Calculus (3) : Integration

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ANTI-DIFFERENTIATION

If we differentiate $y = 3x^2 - 4x + 3$, we obtain $\frac{dy}{dx} = 6x - 4$.

Supposing we were given $\frac{dy}{dx} = 6x - 4$, can we do a reverse operation, i.e. **anti-differentiate**, to find y?

This is easily done for a *single term* as follows.

Start with $y = \frac{ax^{n+1}}{n+1}$. (You will see why we choose this in a moment).

Then $\frac{dy}{dx} = \frac{a(n+1)x^n}{n+1} = ax^n$.

So if we are given $\frac{dy}{dx} = ax^n$, then $y = \frac{ax^{n+1}}{n+1}$.

To obtain this result, the index (n) has been increased by 1 to $n + 1$, and we then divide by the *new* index. Here is the rule for single terms:

> If $\frac{dy}{dx} = ax^n$, $y = \frac{ax^{n+1}}{n+1}$ provided $n \neq -1$

This process of anti-differentiation is actually called **integration**. We **integrate** ax^n wrt x. ax^n is the **integrand** and the result is called the **integral**. A notation for this will be given later.

Example 1

Integrate wrt x   *(a)* x^3 *(b)* $2x^2$ *(c)* $4x$ *(d)* 7 *(e)* $\frac{3}{x^2}$

We show here the steps taken to obtain the integral. With practice these would not be written down, only the result.

(a) Increase the index by 1 to 4: then divide by 4.

     Result $\frac{x^4}{4}$.

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(b) New index is 3: then divide by 3. The factor 2 is left as it is.

Result $\frac{2x^3}{3}$.

(c) $4x = 4x^1$. New index is 2, divide by 2.

Result $\frac{4x^2}{2} = 2x^2$.

Always simplify when possible.

(d) $7 = 7x^0$. New index is 1, divide by 1.

Result $\frac{7x^1}{1} = 7x$.

(e) $\frac{3}{x^2} = 3x^{-2}$. New index is $-2 + 1 = -1$. Divide by -1.

Result $\frac{3x^{-1}}{-1} = -\frac{3}{x}$.

Now check each result by differentiation and verify that the original expression is recovered.

Before we go further there is one important point to note. This is discussed in the next section.

THE ARBITRARY CONSTANT: INDEFINITE INTEGRAL

If $y = x^2 - 3x + c$ where c is any constant, then $\frac{dy}{dx} = 2x - 3$.
Now if we start with $\frac{dy}{dx} = 2x - 3$, then $y = x^2 - 3x$.

But this is not the original expression. The constant c is missing and so it must be added to the result. The correct result is $x^2 - 3x + c$. c is called the **arbitrary constant** as its value is not known, unless we are given further information. **It must always be added to an integral.** Such an integral is called an **indefinite integral**.

It is easy to get confused between differentiation and integration. It may help to remember:

Differentiation : multiply by the index and then **De**crease the index.

Integration : **In**crease the index and then divide by the new index.

As in differentiation, the integral of a sum of terms is the sum of the separate integrals. So we can integrate for example $x^3 - 3x^2 + 1$ or $(x + 2)^2$, provided it is expanded first, but not $\frac{1}{x+1}$.

Example 2

Integrate wrt x (a) $2x^3 - 3x + 1$ (b) $(2x - 3)^2$ (c) $\frac{x^4 - 3x + 1}{2x^2}$

(a) Integrate each term: $\frac{2x^4}{4} - \frac{3x^2}{2} + \frac{x^1}{1} + c = \frac{x^4}{2} - \frac{3x^2}{2} + x + c$

(b) Expand first: $4x^2 - 12x + 9$

Now integrate: $\frac{4x^3}{3} - \frac{12x^2}{2} + 9x + c = \frac{4x^3}{3} - 6x^2 + 9x + c$

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(c) Divide by $2x^3$ first: $\frac{x}{2} - \frac{3x^2}{2} + \frac{x^3}{2}$

Now integrate: $\frac{x^2}{2(2)} - \frac{3x^{-1}}{2(-1)} + \frac{x^{-2}}{2(-2)} + c = \frac{x^2}{4} + \frac{3}{2x} - \frac{1}{4x^2} + c$

Notation

The symbol for integration is $\int$. For example we write $\int 3x \, dx$. This means that the integrand $3x$ is to be integrated wrt x.

So

$$\int 3x \, dx = \frac{3x^2}{2} + c \quad \text{and} \quad \int u^2 \, du = \frac{u^3}{3} + c.$$

If $\frac{dy}{dx} = f(x)$, then $y = \int f(x) \, dx + c$.

Example 3

Find

(a) $\int x^5 \, dx$, (b) $\int dx$, (c) $\int 2t^3 \, dt$, (d) $\int (s^2 - 2s + 3) \, ds$, (e) $\int (p - 1)(2 - p) \, dp$

(a) $\int x^5 \, dx = \frac{x^6}{6} + c$

(b) $\int dx$ means $\int 1 \, dx = \int x^0 \, dx = x + c$

(c) Here the variable is t: $\int 2t^3 \, dt = \frac{2t^4}{4} + c = \frac{t^4}{2} + c$.

(d) If the integrand is a polynomial, it must be placed in brackets between the $\int$ sign and ds.

$\int (s^2 - 2s + 3) \, ds = \frac{s^3}{3} - \frac{2s^2}{2} + 3s + c = \frac{s^3}{3} - s^2 + 3s + c$

(e) Expand first.

$$\int (-2 + 3p - p^2) \, dp = -2p + \frac{3p^2}{2} - \frac{p^3}{3} + c$$

Note that an integral such as $\int 4x \, dy$ is not possible unless x can be expressed in terms of y.

Exercise 11.1 (Answers on page 630.)

