Exponential and Logarithmic Functions

10

Chapter 10 is devoted to the study exponential and logarithmic functions. These functions are used to study many naturally occurring phenomena such as population growth, exponential decay of radioactive matter, and growth of investments.

The following is a Sudoku puzzle. As you work through this chapter, try to simplify the expressions or solve the equations in the clues given below. Use the clues to fill in the boxes labeled a–n. Then fill in the remaining part of the grid so that every row, every column, and every 2×3 box contains the digits 1 through 6.

Clues

a. $\log_2 8$

b. $\ln e$

c. Solution to $3^{2x-10} = 9$

d. Solution to $\ln(x + 3) = \ln 8$

e. $1 + \log 1$

f. $\log_2 16x - \log_2 x$

g. Solution to $\log_{12}(x - 4) = 1 - \log_{12} x$

h. $\dfrac{\ln 2^2}{\ln 2}$

i. $f(f^{-1}(6))$

j. $\ln e^4$

k. $\log_7 7^5$

l. e^0

m. $\left(\dfrac{1}{2}\right)^{-x} = 16$

n. $\log 100$

		a			b
			c	d	
e	f			g	h
	i			j	k
	l	m			
n					

Section 10.1 Algebra and Composition of Functions

Concepts

1. Algebra of Functions
2. Composition of Functions
3. Multiple Operations on Functions

1. Algebra of Functions

Addition, subtraction, multiplication, and division can be used to create a new function from two or more functions. The domain of the new function will be the intersection of the domains of the original functions. Finding domains of functions was first introduced in Section 4.2.

Sum, Difference, Product, and Quotient of Functions

Given two functions f and g, the functions $f + g$, $f - g$, $f \cdot g$, and $\frac{f}{g}$ are defined as

$$(f + g)(x) = f(x) + g(x)$$
$$(f - g)(x) = f(x) - g(x)$$
$$(f \cdot g)(x) = f(x) \cdot g(x)$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad \text{provided } g(x) \neq 0$$

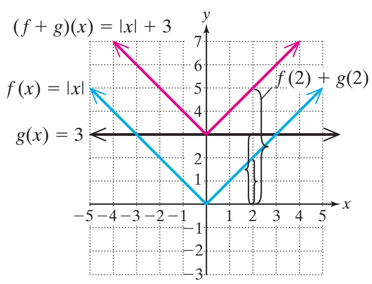


Figure 10-1

For example, suppose $f(x) = |x|$ and $g(x) = 3$. Taking the sum of the functions produces a new function denoted by $(f + g)$. In this case, $(f + g)(x) = |x| + 3$. Graphically, the y-values of the function $(f + g)$ are given by the sum of the corresponding y-values of f and g. This is depicted in Figure 10-1. The function $(f + g)$ appears in red. In particular, notice that $(f + g)(2) = f(2) + g(2) = 2 + 3 = 5$.

Example 1 Adding, Subtracting, and Multiplying Functions

Given: $g(x) = 4x \qquad h(x) = x^2 - 3x \qquad k(x) = \sqrt{x - 2}$

a. Find $(g + h)(x)$ and write the domain of $(g + h)$ in interval notation.

b. Find $(h - g)(x)$ and write the domain of $(h - g)$ in interval notation.

c. Find $(g \cdot k)(x)$ and write the domain of $(g \cdot k)$ in interval notation.

Solution:

a. $(g + h)(x) = g(x) + h(x)$
$= (4x) + (x^2 - 3x)$
$= 4x + x^2 - 3x$
$= x^2 + x$ — The domain is all real numbers $(-\infty, \infty)$.

b. $(h - g)(x) = h(x) - g(x)$
$= (x^2 - 3x) - (4x)$
$= x^2 - 3x - 4x$
$= x^2 - 7x$ — The domain is all real numbers $(-\infty, \infty)$.

c. $(g \cdot k)(x) = g(x) \cdot k(x)$
$= (4x)(\sqrt{x - 2})$
$= 4x\sqrt{x - 2}$ — The domain is $[2, \infty)$ because $x - 2 \geq 0$ for $x \geq 2$.

Skill Practice Given:

$$f(x) = x - 1$$
$$g(x) = 5x^2 + x$$
$$h(x) = \sqrt{5 - x}$$

Perform the indicated operations. Write the domain of the resulting function in interval notation.

1. $(f + g)(x)$ **2.** $(g - f)(x)$ **3.** $(f \cdot h)(x)$

Example 2 Dividing Functions

Given the functions defined by $h(x) = x^2 - 3x$ and $k(x) = \sqrt{x - 2}$, find $\left(\frac{k}{h}\right)(x)$ and write the domain of $\left(\frac{k}{h}\right)$ in interval notation.

Solution:

$$\left(\frac{k}{h}\right)(x) = \frac{\sqrt{x - 2}}{x^2 - 3x}$$

To find the domain, we must consider the restrictions on x imposed by the square root and by the fraction.

- From the numerator we have $x - 2 \geq 0$ or, equivalently, $x \geq 2$.
- From the denominator we have $x^2 - 3x \neq 0$ or, equivalently, $x(x - 3) \neq 0$. Hence, $x \neq 3$ and $x \neq 0$.

Thus, the domain of $\frac{k}{h}$ is the set of real numbers greater than or equal to 2, but not equal to 3 or 0. This is shown graphically in Figure 10-2.

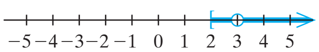


Figure 10-2

The domain is $[2, 3) \cup (3, \infty)$.

Skill Practice Given:

$$f(x) = \sqrt{x + 1}$$
$$g(x) = x^2 + 2x$$

4. Find $\left(\frac{f}{g}\right)(x)$ and write the domain interval notation.

2. Composition of Functions

Composition of Functions

The **composition** of f and g, denoted $f \circ g$, is defined by the rule

$$(f \circ g)(x) = f(g(x)) \qquad \text{provided that } g(x) \text{ is in the domain of } f$$

The composition of g and f, denoted $g \circ f$, is defined by the rule

$$(g \circ f)(x) = g(f(x)) \qquad \text{provided that } f(x) \text{ is in the domain of } g$$

Note: $f \circ g$ is also read as "f compose g," and $g \circ f$ is also read as "g compose f."

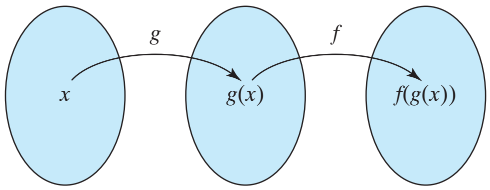
Skill Practice Answers

1. $5x^2 + 2x - 1$; domain: $(-\infty, \infty)$
2. $5x^2 + 1$; domain: $(-\infty, \infty)$
3. $(x - 1)\sqrt{5 - x}$; domain: $(-\infty, 5]$
4. $\frac{\sqrt{x + 1}}{x^2 + 2x}$; domain: $[-1, 0) \cup (0, \infty)$

For example, given $f(x) = 2x - 3$ and $g(x) = x + 5$, we have

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(x + 5) \quad \text{Substitute } g(x) = x + 5 \text{ into the function } f. \\ &= 2(x + 5) - 3 \\ &= 2x + 10 - 3 \\ &= 2x + 7\end{aligned}$$

In this composition, the function g is the innermost operation and acts on x first. Then the output value of function g becomes the domain element of the function f, as shown in the figure.



Example 3 Composing Functions

Given: $f(x) = x - 5$, $g(x) = x^2$, and $n(x) = \sqrt{x + 2}$

a. Find $(f \circ g)(x)$ and write the domain of $(f \circ g)$ in interval notation.

b. Find $(g \circ f)(x)$ and write the domain of $(g \circ f)$ in interval notation.

c. Find $(n \circ f)(x)$ and write the domain of $(n \circ f)$ in interval notation.

Solution:

a. $(f \circ g)(x) = f(g(x))$

$= f(x^2)$ — Evaluate the function f at x^2.

$= (x^2) - 5$ — Replace x by x^2 in the function f.

$= x^2 - 5$ — The domain is all real numbers $(-\infty, \infty)$.

b. $(g \circ f)(x) = g(f(x))$

$= g(x - 5)$ — Evaluate the function g at $(x - 5)$.

$= (x - 5)^2$ — Replace x by $(x - 5)$ in function g.

$= x^2 - 10x + 25$ — The domain is all real numbers $(-\infty, \infty)$.

c. $(n \circ f)(x) = n(f(x))$

$= n(x - 5)$ — Evaluate the function n at $x - 5$.

$= \sqrt{(x - 5) + 2}$ — Replace x by the quantity $(x - 5)$ in function n.

$= \sqrt{x - 3}$ — The domain is $[3, \infty)$.

TIP: Examples 3(a) and 3(b) illustrate that the order in which two functions are composed may result in different functions. That is, $f \circ g$ does not necessarily equal $g \circ f$.

Skill Practice Given $f(x) = 2x^2$, $g(x) = x + 3$, and $h(x) = \sqrt{x - 1}$,

5. Find $(f \circ g)(x)$. Write the domain of $(f \circ g)$ in interval notation.
6. Find $(g \circ f)(x)$. Write the domain of $(g \circ f)$ in interval notation.
7. Find $(h \circ g)(x)$. Write the domain of $(h \circ g)$ in interval notation.

Skill Practice Answers

5. $2x^2 + 12x + 18$; domain: $(-\infty, \infty)$
6. $2x^2 + 3$; domain: $(-\infty, \infty)$
7. $\sqrt{x + 2}$; domain: $[-2, \infty)$

3. Multiple Operations on Functions

Example 4 Combining Functions

Given the functions defined by $f(x) = x - 7$ and $h(x) = 2x^3$, find the function values, if possible.

a. $(f \cdot h)(3)$ **b.** $\left(\frac{h}{f}\right)(7)$ **c.** $(h \circ f)(2)$

Solution:

a. $(f \cdot h)(3) = f(3) \cdot h(3)$ $(f \cdot h)(3)$ is a product (not a composition).

$$= (3 - 7) \cdot 2(3)^3$$
$$= (-4) \cdot 2(27)$$
$$= -216$$

b. The function $\frac{h}{f}$ has restrictions on its domain.

$$\left(\frac{h}{f}\right)(x) = \frac{h(x)}{f(x)} = \frac{2x^3}{x - 7}$$

Therefore, $x = 7$ is not in the domain, and $\left(\frac{h}{f}\right)(7) = \frac{h(7)}{f(7)}$ is undefined.

Avoiding Mistakes:

If you had tried evaluating the function $\frac{h}{f}$ at $x = 7$, the denominator would be zero and the function undefined.

$$\frac{h(7)}{f(7)} = \frac{2(7)^3}{7 - 7}$$

c. $(h \circ f)(2) = h(f(2))$ Evaluate $f(2)$ first. $f(2) = 2 - 7 = -5$.

$$= h(-5)$$ Substitute the result into function h.
$$= 2(-5)^3$$
$$= 2(-125)$$
$$= -250$$

Skill Practice Given:

$$h(x) = x + 4$$
$$k(x) = x^2 - 3$$

8. Find $(h \cdot k)(-2)$. **9.** Find $\left(\frac{h}{k}\right)(4)$. **10.** Find $(k \circ h)(1)$.

Skill Practice Answers

8. 2 **9.** $\frac{8}{13}$ **10.** 22

Example 5 Finding Function Values from a Graph

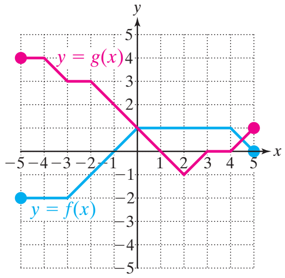
For the functions f and g pictured, find the function values if possible.

a. $g(2)$

b. $(f - g)(-3)$

c. $\left(\frac{g}{f}\right)(5)$

d. $(f \circ g)(4)$



Solution:

a. $g(2) = -1$ — The value $g(2)$ represents the y-value of $y = g(x)$ (the red graph) when $x = 2$. Because the point $(2, -1)$ lies on the graph, $g(2) = -1$.

b. $(f - g)(-3) = f(-3) - g(-3)$ — Evaluate the difference of $f(-3)$ and $g(-3)$.

$= -2 - (3)$ — Estimate function values from the graph.

$= -5$

c. $\left(\frac{g}{f}\right)(5) = \frac{g(5)}{f(5)}$ — Evaluate the quotient of $g(5)$ and $f(5)$.

$= \frac{1}{0}$ (undefined)

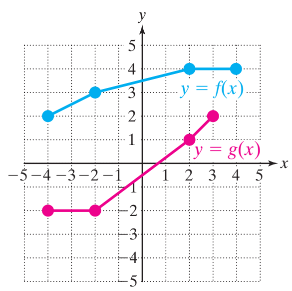
The function $\frac{g}{f}$ is undefined at 5 because the denominator is zero.

d. $(f \circ g)(4) = f(g(4))$ — From the red graph, find the value of $g(4)$ first.

$= f(0)$ — From the blue graph, find the value of f at $x = 0$.

$= 1$

Skill Practice Find the values from the graph.



11. $g(3)$ **12.** $(f + g)(-4)$ **13.** $\left(\frac{f}{g}\right)(2)$ **14.** $(g \circ f)(-2)$

Skill Practice Answers

11. 2 **12.** 0 **13.** 4 **14.** 2

Section 10.1 Practice Exercises

Study Skills Exercise

1. Define the key term **composition of functions**.

Concept 1: Algebra of Functions

For Exercises 2–13, refer to the functions defined below.

$f(x) = x + 4 \qquad g(x) = 2x^2 + 4x$

$h(x) = \sqrt{x - 1} \qquad k(x) = \dfrac{1}{x}$

Find the indicated functions. Write the domain in interval notation.

