 16$ **29.** $(x + 1)^2 + (y + 1)^2 = 1$

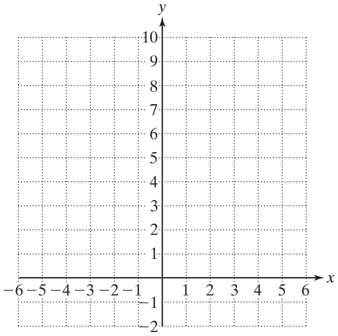


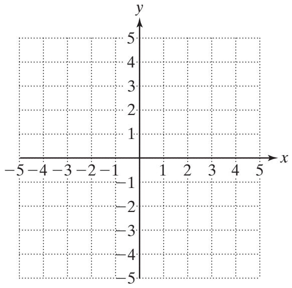




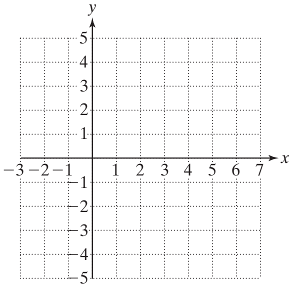
30. $(x - 4)^2 + (y - 4)^2 = 4$ **31.** $x^2 + (y - 5)^2 = 25$ **32.** $(x + 1)^2 + y^2 = 1$



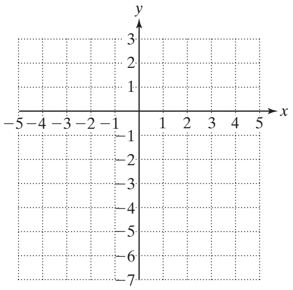




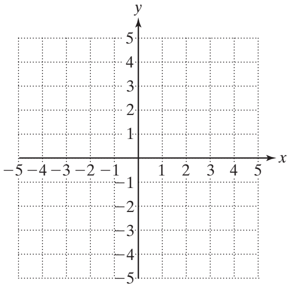
33. $(x - 3)^2 + y^2 = 8$



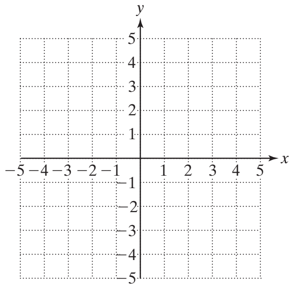
34. $x^2 + (y + 2)^2 = 20$



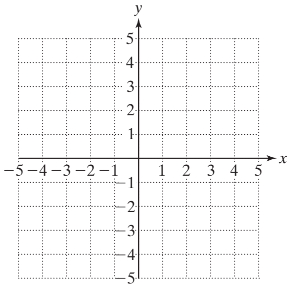
35. $x^2 + y^2 = 6$



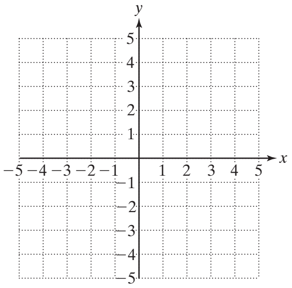
36. $x^2 + y^2 = 15$



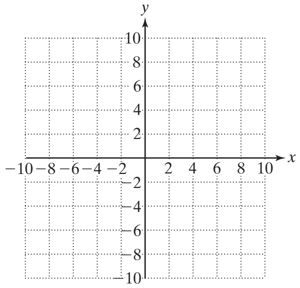
37. $\left(x + \frac{4}{5}\right)^2 + y^2 = \frac{64}{25}$



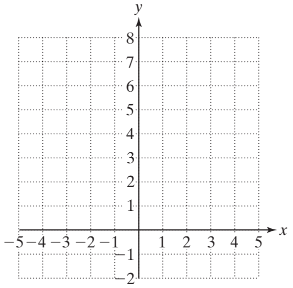
38. $x^2 + \left(y - \frac{5}{2}\right)^2 = \frac{9}{4}$



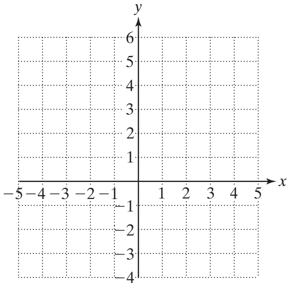
39. $x^2 + y^2 - 2x - 6y - 26 = 0$



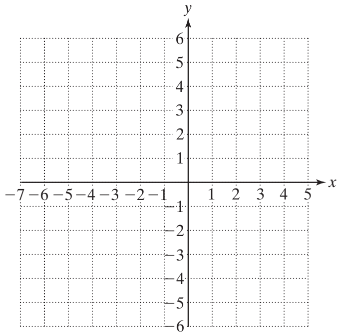
40. $x^2 + y^2 + 4x - 8y + 16 = 0$



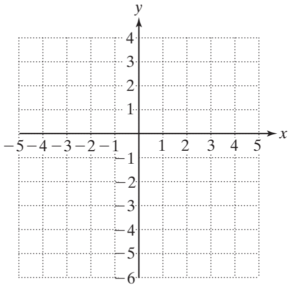
41. $x^2 + y^2 - 6y + 5 = 0$



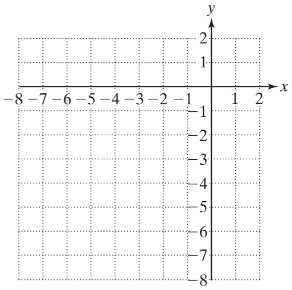
42. $x^2 + 2x + y^2 - 24 = 0$



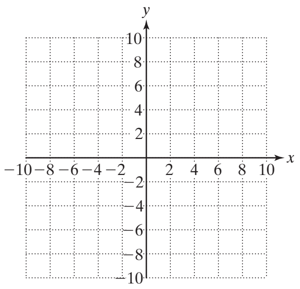
43. $x^2 + y^2 + 6y + \frac{65}{9} = 0$



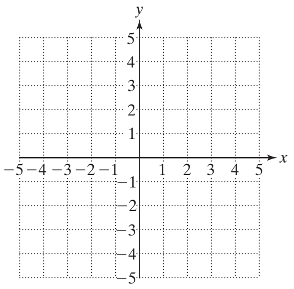
44. $x^2 + y^2 + 12x + \frac{143}{4} = 0$



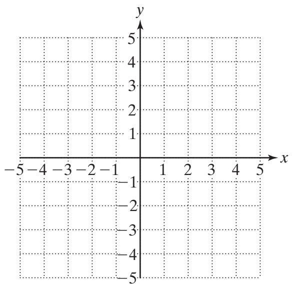
45. $x^2 + y^2 - 12x + 12y + 71 = 0$



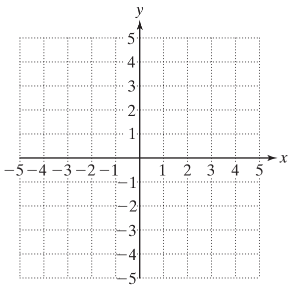
46. $x^2 + y^2 + 2x + 4y - 4 = 0$



47. $2x^2 + 2y^2 = 32$



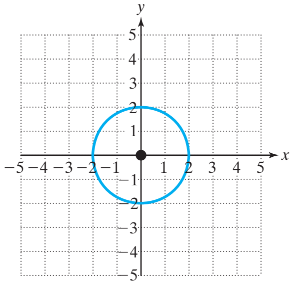
48. $3x^2 + 3y^2 = 3$



Concept 3: Writing an Equation of a Circle

For Exercises 49–54, write an equation that represents the graph of the circle.

49.



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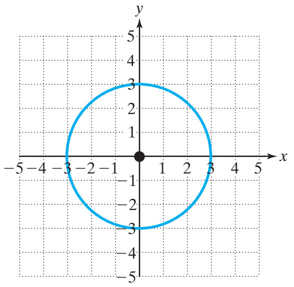
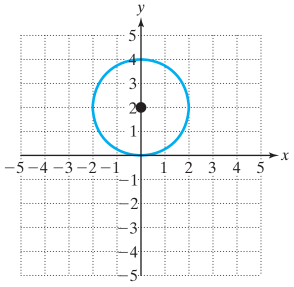


51.



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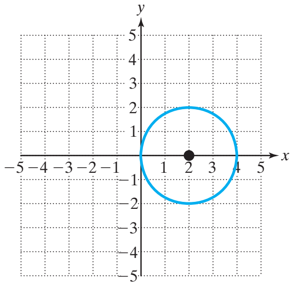

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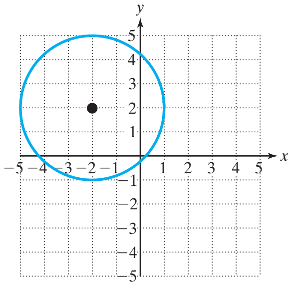

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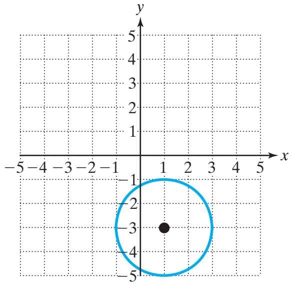


55. Write an equation of a circle centered at the origin with a radius of 7 m.

56. Write an equation of a circle centered at the origin with a radius of 12 m.

57. Write an equation of a circle centered at $(-3, -4)$ with a diameter of 12 ft.

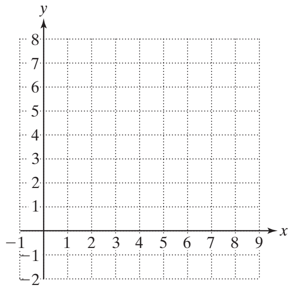
58. Write an equation of a circle centered at $(5, -1)$ with a diameter of 8 ft.

Expanding Your Skills

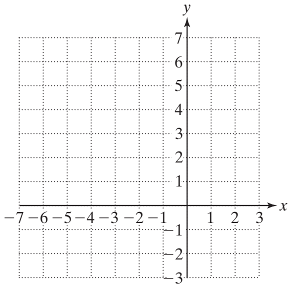
59. Write an equation of a circle that has the points $(-2, 3)$ and $(2, 3)$ as endpoints of a diameter.

60. Write an equation of a circle that has the points $(-1, 3)$ and $(-1, -3)$ as endpoints of a diameter.

61. Write an equation of a circle whose center is at $(4, 4)$ and is tangent to the x- and y-axes. (*Hint:* Sketch the circle first.)



62. Write an equation of a circle whose center is at $(-3, 3)$ and is tangent to the x- and y-axes. (*Hint:* Sketch the circle first.)



63. Write an equation of a circle whose center is at $(1, 1)$ and that passes through the point $(-4, 3)$.

64. Write an equation of a circle whose center is at $(-3, -1)$ and that passes through the point $(5, -2)$.

Graphing Calculator Exercises

For Exercises 65–70, graph the circles from the indicated exercise on a square viewing window, and approximate the center and the radius from the graph.

65. $(x - 4)^2 + (y + 2)^2 = 9$ (Exercise 27)

66. $(x - 3)^2 + (y + 1)^2 = 16$ (Exercise 28)

67. $x^2 + (y - 5)^2 = 25$ (Exercise 31)

68. $(x + 1)^2 + y^2 = 1$ (Exercise 32)

69. $x^2 + y^2 = 6$ (Exercise 35)

70. $x^2 + y^2 = 15$ (Exercise 36)

More on the Parabola

Section 11.2

Concepts

1. Introduction to Conic Sections
2. Parabola—Vertical Axis of Symmetry
3. Parabola—Horizontal Axis of Symmetry
4. Vertex Formula

1. Introduction to Conic Sections

Recall that the graph of a second-degree equation of the form $y = ax^2 + bx + c$ $(a \neq 0)$ is a parabola. In Section 11.1 we learned that the graph of $(x - h)^2 + (y - k)^2 = r^2$ is a circle. These and two other types of figures called ellipses and hyperbolas are called **conic sections**. Conic sections derive their names because each is the intersection of a plane and a double-napped cone (Figure 11-8).

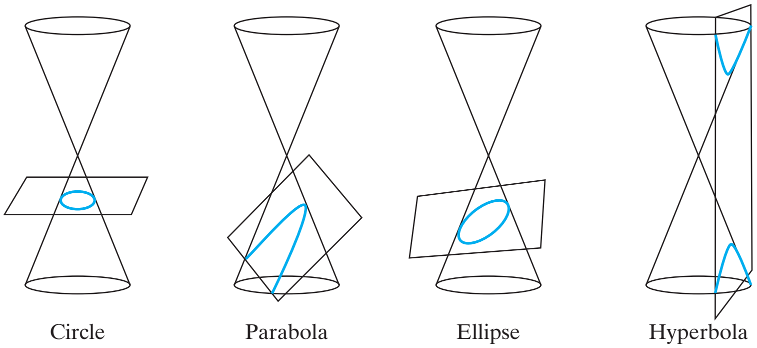
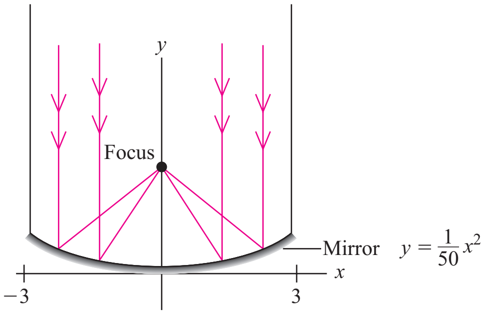


Figure 11-8

2. Parabola—Vertical Axis of Symmetry

A **parabola** is defined by a set of points in a plane that are equidistant from a fixed line (called the directrix) and a fixed point (called the focus) not on the directrix. Parabolas have numerous real-world applications. For example, a reflecting telescope has a mirror with the cross section in the shape of a parabola. A parabolic mirror has the property that incoming rays of light are reflected from the surface of the mirror to the focus.



The graph of the solutions to the quadratic equation $y = ax^2 + bx + c$ is a parabola. We graphed parabolas of this type in Section 8.4 by writing the equation in the form $y = a(x - h)^2 + k$. Recall that the **vertex** (h, k) is the highest or the lowest point of a parabola. The **axis of symmetry** of the parabola is a line that passes through the vertex and is perpendicular to the directrix (Figure 11-9).

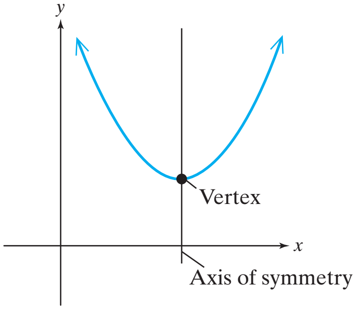


Figure 11-9

Standard Form of the Equation of a Parabola—Vertical Axis of Symmetry

The standard form of the equation of a parabola with vertex (h, k) and vertical axis of symmetry is

$$y = a(x - h)^2 + k \qquad \text{where } a \neq 0$$

If $a > 0$, then the parabola opens upward; and if $a < 0$, the parabola opens downward.

The axis of symmetry is given by $x = h$.

In Section 8.5, we learned that it is sometimes necessary to complete the square to write the equation of a parabola in standard form.

Example 1 Graphing a Parabola by First Completing the Square

Given: the equation of the parabola $y = -2x^2 + 4x + 1$

a. Write the equation in standard form $y = a(x - h)^2 + k$.

b. Identify the vertex and axis of symmetry. Determine if the parabola opens upward or downward.

c. Graph the parabola.

Solution:

a. Complete the square to write the equation in the form $y = a(x - h)^2 + k$.

$y = -2x^2 + 4x + 1$

$y = -2(x^2 - 2x) + 1$ Factor out -2 from the variable terms.

$y = -2(x^2 - 2x + 1 - 1) + 1$ Add and subtract the quantity $\left[\frac{1}{2}(-2)\right]^2 = 1$.

$y = -2(x^2 - 2x + 1) + (-2)(-1) + 1$ Remove the -1 term from within the parentheses by first applying the distributive property. When -1 is removed from the parentheses, it carries with it the factor of -2 from outside the parentheses.

$$y = -2(x - 1)^2 + 2 + 1$$

$$y = -2(x - 1)^2 + 3$$

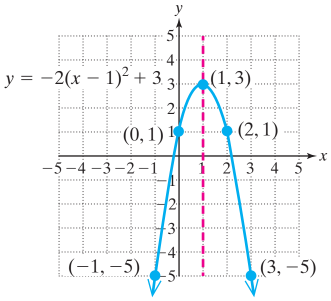
The equation is in the form $y = a(x - h)^2 + k$ where $a = -2$, $h = 1$, and $k = 3$.

b. The vertex is $(1, 3)$. Because a is negative, $(a = -2 < 0)$, the parabola opens downward. The axis of symmetry is $x = 1$.

c. To graph the parabola, we know that its orientation is downward. Furthermore, we know the vertex is $(1, 3)$. To find other points on the parabola, select several values of x and solve for y. Recall that the y-intercept is found by substituting $x = 0$ and solving for y.

x	y	
1	3	← Vertex
−1	−5	
0	1	← y-intercept
2	1	
3	−5	

↑ Choose x. ↑ Solve for y.



Skill Practice

1. Given: the equation of the parabola $y = 2x^2 - 4x + 5$

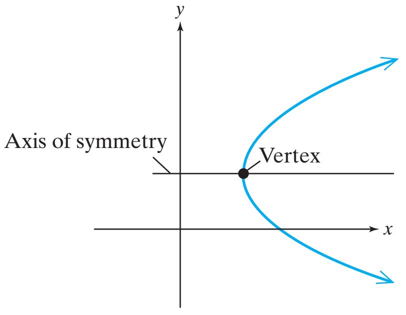
a. Write the equation in standard form.

b. Identify the vertex and axis of symmetry.

c. Graph the parabola.

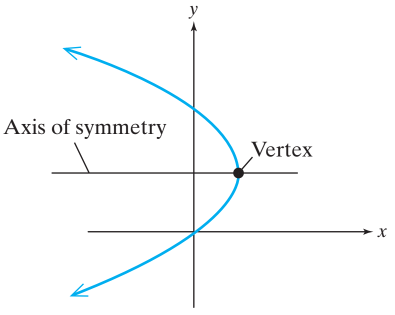
3. Parabola—Horizontal Axis of Symmetry

We have seen that the graph of a parabola $y = ax^2 + bx + c$ opens upward if $a > 0$ and downward if $a < 0$. A parabola can also open to the left or right. In such a case, the "roles" of x and y are essentially interchanged in the equation. Thus, the graph of $x = ay^2 + by + c$ opens to the right if $a > 0$ (Figure 11-10) and to the left if $a < 0$ (Figure 11-11).