1 Find the indefinite integrals wrt of x:

(a) $4x$ (b) $4x^3$ (c) -7

(d) $3x^2$ (e) $3 - x$ (f) $4x^5$

(g) $2x^5$ (h) $x^2 - 3$ (i) $1 - x - x^2$

(j) $x^2 - \frac{x}{4}$ (k) $1 - 3x - 4x^2$ (l) $x^3 - 3x^2$

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(m) $\frac{2}{x^3}$	(n) $(x+2)^2$	(o) $(x-1)^3$
(p) $1 - \frac{1}{3x^4}$	(q) $(2-x)^2$	(r) $x^2 - \frac{1}{x^2}$
(s) $(x+2)(x-3)$	(t) $x + \frac{1}{x^2}$	(u) $\frac{x+1}{3x^4}$
(v) $(x^2+2)(x-1)$		

2 (a) $\int (x-4)\,dx$

    (b) $\int \frac{3}{y^2}\,dy$

    (c) $\int \frac{dx}{3}$ (i.e. $\int \frac{1}{3}\,dx$)

    (d) $\int (2 - \frac{1}{x^3})\,dx$

    (e) $\int (3x-2)\,dx$

    (f) $\int \frac{2t}{3}\,dt$

    (g) $\int \frac{4}{y^3}\,dy$

    (h) $\int (\frac{1}{u^2} + \frac{2}{u^3})\,du$

3 Find

(a) $\int (\frac{u}{3} + \frac{1}{u^3})\,du$

    (b) $\int (3r-2)^3\,dr$

(c) $\int (3p-2)(p-3)\,dp$

    (d) $\int (1-x)^3\,dx$

(e) $\int (\frac{4t^3 - 3t^2 - 1}{3t^2})\,dt$

    (f) $\int (s - \frac{1}{2s})^2\,ds$

(g) $\int (1-4s)(2+3s)\,ds$

    (h) $\int (\frac{x+1}{2})^2\,dx$

(i) $\int (x^3 - \frac{x}{2})\,dx$

    (j) $\int (1-2y)^2\,dy$

(k) $\int (2t^2 - 4t + \frac{1}{3})\,dt$

    (l) $\int (2x + \frac{3}{2x})^2\,dx$

(m) $\int (\frac{2t^4 + t^2 - 2}{t^4})\,dt$

    (n) $\int p(2p+3)(2p-2)\,dp$

The Integral $\int \frac{1}{x}\,dx$

If we use the rule for this integral, $\int x^{-1}\,dx$, the result is $\frac{x^{-1+1}}{0}$ which is not possible. Hence $\int \frac{1}{x}\,dx$ is an exception to the rule.

In Part II of this book, we shall see that a special function is created for this integral.

APPLICATIONS OF INTEGRATION

> ### Example 4
> Find y given that $\frac{dy}{dx} = 2x - 3$ and that $y = -4$ when $x = 1$.
> If $\frac{dy}{dx} = 2x - 3$, then $y = \int (2x-3)\,dx = x^2 - 3x + c$.
> This is the indefinite integral and is illustrated in Fig.11.1. For all values of c, the family of curves $y = x^2 - 3x + c$ are parallel, one vertically above the other. The equations could be $y = x^2 - 3x$ or $y = x^2 - 3x + 5$ etc.
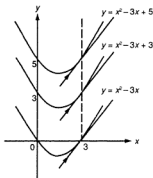
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Fig.11.1

For any given value of x (say 3), the gradient on each curve at $x = 3$ is 3 as $\frac{dy}{dx} = 2x - 3$ and the tangents at these points are parallel.
Further information is therefore needed to identify a particular member of the family. In this example, we have this information to find c.

When $x = 1$, $y = 1 - 3 + c = -4$ so $c = -2$.
Hence $y = x^2 - 3x - 2$.

Example 5

The gradient of the tangent at a point on a curve is given by $x^2 + x - 2$. Find the equation of the curve if it passes through (2,1).

Gradient = $\frac{dy}{dx} = x^2 + x - 2$.

Then $y = \int (x^2 + x - 2) \, dx = \frac{x^3}{3} + \frac{x^2}{2} - 2x + c$.

When $x = 2$, $y = \frac{8}{3} + \frac{4}{2} - 4 + c = 1$.

Hence $c = \frac{1}{3}$.

The equation of the curve is $y = \frac{x^3}{3} + \frac{x^2}{2} - 2x + \frac{1}{3}$ or $6y = 2x^3 + 3x^2 - 12x + 2$.

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Example 6

A curve has a turning point at the point (–1,1). If the gradient is given by $6x^2 + ax - 12$, find the value of a and the equation of the curve.

$\dfrac{dy}{dx} = 6x^2 + ax - 12$

When $x = -1$, $\dfrac{dy}{dx} = 0$.

Then $6 - a - 12 = 0$ so $a = -6$.

So $\dfrac{dy}{dx} = 6x^2 - 6x - 12$.

Hence $y = \int (6x^2 - 6x - 12)\, dx = 2x^3 - 3x^2 - 12x + c$.

When $x = -1$, $y = -2 - 3 + 12 = 1$, so $c = -6$.

The equation is $y = 2x^3 - 3x^2 - 12x - 6$.

Example 7

For a curve $y = f(x)$, $\dfrac{d^2y}{dx^2} = 6x - 2$. Given that $y = 11$ and $\dfrac{dy}{dx} = 10$ when $x = 2$, find the equation of the curve.

$\dfrac{d^2y}{dx^2}$ is obtained by differentiating $\dfrac{dy}{dx}$ wrt x. Then $\dfrac{dy}{dx}$ is found by integrating $\dfrac{d^2y}{dx^2}$ wrt x.

$\dfrac{dy}{dx} = \int \dfrac{d^2y}{dx^2}\, dx = \int (6x - 2)\, dx = 3x^2 - 2x + c$

But $\dfrac{dy}{dx} = 10$ when $x = 2$.

Then $12 - 4 + c = 10$ giving $c = 2$.

$\dfrac{dy}{dx} = 3x^2 - 2x + 2$

Now we integrate again to find y.

$y = \int (3x^2 - 2x + 2)\, dx = x^3 - x^2 + 2x + c_1$

When $x = 2$, $y = 8 - 4 + 4 + c_1 = 11$ so $c_1 = 3$.

Hence the equation is $y = x^3 - x^2 + 2x + 3$.

Example 8

A particle moves in a straight line so that its velocity v m s^{-1} at time t s from the start is given by $v = t^2 - 2t - 3$ ($t \geq 0$).

If it started 3 m from a fixed point O of the line, find

(a) *the value of t when it is at instantaneous rest,*

(b) *its distance from O at that time,*

(c) *for what values of t its acceleration is positive.*

(a) It is instantaneously at rest when $v = 0$.

$v = (t - 3)(t + 1)$ so $t = 3$ (-1 not being allowed).

(b) $v = \dfrac{ds}{dt} = t^2 - 2t - 3$

Then $s = \int (t^2 - 2t - 3)\,dt = \dfrac{t^3}{3} - t^2 - 3t + c$.