2. $(f + g)(x)$

3. $(f - g)(x)$

4. $(g - f)(x)$

5. $(f + h)(x)$

6. $(f \cdot h)(x)$

7. $(h \cdot k)(x)$

8. $(g \cdot f)(x)$

9. $(f \cdot k)(x)$

10. $\left(\dfrac{h}{f}\right)(x)$

11. $\left(\dfrac{g}{f}\right)(x)$

12. $\left(\dfrac{f}{g}\right)(x)$

13. $\left(\dfrac{f}{h}\right)(x)$

Concept 2: Composition of Functions

For Exercises 14–22, find the indicated functions and their domains. Use f, g, h, and k as defined in Exercises 2–13.

14. $(f \circ g)(x)$

15. $(f \circ k)(x)$

16. $(g \circ f)(x)$

17. $(k \circ f)(x)$

18. $(k \circ h)(x)$

19. $(h \circ k)(x)$

20. $(k \circ g)(x)$

21. $(g \circ k)(x)$

22. $(f \circ h)(x)$

23. Based on your answers to Exercises 15 and 17 is it true in general that $(f \circ k)(x) = (k \circ f)(x)$?

24. Based on your answers to Exercises 14 and 16 is it true in general that $(f \circ g)(x) = (g \circ f)(x)$?

For Exercises 25–30, find $(f \circ g)(x)$ and $(g \circ f)(x)$.

25. $f(x) = x^2 - 3x + 1,\ g(x) = 5x$

26. $f(x) = 3x^2 + 8,\ g(x) = 2x - 4$

27. $f(x) = |x|,\ g(x) = x^3 - 1$

28. $f(x) = \dfrac{1}{x + 2},\ g(x) = |x + 2|$

29. For $h(x) = 5x - 4$, find $(h \circ h)(x)$.

30. For $k(x) = -x^2 + 1$, find $(k \circ k)(x)$.

Concept 3: Multiple Operations on Functions

For Exercises 31–44, refer to the functions defined below.

$$m(x) = x^3 \qquad n(x) = x - 3$$

$$r(x) = \sqrt{x + 4} \qquad p(x) = \frac{1}{x + 2}$$

Find the function values if possible.

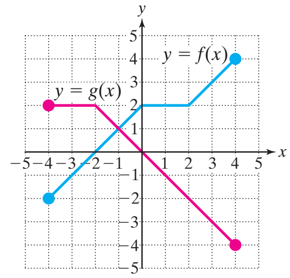
31. $(m \cdot r)(0)$ **32.** $(n \cdot p)(0)$ **33.** $(m + r)(-4)$ **34.** $(n - m)(4)$

35. $(r \circ n)(3)$ **36.** $(n \circ r)(5)$ **37.** $(p \circ m)(-1)$ **38.** $(m \circ n)(5)$

39. $(m \circ p)(2)$ **40.** $(r \circ m)(2)$ **41.** $(r + p)(-3)$ **42.** $(n + p)(-2)$

43. $(m \circ p)(-2)$ **44.** $(r \circ m)(-2)$

For Exercises 45–62, approximate the function values from the graph, if possible.



45. $f(-4)$ **46.** $f(1)$ **47.** $g(-2)$

48. $g(3)$ **49.** $(f + g)(2)$ **50.** $(g - f)(3)$

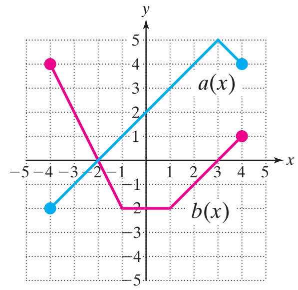
51. $(f \cdot g)(-1)$ **52.** $(g \cdot f)(-4)$ **53.** $\left(\frac{g}{f}\right)(0)$

54. $\left(\frac{f}{g}\right)(-2)$ **55.** $\left(\frac{f}{g}\right)(0)$ **56.** $\left(\frac{g}{f}\right)(-2)$

57. $(g \circ f)(-1)$ **58.** $(f \circ g)(0)$ **59.** $(f \circ g)(-4)$

60. $(g \circ f)(-4)$ **61.** $(g \circ g)(2)$ **62.** $(f \circ f)(-2)$

For Exercises 63–78, approximate the function values from the graph, if possible.



63. $a(-3)$ **64.** $a(1)$ **65.** $b(-1)$

66. $b(3)$ **67.** $(a - b)(-1)$ **68.** $(a + b)(0)$

69. $(b \cdot a)(1)$ **70.** $(a \cdot b)(2)$ **71.** $(b \circ a)(0)$

72. $(a \circ b)(-2)$ **73.** $(a \circ b)(-4)$ **74.** $(b \circ a)(-3)$

75. $\left(\frac{b}{a}\right)(3)$ **76.** $\left(\frac{a}{b}\right)(4)$ **77.** $(a \circ a)(-2)$ **78.** $(b \circ b)(1)$

79. The cost in dollars of producing x toy cars is $C(x) = 2.2x + 1$. The revenue received is $R(x) = 5.98x$. To calculate profit, subtract the cost from the revenue.

a. Write and simplify a function P that represents profit in terms of x.

b. Find the profit of producing 50 toy cars.

80. The cost in dollars of producing x lawn chairs is $C(x) = 2.5x + 10.1$. The revenue for selling x chairs is $R(x) = 6.99x$. To calculate profit, subtract the cost from the revenue.

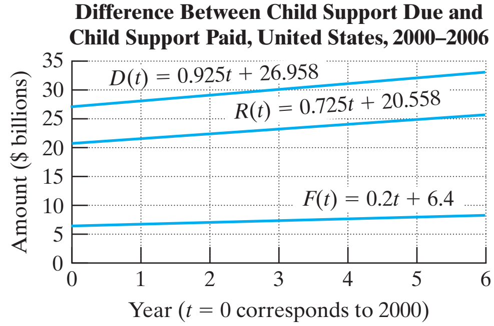
a. Write and simplify a function P that represents profit in terms of x.

b. Find the profit in producing 100 lawn chairs.

81. The functions defined by $D(t) = 0.925t + 26.958$ and $R(t) = 0.725t + 20.558$ approximate the amount of child support (in billions of dollars) that was due and the amount of child support actually received in the United States between the years 2000 and 2006. In each case, $t = 0$ corresponds to the year 2000.

a. Find the function F defined by $F(t) = D(t) - R(t)$. What does F represent in the context of this problem?

b. Find $F(0)$, $F(2)$, and $F(4)$. What do these function values represent in the context of this problem?



(Source: U.S. Bureau of the Census)

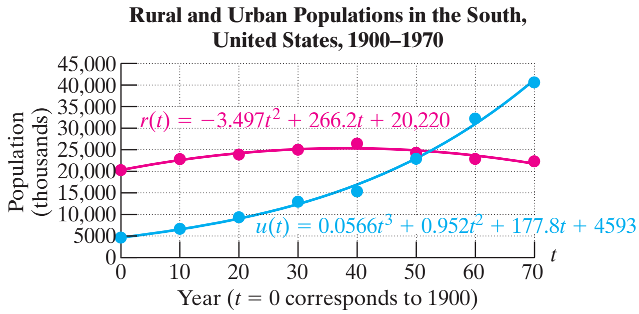
82. If t represents the number of years after 1900, then the rural and urban populations in the South (United States) between the years 1900 and 1970 can be approximated by

$$r(t) = -3.497t^2 + 266.2t + 20{,}220$$

where $t = 0$ corresponds to the year 1900 and $r(t)$ represents the rural population in thousands.

$$u(t) = 0.0566t^3 + 0.952t^2 + 177.8t + 4593$$

where $t = 0$ corresponds to the year 1900 and $u(t)$ represents the urban population in thousands.



(Source: Historical Abstract of the United States)

a. Find the function T defined by $T(t) = r(t) + u(t)$. What does the function T represent in the context of this problem?

b. Use the function T to approximate the total population in the South for the year 1940.

83. Joe rides a bicycle and his wheels revolve at 80 revolutions per minute (rpm). Therefore, the total number of revolutions, r, is given by $r(t) = 80t$, where t is the time in minutes. For each revolution of the wheels of the bike, he travels approximately 7 ft. Therefore, the total distance he travels, D, depends on the total number of revolutions, r, according to the function $D(r) = 7r$.

a. Find $(D \circ r)(t)$ and interpret its meaning in the context of this problem.

b. Find Joe's total distance in feet after 10 min.

84. The area, A, of a square is given by the function $a(x) = x^2$, where x is the length of the sides of the square. If carpeting costs \$9.95 per square yard, then the cost, C, to carpet a square room is given by $C(a) = 9.95a$, where a is the area of the room in square yards.

a. Find $(C \circ a)(x)$ and interpret its meaning in the context of this problem.

b. Find the cost to carpet a square room if its floor dimensions are 15 yd by 15 yd.

Section 10.2 Inverse Functions

Concepts

1. Introduction to Inverse Functions
2. Definition of a One-to-One Function
3. Finding an Equation of the Inverse of a Function
4. Definition of the Inverse of a Function

1. Introduction to Inverse Functions

In Section 4.2, we defined a function as a set of ordered pairs (x, y) such that for every element x in the domain, there corresponds exactly one element y in the range. For example, the function f relates the price, x (in dollars), of a USB Flash drive to the amount of memory that it holds, y (in megabytes).

price, x ($) → ; megabytes, y →

$$f = \{(52, 512), (29, 256), (25, 128)\}$$

That is, the amount of memory depends on how much money a person has to spend. Now suppose we create a new function in which the values of x and y are interchanged. The new function is called the inverse of f and is denoted by f^{-1}. This relates the amount of memory, x, to the cost, y.

megabytes, x → ; price, y ($) →

$$f^{-1} = \{(521, 52), (256, 29), (128, 25)\}$$

Avoiding Mistakes:

f^{-1} denotes the inverse of a function. The -1 does not represent an exponent.

Notice that interchanging the x- and y-values has the following outcome. The domain of f is the same as the range of f^{-1}, and the range of f is the domain of f^{-1}.

2. Definition of a One-to-One Function

A necessary condition for a function f to have an inverse function is that no two ordered pairs in f have different x-coordinates and the same y-coordinate. A function that satisfies this condition is called a **one-to-one function**. The function relating the price of a USB Flash drive to its memory is a one-to-one function. However, consider the function g defined by

$$g = \{(1, 4), (2, 3), (-2, 4)\}$$

same y

different x

This function is not one-to-one because the range element 4 has two different x-coordinates, 1 and -2. Interchanging the x- and y-values produces a relation that violates the definition of a function.

$$\{(4, 1), (3, 2), (4, -2)\}$$

same x

different y

This relation is not a function because for $x = 4$ there are two different y-values, $y = 1$ and $y = -2$.

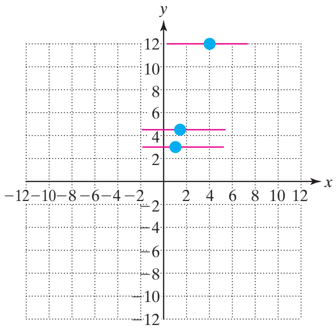
In Section 4.2, you learned the vertical line test to determine visually if a graph represents a function. We use a **horizontal line test** to determine whether a function is one-to-one.

Horizontal Line Test

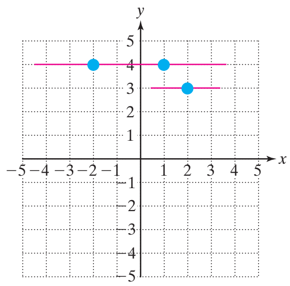
Consider a function defined by a set of points (x, y) in a rectangular coordinate system. The graph of the ordered pairs defines y as a *one-to-one* function of x if no horizontal line intersects the graph in more than one point.

To understand the horizontal line test, consider the functions f and g.

$f = \{(1, 2.99), (1.5, 4.49), (4, 11.96)\}$ $\quad$ $g = \{(1, 4), (2, 3), (-2, 4)\}$



This function is one-to-one.
No horizontal line intersects more than once.

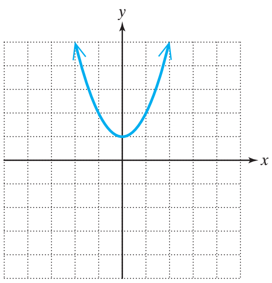


This function is *not* one-to-one.
A horizontal line intersects more than once.

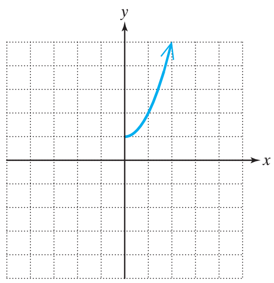
Example 1 Identifying One-to-One Functions

Determine whether the function is one-to-one.

a.

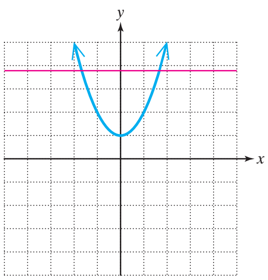


b.

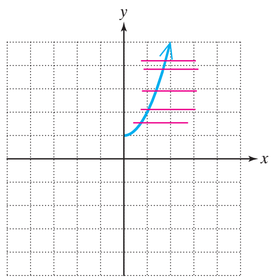


Solution:

a. Function is not one-to-one. A horizontal line intersects in more than one point.

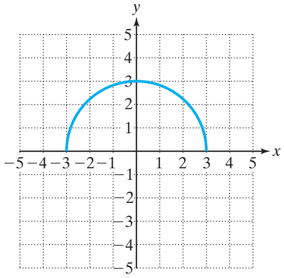


b. Function is one-to-one. No horizontal line intersects more than once.