$x = ay^2 + by + c, a > 0$

Figure 11-10



$x = ay^2 + by + c, a < 0$

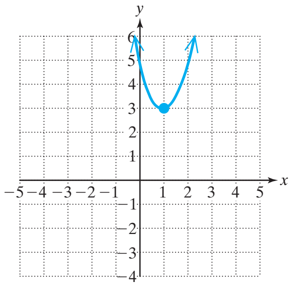
Figure 11-11

Skill Practice Answers

1a. $y = 2(x - 1)^2 + 3$

b. Vertex: $(1, 3)$; axis of symmetry: $x = 1$

c.



Standard Form of the Equation of a Parabola—Horizontal Axis of Symmetry

The standard form of the equation of a parabola with vertex (h, k) and horizontal axis of symmetry is

$$x = a(y - k)^2 + h \qquad \text{where } a \neq 0$$

If $a > 0$, then the parabola opens to the right and if $a < 0$, the parabola opens to the left.

The axis of symmetry is given by $y = k$.

Example 2 Graphing a Parabola with a Horizontal Axis of Symmetry

Given the equation of the parabola $x = 4y^2$,

a. Determine the coordinates of the vertex and the equation of the axis of symmetry.

b. Use the value of a to determine if the parabola opens to the right or left.

c. Plot several points and graph the parabola.

Solution:

a. The equation can be written in the form $x = a(y - k)^2 + h$:

$$x = 4(y - 0)^2 + 0$$

Therefore, $h = 0$ and $k = 0$.

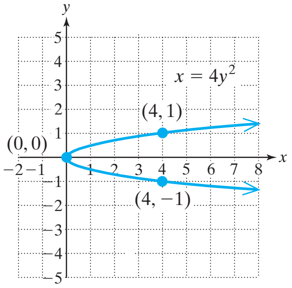
The vertex is $(0, 0)$. The axis of symmetry is $y = 0$ (the x-axis).

b. Because a is positive ($a = 4 > 0$), the parabola opens to the right.

c. The vertex of the parabola is $(0, 0)$. To find other points on the graph, select values for y and solve for x.

x	y	
0	0	⟵Vertex
4	1	
4	−1	
16	2	
16	−2	

Solve for x. Choose y.



Skill Practice

2. Given: the equation $x = -y^2 + 1$

a. Identify the vertex and the axis of symmetry.

b. Determine if the parabola opens to the right or to the left.

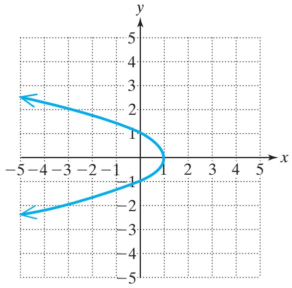
c. Graph the parabola.

Skill Practice Answers

2a. Vertex: $(1, 0)$; axis of symmetry: $y = 0$

b. Parabola opens left.

c.



Calculator Connections

Graphing calculators are designed to graph functions, where y is written as a function of x. The parabola from Example 2 is not a function. It can be graphed, however, as the union of two functions, one representing the top branch and the other representing the bottom branch.

Solving for y in Example 2, we have: $x = 4y^2$

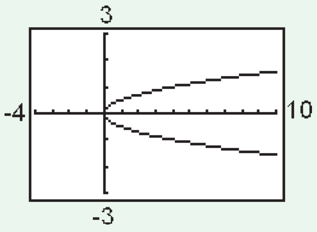
$$y^2 = \frac{x}{4}$$

$$y = \pm\sqrt{\frac{x}{4}}$$

$$y = \pm\frac{\sqrt{x}}{2}$$

Graph these functions as Y_1 and Y_2.

$$Y_1 = \frac{\sqrt{x}}{2} \quad \text{and} \quad Y_2 = -\frac{\sqrt{x}}{2}$$



Example 3 Graphing a Parabola by First Completing the Square

Given the equation of the parabola $x = -y^2 + 8y - 14$,

a. Write the equation in standard form $x = a(y - k)^2 + h$.

b. Identify the vertex and axis of symmetry. Determine if the parabola opens to the right or left.

c. Graph the parabola.

Solution:

a. Complete the square to write the equation in the form $x = a(y - k)^2 + h$.

$x = -y^2 + 8y - 14$

$x = -1(y^2 - 8y) - 14$ Factor out -1 from the variable terms.

$x = -1(y^2 - 8y + 16 - 16) - 14$ Add and subtract the quantity $[\frac{1}{2}(-8)]^2 = 16$.

$x = -1(y^2 - 4y + 16) + (-1)(-16) - 14$ Remove the -16 term from within the parentheses by first applying the distributive property. When -16 is removed from the parentheses, it carries with it the factor of -1 from outside the parentheses.

$x = -1(y - 4)^2 + 16 - 14$

$x = -1(y - 4)^2 + 2$

The equation is in the form $x = a(y - k)^2 + h$, where $a = -1$, $h = 2$, and $k = 4$.

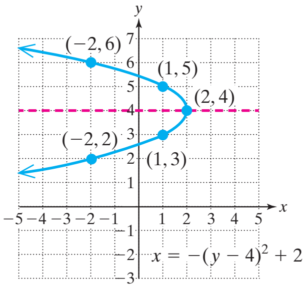
b. The vertex is $(2, 4)$. Because a is negative $(a = -1 < 0)$, the parabola opens to the left. The axis of symmetry is $y = 4$.

c.

x	y
2	4
1	5
1	3
−2	6
−2	2

← Vertex

Solve for x. Choose y.



Skill Practice

3. Given the equation of the parabola $x = \frac{1}{8}(y + 2)^2 + 3$,

a. Determine the coordinates of the vertex.

b. Determine if the parabola opens to the right or to the left.

c. Graph the parabola.

4. Vertex Formula

From Section 8.5, we learned that the vertex formula can also be used to find the vertex of a parabola.

For a parabola defined by $y = ax^2 + bx + c$,

- The x-coordinate of the vertex is given by $x = \frac{-b}{2a}$.
- The y-coordinate of the vertex is found by substituting this value for x into the original equation and solving for y.

For a parabola defined by $x = ay^2 + by + c$,

- The y-coordinate of the vertex is given by $y = \frac{-b}{2a}$.
- The x-coordinate of the vertex is found by substituting this value for y into the original equation and solving for x.

Example 4 Finding the Vertex of a Parabola by Using the Vertex Formula

Find the vertex by using the vertex formula.

a. $x = y^2 + 4y + 5$ **b.** $y = \frac{1}{2}x^2 - 3x + \frac{5}{2}$

Solution:

a. $x = y^2 + 4y + 5$ The parabola is in the form $x = ay^2 + by + c$.

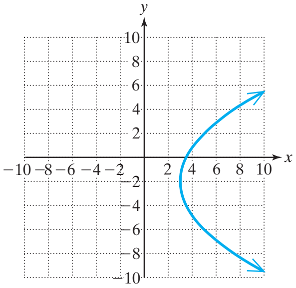
$a = 1 \quad b = 4 \quad c = 5$ Identify a, b, and c.

Skill Practice Answers

3a. Vertex: $(3, -2)$

b. Right

c.



The y-coordinate of the vertex is given by $y = \frac{-b}{2a} = \frac{-(4)}{2(1)} = -2.$

The x-coordinate of the vertex is found by substitution:
$x = (-2)^2 + 4(-2) + 5 = 1.$

The vertex is $(1, -2)$.

b. $y = \frac{1}{2}x^2 - 3x + \frac{5}{2}$ The parabola is in the form $y = ax^2 + bx + c$.

$a = \frac{1}{2}$ $b = -3$ $c = \frac{5}{2}$ Identify a, b, and c.

The x-coordinate of the vertex is given by $x = \frac{-b}{2a} = \frac{-(-3)}{2(\frac{1}{2})} = 3.$

The y-coordinate of the vertex is found by substitution.

$$\begin{aligned} y &= \frac{1}{2}(3)^2 - 3(3) + \frac{5}{2} \\ &= \frac{9}{2} - 9 + \frac{5}{2} \\ &= \frac{14}{2} - 9 \\ &= 7 - 9 \\ &= -2 \end{aligned}$$

The vertex is $(3, -2)$.

Skill Practice Find the vertex by using the vertex formula.

4. $x = -4y^2 + 12y$ **5.** $y = \frac{3}{4}x^2 + 3x + 5$

Skill Practice Answers

4. Vertex: $\left(9, \frac{3}{2}\right)$
5. Vertex: $(-2, 2)$

Section 11.2 Practice Exercises

Study Skills Exercise

1. Define the key terms.

a. Conic sections **b. Parabola** **c. Axis of symmetry** **d. Vertex**

Review Exercises

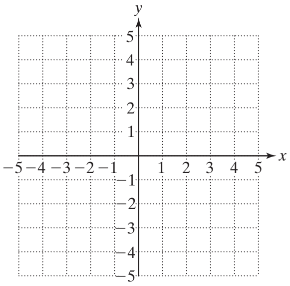
2. Determine the distance between the points $(1, 1)$ and $(2, -2)$.

3. Determine the distance from the origin to the point $(4, -3)$.

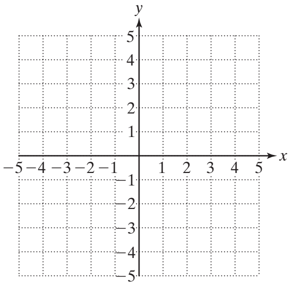
4. Determine the distance between the points $(11, 3)$ and $(5, 3)$.

For Exercises 5–8, identify the center and radius of the circle and then graph the circle.

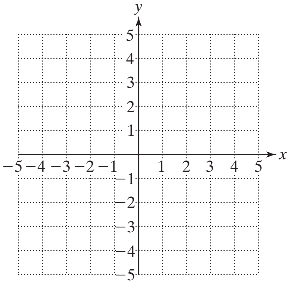
5. $x^2 + (y + 1)^2 = 16$



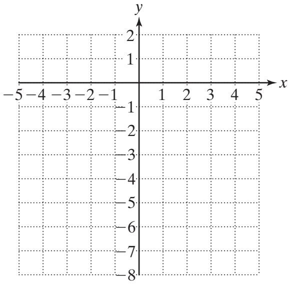
6. $(x - 3)^2 + y^2 = 4$



7. $(x - 3)^2 + (y + 3)^2 = 1$



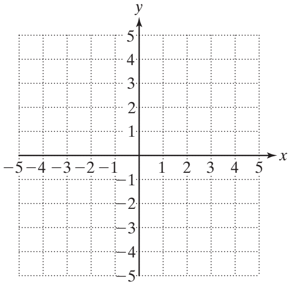
8. $(x + 2)^2 + (y + 4)^2 = 9$



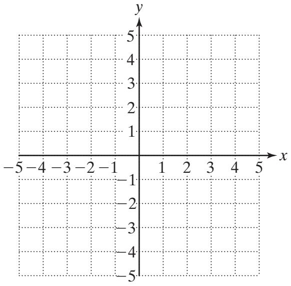
Concept 2: Parabola—Vertical Axis of Symmetry

For Exercises 9–16, use the equation of the parabola in standard form $y = a(x - h)^2 + k$ to determine the coordinates of the vertex and the equation of the axis of symmetry. Then graph the parabola.

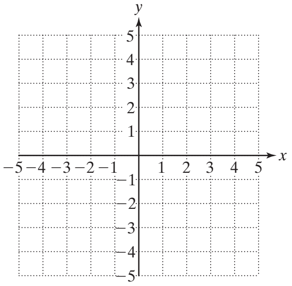
9. $y = (x + 2)^2 + 1$



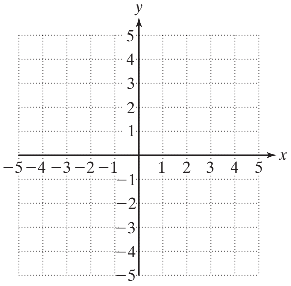
10. $y = (x - 1)^2 + 1$



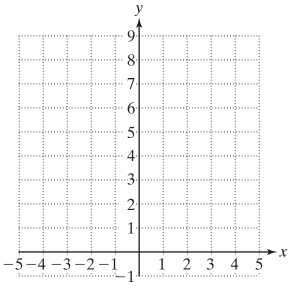
11. $y = x^2 - 4x + 3$



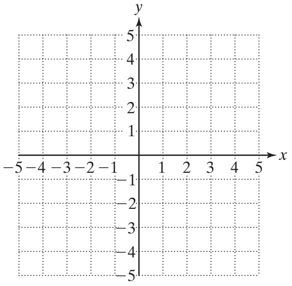
12. $y = x^2 + 6x + 5$



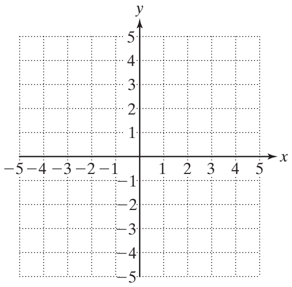
13. $y = -2x^2 + 8x$



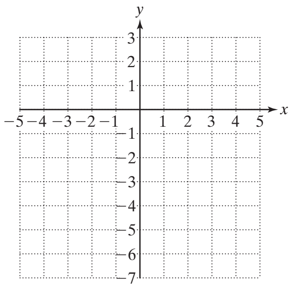
14. $y = -3x^2 - 6x$



15. $y = -x^2 - 3x + 2$



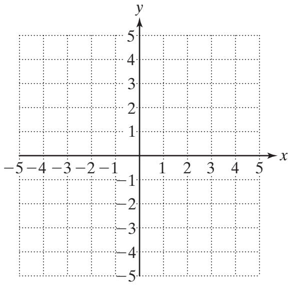
16. $y = -x^2 + x - 4$



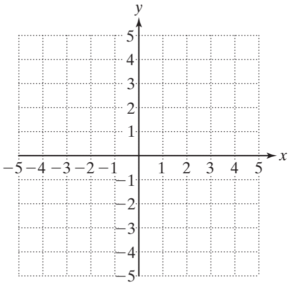
Concept 3: Parabola—Horizontal Axis of Symmetry

For Exercises 17–24, use the equation of the parabola in standard form $x = a(y - k)^2 + h$ to determine the coordinates of the vertex and the axis of symmetry. Then graph the parabola.

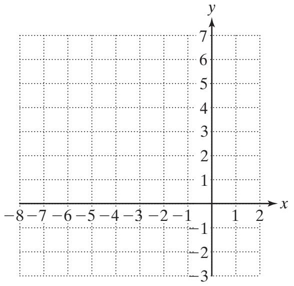
17. $x = y^2 - 3$



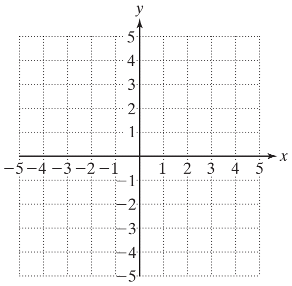
18. $x = y^2 + 1$



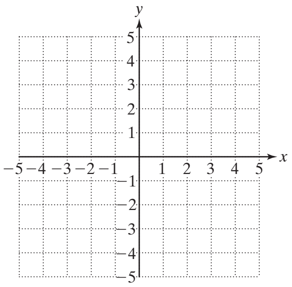
19. $x = -(y - 3)^2 - 3$



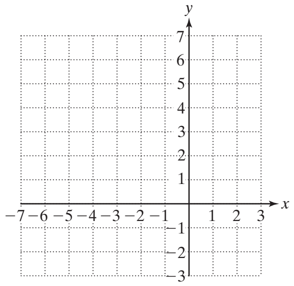
20. $x = -2(y + 2)^2 + 1$



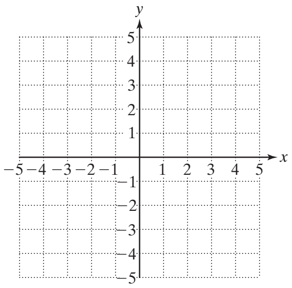
21. $x = -y^2 + 4y - 4$



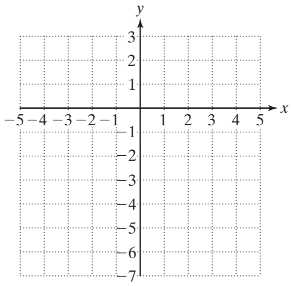
22. $x = -4y^2 - 4y - 2$



23. $x = y^2 - 2y + 2$



24. $x = y^2 + 4y + 1$



Concept 4: Vertex Formula

For Exercises 25–34, determine the vertex by using the vertex formula.

25. $y = x^2 - 4x + 3$

26. $y = x^2 + 6x - 2$

27. $x = y^2 + 2y + 6$

28. $x = y^2 - 8y + 3$

29. $y = -2x^2 + 8x$

30. $y = -3x^2 - 6x$

31. $y = x^2 - 3x + 2$

32. $y = x^2 + x - 4$

33. $x = -2y^2 + 16y + 1$

34. $x = -3y^2 - 6y + 7$

Mixed Exercises

35. Explain how to determine whether a parabola opens upward, downward, left, or right.

36. Explain how to determine whether a parabola has a vertical or horizontal axis of symmetry.

For Exercises 37–48, use the equation of the parabola first to determine whether the axis of symmetry is vertical or horizontal. Then determine if the parabola opens upward, downward, left, or right.

37. $y = (x - 2)^2 + 3$

38. $y = (x - 4)^2 + 2$

39. $y = -2(x + 1)^2 - 4$

40. $y = -3(x + 2)^2 - 1$

41. $x = y^2 + 4$

42. $x = y^2 - 2$

43. $x = -(y + 3)^2 + 2$

44. $x = -2(y - 1)^2 - 3$

45. $y = -2x^2 - 5$

46. $y = -x^2 + 3$

47. $x = 2y^2 + 3y - 2$

48. $x = y^2 - 5y + 1$

Section 11.3 The Ellipse and Hyperbola

Concepts

1. Standard Form of an Equation of an Ellipse
2. Standard Forms of an Equation of a Hyperbola

1. Standard Form of an Equation of an Ellipse

In this section we will study the two remaining conic sections: the ellipse and the hyperbola. An **ellipse** is the set of all points (x, y) such that the sum of the distance between (x, y) and two distinct points is a constant. The fixed points are called the foci (plural of *focus*) of the ellipse.

To visualize an ellipse, consider the following application. Suppose Sonya wants to cut an elliptical rug from a rectangular rug to avoid a stain made by the family dog. She places two tacks along the center horizontal line. Then she ties the ends of a slack piece of rope to each tack. With the rope pulled tight, she traces out a curve. This curve is an ellipse, and the tacks are located at the foci of the ellipse (Figure 11-12).