But $s = 3$ when $t = 0$, so $c = 3$.

Hence $s = \dfrac{t^3}{3} - t^2 - 3t + 3$.

When $t = 3$, $s = 9 - 9 - 9 + 3 = -6$ m.

(c) $a = \dfrac{dv}{dt} = 2t - 2$. Hence $a > 0$ when $t > 1$.

Example 9

For a particle moving in a straight line, its acceleration a m s^{-2} is given by $a = t - \dfrac{5}{2}$ where t is the time in s from the start.

Given that its velocity v at the start was 3 m s^{-1}, find (a) an expression for v in terms of t, (b) the time t when the particle is at instantaneous rest. (c) If the particle started from a fixed point O on the line, how far is it from O after 2 s?

(a) $a = \dfrac{dv}{dt} = t - \dfrac{5}{2}$.

Then $v = \int \left(t - \dfrac{5}{2}\right) dt = \dfrac{t^2}{2} - \dfrac{5t}{2} + c$

When $t = 0$, $v = 3$.

So $c = 3$ and $v = \dfrac{t^2}{2} - \dfrac{5t}{2} + 3$

$= \dfrac{t^2 - 5t + 6}{2} = \dfrac{(t - 3)(t - 2)}{2}$

(b) $v = 0$ when $t = 3$ or $t = 2$.

(c) $v = \dfrac{ds}{dt} = \dfrac{t^2}{2} - \dfrac{5t}{2} + 3$ so $s = \int \left(\dfrac{t^2}{2} - \dfrac{5t}{2} + 3\right) dt = \dfrac{t^3}{6} - \dfrac{5t^2}{4} + 3t + c_1$

When $t = 0$, $s = 0$ so $c_1 = 0$.

Therefore $s = \dfrac{t^3}{6} - \dfrac{5t^2}{4} + 3t$ and if $t = 2$, $s = 2\dfrac{1}{3}$ m.

Exercise 11.2 *(Answers on page 631.)*

1 A curve is given by $\dfrac{dy}{dx} = 2x - 1$. If it passes through the point (2,6), find its equation.

2 If a curve is given by $\dfrac{dy}{dx} = x(x - 1)$, find (a) its equation if it passes through the point (1,0) and (b) the nature and coordinates of its turning points.

3 Given that $\dfrac{dy}{dx} = 1 - 5x$ and that $y = -5$ when $x = 2$, find the value of y when $x = 1$.

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4 The rate of change of a quantity P is given by $\frac{dP}{dt} = t + 2$. If $P = 5$ when $t = 2$, find the value of P when $t = 3$.

5 The velocity v m s^{-1} of a particle P moving in a straight line at time t s is given by $v = 2t^2 - 3t$. Find an expression for its distance s m from a fixed point O on the line if OP = 4 m when $t = 1$ and its acceleration when $t = 1$.

6 Given that $\frac{d^2y}{dx^2} = 2x + 1$ and that $\frac{dy}{dx} = y = 3$ when $x = -1$, find y in terms of x.

7 Given that $\frac{d^2y}{dx^2} = 3$, find y in terms of x if $\frac{dy}{dx} = 4$ and $y = 6$ when $x = 2$.

8 A curve has gradient $x^2 - 4x + 3$ at the point (x,y) on the curve and it passes through the point $(3,-1)$. Find (a) its equation and (b) the types and coordinates of its turning points.

9 For the function $y = f(x)$, $\frac{dy}{dx} = x^2 + kx$ where k is a constant. If y has a turning point at the point $(3,-2)$, find the value of k and the value of y when $x = 4$.

10 If $\frac{dy}{dx} = 1 - \frac{1}{t^2}$, find the value of y when $t = 4$ if $y = 5$ when $t = 1$.

11 If $\frac{dy}{dx} = 6x^2 + 4x - 5$ and $y = 10$ when $x = 2$, find the value of y when $x = 3$.

12 The rate of change of a quantity L with respect to t is given by $\frac{dL}{dt} = 3t - 2$. If $L = 3$ when $t = 4$, find the value of L when $t = 4$.

13 A curve passes through the point $(1,0)$ and its gradient at any point (x,y) on the curve is $3x^2 - 2x - 1$. Find (a) the equation of the curve, and (b) the coordinates of the points where y has a maximum and minimum value, identifying each one. (c) For what range of values of x is the gradient on the curve decreasing?

14 A small body moves in a straight line so that its velocity v m s^{-1} at time t s is given by $v = t^3 - 6t^2 + 9t + 2$. Find (a) the times when its acceleration is zero, and (b) its distance from a fixed point on the line when $t = 2$ given that it started from rest.

15 The velocity v m s^{-1} of a particle moving in a straight line is given by $v = t^2 - 4t$ where t is the time in seconds after starting from a fixed point O on the line. Find
(a) the time the particle is instantaneously at rest,
(b) its velocity and acceleration at the start,
(c) its distance from O when $t = 3$.

16 The velocity v m s^{-1} of a particle moving in a straight line at time t seconds is given by $v = 1 + \frac{9}{t^2}$ for $1 \le t \le 3$.

When $t = 3$, the particle is 6 m from a fixed point on the line.
(a) Find an expression in terms of t for its distance from this fixed point.
(b) How far does it travel between $t = 1$ and $t = 3$?

17 Given that $\frac{d^2y}{dt^2} = 3 - kt$ where k is a constant and that $\frac{dy}{dt} = -6$ when $t = -1$ and 9 when $t = 2$, find the value of k. If $y = -\frac{5}{6}$ when $t = 1$, find y in terms of t.

18 A particle passes a fixed point O on a straight line with a velocity of 10 m s⁻¹ and moves on the line with an acceleration of $(4 - t)$ m s⁻² at time t s after passing through O. Find

(a) its velocity when $t = 4$,

(b) the distance of the particle from O when $t = 2$.

19 A quantity u varies with respect to t so that $\dfrac{du}{dt} = a + bt$ where a and b are constants. Given that it has a maximum value of $5\frac{1}{2}$ when $t = 1$ and that its rate of change when $t = 2$ is -3, find u in terms of t.

Area Under a Curve

An important application of integration is in finding the area under a given curve $y = f(x)$. Up to now, such areas could only be found approximately, for example, by counting squares or by the trapezium rule. Using calculus, we can now find the exact value of areas bounded by curves.