Skill Practice Use the horizontal line test to determine if the functions are one-to-one.

1.



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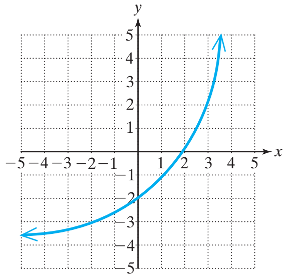


3. Finding an Equation of the Inverse of a Function

Another way to view the construction of the inverse of a function is to find a function that performs the inverse operations in the reverse order. For example, the function defined by $f(x) = 2x + 1$ multiplies x by 2 and then adds 1. Therefore, the inverse function must *subtract* 1 from x and *divide* by 2. We have

$$f^{-1}(x) = \frac{x - 1}{2}$$

To facilitate the process of finding an equation of the inverse of a one-to-one function, we offer the following steps.

Finding an Equation of an Inverse of a Function

For a one-to-one function defined by $y = f(x)$, the equation of the inverse can be found as follows:

1. Replace $f(x)$ by y.
2. Interchange x and y.
3. Solve for y.
4. Replace y by $f^{-1}(x)$.

Example 2 Finding an Equation of the Inverse of a Function

Find the inverse. $f(x) = 2x + 1$

Solution:

We know the graph of f is a nonvertical line. Therefore, $f(x) = 2x + 1$ defines a one-to-one function. To find the inverse we have

$y = 2x + 1$ **Step 1:** Replace $f(x)$ by y.

$x = 2y + 1$ **Step 2:** Interchange x and y.

Skill Practice Answers

1. Not one-to-one
2. One-to-one

$x - 1 = 2y$ **Step 3:** Solve for y. Subtract 1 from both sides.

$\dfrac{x-1}{2} = y$ Divide both sides by 2.

$f^{-1}(x) = \dfrac{x-1}{2}$ **Step 4:** Replace y by $f^{-1}(x)$.

Skill Practice

3. Find the inverse of $f(x) = 4x + 6$.

The key step in determining the equation of the inverse of a function is to interchange x and y. By so doing, a point (a, b) on f corresponds to a point (b, a) on f^{-1}. For this reason, the graphs of f and f^{-1} are symmetric with respect to the line $y = x$ (Figure 10-3). Notice that the point $(-3, -5)$ of the function f corresponds to the point $(-5, -3)$ of f^{-1}. Likewise, $(1, 3)$ of f corresponds to $(3, 1)$ of f^{-1}.

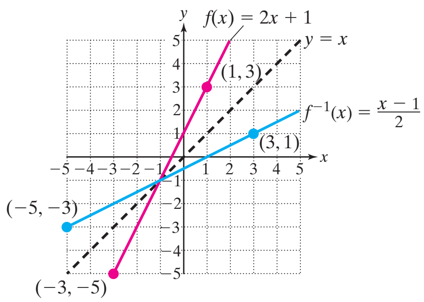


Figure 10-3

Example 3 Finding an Equation of the Inverse of a Function

Find the inverse of the one-to-one function. $g(x) = \sqrt[3]{5x} - 4$

Solution:

$y = \sqrt[3]{5x} - 4$ **Step 1:** Replace $g(x)$ by y.

$x = \sqrt[3]{5y} - 4$ **Step 2:** Interchange x and y.

$x + 4 = \sqrt[3]{5y}$ **Step 3:** Solve for y. Add 4 to both sides.

$(x+4)^3 = (\sqrt[3]{5y})^3$ To eliminate the cube root, cube both sides.

$(x+4)^3 = 5y$ Simplify the right side.

$\dfrac{(x+4)^3}{5} = y$ Divide both sides by 5.

$g^{-1}(x) = \dfrac{(x+4)^3}{5}$ **Step 4:** Replace y by $g^{-1}(x)$.

Skill Practice

4. Find the inverse of $h(x) = \sqrt[3]{2x - 1}$.

Skill Practice Answers

3. $f^{-1}(x) = \dfrac{x-6}{4}$

4. $h^{-1}(x) = \dfrac{x^3+1}{2}$

The graphs of g and g^{-1} from Example 3 are shown in Figure 10-4. Once again we see that the graphs of a function and its inverse are symmetric with respect to the line $y = x$.

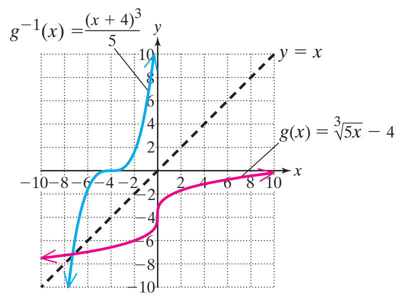


Figure 10-4

For a function that is not one-to-one, sometimes we can restrict its domain to create a new function that is one-to-one. This is demonstrated in Example 4.

Example 4 Finding the Equation of an Inverse of a Function with a Restricted Domain

Given the function defined by $m(x) = x^2 + 4$ for $x \geq 0$, find an equation defining m^{-1}.

Solution:

From Section 8.4, we know that $y = x^2 + 4$ is a parabola with vertex at (0, 4) (Figure 10-5). The graph represents a function that is not one-to-one. However, with the restriction on the domain $x \geq 0$, the graph of $m(x) = x^2 + 4, x \geq 0$, consists of only the "right" branch of the parabola (Figure 10-6). This *is* a one-to-one function.

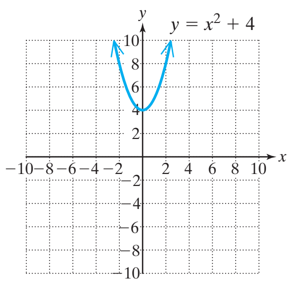


Figure 10-5

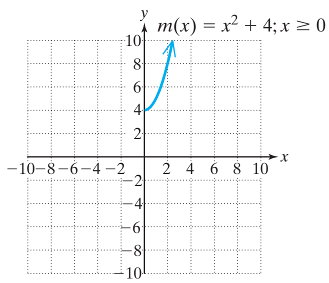


Figure 10-6

To find the inverse, we have

$y = x^2 + 4 \quad x \geq 0$		**Step 1:** Replace $m(x)$ by y.
$x = y^2 + 4 \quad y \geq 0$		**Step 2:** Interchange x and y. Notice that the restriction $x \geq 0$ becomes $y \geq 0$.
$x - 4 = y^2 \quad y \geq 0$		**Step 3:** Solve for y. Subtract 4 from both sides.
$\sqrt{x - 4} = y \quad y \geq 0$		Apply the square root property. Notice that we obtain the *positive* square root of $x - 4$ because of the restriction $y \geq 0$.

$m^{-1}(x) = \sqrt{x - 4}$

Step 4: Replace y by $m^{-1}(x)$. Notice that the domain of m^{-1} has the same values as the range of m.

Figure 10-7 shows the graphs of m and m^{-1}. Compare the domain and range of m and m^{-1}.

Domain of m: $[0, \infty)$
Range of m: $[4, \infty)$

Domain of m^{-1}: $[4, \infty)$
Range of m^{-1}: $[0, \infty)$

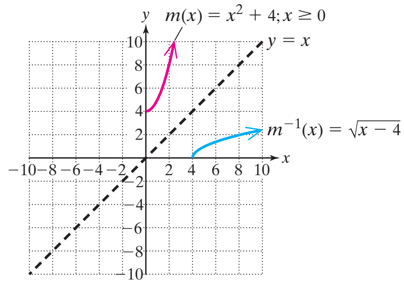


Figure 10-7

Skill Practice

5. Find the inverse. $g(x) = x^2 - 2 \quad x \geq 0$

4. Definition of the Inverse of a Function

Definition of an Inverse Function

If f is a one-to-one function represented by ordered pairs of the form (x, y), then the **inverse function**, denoted f^{-1}, is the set of ordered pairs denoted by ordered pairs of the form (y, x).

An important relationship between a function and its inverse is shown in Figure 10-8.

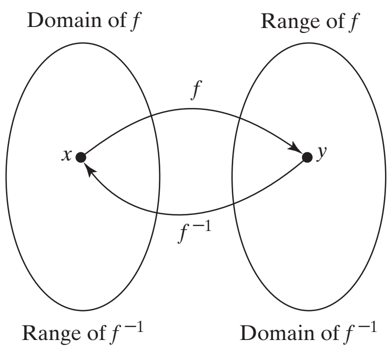


Figure 10-8

Recall that the domain of f is the range of f^{-1} and the range of f is the domain of f^{-1}. The operations performed by f are reversed by f^{-1}. This leads to the inverse function property.

Skill Practice Answers

5. $g^{-1}(x) = \sqrt{x + 2}$

Inverse Function Property

If f is a one-to-one function, then g is the inverse of f if and only if

$$(f \circ g)(x) = x \qquad \text{for all } x \text{ in the domain of } g$$

and

$$(g \circ f)(x) = x \qquad \text{for all } x \text{ in the domain of } f$$

Example 5 Composing a Function with Its Inverse

Show that the functions are inverses.

$$h(x) = 2x + 1 \qquad \text{and} \qquad k(x) = \frac{x-1}{2}$$

Solution:

To show that the functions h and k are inverses, we need to confirm that $(h \circ k)(x) = x$ and $(k \circ h)(x) = x$.

$$\begin{aligned}(h \circ k)(x) = h(k(x)) &= h\left(\frac{x-1}{2}\right)\\ &= 2\left(\frac{x-1}{2}\right) + 1\\ &= x - 1 + 1\\ &= x \checkmark \qquad (h \circ k)(x) = x \text{ as desired.}\end{aligned}$$

$$\begin{aligned}(k \circ h)(x) = k(h(x)) &= k(2x+1)\\ &= \frac{(2x+1) - 1}{2}\\ &= \frac{2x + 1 - 1}{2}\\ &= \frac{2x}{2}\\ &= x \checkmark \qquad (k \circ h)(x) = x \text{ as desired.}\end{aligned}$$

The functions h and k are inverses because $(h \circ k)(x) = x$ and $(k \circ h)(x) = x$ for all real numbers x.

Skill Practice

6. Show that the functions are inverses.

$$f(x) = 3x - 2 \qquad \text{and} \qquad g(x) = \frac{x+2}{3}$$

Skill Practice Answers

6. $(f \circ g)(x) = f(g(x))$
$= 3\left(\frac{x+2}{3}\right) - 2 = x$
$(g \circ f)(x) = g(f(x))$
$= \frac{3x - 2 + 2}{3} = x$

Section 10.2 Practice Exercises

Study Skills Exercise

1. Define the key terms.

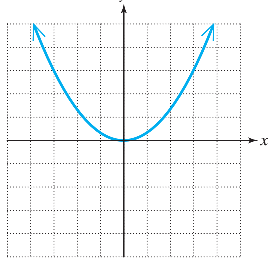
a. Inverse function **b. One-to-one function** **c. Horizontal line test**

Review Exercises

2. Write the domain and range of the relation $\{(3, 4), (5, -2), (6, 1), (3, 0)\}$.

For Exercises 3–8, determine if the relation is a function by using the vertical line test. (See Section 4.2.)

3.



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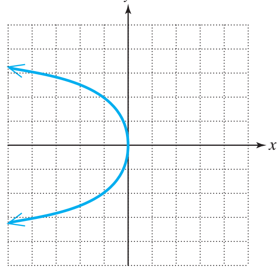

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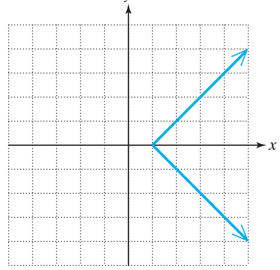

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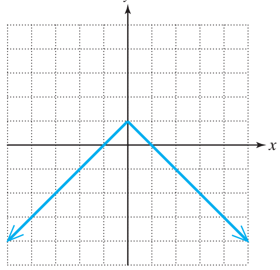

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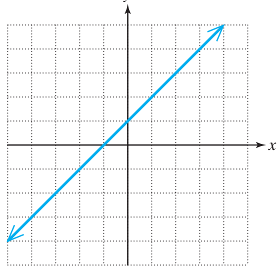

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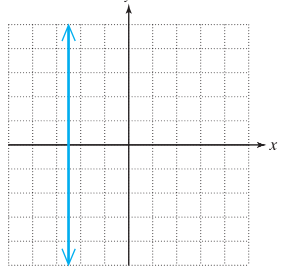


Concept 1: Introduction to Inverse Functions

For Exercises 9–12, write the inverse function for each function.

9. $g = \{(3, 5), (8, 1), (-3, 9), (0, 2)\}$

10. $f = \{(-6, 2), (-9, 0), (-2, -1), (3, 4)\}$

11. $r = \{(a, 3), (b, 6), (c, 9)\}$

12. $s = \{(-1, x), (-2, y), (-3, z)\}$

Concept 2: Definition of a One-to-One Function

13. The table relates a state, x, to the number of representatives in the House of Representatives, y, in the year 2006. Does this relation define a one-to-one function? If so, write a function defining the inverse as a set of ordered pairs.

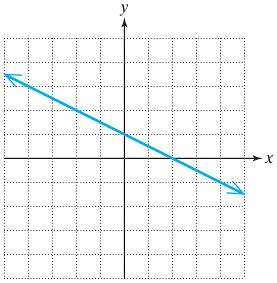
State x	**Number of Representatives** y
Colorado	7
California	53
Texas	32
Louisiana	7
Pennsylvania	19

14. The table relates a city x to its average January temperature y. Does this relation define a one-to-one function? If so, write a function defining the inverse as a set of ordered pairs.