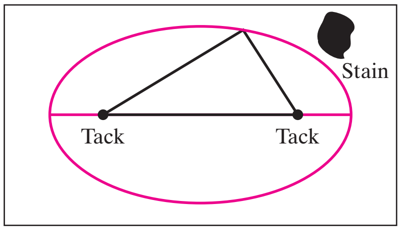


Figure 11-12

We will first graph ellipses that are centered at the origin.

Standard Form of an Equation of an Ellipse Centered at the Origin

An ellipse with the center at the origin has the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where a and b are positive real numbers. In the standard form of the equation, the right side must equal 1.

To graph an ellipse centered at the origin, find the x- and y-intercepts.

To find the x-intercepts, let $y = 0$.

$$\frac{x^2}{a^2} + \frac{0}{b^2} = 1$$

$$\frac{x^2}{a^2} = 1$$

$$x^2 = a^2$$

$$x = \pm\sqrt{a^2}$$

$$x = \pm a$$

The x-intercepts are $(a, 0)$ and $(-a, 0)$.

To find the y-intercepts, let $x = 0$.

$$\frac{0}{a^2} + \frac{y^2}{b^2} = 1$$

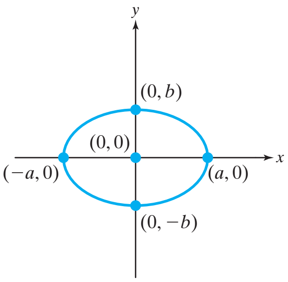
$$\frac{y^2}{b^2} = 1$$

$$y^2 = b^2$$

$$y = \pm\sqrt{b^2}$$

$$y = \pm b$$

The y-intercepts are $(0, b)$ and $(0, -b)$.

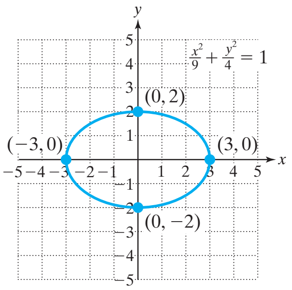


Example 1 Graphing an Ellipse

Graph the ellipse given by the equation. $\frac{x^2}{9} + \frac{y^2}{4} = 1$

Solution:

The equation can be written as $\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$; therefore, $a = 3$ and $b = 2$. Graph the intercepts $(3, 0)$, $(-3, 0)$, $(0, 2)$, $(0, -2)$ and sketch the ellipse.

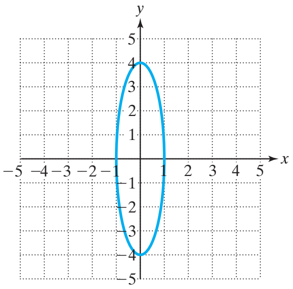


Skill Practice

1. Graph the ellipse given by the equation $x^2 + \frac{y^2}{16} = 1$.

Skill Practice Answers

1.



Example 2 Graphing an Ellipse

Graph the ellipse given by the equation. $25x^2 + y^2 = 25$

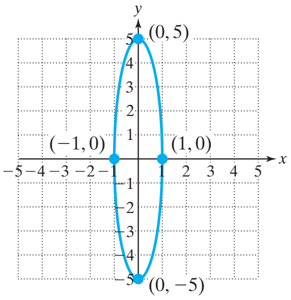
Solution:

First, to write the equation in standard form, divide both sides by 25:

$$x^2 + \frac{y^2}{25} = 1$$

The equation can then be written as $\frac{x^2}{1^2} + \frac{y^2}{5^2} = 1$; therefore, $a = 1$ and $b = 5$.

Graph the intercepts (1, 0), (−1, 0), (0, 5), and (0, −5) and sketch the ellipse.



Skill Practice

2. Graph the ellipse given by the equation $25x^2 + 16y^2 = 400$.

A circle is a special case of an ellipse where $a = b$. Therefore, it is not surprising that we graph an ellipse centered at the point (h, k) in much the same way we graph a circle.

Example 3 Graphing an Ellipse Whose Center is Not at the Origin

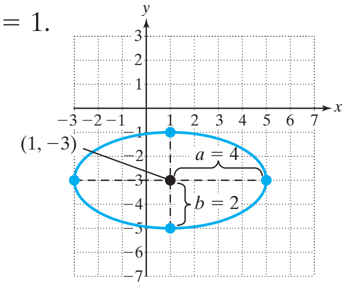
Graph the ellipse give by the equation. $\frac{(x-1)^2}{16} + \frac{(y+3)^2}{4} = 1$

Solution:

Just as we would find the center of a circle, we see that the center of the ellipse is (1, −3). Now we can use the values of a and b to help us plot four strategic points to define the curve.

The equation can be written as $\frac{(x-1)^2}{(4)^2} + \frac{(y+3)^2}{(2)^2} = 1$.

From this, we have $a = 4$ and $b = 2$. To sketch the curve, locate the center at (1, −3). Then move $a = 4$ units to the left and right of the center and plot two points. Similarly, move $b = 2$ units up and down from the center and plot two points.

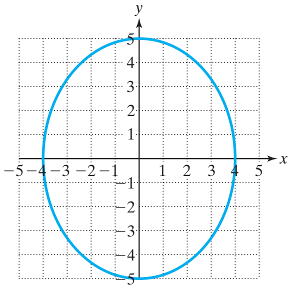


Skill Practice

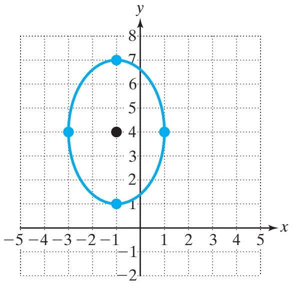
3. Graph the ellipse. $\frac{(x+1)^2}{4} + \frac{(y-4)^2}{9} = 1$

Skill Practice Answers

2.

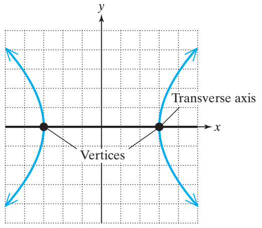


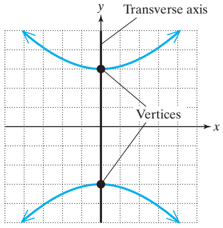
3.



2. Standard Forms of an Equation of a Hyperbola

A **hyperbola** is the set of all points (x, y) such that the *difference* of the distances between (x, y) and two distinct points is a constant. The fixed points are called the foci of the hyperbola. The graph of a hyperbola has two parts, called branches. Each part resembles a parabola but is a slightly different shape. A hyperbola has two vertices that lie on an axis of symmetry called the **transverse axis**. For the hyperbolas studied here, the transverse axis is either horizontal or vertical.





Standard Forms of an Equation of a Hyperbola with Center at the Origin

Let a and b represent positive real numbers.

Horizontal transverse axis:

The standard form of an equation of a hyperbola with a *horizontal transverse axis* and center at the origin is given by $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Note: The x-term is positive. The branches of the hyperbola open left and right.

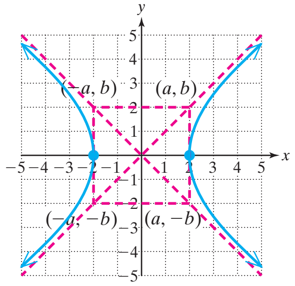
Vertical transverse axis:

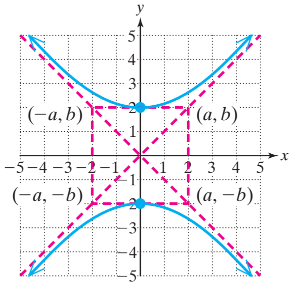
The standard form of an equation of a hyperbola with a *vertical transverse axis* and center at the origin is given by $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$.

Note: The y-term is positive. The branches of the hyperbola open up and down.

In the standard forms of an equation of a hyperbola, the right side must equal 1.

To graph a hyperbola centered at the origin, first construct a reference rectangle. Draw this rectangle by using the points (a, b), $(-a, b)$, $(a, -b)$, and $(-a, -b)$. Asymptotes lie on the diagonals of the rectangle. The branches of the hyperbola are drawn to approach the asymptotes.





Example 4 Graphing a Hyperbola

Graph the hyperbola given by the equation. $\dfrac{x^2}{36} - \dfrac{y^2}{9} = 1$

a. Determine whether the transverse axis is horizontal or vertical.

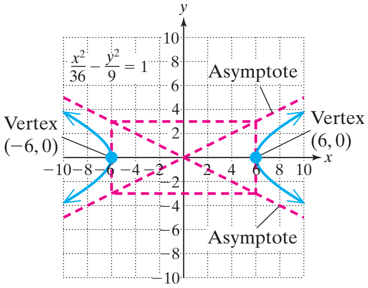
b. Draw the reference rectangle and asymptotes.

c. Graph the hyperbola and label the vertices.

Solution:

a. Since the x-term is positive, the transverse axis is horizontal.

b. The equation can be written $\dfrac{x^2}{6^2} - \dfrac{y^2}{3^2} = 1$; therefore, $a = 6$ and $b = 3$. Graph the reference rectangle from the points (6, 3), (6, −3), (−6, 3), (−6, −3).



c. The coordinates of the vertices are (−6, 0) and (6, 0).

Skill Practice

4. Graph the hyperbola $\dfrac{x^2}{1} - \dfrac{y^2}{9} = 1$.

TIP: In the equation $\dfrac{x^2}{36} - \dfrac{y^2}{9} = 1$, the x-term is positive. Therefore, the hyperbola opens in the x-direction (left/right).

Example 5 Graphing a Hyperbola

Graph the hyperbola given by the equation. $y^2 - 4x^2 - 16 = 0$

a. Write the equation in standard form to determine whether the transverse axis is horizontal or vertical.

b. Draw the reference rectangle and asymptotes.

c. Graph the hyperbola and label the vertices.

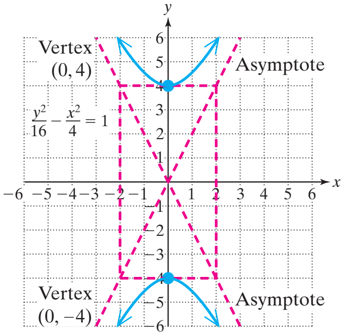
Solution:

a. Isolate the variable terms and divide by 16:

$$\frac{y^2}{16} - \frac{x^2}{4} = 1.$$

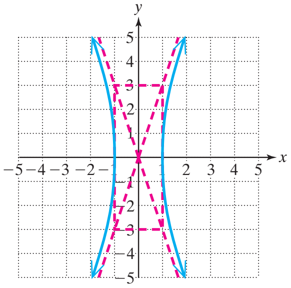
Since the y-term is positive, the transverse axis is vertical.

b. The equation can be written $\dfrac{y^2}{4^2} - \dfrac{x^2}{2^2} = 1$; therefore, $a = 2$ and $b = 4$. Graph the reference rectangle from the points (2, 4), (2, −4), (−2, 4), (−2, −4).



Skill Practice Answers

4.



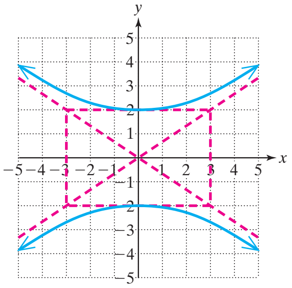
c. The coordinates of the vertices are (0, 4) and (0, −4).

Skill Practice

5. Graph the hyperbola $\frac{y^2}{4} - \frac{x^2}{9} = 1$.

Skill Practice Answers

5.



Section 11.3 Practice Exercises

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Study Skills Exercise

1. Define the key terms.

a. Ellipse **b. Hyperbola** **c. Transverse axis of a hyperbola**

Review Exercises

2. Write the standard form of a circle with center at (h, k) and radius, r.

For Exercises 3–4, identify the center and radius of the circle.

3. $x^2 + y^2 - 16x + 12y = 0$

4. $x^2 + y^2 + 4x + 4y - 1 = 0$

For Exercises 5–6, identify the vertex and the axis of symmetry.

5. $y = 3(x + 3)^2 - 1$

6. $x = -\frac{1}{4}(y - 1)^2 - 6$

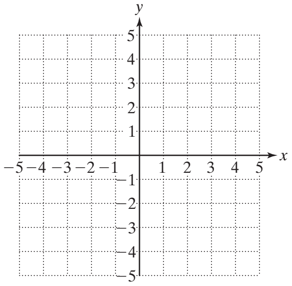
7. Write an equation for a circle whose center has coordinates $(\frac{1}{2}, \frac{5}{2})$ with radius equal to $\frac{1}{2}$.

8. Write the equation for the circle centered at the origin and with radius equal to $\frac{1}{8}$.

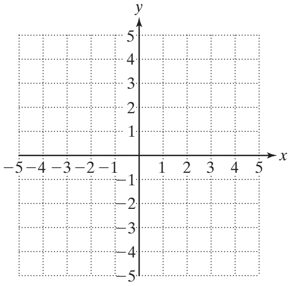
Concept 1: Standard Form of an Equation of an Ellipse

For Exercises 9–16, find the x- and y-intercepts and graph the ellipse.

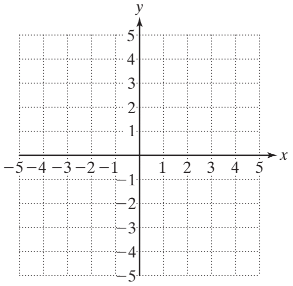
9. $\frac{x^2}{4} + \frac{y^2}{9} = 1$



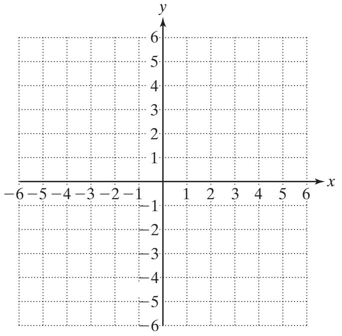
10. $\frac{x^2}{16} + \frac{y^2}{25} = 1$



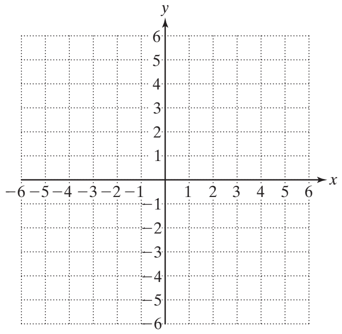
11. $\frac{x^2}{16} + \frac{y^2}{9} = 1$



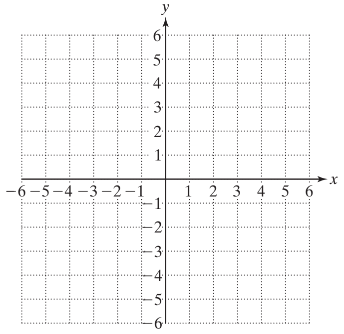
12. $\dfrac{x^2}{36} + \dfrac{y^2}{4} = 1$



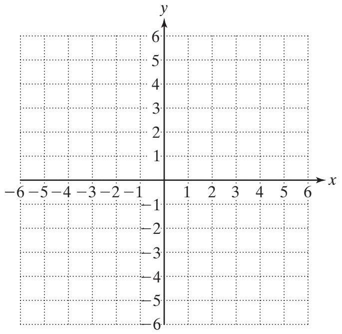
13. $4x^2 + y^2 = 4$



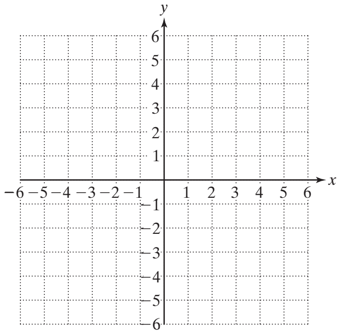
14. $9x^2 + y^2 = 36$



15. $x^2 + 25y^2 - 25 = 0$

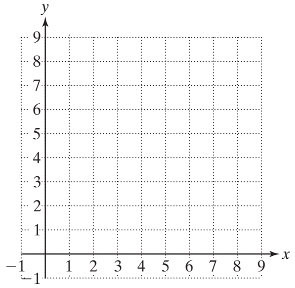


16. $4x^2 + 9y^2 = 144$

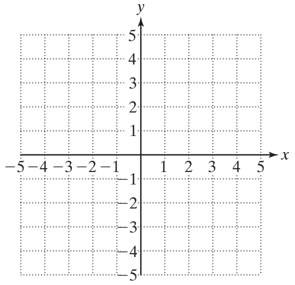


For Exercises 17–24, identify the center (h, k) of the ellipse. Then graph the ellipse.

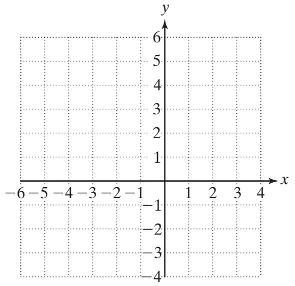
17. $\dfrac{(x - 4)^2}{4} + \dfrac{(y - 5)^2}{9} = 1$



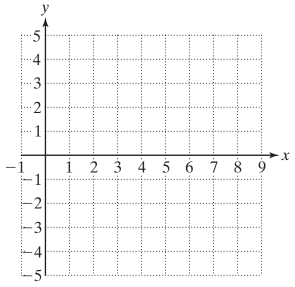
18. $\dfrac{(x + 2)^2}{9} + \dfrac{(y - 1)^2}{16} = 1$



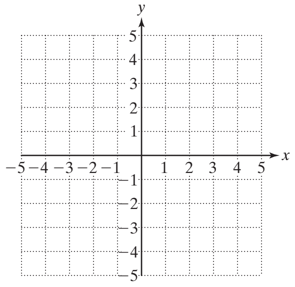
19. $\dfrac{(x + 1)^2}{25} + \dfrac{(y - 2)^2}{9} = 1$



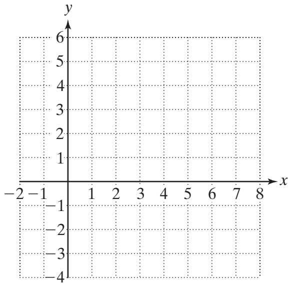
20. $\dfrac{(x - 4)^2}{25} + \dfrac{(y + 1)^2}{16} = 1$



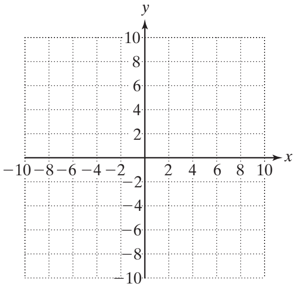
21. $\dfrac{(x - 2)^2}{9} + (y + 3)^2 = 1$



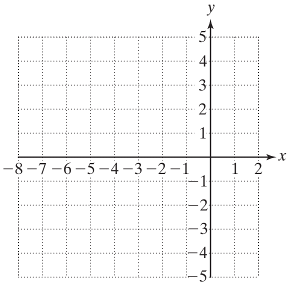
22. $(x - 5)^2 + \dfrac{(y - 3)^2}{4} = 1$



23. $\dfrac{x^2}{36} + \dfrac{(y-1)^2}{25} = 1$



24. $\dfrac{(x+5)^2}{4} + \dfrac{y^2}{16} = 1$



Concept 2: Standard Forms of an Equation of a Hyperbola

For Exercises 25–32, determine whether the transverse axis is horizontal or vertical.