Fig.11.2 shows part of a curve $y = f(x)$. The shaded area A lies between the curve and the x-axis, bounded by the ordinates at a and b. This area is called the area under the curve between a and b.
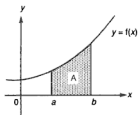
Fig.11.2

We now show a method of finding A. For the moment we can only deal with areas which lie above the x-axis.
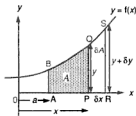
Fig.11.3

Let P be a point on the x-axis where OP $= x$ (Fig.11.3). PQ $= y$, OA $= a$ and AB is perpendicular to the x-axis. The shaded area under the curve from a to x is A.

Now take an increment δx in x to reach R and draw the ordinate RS. $RS = y + \delta y$ and the increment in $A = \delta A =$ the area PRSQ.
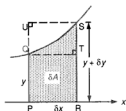

Fig.11.4

QT and US are drawn parallel to the x-axis (Fig.11.4).
Area of rectangle PRTQ $< \delta A <$ area of rectangle PRSU.
i.e. $y\delta x < \delta A < (y + \delta y)\,\delta x$ so so $< \dfrac{\delta A}{\delta x} < y + \delta y$.
Now if $\delta x \to 0$, $\delta y \to 0$ and $y + \delta y \to y$.
The left hand term of the inequality remains fixed at y but the right hand term $\to y$. Hence $\dfrac{\delta A}{\delta x} \to y$ and in the limit $\dfrac{dA}{dx} = y$.

We then have
$$A = \int y \, dx + c = \int f(x)\, dx + c$$

We can find c from the fact that $A = 0$ when $x = a$.

Example 10

Find the area under the curve $y = x^2 + 2$ between $x = 1$ and $x = 3$ (Fig.11.5).
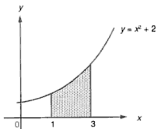
Fig.11.5

$A = \int y \, dx = \int (x^2 + 2)\, dx = \dfrac{x^3}{3} + 2x + c$

Now when $x = 1$, $A = 0$.

Hence $0 = \dfrac{1}{3} + 2 + c$ giving $c = -2\dfrac{1}{3}$.

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Then $A_1^x = \frac{x^3}{3} + 2x - 2\frac{1}{3}$ where A_1^x means the area between the ordinates 1 and x ($x > 1$).

We want the area from 1 to 3. So put $x = 3$.

$A_1^3 = 9 + 6 - 2\frac{1}{3} = 12\frac{2}{3}$ units^2.

If we had required the area from 1 to 2, i.e. A_1^2, then we put $x = 2$ etc.

Example 11

Find the area under the curve $y = 2 + x - x^2$.

The curve meets the x-axis where $y = 0$,
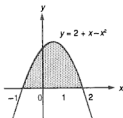i.e. where $(2 - x)(1 + x) = 0$
giving $x = -1$ or $x = 2$ (Fig.11.6).

Fig.11.6

So $A = \int (2 + x - x^2)\, dx = 2x + \frac{x^2}{2} - \frac{x^3}{3} + c$.

But $A = 0$ when $x = -1$.

Hence $0 = -2 + \frac{1}{2} + \frac{1}{3} + c$ giving $c = 1\frac{1}{6}$.

Then $A_{-1}^x = 2x + \frac{x^2}{2} - \frac{x^3}{3} + 1\frac{1}{6}$.

Now put $x = 2$ to obtain the required area.

$A_{-1}^2 = 4 + 2 - \frac{8}{3} + 1\frac{1}{6} = 4\frac{1}{2}$ units^2.

DEFINITE INTEGRALS

We can shorten the above process by using the concept of a **definite integral**.

Suppose A_a^x is the area under $y = f(x)$ from a to x (Fig.11.7).

Then $A = \int f(x)\, dx = g(x) + c$ where $g(x)$ is the indefinite integral of $f(x)$.
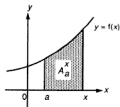
Fig.11.7

Now when $x = a$, $A = 0$ so $0 = g(a) + c$ giving $c = -g(a)$.

Hence $A_a^x = g(x) - g(a)$.

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Now we put $x = b$

and $A_a^b = g(b) - g(a)$

= (value of the integral at b) – (value of the integral at a)

We write this as $\int_a^b f(x) \, dx$ and it is called the **definite integral** of f(x) wrt x between the **limits** a (the lower limit) and b (the upper limit). The arbitrary constant c disappears in the subtraction.

Hence if $y = f(x)$, the area under the curve between the ordinates a and b, where $a < b$, is

$$\int_a^b f(x) \, dx = g(b) - g(a)$$

where g(x) is the indefinite integral of f(x).

At present this is only true if f(x) ≥ 0. We investigate what happens if f(x) < 0 later.

Example 12

Find the area under the curve $y = x^3 + 2$ from $x = -1$ to $x = 2$ (Fig.11.8).
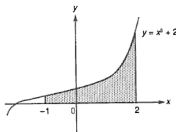
Fig.11.8

$A = \int_1^2 y \, dx$

$= \int_1^2 (x^3 + 2) dx$ $g(x) = \dfrac{x^4}{4} + 2x + c$

$= \left[\dfrac{x^4}{4} + 2x \right]_{-1}^2$ upper limit, lower limit   g(x) is placed in square brackets with the limits at the right. The arbitrary constant is not included.

$= (\dfrac{2^4}{4} + 2 \times 2)$ $(\dfrac{(-1)^4}{4} + 2 \times (-1))$
substitute **upper** limit to obtain g(b), the value when $b = 2$ substitute **lower** limit to obtain g(a), the value when $a = -1$

$= \quad 8 \qquad - \qquad (\tfrac{1}{4} - 2)$

$= \quad 8 \qquad - \qquad (-1\tfrac{3}{4})$

$= 9\tfrac{3}{4}$ units^2

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Example 13

(a) Find the coordinates of (i) the point A where the curves $y = (x + 1)^2$ and $y = (x - 3)^2$ intersect and (ii) the points where the curves meet the x-axis.

(b) Hence find the area of the region enclosed by the curves and the x-axis.

(a) (i) The curves meet where $(x + 1)^2 = (x - 3)^2$ i.e. where $x = 1$.
Hence the coordinates of A are (1,4).

   (ii) $y = (x + 1)^2$ meets $y = 0$ where $x = -1$, i.e. the point $(-1,0)$.
$y = (x - 3)^2$ meets $y = 0$ where $x = 3$, i.e. the point $(3,0)$.