City x	Temperature y (°C)
Gainesville, Florida	13.6
Keene, New Hampshire	−6.0
Wooster, Ohio	−4.0
Rock Springs, Wyoming	−6.0
Lafayette, Louisiana	10.9

For Exercises 15–20, determine if the function is one-to-one by using the horizontal line test.

15.



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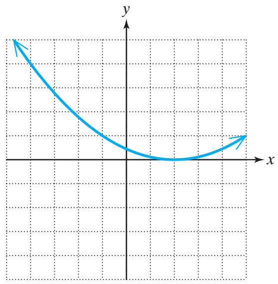

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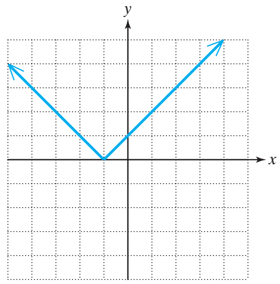

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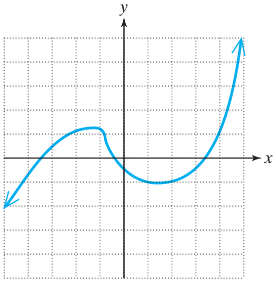

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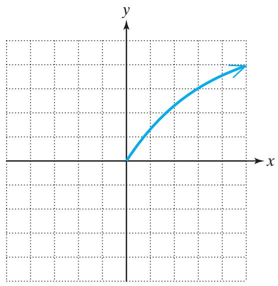

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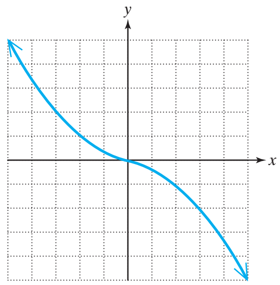


Concept 3: Finding an Equation of the Inverse of a Function

For Exercises 21–30, write an equation of the inverse for each one-to-one function as defined.

21. $h(x) = x + 4$

22. $k(x) = x - 3$

23. $m(x) = \frac{1}{3}x - 2$

24. $n(x) = 4x + 2$

25. $p(x) = -x + 10$

26. $q(x) = -x - \frac{2}{3}$

27. $f(x) = x^3$

28. $g(x) = \sqrt[3]{x}$

29. $g(x) = \sqrt[3]{2x - 1}$

30. $f(x) = x^3 - 4$

31. The function defined by $f(x) = 0.3048x$ converts a length of x feet into $f(x)$ meters.

a. Find the equivalent length in meters for a 4-ft board and a 50-ft wire.

b. Find an equation defining $y = f^{-1}(x)$.

c. Use the inverse function from part (b) to find the equivalent length in feet for a 1500-m race in track and field. Round to the nearest tenth of a foot.

32. The function defined by $s(x) = 1.47x$ converts a speed of x mph to $s(x)$ ft/sec.

a. Find the equivalent speed in feet per second for a car traveling 60 mph.

b. Find an equation defining $y = s^{-1}(x)$.

c. Use the inverse function from part (b) to find the equivalent speed in miles per hour for a train traveling 132 ft/sec. Round to the nearest tenth.

For Exercises 33–39, answer true or false.

33. The function defined by $y = 2$ has an inverse function defined by $x = 2$.

34. The domain of any one-to-one function is the same as the domain of its inverse.

35. All linear functions with a nonzero slope have an inverse function.

36. The function defined by $g(x) = |x|$ is one-to-one.

37. The function defined by $k(x) = x^2$ is one-to-one.

38. The function defined by $h(x) = |x|$ for $x \geq 0$ is one-to-one.

39. The function defined by $L(x) = x^2$ for $x \geq 0$ is one-to-one.

40. Explain how the domain and range of a one-to-one function and its inverse are related.

41. If $(0, b)$ is the y-intercept of a one-to-one function, what is the x-intercept of its inverse?

42. If $(a, 0)$ is the x-intercept of a one-to-one function, what is the y-intercept of its inverse?

43. Can you think of any function that is its own inverse?

44. **a.** Find the domain and range of the function defined by $f(x) = \sqrt{x - 1}$.

b. Find the domain and range of the function defined by $f^{-1}(x) = x^2 + 1, x \geq 0$.

45. **a.** Find the domain and range of the function defined by $g(x) = x^2 - 4, x \leq 0$.

b. Find the domain and range of the function defined by $g^{-1}(x) = -\sqrt{x + 4}$.

For Exercises 46–49, the graph of $y = f(x)$ is given.

a. State the domain of f.

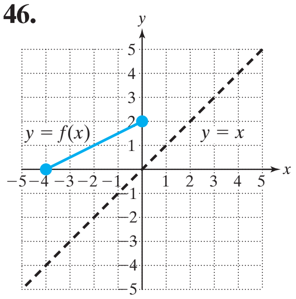
b. State the range of f.

c. State the domain of f^{-1}.

d. State the range of f^{-1}.

e. Graph the function defined by $y = f^{-1}(x)$. The line $y = x$ is provided for your reference.

46.



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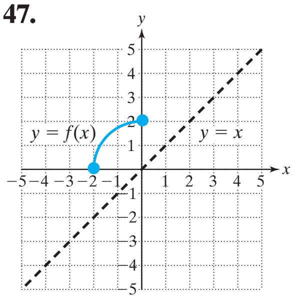

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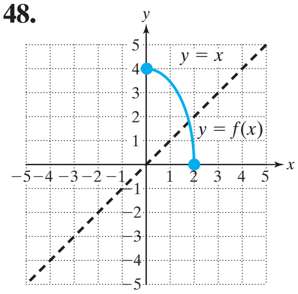

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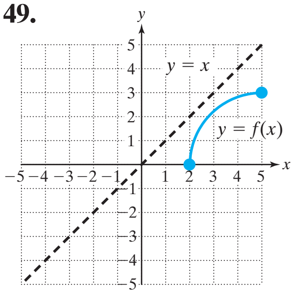


Concept 4: Definition of the Inverse of a Function

For Exercises 50–55, verify that f and g are inverse functions by showing that

a. $(f \circ g)(x) = x$ **b.** $(g \circ f)(x) = x$

50. $f(x) = 6x + 1$ and $g(x) = \dfrac{x - 1}{6}$

51. $f(x) = 5x - 2$ and $g(x) = \dfrac{x + 2}{5}$

52. $f(x) = \dfrac{\sqrt[3]{x}}{2}$ and $g(x) = 8x^3$

53. $f(x) = \dfrac{\sqrt[3]{x}}{3}$ and $g(x) = 27x^3$

54. $f(x) = x^2 + 1,\ x \geq 0$, and $g(x) = \sqrt{x - 1},\ x \geq 1$

55. $f(x) = x^2 - 3,\ x \geq 0$, and $g(x) = \sqrt{x + 3},\ x \geq -3$

Expanding Your Skills

For Exercises 56–67, write an equation of the inverse of the one-to-one function.

56. $f(x) = \dfrac{x - 1}{x + 1}$

57. $p(x) = \dfrac{3 - x}{x + 3}$

58. $t(x) = \dfrac{2}{x - 1}$

59. $w(x) = \dfrac{4}{x + 2}$

60. $g(x) = x^2 + 9 \qquad x \geq 0$

61. $m(x) = x^2 - 1 \qquad x \geq 0$

62. $n(x) = x^2 + 9 \qquad x \leq 0$

63. $g(x) = x^2 - 1 \qquad x \leq 0$

64. $q(x) = \sqrt{x + 4}$

65. $v(x) = \sqrt{x + 16}$

66. $z(x) = -\sqrt{x + 4}$

67. $u(x) = -\sqrt{x + 16}$

Graphing Calculator Exercises

For Exercises 68–71, use a graphing calculator to graph each function on the standard viewing window defined by $-10 \leq x \leq 10$ and $-10 \leq y \leq 10$. Use the graph of the function to determine if the function is one-to-one on the interval $-10 \leq x \leq 10$. If the function is one-to-one, find its inverse and graph both functions on the standard viewing window.

68. $f(x) = \sqrt[3]{x + 5}$

69. $k(x) = x^3 - 4$

70. $g(x) = 0.5x^3 - 2$

71. $m(x) = 3x - 4$

Exponential Functions

Section 10.3

Concepts

1. Definition of an Exponential Function
2. Approximating Exponential Expressions with a Calculator
3. Graphs of Exponential Functions
4. Applications of Exponential Functions

1. Definition of an Exponential Function

The concept of a function was first introduced in Section 4.2. Since then we have learned to recognize several categories of functions, including constant, linear, rational, and quadratic functions. In this section and the next, we will define two new types of functions called exponential and logarithmic functions.

To introduce the concept of an exponential function, consider the following salary plans for a new job. Plan A pays $1 million for a month's work. Plan B starts with 2¢ on Day 1, and every day thereafter the salary is doubled.

At first glance, the million-dollar plan appears to be more favorable. Look, however, at Table 10-1, which shows the daily payments for 30 days under plan B.

Table 10-1

Day	Payment	Day	Payment	Day	Payment
1	2¢	11	$20.48	21	$20,971.52
2	4¢	12	$40.96	22	$41,943.04
3	8¢	13	$81.92	23	$83,886.08
4	16¢	14	$163.84	24	$167,772.16
5	32¢	15	$327.68	25	$335,544.32
6	64¢	16	$655.36	26	$671,088.64
7	$1.28	17	$1310.72	27	$1,342,177.28
8	$2.56	18	$2621.44	28	$2,684,354.56
9	$5.12	19	$5242.88	29	$5,368,709.12
10	$10.24	20	$10,485.76	30	$10,737,418.24

Notice that the salary on the 30th day for plan B is over $10 million. Taking the sum of the payments, we see the total salary for the 30-day period is $21,474,836.46.

The daily salary for plan B can be represented by the function $y = 2^x$, where x is the number of days on the job and y is the salary for that day. An interesting feature of this function is that for every positive 1-unit change in x, the y-value doubles. The function $y = 2^x$ is called an exponential function.

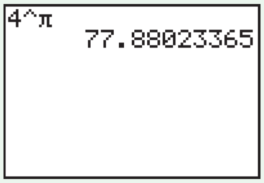
Definition of an Exponential Function

Let b be any real number such that $b > 0$ and $b \neq 1$. Then for any real number x, a function of the form $y = b^x$ is called an **exponential function**.

An exponential function is easily recognized as a function with a constant base and variable exponent. Notice that the base of an exponential function must be a positive real number not equal to 1.

Calculator Connections

On a graphing calculator, use the [^] key to approximate an expression with an irrational exponent.



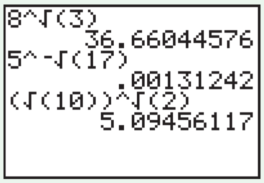
2. Approximating Exponential Expressions with a Calculator

Up to this point, we have evaluated exponential expressions with integer exponents and with rational exponents, for example, $4^3 = 64$ and $4^{1/2} = \sqrt{4} = 2$. However, how do we evaluate an exponential expression with an irrational exponent such as 4^π? In such a case, the exponent is a nonterminating and nonrepeating decimal. The value of 4^π can be thought of as the limiting value of a sequence of approximations using rational exponents:

$$4^{3.14} \approx 77.7084726$$
$$4^{3.141} \approx 77.81627412$$
$$4^{3.1415} \approx 77.87023095$$
$$\vdots$$
$$4^{\pi} \approx 77.88023365$$

An exponential expression can be evaluated at all rational numbers and at all irrational numbers. Hence, the domain of an exponential function is all real numbers.

Calculator Connections



Example 1 Approximating Exponential Expressions with a Calculator

Approximate the expressions. Round the answers to four decimal places.

a. $8^{\sqrt{3}}$ **b.** $5^{-\sqrt{17}}$ **c.** $\sqrt{10}^{\sqrt{2}}$

Solution:

a. $8^{\sqrt{3}} \approx 36.6604$ **b.** $5^{-\sqrt{17}} \approx 0.0013$ **c.** $\sqrt{10}^{\sqrt{2}} \approx 5.0946$

Skill Practice Approximate the value of the expressions. Round the answers to four decimal places.

1. 9^π **2.** $15^{\sqrt{5}}$ **3.** $\sqrt{7}^{\sqrt{3}}$

3. Graphs of Exponential Functions

The functions defined by $f(x) = 2^x$, $g(x) = 3^x$, $h(x) = 5^x$, and $k(x) = (\frac{1}{2})^x$ are all examples of exponential functions. Example 2 illustrates the two general shapes of exponential functions.

Example 2 Graphing an Exponential Function

Graph the functions f and g.

a. $f(x) = 2^x$ **b.** $g(x) = (\frac{1}{2})^x$

Skill Practice Answers

1. 995.0416 **2.** 426.4028
3. 5.3936

Solution:

Table 10-2 shows several function values $f(x)$ and $g(x)$ for both positive and negative values of x. The graph is shown in Figure 10-9.

Table 10-2

x	$f(x) = 2^x$	$g(x) = (\frac{1}{2})^x$
-4	$\frac{1}{16}$	16
-3	$\frac{1}{8}$	8
-2	$\frac{1}{4}$	4
-1	$\frac{1}{2}$	2
0	1	1
1	2	$\frac{1}{2}$
2	4	$\frac{1}{4}$
3	8	$\frac{1}{8}$
4	16	$\frac{1}{16}$

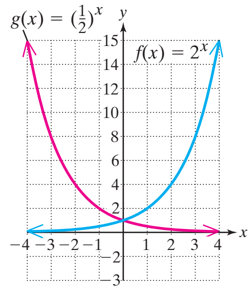


Figure 10-9

Skill Practice Graph the functions.

4. $f(x) = 3^x$ **5.** $f(x) = \left(\frac{1}{3}\right)^x$

The graphs in Figure 10-9 illustrate several important features of exponential functions.