25. $\dfrac{y^2}{6} - \dfrac{x^2}{18} = 1$

26. $\dfrac{y^2}{10} - \dfrac{x^2}{4} = 1$

27. $\dfrac{x^2}{20} - \dfrac{y^2}{15} = 1$

28. $\dfrac{x^2}{12} - \dfrac{y^2}{9} = 1$

29. $x^2 - y^2 = 12$

30. $x^2 - y^2 = 15$

31. $x^2 - 3y^2 = -9$

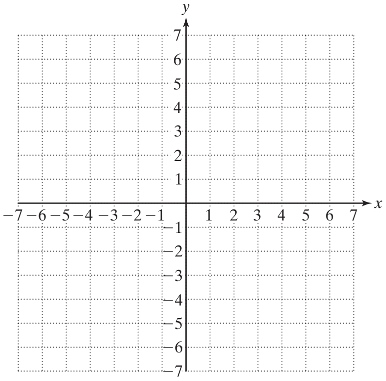
32. $2x^2 - y^2 = -10$

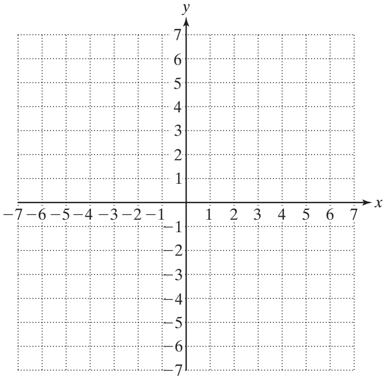
For Exercises 33–40, use the equation in standard form to graph the hyperbola. Label the vertices of the hyperbola.

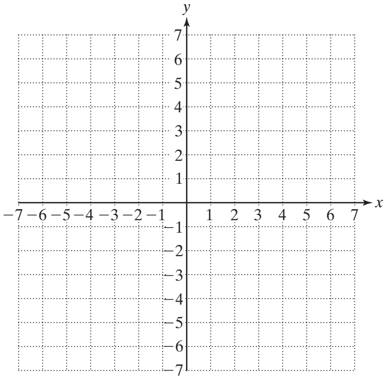
33. $\dfrac{x^2}{25} - \dfrac{y^2}{16} = 1$

34. $\dfrac{x^2}{9} - \dfrac{y^2}{36} = 1$

35. $\dfrac{y^2}{4} - \dfrac{x^2}{4} = 1$



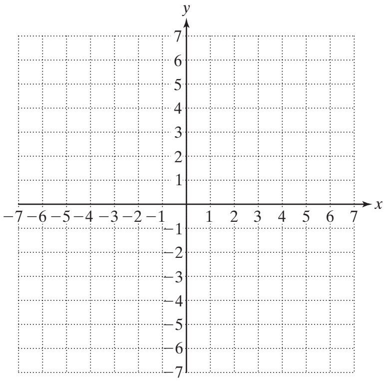


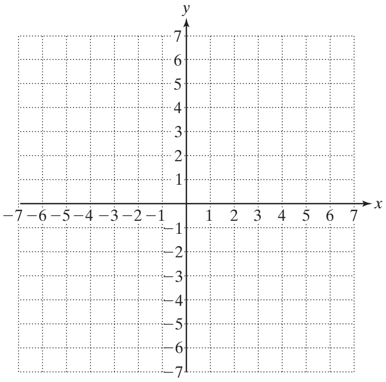


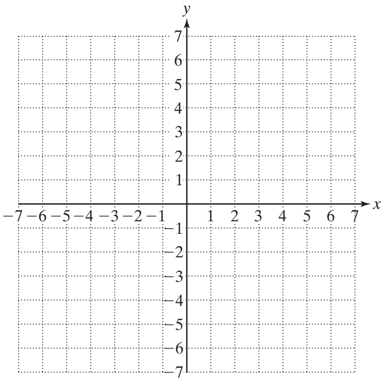
36. $\dfrac{y^2}{9} - \dfrac{x^2}{9} = 1$

37. $36x^2 - y^2 = 36$

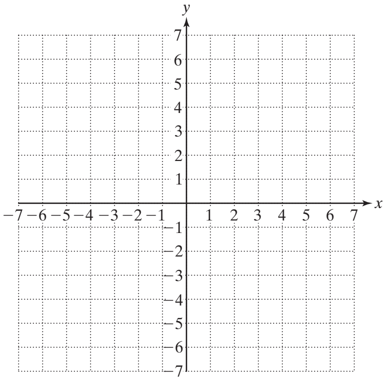
38. $x^2 - 25y^2 = 25$



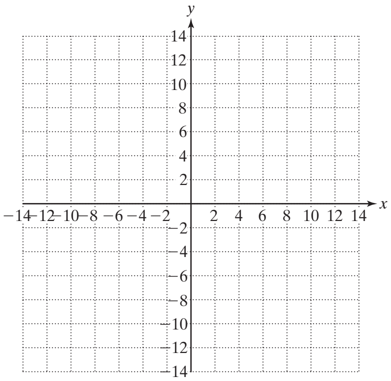




39. $y^2 - 4x^2 - 16 = 0$



40. $y^2 - 4x^2 - 36 = 0$



Mixed Exercises

For Exercises 41–52, determine if the equation represents an ellipse or a hyperbola.

41. $\frac{x^2}{6} - \frac{y^2}{10} = 1$

42. $\frac{x^2}{14} + \frac{y^2}{2} = 1$

43. $\frac{y^2}{4} + \frac{x^2}{16} = 1$

44. $\frac{x^2}{5} + \frac{y^2}{10} = 1$

45. $4x^2 + y^2 = 16$

46. $-3x^2 - 4y^2 = -36$

47. $-y^2 + 2x^2 = -10$

48. $x^2 - y^2 = -1$

49. $5x^2 + y^2 - 10 = 0$

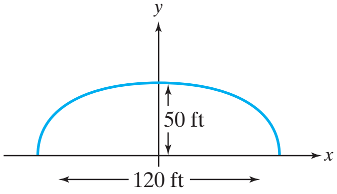
50. $5x^2 - 3y^2 = 15$

51. $y^2 - 6x^2 = 6$

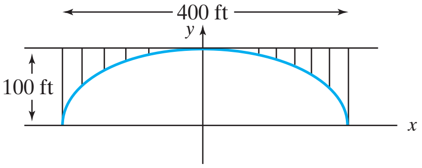
52. $3x^2 + 5y^2 = 15$

Expanding Your Skills

53. An arch for a tunnel is in the shape of a semiellipse. The distance between vertices is 120 ft, and the height to the top of the arch is 50 ft. Find the height of the arch 10 ft from the center. Round to the nearest foot.

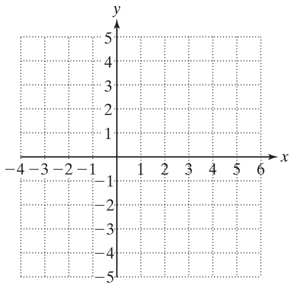


54. A bridge over a gorge is supported by an arch in the shape of a semiellipse. The length of the bridge is 400 ft, and the height is 100 ft. Find the height of the arch 50 ft from the center. Round to the nearest foot.

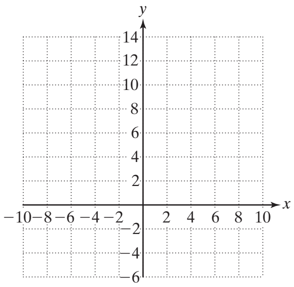


In Exercises 55–58, graph the hyperbola centered at (h, k) by following these guidelines. First locate the center. Then draw the reference rectangle relative to the center (h, k). Using the reference rectangle as a guide, sketch the hyperbola. Label the center and vertices.

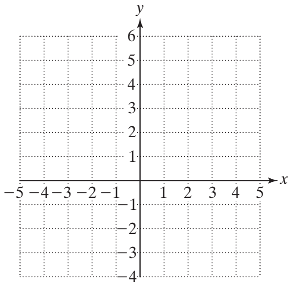
55. $\dfrac{(x-1)^2}{9} - \dfrac{(y+2)^2}{4} = 1$



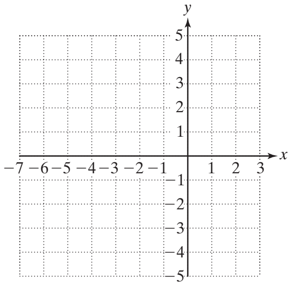
56. $\dfrac{(x+2)^2}{16} - \dfrac{(y-3)^2}{4} = 1$



57. $\dfrac{(y-1)^2}{4} - (x+3)^2 = 1$



58. $(y+2)^2 - \dfrac{(x+3)^2}{4} = 1$



Nonlinear Systems of Equations in Two Variables

Section 11.4

Concepts

1. Solving Nonlinear Systems of Equations by the Substitution Method
2. Solving Nonlinear Systems of Equations by the Addition Method

1. Solving Nonlinear Systems of Equations by the Substitution Method

Recall that a linear equation in two variables x and y is an equation that can be written in the form $ax + by = c$, where a and b are not both zero. In Sections 3.1–3.3, we solved systems of linear equations in two variables by using the graphing method, the substitution method, and the addition method. In this section, we will solve *nonlinear* systems of equations by using the same methods. A **nonlinear system of equations** is a system in which at least one of the equations is nonlinear.

Graphing the equations in a nonlinear system helps to determine the number of solutions and to approximate the coordinates of the solutions. The substitution method is used most often to solve a nonlinear system of equations analytically.

Example 1 Solving a Nonlinear System of Equations

Given the system

$$x - 7y = -25$$
$$x^2 + y^2 = 25$$

a. Solve the system by graphing.

b. Solve the system by the substitution method.

Solution:

a. $x - 7y = -25$ is a line (the slope-intercept form is $y = \frac{1}{7}x + \frac{25}{7}$).

$x^2 + y^2 = 25$ is a circle centered at the origin with radius 5.

From Figure 11-13, we appear to have two solutions $(-4, 3)$ and $(3, 4)$.

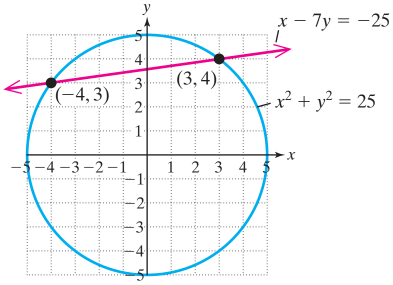


Figure 11-13

b. To use the substitution method, isolate one of the variables from one of the equations. We will solve for x in the first equation.

$$\boxed{A} \qquad x - 7y = -25 \xrightarrow{\text{Solve for } x} x = 7y - 25$$

$$\boxed{B} \qquad x^2 + y^2 = 25$$

$$\boxed{B} \qquad (7y - 25)^2 + y^2 = 25 \qquad \text{Substitute } (7y - 25) \text{ for } x \text{ in the second equation.}$$

$$49y^2 - 350y + 625 + y^2 = 25 \qquad \text{The resulting equation is quadratic in } y.$$

$$50y^2 - 350y + 600 = 0 \qquad \text{Set the equation equal to zero.}$$

$$50(y^2 - 7y + 12) = 0 \qquad \text{Factor.}$$

$$50(y - 3)(y - 4) = 0$$

$$y = 3 \quad \text{or} \quad y = 4$$

For each value of y, find the corresponding x value from the equation $x = 7y - 25$.

$y = 3$: $\quad x = 7(3) - 25 = -4 \qquad$ The solution is $(-4, 3)$.

$y = 4$: $\quad x = 7(4) - 25 = 3 \qquad$ The solution is $(3, 4)$. (See Figure 11-13.)

Skill Practice

1. Given the system

$$2x + y = 5$$
$$x^2 + y^2 = 50$$

a. Solve the system by graphing.

b. Solve the system by the substitution method.

Example 2 Solving a Nonlinear System by the Substitution Method

Given the system

$$y = \sqrt{x}$$
$$x^2 + y^2 = 20$$

a. Sketch the graphs.

b. Solve the system by the substitution method.

Solution:

a. $y = \sqrt{x}$ is one of the six basic functions graphed in Section 4.3.

$x^2 + y^2 = 20$ is a circle centered at the origin with radius $\sqrt{20} \approx 4.5$.

From Figure 11-14, we see that there is one solution.

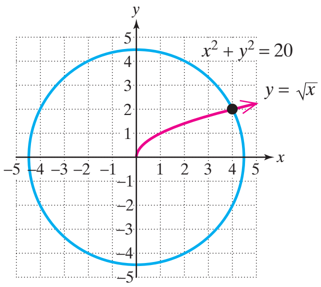


Figure 11-14

b. To use the substitution method, we will substitute $y = \sqrt{x}$ into equation $\boxed{B}$.

$\boxed{A}$ $\quad y = \sqrt{x}$

$\boxed{B}$ $\quad x^2 + y^2 = 20$

$\boxed{B}$ $\quad x^2 + (\sqrt{x})^2 = 20$ Substitute $y = \sqrt{x}$ into the second equation.

$x^2 + x = 20$

$x^2 + x - 20 = 0$ Set the second equation equal to zero.

$(x + 5)(x - 4) = 0$ Factor.

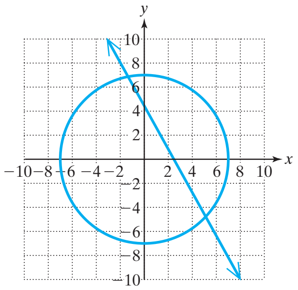
$x \cancel{=} -5$ or $x = 4$ Reject $x = -5$ because it is not in the domain of $y = \sqrt{x}$.

Substitute $x = 4$ into the equation $y = \sqrt{x}$.

If $x = 4$, then $y = \sqrt{4} = 2$. The solution is $(4, 2)$.

Skill Practice Answers

1a.



b. $(-1, 7)$ and $(5, -5)$

Skill Practice

2. Given the system

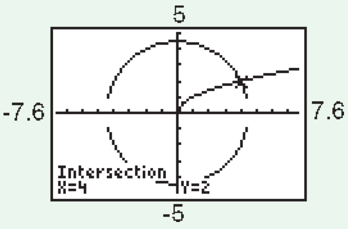
$$y = \sqrt{2x}$$
$$x^2 + y^2 = 8$$

a. Sketch the graphs.
b. Solve the system by using the substitution method.

Calculator Connections

Graph the equations from Example 2 to confirm your solution to the system of equations. Use an *Intersect* feature or *Zoom* and *Trace* to approximate the point of intersection. Recall that the circle must be entered into the calculator as two functions:

$$Y_1 = \sqrt{20 - x^2}$$
$$Y_2 = -\sqrt{20 - x^2}$$
$$Y_3 = \sqrt{x}$$



Example 3 Solving a Nonlinear System by Using the Substitution Method

Solve the system by using the substitution method.

$$y = \sqrt[3]{x}$$
$$y = x$$

Solution:

$\boxed{A}$ $y = \sqrt[3]{x}$

$\boxed{B}$ $y = x$

$\sqrt[3]{x} = x$	Because y is isolated in both equations, we can equate the expressions for y.
$(\sqrt[3]{x})^3 = (x)^3$	To solve the radical equation, raise both sides to the third power.
$x = x^3$	This is a third-degree polynomial equation.
$0 = x^3 - x$	Set the equation equal to zero.
$0 = x(x^2 - 1)$	Factor out the GCF.
$0 = x(x + 1)(x - 1)$	Factor completely.

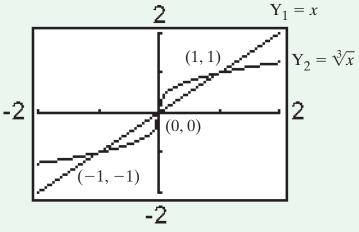
$x = 0$ or $x = -1$ or $x = 1$

For each value of x, find the corresponding y-value from either original equation. We will use equation $\boxed{B}$: $y = x$.

If $x = 0$, then $y = 0$.	The solution is $(0, 0)$.
If $x = -1$, then $y = -1$.	The solution is $(-1, -1)$.
If $x = 1$, then $y = 1$.	T

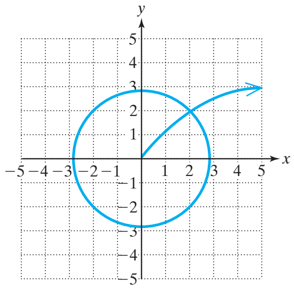
Calculator Connections

Graph the equations $y = \sqrt[3]{x}$ and $y = x$ to support the solutions to Example 3.



Skill Practice Answers

2a.



b. (2, 2)

Skill Practice

3. Solve the system by using the substitution method.

$$\begin{aligned} y &= \sqrt[3]{9x} \\ y &= x \end{aligned}$$

2. Solving Nonlinear Systems of Equations by the Addition Method

The substitution method is used most often to solve a system of nonlinear equations. In some situations, however, the addition method offers an efficient means of finding a solution. Example 4 demonstrates that we can eliminate a variable from both equations provided the terms containing that variable are *like* terms.

Example 4 Solving a Nonlinear System of Equations by the Addition Method

Solve the system.

$$\begin{aligned} 2x^2 + y^2 &= 17 \\ x^2 + 2y^2 &= 22 \end{aligned}$$

Solution:

$\boxed{A}$ $2x^2 + y^2 = 17$ Notice that the y^2-terms are *like* in each equation.

$\boxed{B}$ $x^2 + 2y^2 = 22$ To eliminate the y^2-terms, multiply the first equation by −2.

$$\begin{array}{lrcl} \boxed{A} & 2x^2 + y^2 = 17 & \xrightarrow{\text{Multiply by } -2.} & -4x^2 - 2y^2 = -34 \\ \boxed{B} & x^2 + 2y^2 = 22 & \xrightarrow{\hspace{2cm}} & \underline{x^2 + 2y^2 = 22} \\ & & & -3x^2 = -12 \end{array}$$

Eliminate the y^2 term.

$$\frac{-3x^2}{-3} = \frac{-12}{-3}$$

$$x^2 = 4$$

$$x = \pm 2$$

Substitute each value of x into one of the original equations to solve for y. We will use equation $\boxed{A}$: $2x^2 + y^2 = 17$.