(b) The curves are shown in Fig.11.9. The area required is divided into two parts because the boundary changes at A ($x = 1$).
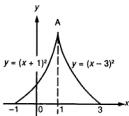
Fig.11.9

$$\text{Total area} = \int_{-1}^{1} (x + 1)^2 \, dx + \int_{1}^{3} (x - 3)^2 \, dx$$

$$= \int_{-1}^{1} (x^2 + 2x + 1) \, dx + \int_{1}^{3} (x^2 - 6x + 9) \, dx$$

$$= \left[\frac{x^3}{3} + x^2 + x\right]_{-1}^{1} + \left[\frac{x^3}{3} - 3x^2 + 9x\right]_{1}^{3}$$

$$= (\tfrac{1}{3} + 1 + 1) - (-\tfrac{1}{3} + 1 - 1) + (9 - 27 + 27) - (\tfrac{1}{3} - 3 + 9)$$

$$\underset{\substack{\text{substitute} \\ \text{upper limit 1}}}{} \quad \underset{\substack{\text{substitute} \\ \text{lower limit } -1}}{} \quad \underset{\substack{\text{substitute} \\ \text{upper limit 3}}}{} \quad \underset{\substack{\text{substitute} \\ \text{lower limit 1}}}{}$$

$$= \tfrac{7}{3} + \tfrac{1}{3} + 9 - \tfrac{19}{3}$$

$$= \tfrac{16}{3} \text{ units}^2$$

In the next two examples, only the *value* of the definite integral is to be found.

Example 14

Find $\int_{-4}^{-2} (x - \frac{1}{x^2})\, dx$.

$$\int_{-4}^{-2} (x - \frac{1}{x^2})\, dx = \left[\frac{x^2}{2} + \frac{1}{x} \right]_{-4}^{-2}$$

$$= \underbrace{\left(\frac{(-2)^2}{2} + \frac{1}{-2} \right)}_{\text{substitute the upper limit } (-2)} - \underbrace{\left(\frac{(-4)^2}{2} + \frac{1}{-4} \right)}_{\text{substitute the lower limit } (-4)}$$

$$= (2 - \tfrac{1}{2}) - (8 - \tfrac{1}{4})$$

$$= -6\tfrac{1}{4}$$

Example 15

Evaluate $\int_{-2}^{0} (1 - t - t^2)\, dt$.

$$\int_{-2}^{0} (1 - t - t^2)\, dt = \left[t - \frac{t^2}{2} - \frac{t^3}{3} \right]_{-2}^{0}$$

$$= [0] - [-2 - \frac{(-2)^2}{2} - \frac{(-2)^3}{3}]$$

$$= 0 - (-1\tfrac{1}{3}) = 1\tfrac{1}{3}$$

Example 16

The volume V of the liquid in a container leaks out at the rate of $30t$ cm³ s⁻¹ where t is the time in seconds. Find the amount of liquid lost in the third second.

$$\frac{dV}{dt} = -30t \quad \text{(decreasing).}$$

The third second is between $t = 2$ and $t = 3$.
So we find the (value of V when $t = 3$) – (value of V when $t = 2$) using a definite integral.

Change of volume $= \int_{2}^{3} \frac{dV}{dt}\, dt = \int_{2}^{3} (-30t)\, dt = \left[-15t^2 \right]_{2}^{3} = -135 - (-60)$

$$= -75$$

Hence 75 cm³ of liquid was lost in that time.

Example 17

The velocity v of a particle moving in a straight line is given by $v = t^2 - 3t$ where t is the time after the start. What is the displacement of the particle between the times t = 2 and t = 4?

$$v = \frac{dx}{dt} = t^2 - 3t \quad \text{so } s = \int (t^2 - 3t)\, dt$$

The displacement is the distance between the positions of the particle at times $t = 2$ and $t = 4$, so it is the value of the definite integral

$$\int_2^4 (t^2 - 3t)\, dt = \left[\frac{t^3}{3} - \frac{3t^2}{2}\right]_2^4$$

$$= (\tfrac{64}{3} - \tfrac{48}{2}) - (\tfrac{8}{3} - \tfrac{12}{2}) = \tfrac{2}{3}$$

Note: As we have seen in Chapter 10, this is not necessarily equal to the actual distance travelled by the particle. It may have gone, for example, 8 units to the left followed by $8\frac{2}{3}$ units to the right.

Example 18

(a) Show from a diagram that

$$\int_a^c f(x)\, dx = \int_a^b f(x)\, dx + \int_b^c f(x)\, dx \text{ where } a < b < c.$$

(b) Given that $\int_2^4 f(x)\, dx = 10$, find (if possible) the values of

    (i) $\int_2^4 2f(x)\, dx$,

    (ii) $\int_2^4 [f(x) + 1]\, dx$,

    (iii) $\int_2^3 [f(x) - 2]\, dx + \int_3^4 f(x)\, dx$,

    (iv) $\int_2^4 [1 - \tfrac{1}{2} f(x)]\, dx$,

    (v) $\int_2^4 [f(x)]^2\, dx$

(a)
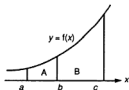
Fig.11.10

From Fig.11.10, $\int_a^c f(x)\, dx = \text{area A} + \text{area B}$

$$= \int_a^b f(x)\, dx + \int_b^c f(x)\, dx$$

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(b) (i) $\int_2^5 2f(x)\,dx = 2\int_2^5 f(x)\,dx = 2 \times 10 = 20$

    (ii) $\int_2^5 [f(x) + 1]\,dx = \int_2^5 f(x)\,dx + \int_2^5 1\,dx$

               $= 10 + [x]_2^5$

               $= 10 + (5) - (2) = 13$

    (iii) This equals $\int_2^3 f(x)\,dx - \int_3^5 2\,dx + \int_3^5 f(x)\,dx$

                 $= \int_2^5 f(x)\,dx - [2x]_3^5$

                 $= 10 - 2 = 8$

    (iv) $\int_2^5 [1 - \tfrac{1}{2}f(x)]\,dx = \int_2^5 1\,dx - \tfrac{1}{2}\int_2^5 f(x)\,dx$

                 $= 3 - 5 = -2$

    (v) Not possible, as f(x) is not known.