Graphs of $f(x) = b^x$

The graph of an exponential function defined by $f(x) = b^x$ ($b > 0$ and $b \neq 1$) has the following properties.

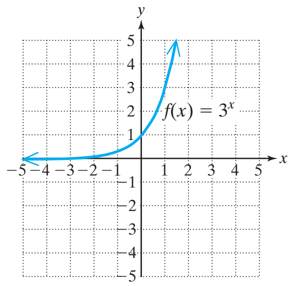
1. If $b > 1$, f is an *increasing* exponential function, sometimes called an **exponential growth function**.

   If $0 < b < 1$, f is a *decreasing* exponential function, sometimes called an **exponential decay function**.
2. The domain is the set of all real numbers, $(-\infty, \infty)$.
3. The range is $(0, \infty)$.
4. The x-axis is a horizontal asymptote.
5. The function passes through the point $(0, 1)$ because $f(0) = b^0 = 1$.

These properties indicate that the graph of an exponential function is an *increasing function* if the base is greater than 1. Furthermore, the base affects its "steepness." Consider the graphs of $f(x) = 2^x$, $h(x) = 3^x$, and $k(x) = 5^x$ (Figure 10-10). For every positive 1-unit change in x, $f(x) = 2^x$ increases by 2 times, $h(x) = 3^x$ increases by 3 times, and $k(x) = 5^x$ increases by 5 times (Table 10-3).

Skill Practice Answers

4.



5.

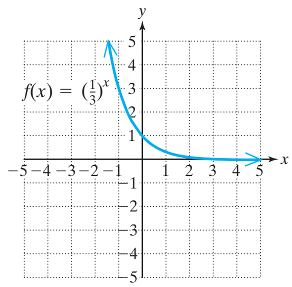


Table 10-3

x	$f(x) = 2^x$	$h(x) = 3^x$	$k(x) = 5^x$
-3	$\frac{1}{8}$	$\frac{1}{27}$	$\frac{1}{125}$
-2	$\frac{1}{4}$	$\frac{1}{9}$	$\frac{1}{25}$
-1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{5}$
0	1	1	1
1	2	3	5
2	4	9	25
3	8	27	125

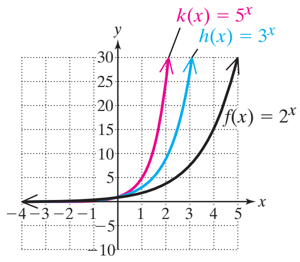


Figure 10-10

The graph of an exponential function is a *decreasing function* if the base is between 0 and 1. Consider the graphs of $g(x) = (\frac{1}{2})^x$, $m(x) = (\frac{1}{3})^x$, and $n(x) = (\frac{1}{5})^x$ (Table 10-4 and Figure 10-11).

Table 10-4

x	$g(x) = (\frac{1}{2})^x$	$m(x) = (\frac{1}{3})^x$	$n(x) = (\frac{1}{5})^x$
-3	8	27	125
-2	4	9	25
-1	2	3	5
0	1	1	1
1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{5}$
2	$\frac{1}{4}$	$\frac{1}{9}$	$\frac{1}{25}$
3	$\frac{1}{8}$	$\frac{1}{27}$	$\frac{1}{125}$

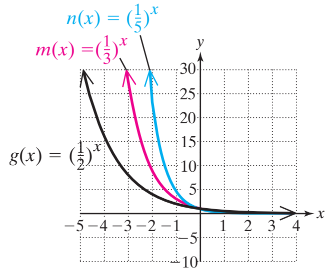


Figure 10-11

4. Applications of Exponential Functions

Exponential growth and decay can be found in a variety of real-world phenomena; for example,

- Population growth can often be modeled by an exponential function.
- The growth of an investment under compound interest increases exponentially.
- The mass of a radioactive substance decreases exponentially with time.
- The temperature of a cup of coffee decreases exponentially as it approaches room temperature.

Marie and Pierre Curie

A substance that undergoes radioactive decay is said to be radioactive. The *half-life* of a radioactive substance is the amount of time it takes for one-half of the original amount of the substance to change into something else. That is, after each half-life the amount of the original substance decreases by one-half.

In 1898, Marie Curie discovered the highly radioactive element radium. She shared the 1903 Nobel Prize in physics for her research on radioactivity and was awarded the 1911 Nobel Prize in chemistry for her discovery of radium and polonium. Radium 226 (an isotope of radium) has a half-life of 1620 years and decays into radon 222 (a radioactive gas).

Example 3 Applying an Exponential Decay Function

In a sample originally having 1 g of radium 226, the amount of radium 226 present after t years is given by

$$A(t) = \left(\tfrac{1}{2}\right)^{t/1620}$$

where A is the amount of radium in grams and t is the time in years.

a. How much radium 226 will be present after 1620 years?

b. How much radium 226 will be present after 3240 years?

c. How much radium 226 will be present after 4860 years?

Solution:

$$A(t) = \left(\frac{1}{2}\right)^{t/1620}$$

a. $A(1620) = \left(\frac{1}{2}\right)^{1620/1620}$ Substitute $t = 1620$.

$= \left(\frac{1}{2}\right)^{1}$

$= 0.5$

After 1620 years (1 half-life), 0.5 g remains.

b. $A(3240) = \left(\frac{1}{2}\right)^{3240/1620}$ Substitute $t = 3240$.

$= \left(\frac{1}{2}\right)^{2}$

$= 0.25$

After 3240 years (2 half-lives), the amount of the original substance is reduced by one-half, 2 times: 0.25 g remains.

c. $A(4860) = \left(\frac{1}{2}\right)^{4860/1620}$ Substitute $t = 4860$.

$= \left(\frac{1}{2}\right)^{3}$

$= 0.125$

After 4860 years (3 half-lives), the amount of the original substance is reduced by one-half, 3 times: 0.125 g remains.

Skill Practice

6. Cesium 137 is a radioactive metal with a short half-life of 30 years. In a sample originally having 1 g of cesium 137, the amount of cesium 137 present after t years is given by

$$A(t) = \left(\frac{1}{2}\right)^{t/30}$$

a. How much cesium 137 will be present after 30 years?
b. How much cesium 137 will be present after 90 years?

Skill Practice Answers

6a. 0.5 g **b.** 0.125 g

Exponential functions are often used to model population growth. Suppose the initial value of a population at some time $t = 0$ is P_0. If the rate of increase of a population is r, then after 1, 2, and 3 years, the new population can be found as follows:

After 1 year:
$$\begin{pmatrix}\text{Total}\\\text{population}\end{pmatrix} = \begin{pmatrix}\text{initial}\\\text{population}\end{pmatrix} + \begin{pmatrix}\text{increase in}\\\text{population}\end{pmatrix}$$
$$= P_0 + P_0 r$$
$$= P_0(1 + r) \qquad \text{Factor out } P_0.$$

After 2 years:
$$\begin{pmatrix}\text{Total}\\\text{population}\end{pmatrix} = \begin{pmatrix}\text{population}\\\text{after 1 year}\end{pmatrix} + \begin{pmatrix}\text{increase in}\\\text{population}\end{pmatrix}$$
$$= P_0(1 + r) + P_0(1 + r)r$$
$$= P_0(1 + r)1 + P_0(1 + r)r$$
$$= P_0(1 + r)(1 + r) \qquad \text{Factor out } P_0(1 + r).$$
$$= P_0(1 + r)^2$$

After 3 years:
$$\begin{pmatrix}\text{Total}\\\text{population}\end{pmatrix} = \begin{pmatrix}\text{population}\\\text{after 2 years}\end{pmatrix} + \begin{pmatrix}\text{increase in}\\\text{population}\end{pmatrix}$$
$$= P_0(1 + r)^2 + P_0(1 + r)^2 r$$
$$= P_0(1 + r)^2 1 + P_0(1 + r)^2 r$$
$$= P_0(1 + r)^2(1 + r) \qquad \text{Factor out } P_0(1 + r)^2.$$
$$= P_0(1 + r)^3$$

This pattern continues, and after t years, the population $P(t)$ is given by

$$P(t) = P_0(1 + r)^t$$

Example 4 Applying an Exponential Growth Function

The population of the Bahamas in 2000 was estimated at 300,000 with an annual rate of increase of 2%.

a. Find a mathematical model that relates the population of the Bahamas as a function of the number of years after 2000.

b. If the annual rate of increase remains the same, use this model to predict the population of the Bahamas in the year 2010. Round to the nearest thousand.

Solution:

a. The initial population is $P_0 = 300{,}000$, and the rate of increase is $r = 2\%$.

$$P(t) = P_0(1 + r)^t \qquad \text{Substitute } P_0 = 300{,}000 \text{ and } r = 0.02$$
$$= 300{,}000(1 + 0.02)^t$$
$$= 300{,}000(1.02)^t \qquad \text{Here } t = 0 \text{ corresponds to the year 2000.}$$

b. Because the initial population ($t = 0$) corresponds to the year 2000, we use $t = 10$ to find the population in the year 2010.

$$P(10) = 300{,}000(1.02)^{10}$$
$$\approx 366{,}000$$

Skill Practice

7. The population of Colorado in 2000 was approximately 4,300,000 with an annual increase of 2.6%.

a. Find a mathematical model that relates the population of Colorado as function of the number of years after 2000.

b. If the annual increase remains the same, use this model to predict the population of Colorado in 2010.

Skill Practice Answers

7a. $P(t) = 4{,}300{,}000(1.026)^t$

b. Approximately 5,558,000

Section 10.3 Practice Exercises

Study Skills Exercise

1. Define the key terms.

a. **Exponential function** **b.** **Exponential growth function** **c.** **Exponential decay function**

Review Exercises

For Exercises 2–7, find the functions, using f and g as given. $f(x) = 2x^2 + x + 2$ $g(x) = 3x - 1$

2. $(f + g)(x)$ **3.** $(g - f)(x)$ **4.** $(f \cdot g)(x)$

5. $\left(\frac{g}{f}\right)(x)$ **6.** $(f \circ g)(x)$ **7.** $(g \circ f)(x)$

For Exercises 8–10, find the inverse function.

8. $\{(2, 3), (0, 0), (-8, 4), (10, 12)\}$

9. $\left\{(-13, 14), \left(\frac{1}{2}, -\frac{1}{2}\right), (6, 30), \left(-\frac{5}{3}, 0\right), (0, 1)\right\}$

10. $\{(a, b), (c, d), (e, f)\}$

For Exercises 11–18, evaluate the expression without the use of a calculator.

11. 5^2 **12.** 2^{-3} **13.** 10^{-3} **14.** 3^4

15. $36^{1/2}$ **16.** $27^{1/3}$ **17.** $16^{3/4}$ **18.** $8^{2/3}$

Concept 2: Approximating Exponential Expressions with a Calculator

For Exercises 19–26, evaluate the expression by using a calculator. Round to 4 decimal places.

19. $5^{1.1}$ **20.** $2^{\sqrt{3}}$ **21.** 10^{π} **22.** $3^{4.8}$

23. $36^{-\sqrt{2}}$ **24.** $27^{-0.5126}$ **25.** $16^{-0.04}$ **26.** $8^{-0.61}$

27. Solve for x.

a. $3^x = 9$ **b.** $3^x = 27$ **c.** Between what two consecutive integers must the solution to $3^x = 11$ lie?

28. Solve for x.

a. $5^x = 125$ **b.** $5^x = 625$ **c.** Between what two consecutive integers must the solution to $5^x = 130$ lie?

29. Solve for x.

a. $2^x = 16$ **b.** $2^x = 32$ **c.** Between what two consecutive integers must the solution to $2^x = 30$ lie?

30. Solve for x.

a. $4^x = 16$ **b.** $4^x = 64$ **c.** Between what two consecutive integers must the solution to $4^x = 20$ lie?

31. For $f(x) = \left(\frac{1}{5}\right)^x$ find $f(0), f(1), f(2), f(-1)$, and $f(-2)$.

32. For $g(x) = \left(\frac{2}{3}\right)^x$ find $g(0), g(1), g(2), g(-1)$, and $g(-2)$.

33. For $h(x) = \pi^x$ use a calculator to find $h(0), h(1), h(-1), h(\sqrt{2})$, and $h(\pi)$. Round to two decimal places.

34. For $k(x) = (\sqrt{5})^x$ use a calculator to find $k(0), k(1), k(-1), k(\pi)$, and $k(\sqrt{2})$. Round to two decimal places.

35. For $r(x) = 3^{x+2}$ find $r(0), r(1), r(2), r(-1), r(-2)$, and $r(-3)$.

36. For $s(x) = 2^{2x-1}$ find $s(0), s(1), s(2), s(-1)$, and $s(-2)$.

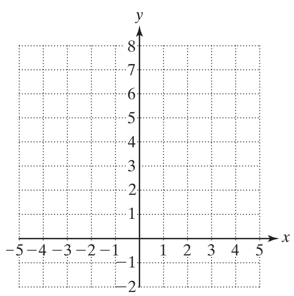
Concept 3: Graphs of Exponential Functions

37. How do you determine whether the graph of $f(x) = b^x$ $(b > 0, b \neq 1)$ is increasing or decreasing?

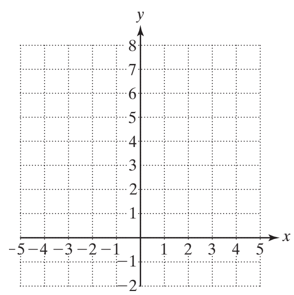
38. For $f(x) = b^x$ $(b > 0, b \neq 1)$, find $f(0)$.

Graph the functions defined in Exercises 39–46. Plot at least three points for each function.