$x = 2$: $\boxed{A}$

$$\begin{aligned} 2(2)^2 + y^2 &= 17 \\ 8 + y^2 &= 17 \\ y^2 &= 9 \\ y &= \pm 3 \end{aligned}$$

The solutions are (2, 3) and (2, −3).

$x = -2$: $\boxed{A}$

$$\begin{aligned} 2(-2)^2 + y^2 &= 17 \\ 8 + y^2 &= 17 \\ y^2 &= 9 \\ y &= \pm 3 \end{aligned}$$

The solutions are (−2, 3) and (−2, −3).

TIP: In Example 4, the x^2-terms are also *like* in both equations. We could have eliminated the x^2-terms by multiplying equation $\boxed{B}$ by −2.

Skill Practice Answers

3. (0, 0); (3, 3); (−3, −3)

Skill Practice

4. Solve the system by using the addition method.

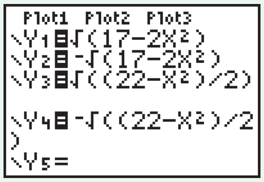
$$x^2 - y^2 = 24$$
$$3x^2 + y^2 = 76$$

Calculator Connections

The solutions to Example 4 can be checked from the graphs of the equations.

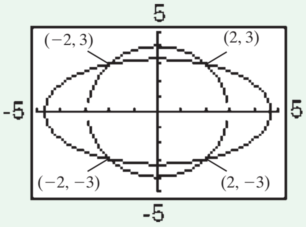
For the equation $2x^2 + y^2 = 17$, we have $\quad y = \pm\sqrt{17 - 2x^2}$

For the equation $x^2 + 2y^2 = 22$, we have $\quad y = \pm\sqrt{\dfrac{22 - x^2}{2}}$



```
Plot1 Plot2 Plot3
\Y1=√(17-2X²)
\Y2=-√(17-2X²)
\Y3=√((22-X²)/2)

\Y4=-√((22-X²)/2
)
\Y5=
```



TIP: It is important to note that the addition method can be used only if two equations share a pair of *like* terms. The substitution method is effective in solving a wider range of systems of equations. The system in Example 4 could also have been solved by using substitution.

$\boxed{\text{A}}$ $\quad 2x^2 + y^2 = 17 \xrightarrow{\text{Solve for } y^2.} y^2 = 17 - 2x^2$

$\boxed{\text{B}}$ $\quad x^2 + 2y^2 = 22$

$\boxed{\text{B}}$ $\quad x^2 + 2(17 - 2x^2) = 22$

$x^2 + 34 - 4x^2 = 22$

$-3x^2 = -12$

$x^2 = 4$

$x = \pm 2$

$x = 2$: $\quad y^2 = 17 - 2(2)^2$

$y^2 = 9$

$y = \pm 3$ The solutions are (2, 3) and (2, −3).

$x = -2$: $\quad y^2 = 17 - 2(-2)^2$

$y^2 = 9$

$y = \pm 3$ The solutions are (−2, 3) and (−2, −3).

Skill Practice Answers

4. (5, 1); (5, −1); (−5, 1); (−5, −1)

Section 11.4 Practice Exercises

Study Skills Exercise

1. Define the key term **nonlinear system of equations**.

Review Exercises

2. Write the distance formula between two points (x_1, y_1) and (x_2, y_2) from memory.

3. Find the distance between the two points $(8, -1)$ and $(1, -8)$.

4. Write an equation representing the set of all points 2 units from the point $(-1, 1)$.

5. Write an equation representing the set of all points 8 units from the point $(-5, 3)$.

For Exercises 6–13, determine if the equation represents a parabola, circle, ellipse, or hyperbola.

6. $x^2 + y^2 = 15$

7. $\dfrac{x^2}{4} - \dfrac{y^2}{2} = 1$

8. $y = (x - 6)^2 + 4$

9. $\dfrac{(x + 1)^2}{2} + \dfrac{(y + 1)^2}{5} = 1$

10. $\dfrac{y^2}{3} - \dfrac{x^2}{3} = 1$

11. $3x^2 + 3y^2 = 1$

12. $\dfrac{x^2}{9} + \dfrac{y^2}{12} = 1$

13. $x = (y + 2)^2 - 5$

Concept 1: Solving Nonlinear Systems of Equations by the Substitution Method

For Exercises 14–21, use sketches to explain.

14. How many points of intersection are possible between a line and a parabola?

15. How many points of intersection are possible between a line and a circle?

16. How many points of intersection are possible between two distinct circles?

17. How many points of intersection are possible between two distinct parabolas of the form $y = ax^2 + bx + c$, $a \neq 0$?

18. How many points of intersection are possible between a circle and a parabola?

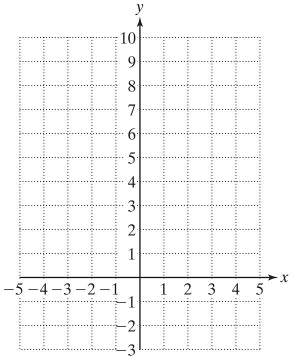
19. How many points of intersection are possible between two distinct lines?

20. How many points of intersection are possible with an ellipse and a hyperbola?

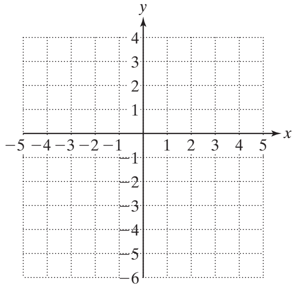
21. How many points of intersection are possible with an ellipse and a parabola?

For Exercises 22–27, sketch each system of equations. Then solve the system by the substitution method.

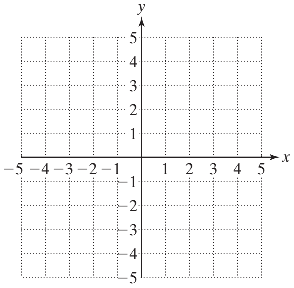
22. $y = x + 3$
$x^2 + y = 9$



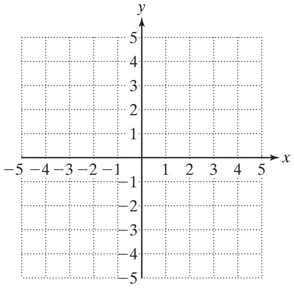
23. $y = x - 2$
$x^2 + y = 4$



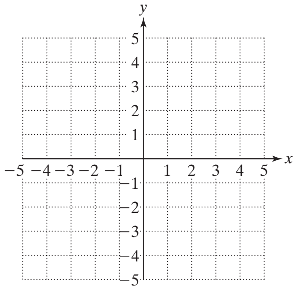
24. $x^2 + y^2 = 1$
$y = x + 1$



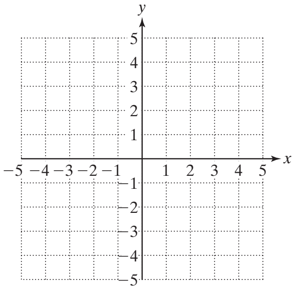
25. $x^2 + y^2 = 25$
$y = 2x$



26. $x^2 + y^2 = 6$
$y = x^2$



27. $x^2 + y^2 = 12$
$y = x^2$



For Exercises 28–36, solve the system by the substitution method.

28. $x^2 + y^2 = 20$
$y = \sqrt{x}$

29. $x^2 + y^2 = 30$
$y = \sqrt{x}$

30. $y = x^2$
$y = -\sqrt{x}$

31. $y = -x^2$
$y = -\sqrt{x}$

32. $y = x^2$
$y = (x - 3)^2$

33. $y = (x + 4)^2$
$y = x^2$

34. $y = x^2 + 6x$
$y = 4x$

35. $y = 3x^2 - 6x$
$y = 3x$

36. $x^2 - 5x + y = 0$
$y = 3x + 1$

Concept 2: Solving Nonlinear Systems of Equations by the Addition Method

For Exercises 37–50, solve the system of nonlinear equations by the addition method.

37. $x^2 + y^2 = 13$
$x^2 - y^2 = 5$

38. $4x^2 - y^2 = 4$
$4x^2 + y^2 = 4$

39. $9x^2 + 4y^2 = 36$
$x^2 + y^2 = 9$

40. $x^2 + y^2 = 4$
$2x^2 + y^2 = 8$

41. $3x^2 + 4y^2 = 16$
$2x^2 - 3y^2 = 5$

42. $2x^2 - 5y^2 = -2$
$3x^2 + 2y^2 = 35$

43. $y = x^2 - 2$

$y = -x^2 + 2$

44. $y = x^2$

$y = -x^2 + 8$

45. $\frac{x^2}{4} + \frac{y^2}{9} = 1$

$x^2 + y^2 = 4$

46. $\frac{x^2}{16} + \frac{y^2}{4} = 1$

$x^2 + y^2 = 4$

47. $x^2 + 6y^2 = 9$

$\frac{x^2}{9} + \frac{y^2}{12} = 1$

48. $\frac{x^2}{10} + \frac{y^2}{10} = 1$

$2x^2 + y^2 = 11$

49. $x^2 - xy = -4$

$2x^2 - xy = 12$

50. $x^2 - xy = 3$

$2x^2 + xy = 6$

Expanding Your Skills

51. The sum of two numbers is 7. The sum of the squares of the numbers is 25. Find the numbers.

52. The sum of the squares of two numbers is 100. The sum of the numbers is 2. Find the numbers.

53. The sum of the squares of two numbers is 32. The difference of the squares of the numbers is 18. Find the numbers.

54. The sum of the squares of two numbers is 24. The difference of the squares of the numbers is 8. Find the numbers.

Graphing Calculator Exercises

For Exercises 55–58, use the *Intersect* feature or *Zoom* and *Trace* to approximate the solutions to the system.

55. $y = x + 3$ (Exercise 22)

$x^2 + y = 9$

56. $y = x - 2$ (Exercise 23)

$x^2 + y = 4$

57. $y = x^2$ (Exercise 30)

$y = -\sqrt{x}$

58. $y = -x^2$ (Exercise 31)

$y = -\sqrt{x}$

For Exercises 59–60, graph the system on a square viewing window. What can be said about the solution to the system?

59. $x^2 + y^2 = 4$

$y = x^2 + 3$

60. $x^2 + y^2 = 16$

$y = -x^2 - 5$

Section 11.5 Nonlinear Inequalities and Systems of Inequalities

Concepts

1. Nonlinear Inequalities in Two Variables
2. Systems of Nonlinear Inequalities in Two Variables

1. Nonlinear Inequalities in Two Variables

In Section 9.5 we graphed the solution sets to linear inequalities in two variables, such as $y \leq 2x + 1$. This involved graphing the related equation (a line in the xy-plane) and then shading the appropriate region above or below the line. See Figure 11-15. In this section, we will employ the same technique to solve nonlinear inequalities in two variables.

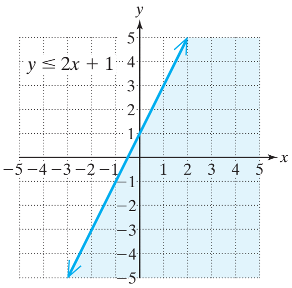


Figure 11-15

Example 1 Graphing a Nonlinear Inequality in Two Variables

Graph the solution set of the inequality $x^2 + y^2 < 16$.

Solution:

The related equation $x^2 + y^2 = 16$ is a circle of radius 4, centered at the origin. Graph the related equation by using a dashed curve because the points satisfying the equation $x^2 + y^2 = 16$ are not part of the solution to the strict inequality $x^2 + y^2 < 16$. See Figure 11-16.

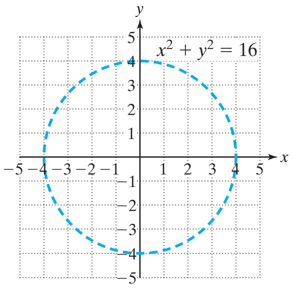


Figure 11-16

Notice that the dashed curve separates the xy-plane into two regions, one "inside" the circle, the other "outside" the circle. Select a test point from each region and test the point in the original inequality.

Test Point "Inside": (0, 0)	**Test Point "Outside": (4, 4)**
$x^2 + y^2 < 16$	$x^2 + y^2 < 16$
$(0)^2 + (0)^2 \overset{?}{<} 16$	$(4)^2 + (4)^2 \overset{?}{<} 16$
$0 \overset{?}{<} 16$ True	$32 \overset{?}{<} 16$ False

The inequality $x^2 + y^2 < 16$ is true at the test point (0, 0). Therefore, the solution set is the region "inside" the circle. See Figure 11-17.

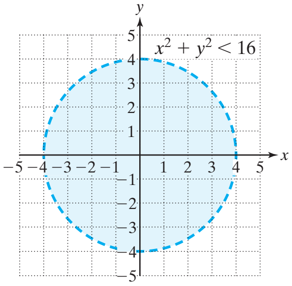


Figure 11-17

Skill Practice

1. Graph the solution set of the inequality $x^2 + y^2 \geq 9$.

Example 2 Graphing a Nonlinear Inequality in Two Variables

Graph the solution set of the inequality $9y^2 \geq 36 + 4x^2$.

Solution:

First graph the related equation $9y^2 = 36 + 4x^2$. Notice that the equation can be written in the standard form of a hyperbola.

$$9y^2 = 36 + 4x^2$$

$$9y^2 - 4x^2 = 36 \qquad \text{Subtract } 4x^2 \text{ from both sides.}$$

$$\frac{y^2}{4} - \frac{x^2}{9} = 1 \qquad \text{Divide both sides by 36.}$$

Graph the hyperbola as a solid curve, because the original inequality includes equality. See Figure 11-18.

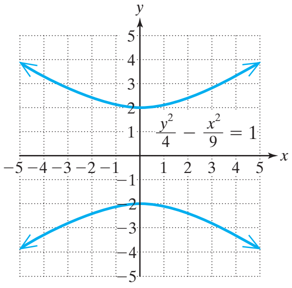


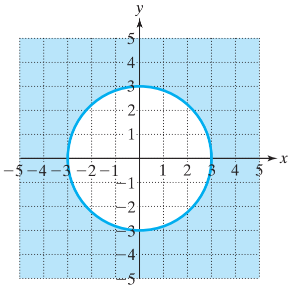
Figure 11-18

The hyperbola divides the xy-plane into three regions: a region above the upper branch, a region between the branches, and a region below the lower branch. Select a test point from each region.

$$9y^2 \geq 36 + 4x^2$$

Skill Practice Answers

1.



Test: (0, 3)	**Test: (0, 0)**	**Test: (0, −3)**
$9(3)^2 \overset{?}{\geq} 36 + 4(0)^2$	$9(0)^2 \overset{?}{\geq} 36 + 4(0)^2$	$9(-3)^2 \overset{?}{\geq} 36 + 4(0)^2$
$81 \overset{?}{\geq} 36$ True	$0 \overset{?}{\geq} 36$ False	$81 \overset{?}{\geq} 36$ True

Shade the regions above the top branch and below the bottom branch of the hyperbola. See Figure 11-19.

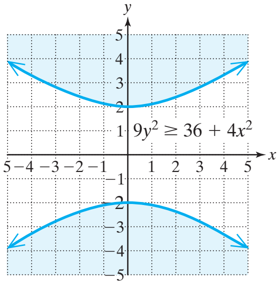


Figure 11-19

Skill Practice

2. Graph the solution set of the inequality $9x^2 < 144 - 16y^2$.

2. Systems of Nonlinear Inequalities in Two Variables

In Section 11.4 we solved systems of nonlinear equations in two variables. The solution set for such a system is the set of ordered pairs that satisfy both equations simultaneously. We will now solve systems of nonlinear inequalities in two variables. Similarly, the solution set is the set of all ordered pairs that simultaneously satisfy each inequality. To solve a system of inequalities, we will graph the solution to each individual inequality and then take the intersection of the solution sets.

Example 3 Graphing a System of Nonlinear Inequalities in Two Variables

Graph the solution set.

$$y > e^x$$
$$y < -x^2 + 4$$

Solution:

The solution to $y > e^x$ is the set of points above the curve $y = e^x$. See Figure 11-20. The solution to $y < -x^2 + 4$ is the set of points below the parabola $y = -x^2 + 4$. See Figure 11-21.

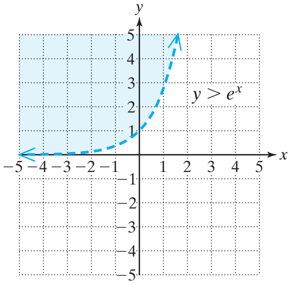


Figure 11-20

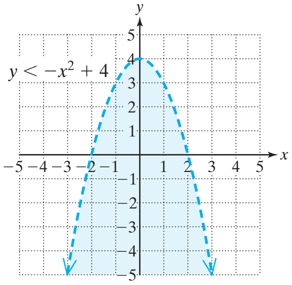
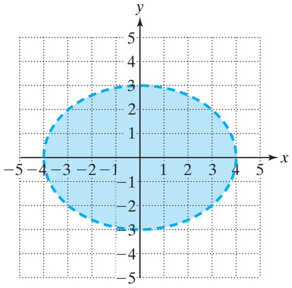


Figure 11-21

Skill Practice Answers

2.



The solution to the system of inequalities is the intersection of the solution sets of the individual inequalities. See Figure 11-22.

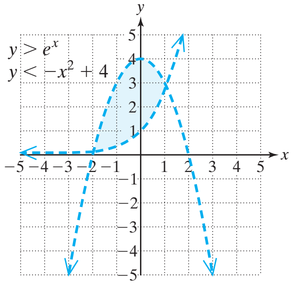


Figure 11-22

Skill Practice

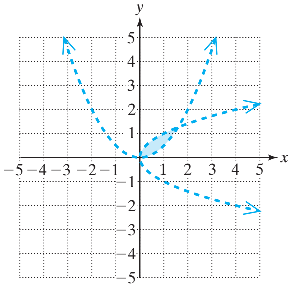
3. Graph the solution set.

$$y > \frac{1}{2}x^2$$

$$x > y^2$$

Skill Practice Answers

3.



Section 11.5 Practice Exercises

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Review Exercises

For Exercises 1–11, match the equation with its graph.