Exercise 11.3 *(Answers on page 631.)*

1 (a) $\int_{-2}^1 dx$ (b) $\int_{-1}^0 x\,dx$

  (c) $\int_{-2}^1 x^2\,dx$ (d) $\int_0^3 (2x - 1)\,dx$

  (e) $\int_{-2}^{-1} (1 - x)\,dx$ (f) $\int_{-1}^2 x^3\,dx$

  (g) $\int_1^3 \frac{1}{x^2}\,dx$ (h) $\int_{-2}^{-1} (x - x^2)\,dx$

  (i) $\int_1^2 (p - 2)(p - 3)\,dp$ (j) $\int_{-4}^0 dx$

  (k) $\int_1^{2y} (u^2 - 3)\,du$ (l) $\int_1^3 \frac{x-3}{x^3}\,dx$

  (m) $\int_{-2}^0 t(t^2 - 1)\,dt$ (n) $\int_{-3}^{-2} (\frac{1}{x^2} - \frac{1}{x^3})\,dx$

  (o) $\int_{-2}^1 3x^2\,dx$ (p) $\int_1^2 x^3\,dx$

  (q) $\int_0^3 x(3 - x)\,dx$ (r) $\int_{-4}^0 (3 - 2x)\,dx$

  (s) $\int_1^2 (x + 1)(x + 2)\,dx$ (t) $\int_1^2 (x - \frac{1}{x^2})\,dx$

  (u) $\int_{-2}^2 (x^3 - x^2 + x)\,dx$

2 If $\int_4^a (x - 4)\,dx = 10$, find the value of a.

3 Given that $\int_0^t (2x - 1)\,dx = 12$, find the values of t.

4 Given that $\int_1^2 (x + p)\,dx = 3$, find the value of p.

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5 If $\int_0^1 (t^2 + \mu) \, dt = 3$, find the value of μ.

6 Find the value of u if $\int_u^{2u} \frac{1}{x^2} \, dx = \frac{1}{4}$.

7 Find the areas under the following curves between the coordinates given:
(a) $y = 4 - x^2$; $x = -2$, $x = 0$
(b) $y = x(3 - x)$; $x = 0$, $x = 3$
(c) $y = \frac{1}{2}x^2$; $x = 1$, $x = 2$
(d) $y = 3 - 2x - x^2$; $x = -3$, $x = 1$
(e) $y = 2 - x^2$; $x = -1$, $x = 1$
(f) $y = 1 + x^2$; $x = -1$, $x = 1$
(g) $y = x^3 + 2$; $x = 0$, $x = 2$
(h) $y = x^2 - 2$; $x = -3$, $x = -1$

8 Find the area bounded by the curve $y = 2x - 2x^2$ and the positive x– and y-axes.

9 Find the area under the curve $y = x^2 + 3$ between the ordinates (i) $x = 0$ and $x = 2$, (ii) $x = -2$ and $x = 2$. Using a sketch of the curve explain the relation between the two areas.

10 The area under the curve $y = x^2 + ax - 5$ between the lines $x = 1$ and $x = 3$ is $14\frac{2}{3}$. Find the value of a.

11 If the area under the curve $y = \frac{x^2}{3}$ between $x = 2$ and $x = k$, where k is a constant, is 8 times the area under the curve between $x = 1$ and $x = 2$, find the value of k.

12 Given that $\frac{dA}{dt} = 2t^2 - t + 5$, find the change in the value of A between $t = 1$ and $t = 3$.

13 If $\frac{dT}{dt} = t^2 - t + 1$, find the change in T as t changes from 1 to 2.

14 The rate of change of a quantity P is given by $\frac{dP}{dt} = \frac{10}{t^2} + t$ for $t > 2$. Find the change in the value of P when t increases from 3 to 5.

15 If $\frac{d^2y}{dx^2} = 2x - 1$, find the increase in y as x increases from 2 to 4 given that $\frac{dy}{dx} = 6$ when $x = 2$.

16 The curve $y = ax^2 + bx + c$ passes through the points $(0, -2)$ and $(1, -3)$ and its gradient where $x = 2$ is 5. Find (a) the value of a, of b and of c (b) the area under the curve between the lines $x = 2$ and $x = 3$.

17 (a) If $\int_a^b (x - 1) \, dx = \frac{1}{2} \int_0^a (x + 1) \, dx$, find the value of a.
(b) Given that $\int_1^6 f(x) \, dx = 7$, evaluate (i) $\int_1^6 3f(x) \, dx$, (ii) $\int_1^6 [2 - f(x)] \, dx$,
(iii) $\int_1^3 [f(x) - 2] \, dx + \int_3^6 f(x) \, dx$.

18 A particle starts from a fixed point O and moves in a straight line. Its distance s from O at time t seconds from the start is given by $s = \frac{1}{3}t^3 - 2t^2 + 3t$. Find an expression for the velocity v of the particle in terms of t. At what times from the start is the particle at instantaneous rest? What is its displacement between those times?

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19 Fig.11.11 shows part of the line $y + 2x = 5$ and the curve $y = x(4 - x)$, which meet at A.

(a) Find the coordinates of A.
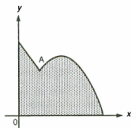
(b) Hence find the area of the shaded region.

Fig.11.11

20 The curve $y = 4 - x^2$ meets the positive x-axis at B and the curve $y = x(4 - x)$ meets the positive x-axis at C. The curves intersect at A. Find

(a) the coordinates of A, B and C,

(b) the area of the region ABC bounded by the curves and the x-axis.

21 (a) If $y = x^2 - 4x + 4$, find (i) where the curve meets the y-axis and (ii) the x-coordinate m of the minimum point on the curve.

(b) Sketch on the same diagram the graph of $y = 4 - x^2$ for $-2 \leq x \leq 0$ and the graph of $y = x^2 - 4x + 4$ for $0 \leq x \leq m$.

(c) Hence find the total area under the two curves.

Further Notes on Areas

I Area between a curve and the y-axis

The area between $y = f(x)$ and the x-axis for $a \leq x \leq b$ is $\int_a^b y \, dx$.

Similarly, the area between $y = f(x)$ and the y-axis is $\int_c^d x \, dy$ where c and d are the limits on the y-axis and the equation of the curve is expressed in the form $x = g(y)$ (Fig.11.12).
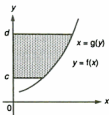
Fig.11.12

Example 19

Fig.11.13 shows part of the curve $(y - 1)^2 = x - 1$. Find the area of the shaded region.
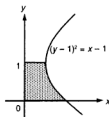
Fig.11.13

The area $= \int_0^1 x \, dy$ as 0 and 1 are the limits for y.