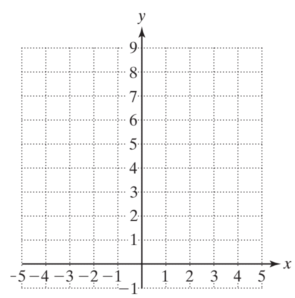
39. $f(x) = 4^x$



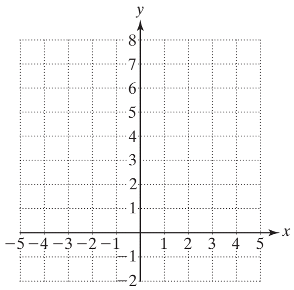
40. $g(x) = 6^x$



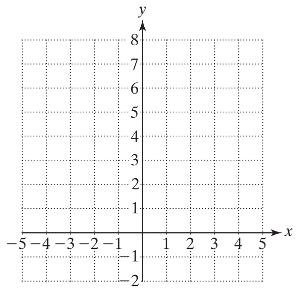
41. $m(x) = \left(\frac{1}{8}\right)^x$



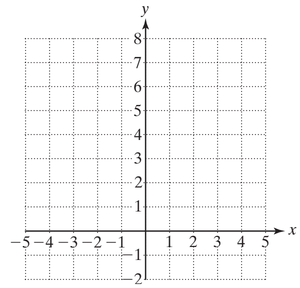
42. $n(x) = \left(\frac{1}{3}\right)^x$



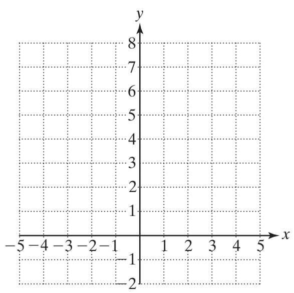
43. $h(x) = 2^{x+1}$



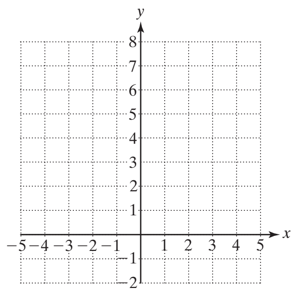
44. $k(x) = 5^{x-1}$



45. $g(x) = 5^{-x}$



46. $f(x) = 2^{-x}$



Concept 4: Applications of Exponential Functions

47. The half-life of the element radon (Rn 86) is 3.8 days. In a sample originally containing 1 g of radon, the amount left after t days is given by $A(t) = (0.5)^{t/3.8}$. (Round to two decimal places, if necessary.)

a. How much radon will be present after 7.6 days?

b. How much radon will be present after 10 days?

48. Nobelium, an element discovered in 1958, has a half-life of 10 min under certain conditions. In a sample containing 1 g of nobelium, the amount left after t min is given by $A(t) = (0.5)^{t/10}$. (Round to three decimal places.)

a. How much nobelium is left after 5 min?

b. How much nobelium is left after 1 hr?

49. Once an antibiotic is introduced to bacteria, the number of bacteria decreases exponentially. For example, beginning with 1 million bacteria, the amount present t days from the time penicillin is introduced is given by the function $A(t) = 1{,}000{,}000(2)^{-t/5}$. Rounding to the nearest thousand, determine how many bacteria are present after

a. 2 days **b.** 1 week **c.** 2 weeks

50. Once an antibiotic is introduced to bacteria, the number of bacteria decreases exponentially. For example, beginning with 1 million bacteria, the amount present t days from the time streptomycin is introduced is given by the function $A(t) = 1{,}000{,}000(2)^{-t/10}$. Rounding to the nearest thousand, determine how many bacteria are present after

a. 5 days **b.** 1 week **c.** 2 weeks

51. The population of Bangladesh was 141,340,000 in 2004 with an annual growth rate of 1.5%.

a. Find a mathematical model that relates the population of Bangladesh as a function of the number of years after 2004.

b. If the annual rate of increase remains the same, use this model to predict the population of Bangladesh in the year 2050. Round to the nearest million.

52. The population of Figi was 886,000 in 2004 with an annual growth rate of 1.07%.

a. Find a mathematical model that relates the population of Figi as a function of the number of years after 2004.

b. If the annual rate of increase remains the same, use this model to predict the population of Figi in the year 2050. Round to the nearest thousand.

53. Suppose \$1000 is initially invested in an account and the value of the account grows exponentially. If the investment doubles in 7 years, then the amount in the account t years after the initial investment is given by

$$A(t) = 1000(2)^{t/7}$$

where t is expressed in years and $A(t)$ is the amount in the account.

a. Find the amount in the account after 5 years.

b. Find the amount in the account after 10 years.

c. Find $A(0)$ and $A(7)$ and interpret the answers in the context of this problem.

54. Suppose \$1500 is initially invested in an account and the value of the account grows exponentially. If the investment doubles in 8 years, then the amount in the account t years after the initial investment is given by

$$A(t) = 1500(2)^{t/8}$$

where t is expressed in years and $A(t)$ is the amount in the account.

a. Find the amount in the account after 5 years.

b. Find the amount in the account after 10 years.

c. Find $A(0)$ and $A(8)$ and interpret the answers in the context of this problem.

Graphing Calculator Exercises

For Exercises 55–62, graph the functions on your calculator to support your solutions to the indicated exercises.

55. $f(x) = 4^x$
(see Exercise 39)

56. $g(x) = 6^x$
(see Exercise 40)

57. $m(x) = (\frac{1}{8})^x$
(see Exercise 41)

58. $n(x) = (\frac{1}{3})^x$
(see Exercise 42)

59. $h(x) = 2^{x+1}$
(see Exercise 43)

60. $k(x) = 5^{x-1}$
(see Exercise 44)

61. $g(x) = 5^{-x}$ (see Exercise 45)

62. $f(x) = 2^{-x}$ (see Exercise 46)

63. The function defined by $A(x) = 1000(2)^{x/7}$ represents the total amount A in an account x years after an initial investment of \$1000.

a. Graph $y = A(x)$ on the window where $0 \le x \le 25$ and $0 \le y \le 10{,}000$.

b. Use *Zoom* and *Trace* to approximate the times required for the account to reach \$2000, \$4000, and \$8000.

64. The function defined by $A(x) = 1500(2)^{x/8}$ represents the total amount A in an account x years after an initial investment of \$1500.

a. Graph $y = A(x)$ on the window where $0 \le x \le 40$ and $0 \le y \le 25{,}000$.

b. Use *Zoom* and *Trace* to approximate the times required for the account to reach \$3000, \$6000, and \$12,000.

Logarithmic Functions

Section 10.4

Concepts

1. Definition of a Logarithmic Function

Consider the following equations in which the variable is located in the exponent of an expression. In some cases the solution can be found by inspection because the constant on the right-hand side of the equation is a perfect power of the base.

Equation		Solution
$5^x = 5$	⟶	$x = 1$
$5^x = 20$	⟶	$x = ?$
$5^x = 25$	⟶	$x = 2$
$5^x = 60$	⟶	$x = ?$
$5^x = 125$	⟶	$x = 3$

The equation $5^x = 20$ cannot be solved by inspection. However, we might suspect that x is between 1 and 2. Similarly, the solution to the equation $5^x = 60$ is between 2 and 3. To solve an exponential equation for an unknown exponent, we must use a new type of function called a logarithmic function.

Definition of a Logarithm Function

If x and b are positive real numbers such that $b \neq 1$, then $y = \log_b x$ is called the **logarithmic function** with base b and

$$y = \log_b x \quad \text{is equivalent to} \quad b^y = x$$

Note: In the expression $y = \log_b x$, y is called the **logarithm**, b is called the **base**, and x is called the **argument**.

The expression $y = \log_b x$ is equivalent to $b^y = x$ and indicates that *the logarithm y is the exponent to which b must be raised to obtain x*. The expression $y = \log_b x$ is called the logarithmic form of the equation, and the expression $b^y = x$ is called the exponential form of the equation.

The definition of a logarithmic function suggests a close relationship with an exponential function of the same base. In fact, a logarithmic function is the inverse of the corresponding exponential function. For example, the following steps find the inverse of the exponential function defined by $f(x) = b^x$.

$f(x) = b^x$	
$y = b^x$	Replace $f(x)$ by y.
$x = b^y$	Interchange x and y.
$y = \log_b x$	Solve for y using the definition of a logarithmic function.
$f^{-1}(x) = \log_b x$	Replace y by $f^{-1}(x)$.

Example 1 Converting from Logarithmic Form to Exponential Form

Rewrite the logarithmic equations in exponential form.

a. $\log_2 32 = 5$ **b.** $\log_{10}\left(\frac{1}{1000}\right) = -3$ **c.** $\log_5 1 = 0$

Solution:

	Logarithmic Form		Exponential Form
a.	$\log_2 32 = 5$	$\Leftrightarrow$	$2^5 = 32$
b.	$\log_{10}\left(\frac{1}{1000}\right) = -3$	$\Leftrightarrow$	$10^{-3} = \frac{1}{1000}$
c.	$\log_5 1 = 0$	$\Leftrightarrow$	$5^0 = 1$

Skill Practice Rewrite the logarithmic equations in exponential form.

1. $\log_3 9 = 2$ **2.** $\log_{10}\left(\frac{1}{100}\right) = -2$ **3.** $\log_8 1 = 0$

Skill Practice Answers

1. $3^2 = 9$ **2.** $10^{-2} = \frac{1}{100}$
3. $8^0 = 1$

2. Evaluating Logarithmic Expressions

Example 2 Evaluating Logarithmic Expressions

Evaluate the logarithmic expressions.

a. $\log_{10} 10{,}000$ **b.** $\log_5\left(\frac{1}{125}\right)$ **c.** $\log_{1/2}\left(\frac{1}{8}\right)$

d. $\log_b b$ **e.** $\log_c\left(c^7\right)$ **f.** $\log_3\left(\sqrt[4]{3}\right)$

Solution:

a. $\log_{10} 10{,}000$ is the exponent to which 10 must be raised to obtain 10,000.

$y = \log_{10} 10{,}000$ Let y represent the value of the logarithm.

$10^y = 10{,}000$ Rewrite the expression in exponential form.

$y = 4$

Therefore, $\log_{10} 10{,}000 = 4$.

b. $\log_5\left(\frac{1}{125}\right)$ is the exponent to which 5 must be raised to obtain $\frac{1}{125}$.

$y = \log_5\left(\frac{1}{125}\right)$ Let y represent the value of the logarithm.

$5^y = \frac{1}{125}$ Rewrite the expression in exponential form.

$y = -3$

Therefore, $\log_5\left(\frac{1}{125}\right) = -3$.

c. $\log_{1/2}\left(\frac{1}{8}\right)$ is the exponent to which $\frac{1}{2}$ must be raised to obtain $\frac{1}{8}$.

$y = \log_{1/2}\left(\frac{1}{8}\right)$ Let y represent the value of the logarithm.

$\left(\frac{1}{2}\right)^y = \frac{1}{8}$ Rewrite the expression in exponential form.

$y = 3$

Therefore, $\log_{1/2}\left(\frac{1}{8}\right) = 3$.

d. $\log_b b$ is the exponent to which b must be raised to obtain b.

$y = \log_b b$ Let y represent the value of the logarithm.

$b^y = b$ Rewrite the expression in exponential form.

$y = 1$

Therefore, $\log_b b = 1$.

e. $\log_c\left(c^7\right)$ is the exponent to which c must be raised to obtain c^7.

$y = \log_c\left(c^7\right)$ Let y represent the value of the logarithm.

$c^y = c^7$ Rewrite the expression in exponential form.

$y = 7$

Therefore, $\log_c\left(c^7\right) = 7$.

f. $\log_3\left(\sqrt[4]{3}\right) = \log_3\left(3^{1/4}\right)$ is the exponent to which 3 must be raised to obtain $3^{1/4}$.

$y = \log_3(3^{1/4})$ Let y represent the value of the logarithm.

$3^y = 3^{1/4}$ Rewrite the expression in exponential form.

$y = \dfrac{1}{4}$

Therefore, $\log_3\left(\sqrt[4]{3}\right) = \frac{1}{4}$

Skill Practice Evaluate the logarithmic expressions.

4. $\log_{10} 1000$ **5.** $\log_4\left(\dfrac{1}{16}\right)$ **6.** $\log_{1/3} 3$

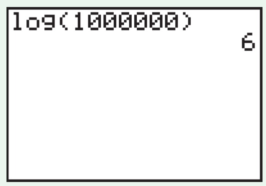
7. $\log_x x$ **8.** $\log_b(b^{10})$ **9.** $\log_5(\sqrt[3]{5})$

3. The Common Logarithmic Function

The logarithmic function with base 10 is called the **common logarithmic function** and is denoted by $y = \log x$. Notice that the base is not explicitly written but is understood to be 10. That is, $y = \log_{10} x$ is written simply as $y = \log x$.

Calculator Connections

On most calculators, the [log] key is used to compute logarithms with base 10. For example, we know the expression $\log(1{,}000{,}000) = 6$ because $10^6 = 1{,}000{,}000$. Use the [log] key to show this result on a calculator.



Example 3 Evaluating Common Logarithms on a Calculator

Evaluate the common logarithms. Round the answers to four decimal places.

a. $\log 420$ **b.** $\log\left(8.2 \times 10^9\right)$ **c.** $\log(0.0002)$

Solution:

a. $\log 420 \approx 2.6232$

b. $\log\left(8.2 \times 10^9\right) \approx 9.9138$

c. $\log(0.0002) \approx -3.6990$

Skill Practice Evaluate the common logarithms. Round answers to 4 decimal places.