1. $y = \left(\frac{1}{3}\right)^x$

2. $y = 4x - 1$

3. $y = -4x^2$

4. $y = e^x$

5. $y = x^3$

6. $\frac{x^2}{4} + \frac{y^2}{9} = 1$

7. $\frac{x^2}{4} - \frac{y^2}{9} = 1$

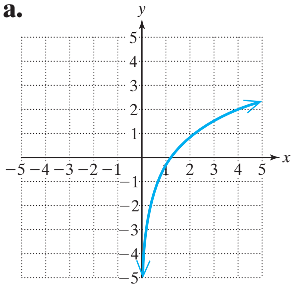
8. $y = \frac{1}{x}$

9. $y = \log_2(x)$

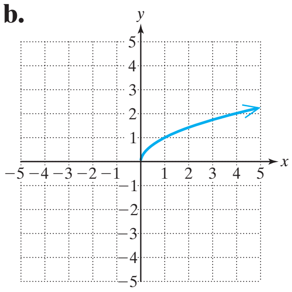
10. $y = \sqrt{x}$

11. $(x + 2)^2 + (y - 1)^2 = 4$

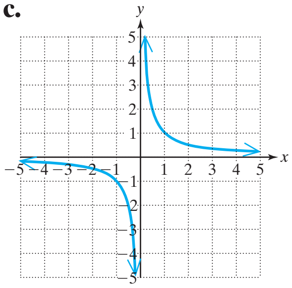
a.



b.



c.



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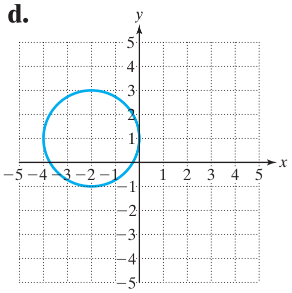
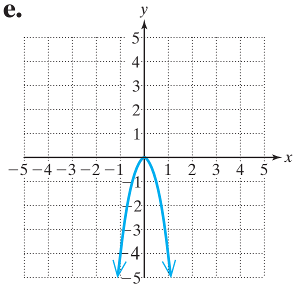
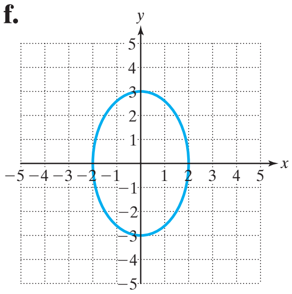


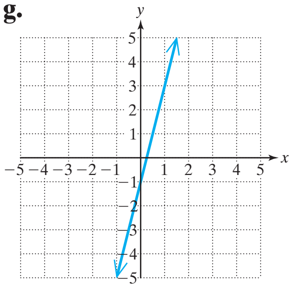
e.



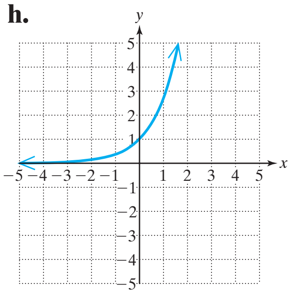
f.



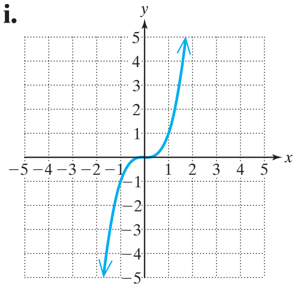
g.



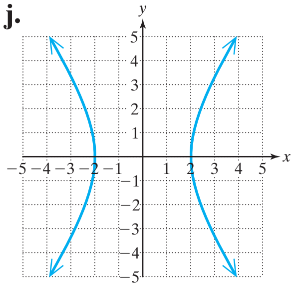
h.



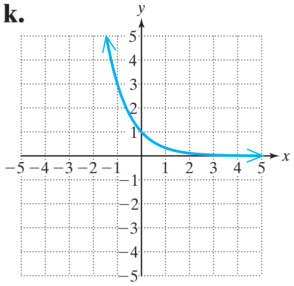
i.



j.



k.



Concept 1: Nonlinear Inequalities in Two Variables

12. True or false? The point (1, 1) satisfies the inequality $-x^2 + y^3 > 1$.

13. True or false? The point (4, −2) satisfies the inequality $4x^2 - 2x + 1 + y^2 < 3$.

14. True or false? The point (5, 4) satisfies the system of inequalities.

$$\frac{x^2}{36} + \frac{y^2}{25} < 1$$
$$x^2 + y^2 \geq 4$$

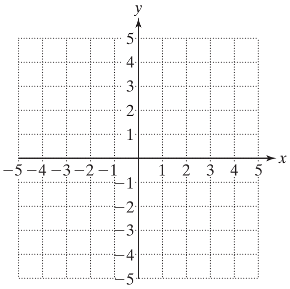
15. True or false? The point (1, −2) satisfies the system of inequalities.

$$y < x^2$$
$$y > x^2 - 4$$

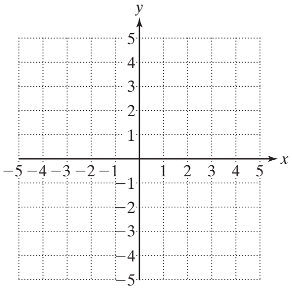
16. True or false? The point (−3, 5) satisfies the system of inequalities.

$$(x + 3)^2 + (y - 5)^2 \leq 2$$
$$y > x^2$$

17. **a.** Graph the solution set of $x^2 + y^2 \leq 9$.



18. **a.** Graph the solution set of $\frac{x^2}{4} + \frac{y^2}{9} \geq 1$.



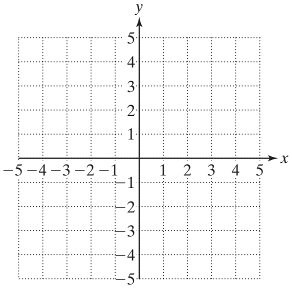
b. Describe the solution set for the inequality $x^2 + y^2 \geq 9$.

c. Describe the solution set of the equation $x^2 + y^2 = 9$

b. Describe the solution set for the inequality $\frac{x^2}{4} + \frac{y^2}{9} \leq 1$.

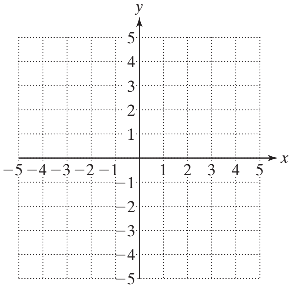
c. Describe the solution set for the equation $\frac{x^2}{4} + \frac{y^2}{9} = 1$

19. **a.** Graph the solution set of $y \geq x^2 + 1$.



b. How would the solution change for the strict inequality $y > x^2 + 1$?

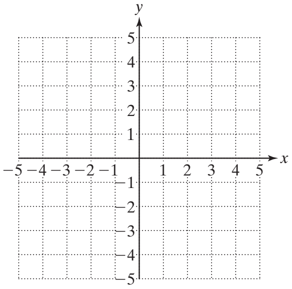
20. **a.** Graph the solution set of $\frac{x^2}{4} - \frac{y^2}{9} \leq 1$.



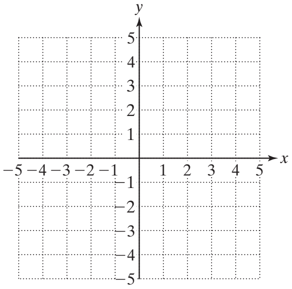
b. How would the solution change for the strict inequality $\frac{x^2}{4} - \frac{y^2}{9} < 1$?

For Exercises 21–36, graph the solution set.

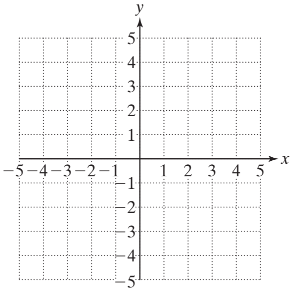
21. $2x + y \geq 1$



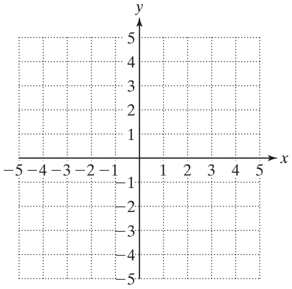
22. $3x + 2y \geq 6$



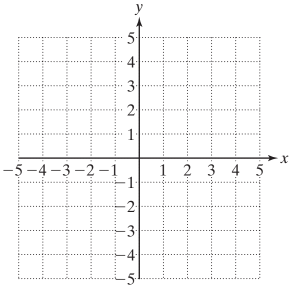
23. $x \leq y^2$



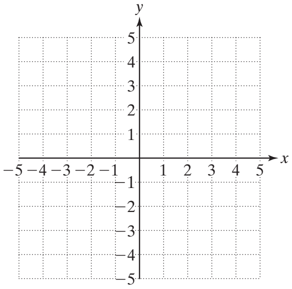
24. $y \leq -x^2$



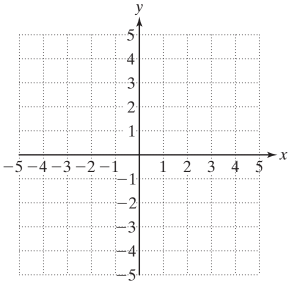
25. $(x - 1)^2 + (y + 2)^2 > 9$



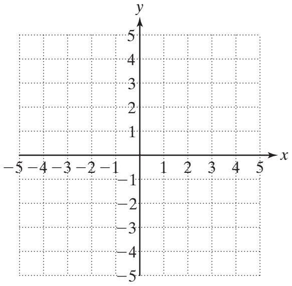
26. $(x + 1)^2 + (y - 4)^2 > 1$



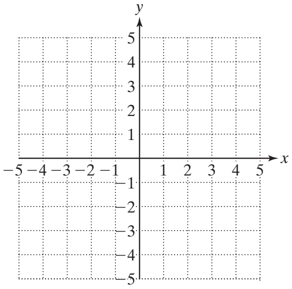
27. $x + y^2 \geq 4$



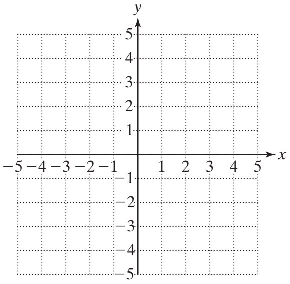
28. $x^2 + 2x + y - 1 \leq 0$



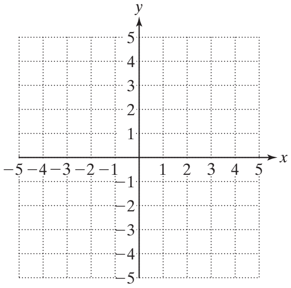
29. $9x^2 - y^2 > 9$



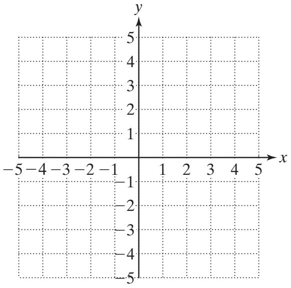
30. $y^2 - 4x^2 \leq 4$



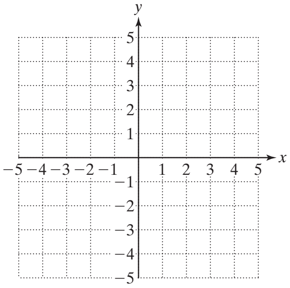
31. $x^2 + 16y^2 \leq 16$



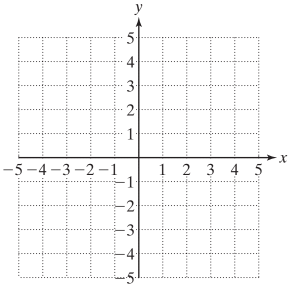
32. $4x^2 + y^2 \leq 4$



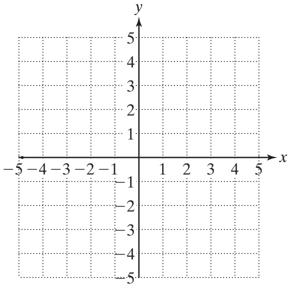
33. $y \leq \ln x$



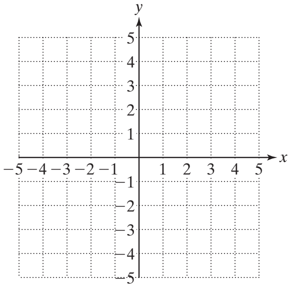
34. $y \leq \log x$



35. $y > 5^x$



36. $y \geq \left(\frac{1}{3}\right)^x$



Concept 2: Systems of Nonlinear Inequalities in Two Variables

For Exercises 37–50, graph the solution set to the system of inequalities.

37. $y < \sqrt{x}$

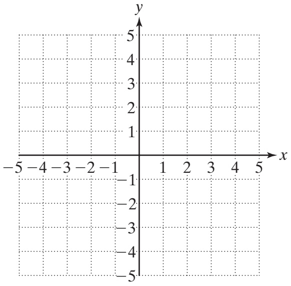
$x > 1$

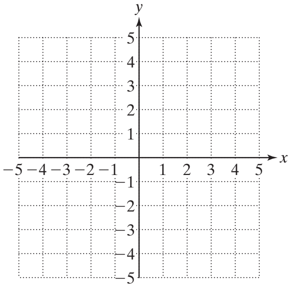
38. $y \geq \sqrt{x}$

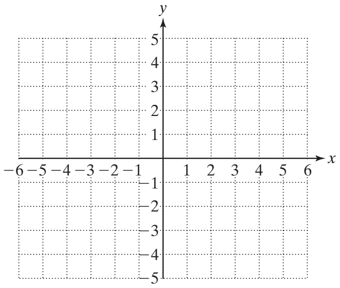
$x \geq 0$

39. $\frac{x^2}{36} + \frac{y^2}{25} < 1$

$x^2 + y^2 \geq 4$







40. $x^2 - y^2 \geq 1$

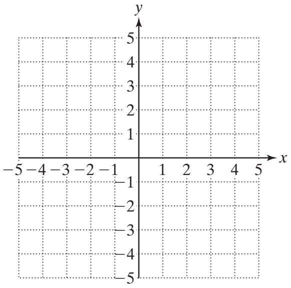
$x \leq 0$

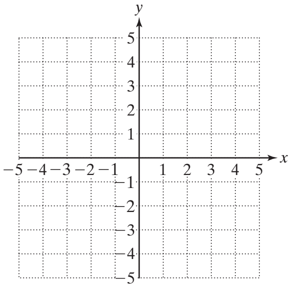
41. $y < x^2$

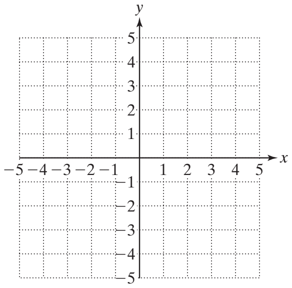
$y > x^2 - 4$

42. $y^2 - x^2 \geq 1$

$y \geq 0$







43. $y < \frac{1}{x}$

$y > 0$

$y < x$

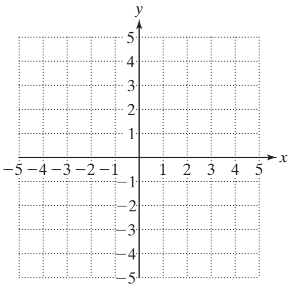
44. $y > x^3$

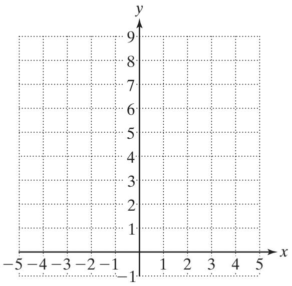
$y < 8$

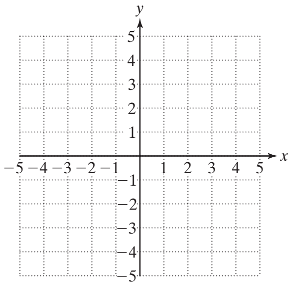
$x > 0$

45. $x^2 + y^2 \geq 25$

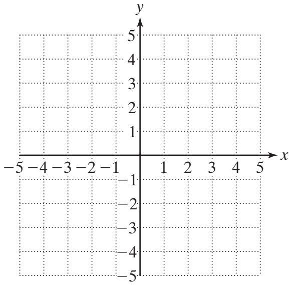
$x^2 + y^2 \leq 9$



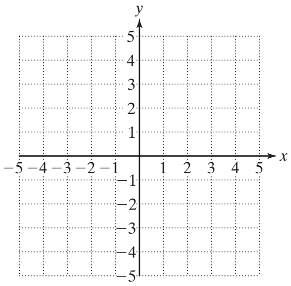




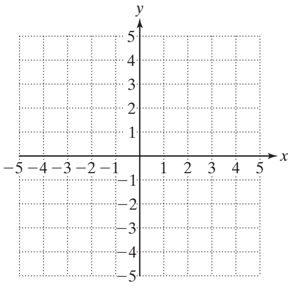
46. $\frac{x^2}{4} + \frac{y^2}{25} \geq 1$
$x^2 + \frac{y^2}{4} \leq 1$



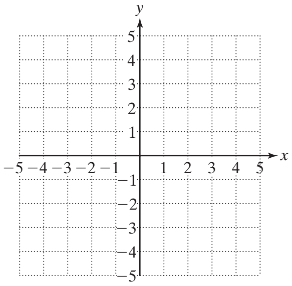
47. $x < -(y - 1)^2 + 3$
$x + y > 2$



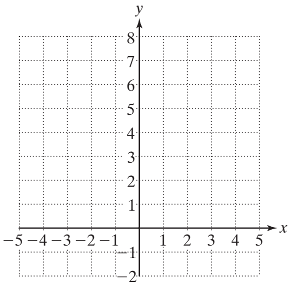
48. $x > (y - 2)^2 + 1$
$x - y < 1$



49. $x^2 + y^2 < 25$
$y < \frac{4}{3}x$
$y > -\frac{4}{3}x$



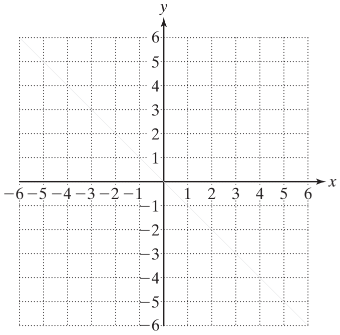
50. $y < e^x$
$y > 1$
$x < 2$



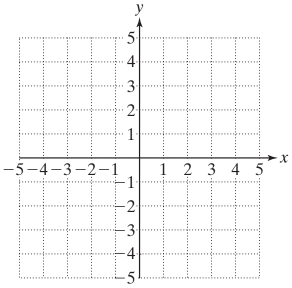
Expanding Your Skills

For Exercises 51–54, graph the compound inequalities.