The equation of the curve is rewritten as $x = (y - 1)^2 + 1 = y^2 - 2y + 2$.

So the area is $\int_0^1 (y^2 - 2y + 2) \, dy = \left[\frac{y^3}{3} - y^2 + 2y \right]_0^1$

$$= (\tfrac{1}{3} - 1 + 2) - (0) = 1\tfrac{1}{3} \text{ units}^2$$

II Area under the x-axis

The curve $y = x^2 - 3x + 2 = (x - 2)(x - 1)$ meets the x-axis where $x = 1$ and $x = 2$ (Fig.11.14).
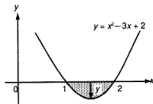
Fig.11.14

For all points in the domain $1 < x < 2$, y will be negative. So $\int y \, dx$ will also be negative for this domain.

$$\int_1^2 y \, dx = \int_1^2 (x^2 - 3x + 2) \, dx = \left[\frac{x^3}{3} - \frac{3x^2}{2} + 2x \right]_1^2$$

$$= (\tfrac{8}{3} - 6 + 4) - (\tfrac{1}{3} - \tfrac{3}{2} + 2)$$

$$= \tfrac{2}{3} - \tfrac{5}{6}$$

$$= -\tfrac{1}{6} \text{ which is negative as expected.}$$

The numerical value of $\int y \, dx$ is $-\int y \, dx = \tfrac{1}{6}$ and this is the area below the x-axis.

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**If part of a curve lies below the x-axis, the area
between that part and the x-axis is − ∫ y dx (Fig.11.15).**
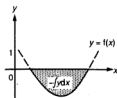
Fig. 11.15

If a curve lies partly above and partly below the x-axis (Fig.11.16), the total area will be $\int_a^b y\,dx - \int_b^c y\,dx$.
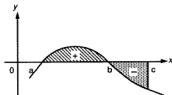

Fig. 11.16

The two parts are evaluated separately. Hence a sketch of the curve must be made to check if any part is below the x-axis.

Similarly the area of a region on the left of the y-axis will be negative. Its numerical value is − ∫ x dy.

Example 20

Find the area between the curve $y = x(x − 2)$ and the x-axis from $x = −1$ to $x = 2$.

The curve meets the x-axis at $x = 0$ and $x = 2$ (Fig.11.17).

Fig.11.17

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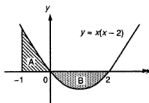
Area $A = \int_1^0 (x^2 - 2x)\, dx = \left[\frac{x^3}{3} - x^2\right]_1^0 = (0) - (-\frac{1}{3} - 1) = 1\frac{1}{3}$

Area B $= -\int_0^2 (x^2 - 2x)\, dx = -\left[\frac{x^3}{3} - x^2\right]_0^2 = -\left[\left(\frac{8}{3} - 4\right) - (0)\right] = 1\frac{1}{3}$

Hence the total area is $1\frac{1}{3} + 1\frac{1}{3} = 2\frac{2}{3}$

Note that $\int_{-1}^2 (x^2 - 2x)\, dx = \left[\frac{x^3}{3} - x^2\right]_{-1}^2$

$= (\frac{8}{3} - 4) - (-\frac{1}{3} - 1) = 0$

which is the correct value for the *integral* but not for the *area*.

III Area between two curves

Example 21

Find the area enclosed by the curve $y = 5 + x - x^2$ and the line $y = x + 4$.

First we find where these intersect:

$5 + x - x^2 = x + 4$

i.e. $x^2 = 1$ giving $x = 1$ or -1.

Then we require the area of the shaded region in Fig.11.18 with limits $x = -1$ and $x = 1$.
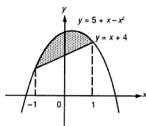
Fig.11.18

Now the area under the curve

$= \int_{-1}^1 (5 + x - x^2)\, dx$

and the area under the line

$= \int_{-1}^1 (x + 4)\, dx$.

Hence the shaded area $= \int_{-1}^1 (5 + x - x^2)\, dx - \int_{-1}^1 (x + 4)\, dx$.

Since these two definite integrals have the same limits they can be combined into one definite integral:

$\int_{-1}^1 [(5 + x - x^2) - (x + 4)]\, dx = \int_{-1}^1 (5 + x - x^2 - x - 4)\, dx$

$= \int_{-1}^1 (1 - x^2)\, dx$

$= \left[x - \frac{x^3}{3}\right]_{-1}^1$

$= (1 - \frac{1}{3}) - (-1 + \frac{1}{3}) = \frac{4}{3}$

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In Fig.11.19, y = f(x) and y = g(x) are two curves such that f(x) > g(x) for $a \le x \le b$.
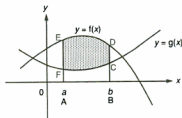

Fig. 11.19

Then $\int_a^b f(x)\, dx$ = area ABDE and $\int_a^b g(x)\, dx$ = area ABCF.

Hence the area between the curves i.e. the shaded area

= ABDE – ABCF

= $\int_a^b [f(x) - g(x)]\, dx$

> Area between y = f(x) and y = g(x) for $a \le x \le b$
>
> $$= \int_a^b [f(x) - g(x)]\, dx$$
>
> where f(x) > g(x)

This rule is still true if parts of either curve are below the x-axis (provided f(x) > g(x)) as the next example shows.

Example 22

Find the area enclosed by the curves $y_1 = x^2$ and $y_2 = x^2 - 2x$ and the lines $x = 1$ and $x = 3$ (Fig.11.20).

Fig.11.20

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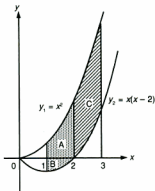
We require the shaded area.

The curve y_2 meets the x-axis at $x = 0$ and $x = 2$.

Consider the regions A, B and C.

$A = \int_1^2 y_1 \, dx.$

$B = -\int_1^2 y_2 \, dx$ (as y_2 is below the x-axis in this interval).

So $\int_1^2 y_1 \, dx - \int_1^2 y_2 \, dx = A + B.$

Hence the rule is true for areas crossing the x-axis.

$C = \int_2^3 y_1 \, dx - \int_2^3 y_2 \, dx$ (as both y_1 and y_2 are above the x-axis).