10. $\log 1200$ **11.** $\log\left(6.3 \times 10^5\right)$ **12.** $\log(0.00025)$

Skill Practice Answers

4. 3 **5.** -2 **6.** -1

7. 1 **8.** 10 **9.** $\dfrac{1}{3}$

10. 3.0792 **11.** 5.7993

12. -3.6021

4. Graphs of Logarithmic Functions

In Section 10.3 we studied the graphs of exponential functions. In this section, we will graph logarithmic functions. First, recall that $f(x) = \log_b x$ is the inverse of $g(x) = b^x$. Therefore, the graph of $y = f(x)$ is symmetric to the graph of $y = g(x)$ about the line $y = x$, as shown in Figures 10-12 and 10-13.

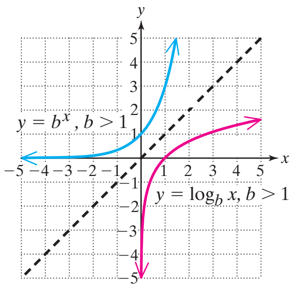


Figure 10-12

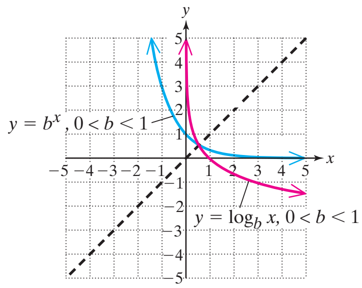


Figure 10-13

From Figures 10-12 and 10-13, we see that the range of $y = b^x$ is the set of positive real numbers. As expected, the domain of its inverse, the logarithmic function $y = \log_b x$, is also the set of positive real numbers. Therefore, the **domain of the logarithmic function** $y = \log_b x$ is the set of positive real numbers.

Example 4 Graphing Logarithmic Functions

Graph the functions and compare the graphs to examine the effect of the base on the shape of the graph.

a. $y = \log_2 x$ **b.** $y = \log x$

Solution:

We can write each equation in its equivalent exponential form and create a table of values (Table 10-5). To simplify the calculations, choose integer values of y and then solve for x.

$$y = \log_2 x \quad \text{or} \quad 2^y = x \qquad y = \log x \quad \text{or} \quad 10^y = x$$

Choose values for y.

Table 10-5

$x = 2^y$	$x = 10^y$	y
$\frac{1}{8}$	$\frac{1}{1000}$	-3
$\frac{1}{4}$	$\frac{1}{100}$	-2
$\frac{1}{2}$	$\frac{1}{10}$	-1
1	1	0
2	10	1
4	100	2
8	1000	3

Solve for x.

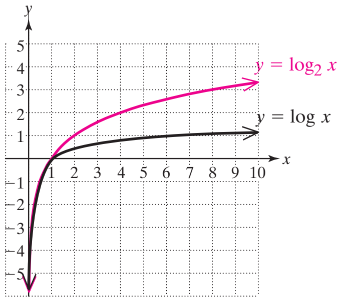


Figure 10-14

The graphs of $y = \log_2 x$ and $y = \log x$ are shown in Figure 10-14. Both graphs exhibit the same general behavior, and the steepness of the curve is affected by the base. The function $y = \log x$ requires a 10-fold increase in x to increase the y-value by 1 unit. The function $y = \log_2 x$ requires a 2-fold increase in x to increase the y-value by 1 unit. In addition, the following characteristics are true for both graphs.

- The domain is the set of real numbers x such that $x > 0$.
- The range is the set of real numbers.
- The y-axis is a vertical asymptote.
- Both graphs pass through the point $(1, 0)$.

Skill Practice Graph the functions

13. $y = \log_3 x$.

Example 4 illustrates that a logarithmic function with base $b > 1$ is an increasing function. In Example 5, we see that if the base b is between 0 and 1, the function decreases over its entire domain.

Example 5 Graphing a Logarithmic Function

Graph $y = \log_{1/4} x$.

Solution:

The equation $y = \log_{1/4} x$ can be written in exponential form as $(\frac{1}{4})^y = x$. By choosing several values for y, we can solve for x and plot the corresponding points (Table 10-6).

The expression $y = \log_{1/4} x$ defines a decreasing logarithmic function (Figure 10-15). Notice that the vertical asymptote, domain, and range are the same for both increasing and decreasing logarithmic functions.

Table 10-6

$x = (\frac{1}{4})^y$	y
64	−3
16	−2
4	−1
1	0
$\frac{1}{4}$	1
$\frac{1}{16}$	2
$\frac{1}{64}$	3

Solve for x.

Choose values for y.

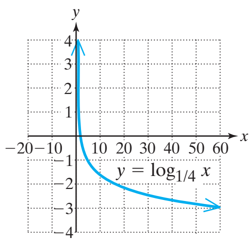


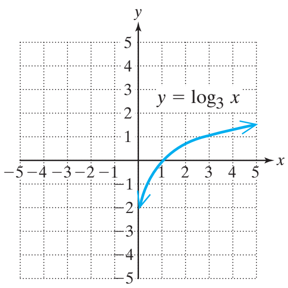
Figure 10-15

Skill Practice

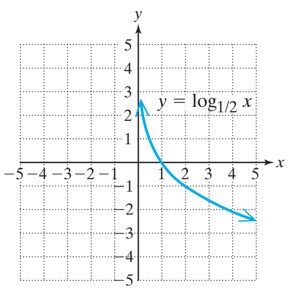
14. Graph $y = \log_{1/2} x$.

Skill Practice Answers

13.



14.



When graphing a logarithmic equation, it is helpful to know its domain.

Example 6 Identifying the Domain of a Logarithmic Function

Find the domain of the functions.

a. $f(x) = \log(4 - x)$ **b.** $g(x) = \log(2x + 6)$

Solution:

The domain of the function $y = \log_b(x)$ is the set of positive real numbers. That is, the argument x must be greater than zero: $x > 0$.

a. $f(x) = \log(4 - x)$ The argument is $4 - x$.

$4 - x > 0$ The argument of the logarithm must be greater than zero.

$-x > -4$ Solve for x.

$x < 4$ Divide by -1 and reverse the inequality sign.

The domain is $(-\infty, 4)$.

b. $g(x) = \log(2x + 6)$ The argument is $2x + 6$.

$2x + 6 > 0$ The argument of the logarithm must be greater than zero.

$2x > -6$ Solve for x.

$x > -3$

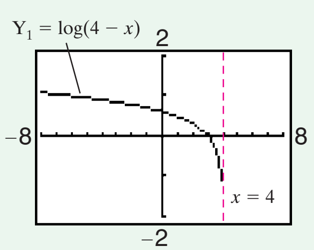
The domain is $(-3, \infty)$.

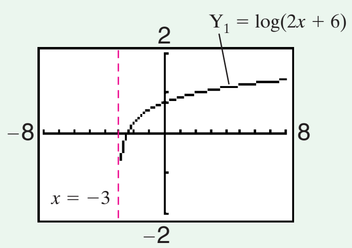
Skill Practice Find the domain of the functions.

15. $f(x) = \log_3(x + 7)$ **16.** $g(x) = \log(4 - 8x)$

Calculator Connections

The graphs of $Y_1 = \log(4 - x)$ and $Y_2 = \log(2x + 6)$ are shown here and can be used to confirm the solutions to Example 6. Notice that each function has a vertical asymptote at the value of x where the argument equals zero.





The general shape and important features of exponential and logarithmic functions are summarized as follows.

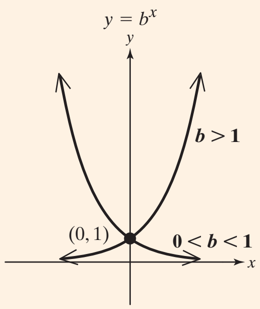
Skill Practice Answers

15. Domain: $(-7, \infty)$

16. Domain: $\left(-\infty, \frac{1}{2}\right)$

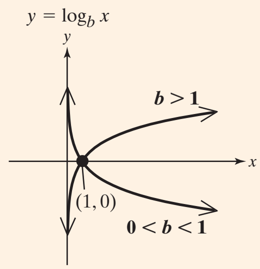
Graphs of Exponential and Logarithmic Functions—A Summary

Exponential Functions



Domain: $(-\infty, \infty)$
Range: $(0, \infty)$
Horizontal asymptote: $y = 0$
Passes through $(0, 1)$
If $b > 1$, the function is increasing.
If $0 < b < 1$, the function is decreasing.

Logarithmic Functions



Domain: $(0, \infty)$
Range: $(-\infty, \infty)$
Vertical asymptote: $x = 0$
Passes through $(1, 0)$
If $b > 1$, the function is increasing.
If $0 < b < 1$, the function is decreasing.

Notice that the roles of x and y are interchanged for the functions $y = b^x$ and $b^y = x$. Therefore, it is not surprising that the domain and range are reversed between exponential and logarithmic functions. Moreover, an exponential function passes through $(0, 1)$, whereas a logarithmic function passes through $(1, 0)$. An exponential function has a horizontal asymptote at $y = 0$, whereas a logarithmic function has a vertical asymptote at $x = 0$.

5. Applications of the Common Logarithmic Function

Example 7 Applying a Common Logarithm to Compute pH

The pH (hydrogen potential) of a solution is defined as

$$\text{pH} = -\log [\text{H}^+]$$

where $[\text{H}^+]$ represents the concentration of hydrogen ions in moles per liter (mol/L). The pH scale ranges from 0 to 14. The midpoint of this range, 7, represents a neutral solution. Values below 7 are progressively more acidic, and values above 7 are progressively more alkaline. Based on the equation $\text{pH} = -\log [\text{H}^+]$, a 1-unit change in pH means a 10-fold change in hydrogen ion concentration.

a. Normal rain has a pH of 5.6. However, in some areas of the northeastern United States the rainwater is more acidic. What is the pH of a rain sample for which the concentration of hydrogen ions is 0.0002 mol/L?

b. Find the pH of household ammonia if the concentration of hydrogen ions is 1.0×10^{-11} mol/L.

Solution:

a. $\text{pH} = -\log [\text{H}^+]$

$= -\log (0.0002)$ Substitute $[\text{H}^+] = 0.0002$.

≈ 3.7 (To compare this value with a familiar substance,

b. $\text{pH} = -\log[\text{H}^+]$

$= -\log(1.0 \times 10^{-11})$ Substitute $[\text{H}^+] = 1.0 \times 10^{-11}$.

$= -\log(10^{-11})$

$= -(-11)$

$= 11$

The pH of household ammonia is 11.

Skill Practice

17. A new all-natural shampoo on the market claims its hydrogen ion concentration is 5.88×10^{-7} mol/L. Use the formula $\text{pH} = -\log[\text{H}^+]$ to calculate the pH level of the shampoo.

Example 8 Applying Logarithmic Functions to a Memory Model

One method of measuring a student's retention of material after taking a course is to retest the student at specified time intervals after the course has been completed. A student's score on a calculus test *t* months after completing a course in calculus is approximated by

$$S(t) = 85 - 25\log(t + 1)$$

where *t* is the time in months after completing the course and $S(t)$ is the student's score as a percent.

a. What was the student's score at $t = 0$?

b. What was the student's score after 2 months?

c. What was the student's score after 1 year?

Solution:

a. $S(t) = 85 - 25\log(t + 1)$

$S(0) = 85 - 25\log(0 + 1)$ Substitute $t = 0$.

$= 85 - 25\log 1$ $\log 1 = 0$ because $10^0 = 1$.

$= 85 - 25(0)$

$= 85 - 0$

$= 85$ The student's score at the time the course was completed was 85%.

b. $S(t) = 85 - 25\log(t + 1)$

$S(2) = 85 - 25\log(2 + 1)$

$= 85 - 25\log 3$ Use a calculator to approximate log 3.

≈ 73.1 The student's score dropped to 73.1%.

c. $S(t) = 85 - 25\log(t + 1)$

$S(12) = 85 - 25\log(12 + 1)$

$= 85 - 25\log 13$ Use a calculator to approximate log 13.

≈ 57.2 The student's score dropped to 57.2%.

Skill Practice Answers

17. $\text{pH} \approx 6.23$

Skill Practice

18. The memory model for a student's score on a statistics test t months after completion of the course in statistics is approximated by

$$S(t) = 92 - 28 \log(t + 1)$$

a. What was the student's score at the time the course was completed $(t = 0)$?

b. What was her score after 1 month?

c. What was the score after 2 months?

Skill Practice Answers

18a. 92 **b.** 83.6 **c.** 78.6

Section 10.4 Practice Exercises

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Study Skills Exercise

1. Define the key terms.

a. Logarithmic function **b. Logarithm** **c. Base of a logarithm**

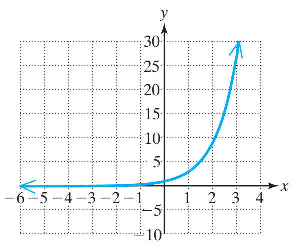
d. Argument **e. Common logarithmic function** **f. Domain of a logarithmic function**

Review Exercises

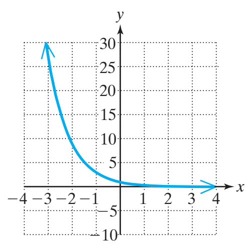
2. For which graph of $y = b^x$ is $0 < b < 1$?

3. For which graph of $y = b^x$ is $b > 1$?

i.



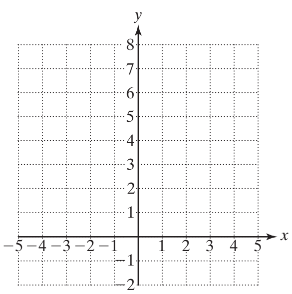
ii.