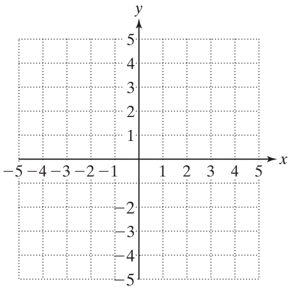
51. $x^2 + y^2 \leq 36$ or $x + y \geq 0$



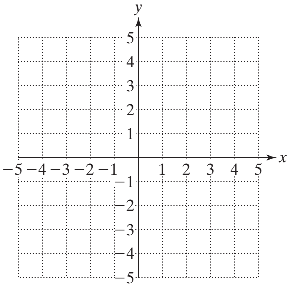
52. $y \leq -x^2 + 4$ or $y \geq x^2 - 4$



53. $y + 1 \geq x^2$ or $y + 1 \leq -x^2$



54. $(x + 2)^2 + (y + 3)^2 \leq 4$ or $x \geq y^2$



Chapter 11 SUMMARY

Section 11.1 Distance Formula and Circles

Key Concepts

The **distance between two points** (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The standard equation of a **circle** with center (h, k) and radius r is

$$(x - h)^2 + (y - k)^2 = r^2$$

Examples

Example 1

Find the distance between two points.

$(5, -2)$ and $(-1, -6)$

$$\begin{aligned} d &= \sqrt{(-1 - 5)^2 + [-6 - (-2)]^2} \\ &= \sqrt{(-6)^2 + (-4)^2} \\ &= \sqrt{36 + 16} \\ &= \sqrt{52} = 2\sqrt{13} \end{aligned}$$

Example 2

Find the center and radius of the circle.

$$\begin{aligned} x^2 + y^2 - 8x + 6y &= 0 \\ (x^2 - 8x + 16) + (y^2 + 6y + 9) &= 16 + 9 \\ (x - 4)^2 + (y + 3)^2 &= 25 \end{aligned}$$

The center is $(4, -3)$ and the radius is 5.

Section 11.2 More on the Parabola

Key Concepts

A **parabola** is the set of points in a plane that are equidistant from a fixed line (called the directrix) and a fixed point (called the focus) not on the directrix.

The standard form of an equation of a parabola with **vertex** (h, k) and vertical **axis of symmetry** is

$$y = a(x - h)^2 + k$$

- The equation of the axis of symmetry is $x = h$.
- If $a > 0$, then the parabola opens upward.
- If $a < 0$, the parabola opens downward.

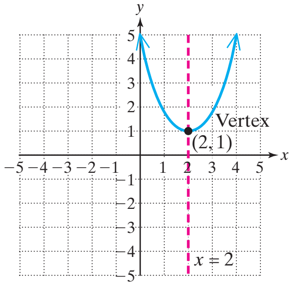
Examples

Example 1

Given the parabola, $y = (x - 2)^2 + 1$

The vertex is $(2, 1)$.

The axis of symmetry is $x = 2$.



The standard form of an equation of a parabola with vertex (h, k) and horizontal axis of symmetry is

$$x = a(y - k)^2 + h$$

- The equation of the axis of symmetry is $y = k$.
- If $a > 0$, then the parabola opens to the right.
- If $a < 0$, the parabola opens to the left.

Example 2

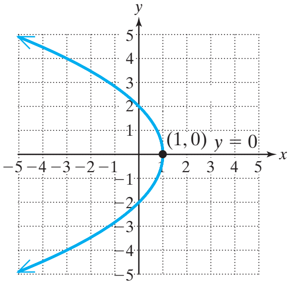
Given the parabola $x = -\frac{1}{4}y^2 + 1$,

determine the coordinates of the vertex and the equation of the axis of symmetry.

$$x = -\frac{1}{4}(y - 0)^2 + 1$$

The vertex is $(1, 0)$.

The axis of symmetry is $y = 0$.



Section 11.3 The Ellipse and Hyperbola

Key Concepts

An **ellipse** is the set of all points (x, y) such that the sum of the distances between (x, y) and two distinct points (called foci) is constant.

Standard Form of an Ellipse with Center at the Origin

An ellipse with the center at the origin has the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

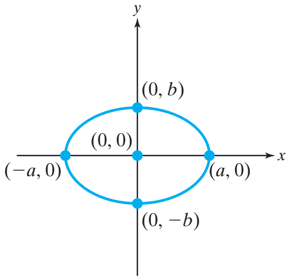
where a and b are positive real numbers.

For an ellipse centered at the origin, the x-intercepts are given by $(a, 0)$ and $(-a, 0)$, and the y-intercepts are given by $(0, b)$ and $(0, -b)$.

A **hyperbola** is the set of all points (x, y) such that the difference of the distances between (x, y) and two distinct points is a constant. The fixed points are called the foci of the hyperbola.

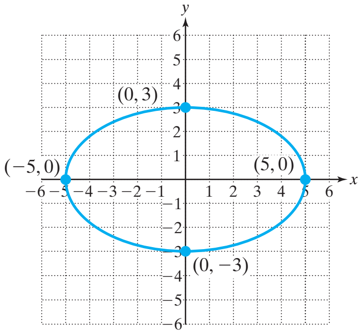
Examples

Example 1



Example 2

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$



Standard Forms of an Equation of a Hyperbola

Let a and b represent positive real numbers.

Horizontal Transverse Axis. The standard form of an equation of a hyperbola with a horizontal transverse axis and center at the origin is given by

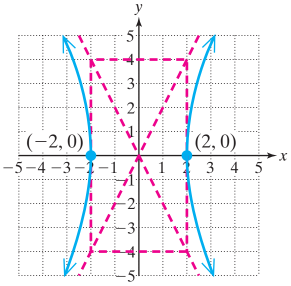
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Vertical Transverse Axis. The standard form of an equation of a hyperbola with a vertical transverse axis and center at the origin is given by

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

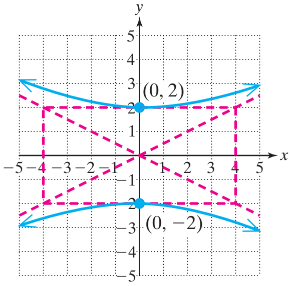
Example 3

$$\frac{x^2}{4} - \frac{y^2}{16} = 1$$



Example 4

$$\frac{y^2}{4} - \frac{x^2}{16} = 1$$



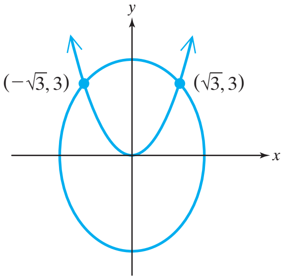
Section 11.4 Nonlinear Systems of Equations in Two Variables

Key Concepts

A **nonlinear system of equations** can be solved by graphing or by the substitution method.

$2x^2 + y^2 = 15$

$x^2 - y = 0$



Examples

Example 1

$\boxed{\text{A}}$ $2x^2 + y^2 = 15$

$\boxed{\text{B}}$ $x^2 - y = 0$ Solve for y: $y = x^2$

$\boxed{\text{A}}$ $2x^2 + (x^2)^2 = 15$ Substitute in first equation.

$$2x^2 + x^4 = 15$$

$$x^4 + 2x^2 - 15 = 0$$

$$(x^2 + 5)(x^2 - 3) = 0$$

$x^2 + 5 = 0$ or $x^2 - 3 = 0$

$\cancel{x^2 = -5}$ or $x^2 = 3$

$x = \pm\sqrt{3}$

If $x = \sqrt{3}$ then $y = (\sqrt{3})^2 = 3$.

If $x = -\sqrt{3}$ then $y = (-\sqrt{3})^2 = 3$.

Points of intersection are $(\sqrt{3}, 3)$ and $(-\sqrt{3}, 3)$.

A nonlinear system may also be solved by using the *addition method* when the equations share *like* terms.

Example 2

$$\begin{array}{rcr} 2x^2 + y^2 = 4 & \xrightarrow{\text{Mult. by } -5.} & -10x^2 - 5y^2 = -20 \\ 3x^2 + 5y^2 = 13 & \longrightarrow & \underline{3x^2 + 5y^2 = 13} \\ & & -7x^2 \qquad = -7 \end{array}$$

$$\frac{-7x^2}{-7} = \frac{-7}{-7}$$

$$x^2 = 1 \longrightarrow x = \pm 1$$

$$\begin{array}{lr} \text{If } x = 1, & 2(1)^2 + y^2 = 4 \\ & y^2 = 2 \\ & y = \pm\sqrt{2} \\ \text{If } x = -1, & 2(-1)^2 + y^2 = 4 \\ & y^2 = 2 \\ & y = \pm\sqrt{2} \end{array}$$

The points of intersection are $(1, \sqrt{2})$, $(1, -\sqrt{2})$, $(-1, \sqrt{2})$, and $(-1, -\sqrt{2})$.

Section 11.5 Nonlinear Inequalities and Systems of Inequalities

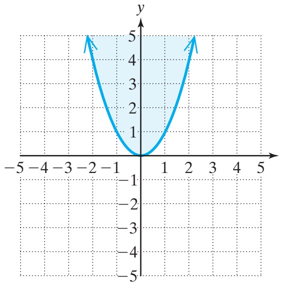
Key Concepts

Graph a nonlinear inequality by using the test point method. That is, graph the related equation. Then choose test points in each region to determine where the inequality is true.

Examples

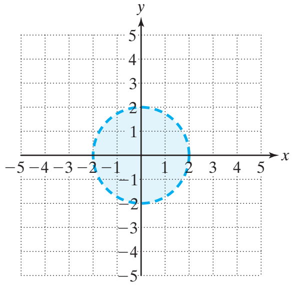
Example 1

$y \geq x^2$



Example 2

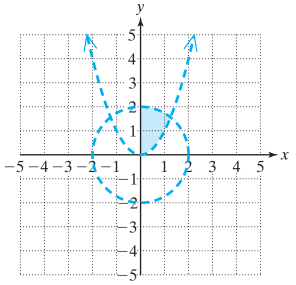
$x^2 + y^2 < 4$



Graph a system of nonlinear inequalities by finding the intersection of the solution sets. That is, graph the solution set for each individual inequality, then take the intersection.

Example 3

$x \geq 0, \quad y > x^2, \quad \text{and} \quad x^2 + y^2 < 4$



Chapter 11 Review Exercises

Section 11.1

For Exercises 1–4, find the distance between the two points by using the distance formula.

1. $(-6, 3)$ and $(0, 1)$

2. $(4, 13)$ and $(-1, 5)$

3. Find x such that $(x, 5)$ is 5 units from $(2, 9)$.

4. Find x such that $(-3, 4)$ is 3 units from $(x, 1)$.

Points are said to be collinear if they all lie on the same line. If three points are collinear, then the distance between the outermost points will equal the sum of the distances between the middle point and each of the outermost points. For Exercises 5–6, determine if the three points are collinear.

5. $(-2, -3)$, $(1, 3)$, and $(5, 11)$

6. $(-2, 11)$, $(0, 5)$, and $(4, -7)$

For Exercises 7–10, find the center and the radius of the circle.

7. $(x - 12)^2 + (y - 3)^2 = 16$

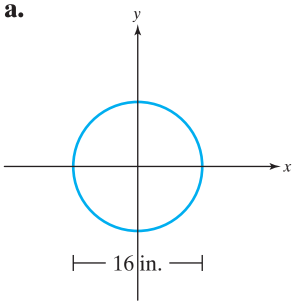
8. $(x + 7)^2 + (y - 5)^2 = 81$

9. $(x + 3)^2 + (y + 8)^2 = 20$

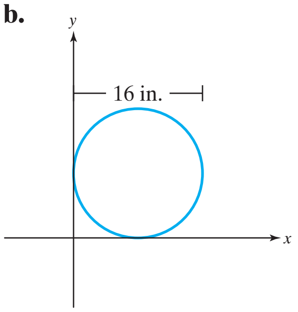
10. $(x - 1)^2 + (y + 6)^2 = 32$

11. A stained glass window is in the shape of a circle with a 16-in. diameter. Find an equation of the circle relative to the origin for each of the following graphs.

a.



b.



For Exercises 12–15, write the equation of the circle in standard form by completing the square.

12. $x^2 + y^2 + 12x - 10y + 51 = 0$

13. $x^2 + y^2 + 4x + 16y + 60 = 0$

14. $x^2 + y^2 - x - 4y + \frac{1}{4} = 0$

15. $x^2 + y^2 - 6x - \frac{2}{3}y + \frac{1}{9} = 0$

16. Write an equation of a circle with center at the origin and a diameter of 7 m.

17. Write an equation of a circle with center at $(0, 2)$ and a diameter of 6 m.

Section 11.2

For Exercises 18–21, determine whether the axis of symmetry is vertical or horizontal and if the parabola opens upward, downward, left, or right.

18. $y = -2(x - 3)^2 + 2$

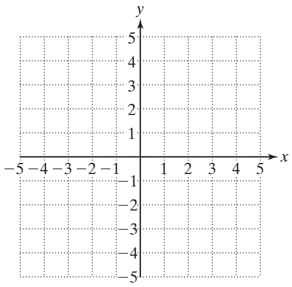
19. $x = 3(y - 9)^2 + 1$

20. $x = -(y + 4)^2 - 8$

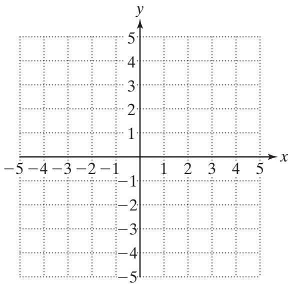
21. $y = (x + 3)^2 - 10$

For Exercises 22–25, determine the coordinates of the vertex and the equation of the axis of symmetry. Then use this information to graph the parabola.

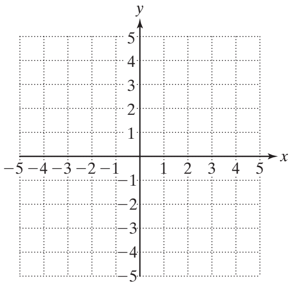
22. $x = -(y - 1)^2$



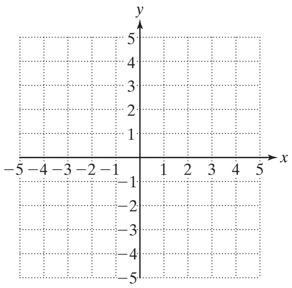
23. $y = (x + 2)^2$



24. $y = -\frac{1}{4}x^2$



25. $x = 2y^2 - 1$



For Exercises 26–29, write the equation in standard form $y = a(x - h)^2 + k$ or $x = a(y - k)^2 + h$. Then identify the vertex, and axis of symmetry.

26. $y = x^2 - 6x + 5$

27. $x = y^2 + 4y + 2$

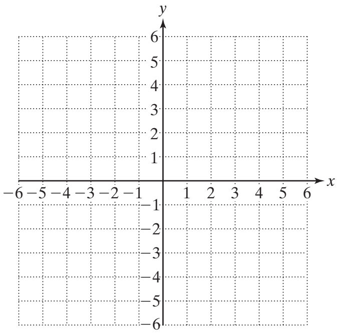
28. $x = -4y^2 + 4y$

29. $y = -2x^2 - 2x$

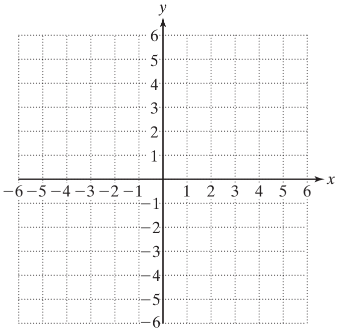
Section 11.3

For Exercises 30–31, identify the x- and y-intercepts. Then graph the ellipse.

30. $\frac{x^2}{9} + \frac{y^2}{25} = 1$

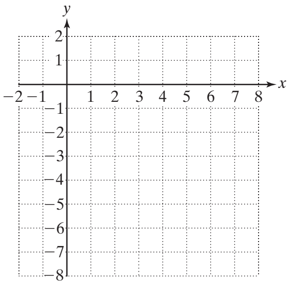


31. $x^2 + 4y^2 = 36$

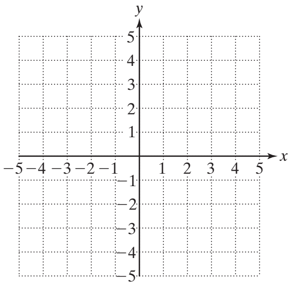


For Exercises 32–33, identify the center of the ellipse and graph the ellipse.

32. $\frac{(x - 5)^2}{4} + \frac{(y + 3)^2}{16} = 1$



33. $\frac{x^2}{25} + \frac{(y - 2)^2}{9} = 1$



For Exercises 34–37, determine whether the transverse axis is horizontal or vertical.

34. $\frac{x^2}{12} - \frac{y^2}{16} = 1$

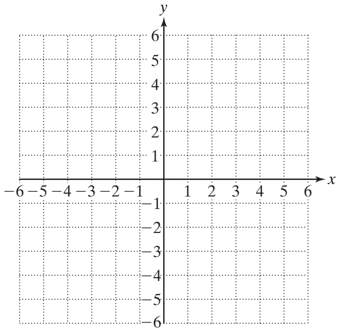
35. $\frac{y^2}{9} - \frac{x^2}{9} = 1$

36. $\frac{y^2}{24} - \frac{x^2}{10} = 1$

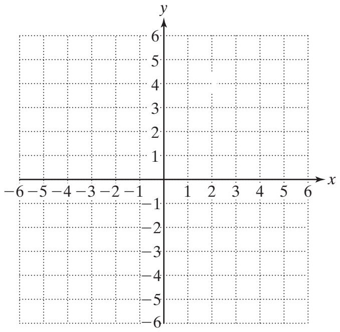
37. $\frac{x^2}{6} - \frac{y^2}{16} = 1$

For Exercises 38–39, graph the hyperbola by first drawing the reference rectangle and the asymptotes. Label the vertices.

38. $\frac{x^2}{4} - y^2 = 1$



39. $y^2 - x^2 = 16$



For Exercises 40–43, identify the equations as representing an ellipse or a hyperbola.

40. $\frac{x^2}{4} - \frac{y^2}{9} = 1$

41. $\frac{x^2}{16} + \frac{y^2}{9} = 1$

42. $\frac{x^2}{4} + \frac{y^2}{1} = 1$

43. $\frac{y^2}{1} - \frac{x^2}{16} = 1$

Section 11.4

For Exercises 44–47,

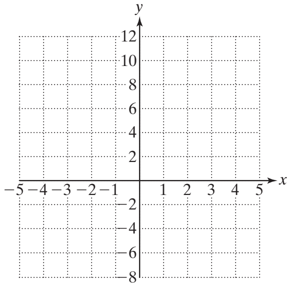
a. Identify each equation as representing a line, a parabola, a circle, an ellipse, or a hyperbola.

b. Graph both equations on the same coordinate system.

c. Solve the system analytically and verify the answers from the graph.