Hence the total shaded area A + B + C

$$= \int_1^2 (y_1 - y_2) \, dx + \int_2^3 (y_1 - y_2) \, dx$$
$$= \int_1^3 (y_1 - y_2) \, dx$$
$$= \int_1^3 (x^2 - x^2 + 2x) \, dx$$
$$= \left[x^2 \right]_1^3 = 9 - 1 = 8 \text{ units}^2$$

Example 23

The tangents at $x = 0$ and $x = 3$ on the curve $y = 2x - x^2 - 1$ meet at T.

(a) Find the equations of these tangents and the coordinates of T.

(b) Calculate the area of the region bounded by the curve and the tangents.

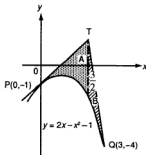The curve and the tangents are shown in Fig.11.21.

Fig.11.21

If $y = 2x - x^2 - 1$, then $\dfrac{dy}{dx} = 2 - 2x.$

(a) The tangent at P(0,−1) has gradient 2.

Hence its equation is $y + 1 = 2x.$

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The tangent at Q(3,−4) has gradient 2 − 6 = −4.

Hence its equation is y + 4 = −4(x − 3) i.e. y = −4x + 8.

These lines intersect where 2x − 1 = −4x + 8 i.e. x = 1½.

When x = 1½, y = 2x − 1 = 2.

So the coordinates of T are (1½, 2).

(b) The shaded area is divided into two parts A and B as the boundary line changes at T (x = $\frac{3}{2}$).

Area of A = $\int_0^{3/2} [(2x − 1) − (2x − x^2 − 1)] \, dx$

$= \int_0^{3/2} x^2 \, dx = \frac{27}{24} = \frac{9}{8}$

Area of B = $\int_{3/2}^3 [(−4x + 8) − (2x − x^2 − 1)] \, dx$

$= \int_{3/2}^3 (x^2 − 6x + 9) \, dx$

$= \left[\frac{x^3}{3} − 3x^2 + 9x \right]_{3/2}^3$

$= (9 − 27 + 27) − (\frac{27}{8} − \frac{27}{4} + \frac{27}{2}) = \frac{27}{24} = \frac{9}{8}$

Hence the total shaded area = $\frac{9}{4}$ units².

Exercise 11.4 (Answers on page 631.)

1 Find the area of the region bounded by the curve y = x² − 9 and the x-axis.

2 Calculate the area enclosed by the curve y = 3x − x², the x-axis and the lines x = −1, x = 2.

3 Find where the curve y = x² − x − 1 meets the line y = 5. Hence find the area of the region bounded by the curve and the line y = 5.

4 Part of the curve y = x(x − 1)(x − 2) is shown in Fig.11.22. Find the values of a and b. Hence find the area of the region enclosed by the curve and the x-axis from x = 0 to x = b.
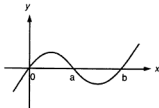

Fig.11.22

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5 Find the area of the region enclosed by the following curves or lines:

(a) $y = 2x$, $y = x^2$
(b) $y = x^2$, $y = 4$
(c) $y = x^2 - 2$, $y = \frac{1}{2}x^2$
(d) $y = x^2$, $y = x^3$
(e) $y = 2x^2$, $y = x + 1$
(f) $y = x(2 - x)$, $y = x$
(g) $y = 2 - x^2$, $y = -2$
(h) $y = x^2 + 3$, $y = 5 - x$
(i) $y = x^2 - 1$, $y = x + 1$

6 In Fig.11.23, the curve $(y - 1)^2 = x + 4$ meets the y-axis at A and B.
(a) Find the coordinates of A and B and (b) calculate the areas of
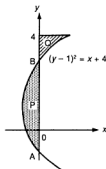(i) the shaded region P, (ii) the shaded region Q.

Fig.11.23

7 For a curve, $\dfrac{dy}{dx} = 2x + k$ where k is a constant, and the curve has a turning point where $x = 2$.
(a) If it passes through the point $(-1,8)$, find its equation.
(b) The line $y = x + 3$ meets the curve at points A and B. Find the coordinates of A and B.
(c) Hence find the area of the region enclosed by the curve and the line.
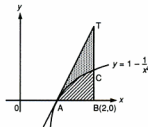
8 (a) Sketch the curve $y = x(3 - x)$.
(b) Find the equation of the normal to the curve at the origin and the x-coordinate of the point where this normal meets the curve again.
(c) Find the area of the region bounded by the curve and the normal.

9 The normal at the point A($x = 0$) on the curve $y = 2 - x - x^2$ meets the curve again at B. Find (a) the coordinates of B, and (b) the area of the region bounded by the curve and the normal.

**10** Fig.11.24 shows part of the curve $y = 1 - \frac{1}{x^2}$.

Find (a) the coordinates of the point A where the curve meets the *x*-axis and (b) the equation of the tangent to the curve at A. (c) The line through B(2,0) parallel to the *y*-axis meets the curve at C and the tangent at T. Find the ratio of the areas of the shaded regions ABC and ACT.

Fig.11.24

11 Fig.11.25 shows part of the curve $y = x^2$ and the line $y = 4$. The line AB is drawn through A(0,2) with gradient −1 to meet the curve at B. Find (a) the coordinates of B, and (b) the ratio of the shaded areas P and Q.

Fig.11.25
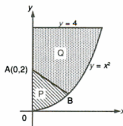
12 (a) Sketch the curve $y = x(4 - x)$.

(b) Find the equations of the tangents to the curve at the origin O and at the point where $x = 3$.

(c) If these tangents meet at T, find the *x*-coordinate of T and the area of the region enclosed by the tangents and the curve.

266

13 Fig.11.26 shows part of the curve $y = x^2 - x + 2$ and a line UV.
   (a) Find the coordinates of U and V and the equation of UV.
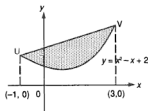   (b) Hence find the area of the shaded region.

Fig.11.26

14 Fig.11.27 shows part of the curve $y = 5 - x - x^2$ and the line $y = 2x + 1$ which meet at A and B. Find
   (a) the x-coordinate of A and of B, and
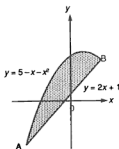   (b) the area of the shaded region.

Fig.11.27

15 Fig.11.28 shows part of the curves $y = \frac{2}{x^2}$ and $y = x^2 - 4x$.
   A is the point $(1,2)$ and BC is part of the line $x = 3$. Find the area of the shaded region.
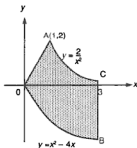

Fig.11.28