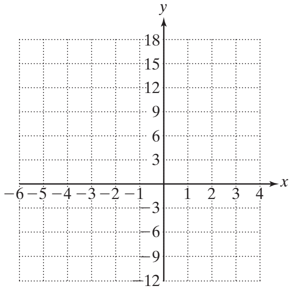
44. $3x + 2y = 10$

$y = x^2 - 5$



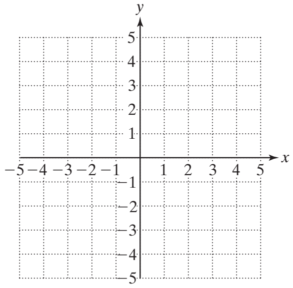
45. $4x + 2y = 10$

$y = x^2 - 10$



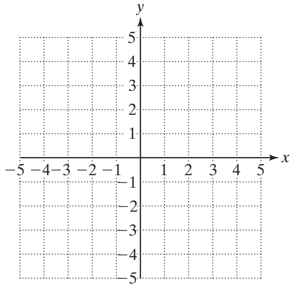
46. $x^2 + y^2 = 9$

$2x + y = 3$



47. $x^2 + y^2 = 16$

$x - 2y = 8$



For Exercises 48–53, solve the system of nonlinear equations by using either the substitution method or the addition method.

48. $x^2 + 2y^2 = 8$
$2x - y = 2$

49. $x^2 + 4y^2 = 29$
$x - y = -4$

50. $x - y = 4$
$y^2 = 2x$

51. $y = x^2$
$6x^2 - y^2 = 8$

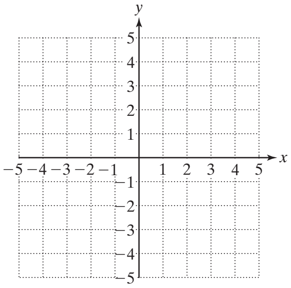
52. $x^2 + y^2 = 10$
$x^2 + 9y^2 = 18$

53. $x^2 + y^2 = 61$
$x^2 - y^2 = 11$

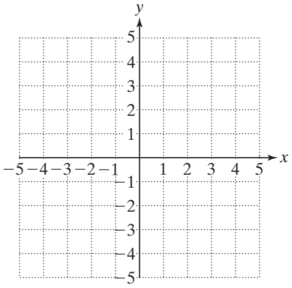
Section 11.5

For Exercises 54−61, graph the solution set to the inequality.

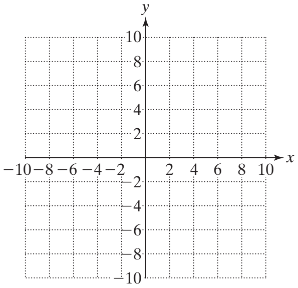
54. $3x + y \leq 4$



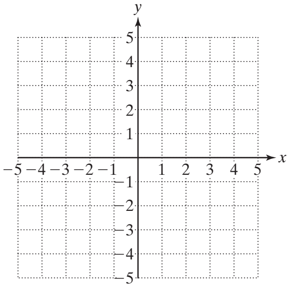
55. $x - 2y \geq -2$



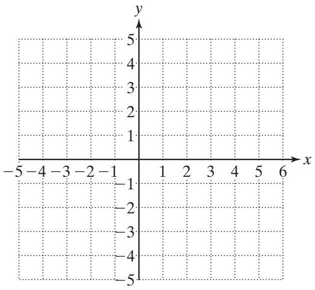
56. $\dfrac{x^2}{16} + \dfrac{y^2}{81} < 1$



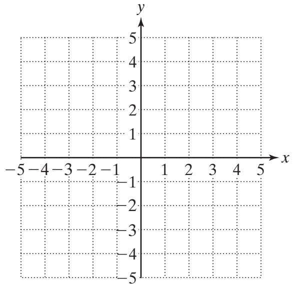
57. $\dfrac{x^2}{25} + \dfrac{y^2}{4} > 1$



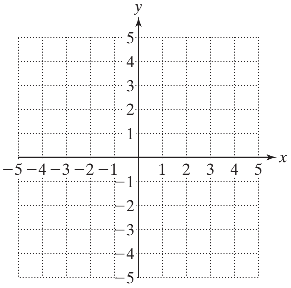
58. $(x - 3)^2 + (y + 1)^2 \geq 9$



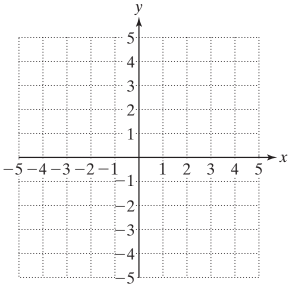
59. $(x + 2)^2 + (y + 1)^2 \leq 4$



60. $y > (x - 1)^2$

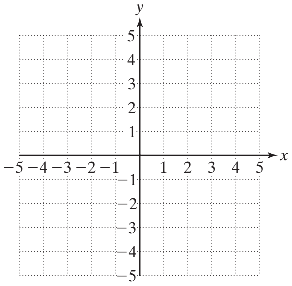


61. $y > x^2 - 1$

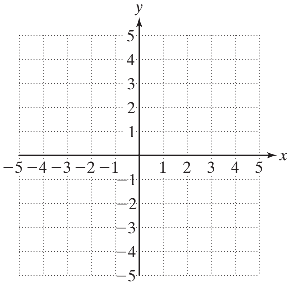


For Exercises 62−64, graph the solution set to the system of nonlinear inequalities.

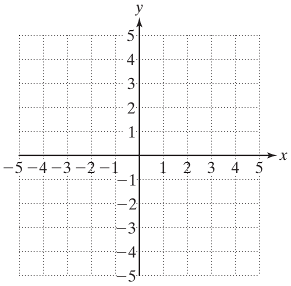
62. $y > 2^x$
$x^2 + y^2 < 4$



63. $y < 2^x$
$x^2 + y^2 < 9$



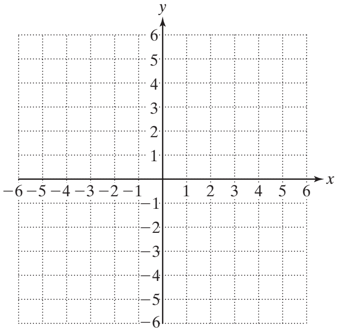
64. $\dfrac{x^2}{4} - y^2 < 1$
$x^2 + y^2 < 9$



Chapter 11 Test

1. Determine the coordinates of the vertex and the equation of the axis of symmetry. Then graph the parabola.

$$x = -(y-2)^2 + 3$$

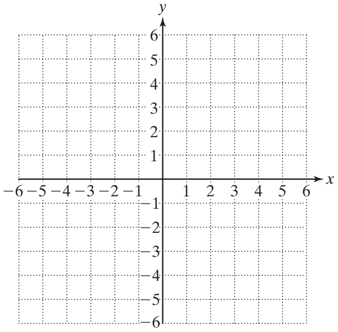


2. Determine the coordinates of the center and radius of the circle.

$$\left(x-\frac{5}{6}\right)^2+\left(y+\frac{1}{3}\right)^2=\frac{25}{49}$$

3. Write the equation in standard form $y = a(x-h)^2 + k$, and graph the parabola.

$$y = x^2 + 4x + 5$$



4. Use the distance formula to find the distance between the two points $(-5, 19)$ and $(-3, 13)$.

5. Determine the center and radius of the circle.

$$x^2 + y^2 - 4y - 5 = 0$$

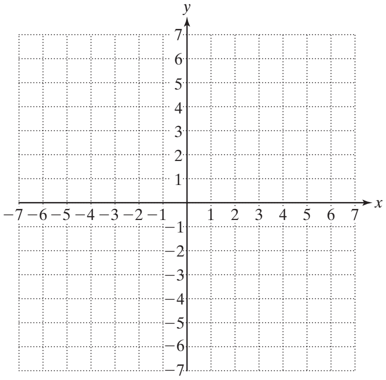
6. Let $(0, 4)$ be the center of a circle that passes through the point $(-2, 5)$.

a. What is the radius of the circle?

b. Write the equation of the circle in standard form.

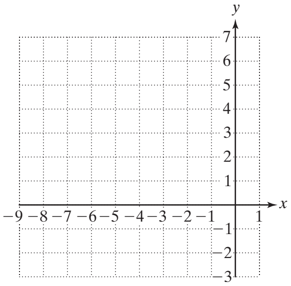
7. Graph the ellipse.

$$\frac{x^2}{16}+\frac{y^2}{49}=1$$



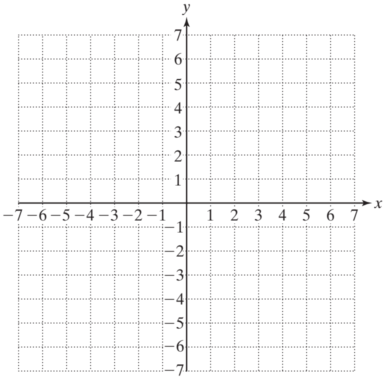
8. Graph the ellipse.

$$\frac{(x+4)^2}{25}+(y-3)^2=1$$



9. Graph the hyperbola.

$$y^2-\frac{x^2}{4}=1$$

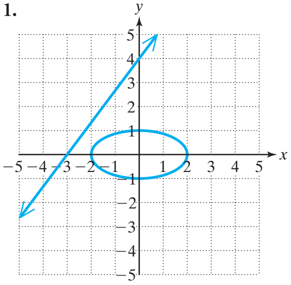


10. Solve the systems and identify the correct graph of the equations.

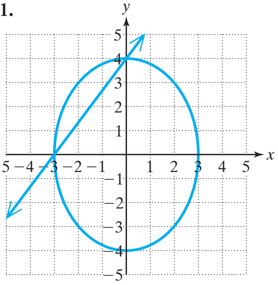
a. $16x^2 + 9y^2 = 144$
$4x - 3y = -12$

b. $x^2 + 4y^2 = 4$
$4x - 3y = -12$

i.



ii.



11. Describe the circumstances in which a nonlinear system of equations can be solved by using the addition method.

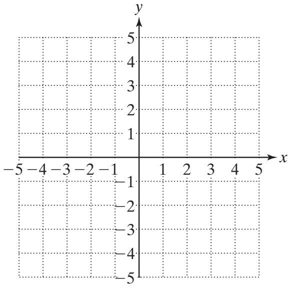
12. Solve the system by using either the substitution method or the addition method.

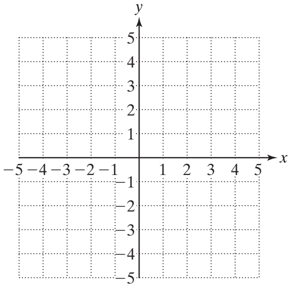
$$25x^2 + 4y^2 = 100$$
$$25x^2 - 4y^2 = 100$$

For Exercises 13–16, graph the solution set.

13. $x \le y^2 + 1$

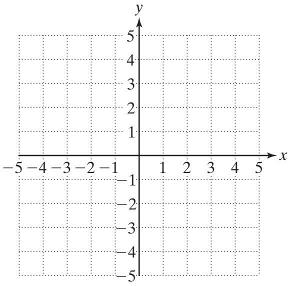
14. $y \ge -\frac{1}{3}x + 1$

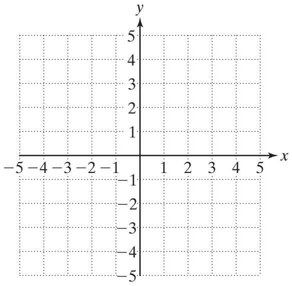




15. $x < y^2 + 1$
$y > -\frac{1}{3}x + 1$

16. $y < \sqrt{x}$
$y > x - 2$
$x > 0$





Chapters 1–11 Cumulative Review Exercises

1. Solve the equation.

$$5(2y - 1) = 2y - 4 + 8y - 1$$

2. Solve the inequality. Graph the solution and write the solution in interval notation.

$$4(x - 1) + 2 > 3x + 8 - 2x$$

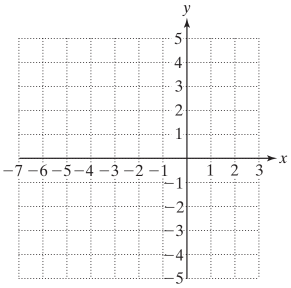
3. The product of two integers is 150. If one integer is 5 less than twice the other, find the integers.

4. For $5y - 3x - 15 = 0$:

a. Find the x- and y-intercepts.

b. Find the slope.

c. Graph the line.



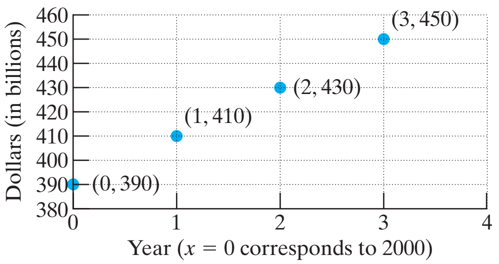
5. The amount of money spent on motor vehicles and parts each year since the year 2000 is shown in the graph. Let $x = 0$ correspond to the year 2000. Let y represent the amount of money spent on motor vehicles and parts (in billions of dollars).

a. Use any two data points to find the slope of the line.

b. Find an equation of the line through the points. Write the answer in slope-intercept form.

c. Use the linear model found in part (b) to predict the amount spent on motor vehicles and parts in the year 2010.

Amount Spent (in $ billions) on Motor Vehicles and Parts in United States



Source: Bureau of Economic Analysis, U.S. Department of Commerce

6. Find the slope and y-intercept of $3x - 4y = 6$ by first writing the equation in slope-intercept form.

7. A collection of dimes and quarters has a total value of \$2.45. If there are 17 coins, how many of each type are there?

8. Solve the system.

$$\begin{aligned} x + y \phantom{{}- z} &= -1 \\ 2x \phantom{{}+ y} - z &= 3 \\ y + 2z &= -1 \end{aligned}$$

9. **a.** Given the matrix $\mathbf{A} = \left[\begin{array}{cc|c} 1 & -2 & -8 \\ 0 & 3 & 6 \end{array}\right]$, write the matrix obtained by multiplying the elements in the second row by $\frac{1}{3}$.

b. Using the matrix obtained from part (a), write the matrix obtained by multiplying the second row by 2 and adding the result to the first row.

10. Solve the following system.

$$\begin{aligned} 4x - 2y &= 7 \\ -3x + 5y &= 0 \end{aligned}$$

11. Solve using Cramer's rule:

$$\begin{aligned} 3x - 4y &= 6 \\ x + 2y &= 12 \end{aligned}$$

12. For $f(x) = 3x - x^2 - 12$, find the function values $f(0), f(-1), f(2)$, and $f(4)$.

13. For $g = \{(2, 5), (8, -1), (3, 0), (-5, 5)\}$ find the function values $g(2), g(8), g(3)$, and $g(-5)$.

14. The quantity z varies jointly as y and as the square of x. If z is 80 when x is 5 and y is 2, find z when $x = 2$ and $y = 5$.

15. For $f(x) = \sqrt{x + 1}$ and $g(x) = x^2 + 6$, find $(g \circ f)(x)$.

16. **a.** Find the value of the expression $x^3 + x^2 + x + 1$ for $x = -2$.

b. Factor the expression $x^3 + x^2 + x + 1$ and find the value when x is -2.

c. Compare the values for parts (a) and (b).

17. Factor completely.

$$x^2 - y^2 - 6x - 6y$$

18. Multiply: $(x - 4)(x^2 + 2x + 1)$

19. Solve for x: $2x(x - 7) = x - 18$

20. Reduce the expression: $\dfrac{3a^2 - a - 2}{3a^2 + 8a + 4}$

21. Subtract: $\dfrac{2}{x + 3} - \dfrac{x}{x - 2}$

22. Solve: $\dfrac{2}{x + 3} - \dfrac{x}{x - 2} = \dfrac{-4}{x^2 + x - 6}$

23. Solve the radical equations.

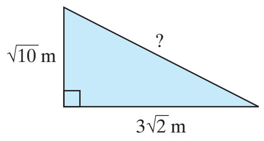
a. $\sqrt{2x - 5} = -3$

b. $\sqrt[3]{2x - 5} = -3$

24. Perform the indicated operations with complex numbers.

a. $6i(4 + 5i)$ **b.** $\dfrac{3}{4 - 5i}$

25. Find the length of the missing side.



26. An automobile starts from rest and accelerates at a constant rate for 10 sec. The distance, $d(t)$, in feet traveled by the car is given by

$$d(t) = 4.4t^2$$

where $0 \le t \le 10$ is the time in seconds.

a. How far has the car traveled after 2, 3, and 4 sec, respectively?

b. How long will it take for the car to travel 281.6 ft?

27. Solve the equation $125w^3 + 1 = 0$ by factoring and using the quadratic formula. (*Hint:* You will find one real solution and two imaginary solutions.)

28. Solve the rational equation.

$$\frac{x}{x+2} - \frac{3}{x-1} = \frac{1}{x^2 + x - 2}$$

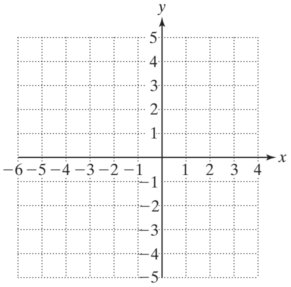
29. Find the coordinates of the vertex of the parabola defined by $f(x) = x^2 + 10x - 11$ by completing the square.

30. Graph the quadratic function defined by $g(x) = -x^2 - 2x + 3$.

a. Label the x-intercepts.

b. Label the y-intercept.

c. Label the vertex.



31. Solve the inequality and write the answer in interval notation.

$$|x - 9| - 3 < 7$$

32. Solve the inequality: $|2x - 5| \geq 4$

33. Write the expression in logarithmic form. $8^{5/3} = 32$

34. Solve the equation: $5^2 = 125^x$

35. For $h(x) = x^3 - 1$, find $h^{-1}(x)$.

36. Write an equation representing the set of all points 4 units from the point $(0, 5)$.

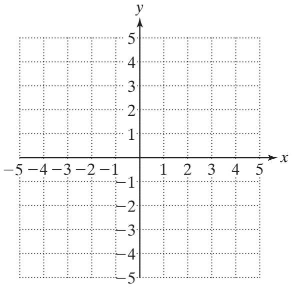
37. Can a circle and a parabola intersect in only one point? Explain.

38. Solve the system of nonlinear equations.

$$x^2 + y^2 = 16$$
$$y = -x^2 - 4$$

39. Graph the solution set.

$$y^2 - x^2 < 1$$



40. Graph the solution set to this system.

$$y > \left(\frac{1}{2}\right)^x$$
$$x < 0$$