4. Let $f(x) = 6^x$.

a. Find $f(-2), f(-1), f(0), f(1)$, and $f(2)$.

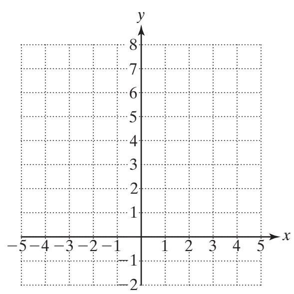
b. Graph $y = f(x)$.



5. Let $g(x) = 3^x$.

a. Find $g(-2), g(-1), g(0), g(1)$, and $g(2)$.

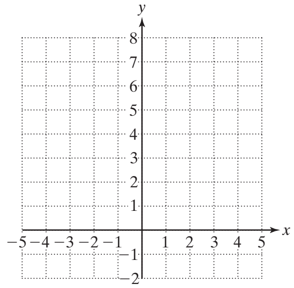
b. Graph $y = g(x)$.



6. Let $r(x) = \left(\frac{3}{4}\right)^x$.

a. Find $r(-2)$, $r(-1)$, $r(0)$, $r(1)$, and $r(2)$.

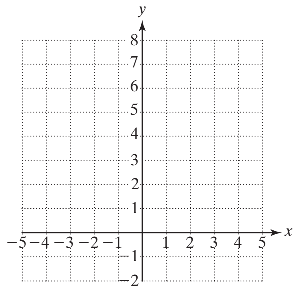
b. Graph $y = r(x)$.



7. Let $s(x) = \left(\frac{2}{5}\right)^x$.

a. Find $s(-2)$, $s(-1)$, $s(0)$, $s(1)$, and $s(2)$.

b. Graph $y = s(x)$.



Concept 1: Definition of a Logarithmic Function

8. For the equation $y = \log_b x$, identify the base, the argument, and the logarithm.

9. Rewrite the equation in exponential form. $y = \log_b x$

For Exercises 10–21, write the equation in logarithmic form.

10. $3^x = 81$ **11.** $10^3 = 1000$ **12.** $5^2 = 25$ **13.** $8^{1/3} = 2$

14. $7^{-1} = \dfrac{1}{7}$ **15.** $8^{-2} = \dfrac{1}{64}$ **16.** $b^x = y$ **17.** $b^y = x$

18. $e^x = y$ **19.** $e^y = x$ **20.** $\left(\dfrac{1}{3}\right)^{-2} = 9$ **21.** $\left(\dfrac{5}{2}\right)^{-1} = \dfrac{2}{5}$

For Exercises 22–33, write the equation in exponential form.

22. $\log_5 625 = 4$ **23.** $\log_{125} 25 = \dfrac{2}{3}$ **24.** $\log_{10}(0.0001) = -4$ **25.** $\log_{25}\left(\dfrac{1}{5}\right) = -\dfrac{1}{2}$

26. $\log_6 36 = 2$ **27.** $\log_2 128 = 7$ **28.** $\log_b 15 = x$ **29.** $\log_b 82 = y$

30. $\log_3 5 = x$ **31.** $\log_2 7 = x$ **32.** $\log_{1/4} x = 10$ **33.** $\log_{1/2} x = 6$

Concept 2: Evaluating Logarithmic Expressions

For Exercises 34–49, find the logarithms without the use of a calculator.

34. $\log_7 49$ **35.** $\log_3 27$ **36.** $\log_{10} 0.1$ **37.** $\log_2\left(\dfrac{1}{16}\right)$

38. $\log_{16} 4$ **39.** $\log_8 2$ **40.** $\log_{7/2} 1$ **41.** $\log_{1/2} 2$

42. $\log_3 3^5$ **43.** $\log_9 9^3$ **44.** $\log_{10} 10$ **45.** $\log_7 1$

46. $\log_a a^3$ **47.** $\log_r r^4$ **48.** $\log_x \sqrt{x}$ **49.** $\log_y \sqrt[3]{y}$

Concept 3: The Common Logarithmic Function

For Exercises 50–58, find the common logarithm without the use of a calculator.

50. log 10 **51.** log 100 **52.** log 1000

53. log 10,000 **54.** $\log(1.0 \times 10^6)$ **55.** log 0.1

56. log (0.01) **57.** log (0.001) **58.** $\log(1.0 \times 10^{-6})$

For Exercises 59–70, use a calculator to approximate the logarithms. Round to four decimal places.

59. log 6 **60.** log 18 **61.** $\log \pi$ **62.** $\log\left(\frac{1}{8}\right)$

63. $\log\left(\frac{1}{32}\right)$ **64.** $\log\sqrt{5}$ **65.** log (0.0054) **66.** log (0.0000062)

67. $\log(3.4 \times 10^5)$ **68.** $\log(4.78 \times 10^9)$ **69.** $\log(3.8 \times 10^{-8})$ **70.** $\log(2.77 \times 10^{-4})$

71. Given: log 10 = 1 and log 100 = 2

a. Estimate log 93.

b. Estimate log 12.

c. Evaluate the logarithms in parts (a) and (b) on a calculator and compare to your estimates.

72. Given: $\log\left(\frac{1}{10}\right) = -1$ and $\log 1 = 0$

a. Estimate $\log\left(\frac{9}{10}\right)$.

b. Estimate $\log\left(\frac{1}{5}\right)$.

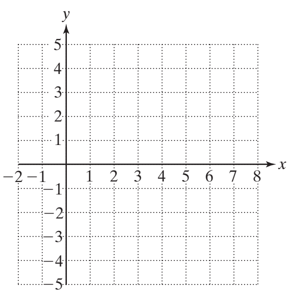
c. Evaluate the logarithms in parts (a) and (b) on a calculator and compare to your estimates.

Concept 4: Graphs of Logarithmic Functions

73. Let $f(x) = \log_4 x$.

a. Find the values of $f(\frac{1}{64}), f(\frac{1}{16}), f(\frac{1}{4}), f(1), f(4), f(16)$, and $f(64)$.

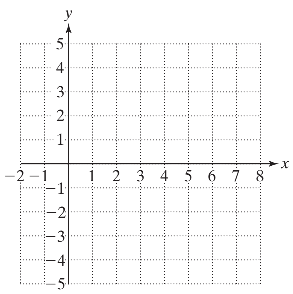
b. Graph $y = f(x)$.



74. Let $g(x) = \log_2 x$.

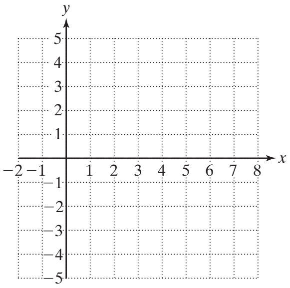
a. Find the values of $g(\frac{1}{8}), g(\frac{1}{4}), g(\frac{1}{2}), g(1), g(2), g(4)$, and $g(8)$.

b. Graph $y = g(x)$.



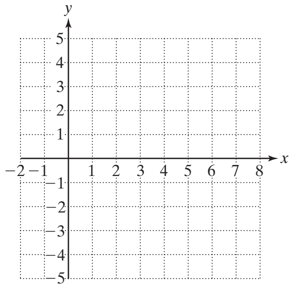
Graph the logarithmic functions in Exercises 75–78 by writing the function in exponential form and making a table of points.

75. $y = \log_3 x$



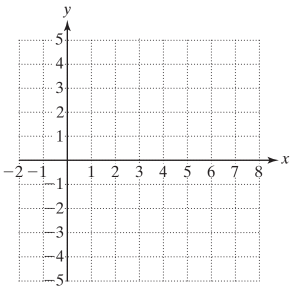
x	y

76. $y = \log_5 x$



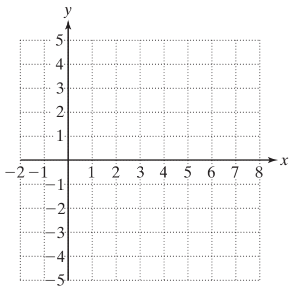
x	y

77. $y = \log_{1/2} x$



x	y

78. $y = \log_{1/3} x$



x	y

For Exercises 79–84, find the domain of the function and express the domain in interval notation.

79. $y = \log_7 (x - 5)$

80. $y = \log_3 (2x + 1)$

81. $y = \log_3 (x + 1.2)$

82. $y = \log\left(x - \frac{1}{2}\right)$

83. $y = \log x^2$

84. $y = \log (x^2 + 1)$

Concept 5: Applications of the Common Logarithmic Function

For Exercises 85–86, use the formula $\text{pH} = -\log[\text{H}^+]$, where $[\text{H}^+]$ represents the concentration of hydrogen ions in moles per liter. Round to the hundredths place.

85. Normally, the level of hydrogen ions in the blood is approximately 4.47×10^{-8} mol/L. Find the pH level of blood.

86. The normal pH level for streams and rivers is between 6.5 and 8.5. A high level of bacteria in a particular stream caused environmentalists to test the water. The level of hydrogen ions was found to be 0.006 mol/L. What is the pH level of the stream?

87. A graduate student in education is doing research to compare the effectiveness of two different teaching techniques designed to teach vocabulary to sixth-graders. The first group of students (group 1) was taught with method I, in which the students worked individually to complete the assignments in a workbook. The second group (group 2) was taught with method II, in which the students worked cooperatively in groups of four to complete the assignments in the same workbook.

None of the students knew any of the vocabulary words before the study began. After completing the assignments in the workbook, the students were then tested on the vocabulary at 1-month intervals to assess how much material they had retained over time. The students' average score t months after completing the assignments are given by the following functions:

Method I: $S_1(t) = 91 - 30\log(t + 1)$, where t is the time in months and $S_1(t)$ is the average score of students in group 1.

Method II: $S_2(t) = 88 - 15\log(t + 1)$, where t is the time in months and $S_2(t)$ is the average score of students in group 2.

a. Complete the table to find the average scores for each group of students after the indicated number of months. Round to one decimal place.

***t* (months)**	0	1	2	6	12	24
$S_1(t)$						
$S_2(t)$						

b. Based on the table of values, what were the average scores for each group immediately after completion of the assigned material ($t = 0$)?

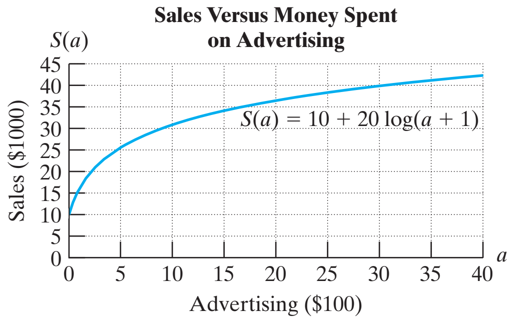
c. Based on the table of values, which teaching method helped students retain the material better for a long time?

88. Generally, the more money a company spends on advertising, the higher the sales. Let a represent the amount of money spent on advertising (in \$100s). Then the amount of money in sales $S(a)$ (in \$1000s) is given by

$$S(a) = 10 + 20\log(a + 1) \text{ where } a \geq 0.$$

a. The value of $S(1) \approx 16.0$ means that if \$100 is spent on advertising, \$16,000 is returned in sales. Find the values of $S(11)$, $S(21)$, and $S(31)$. Round to 1 decimal place. Interpret the meaning of each function value in the context of this problem.

b. The graph of $y = S(a)$ is shown here. Use the graph and your answers from part (a) to explain why the money spent in advertising becomes less "efficient" as it is used in larger amounts.



Graphing Calculator Exercises

For Exercises 89–94, graph the function on an appropriate viewing window. From the graph, identify the domain of the function and the location of the vertical asymptote.

89. $y = \log(x + 6)$ **90.** $y = \log(2x + 4)$ **91.** $y = \log(0.5x - 1)$

92. $y = \log(x + 8)$ **93.** $y = \log(2 - x)$ **94.** $y = \log(3 - x)$

Properties of Logarithms

Section 10.5

Concepts

1. Properties of Logarithms
2. Expanded Logarithmic Expressions
3. Single Logarithmic Expressions

1. Properties of Logarithms

You have already been exposed to certain properties of logarithms that follow directly from the definition. Recall

$y = \log_b x$ is equivalent to $b^y = x$ for $x > 0$, $b > 0$, and $b \neq 1$

The following properties follow directly from the definition.

$\log_b 1 = 0$ Property 1

$\log_b b = 1$ Property 2

$\log_b b^p = p$ Property 3

$b^{\log_b x} = x$ Property 4

Example 1 Applying the Properties of Logarithms to Simplify Expressions

Use the properties of logarithms to simplify the expressions. Assume that all variable expressions within the logarithms represent positive real numbers.

a. $\log_8 8 + \log_8 1$ **b.** $10^{\log(x+2)}$ **c.** $\log_{1/2}\left(\frac{1}{2}\right)^x$

Solution:

a. $\log_8 8 + \log_8 1 = 1 + 0 = 1$ Properties 2 and 1

b. $10^{\log(x+2)} = x + 2$ Property 4

c. $\log_{1/2}\left(\frac{1}{2}\right)^x = x$ Property 3

Skill Practice Use the properties of logarithms to simplify the expressions.

1. $\log_5 1 + \log_5 5$ **2.** $15^{\log_{15} 7}$ **3.** $\log_{1/3}\left(\frac{1}{3}\right)^c$

Three additional properties are useful when simplifying logarithmic expressions. The first is the product property for logarithms.

Product Property for Logarithms

Let b, x, and y be positive real numbers where $b \neq 1$. Then

$$\log_b (xy) = \log_b x + \log_b y$$

The logarithm of a product equals the sum of the logarithms of the factors.

Proof:

Let $M = \log_b x$, which implies $b^M = x$.

Let $N = \log_b y$, which implies $b^N = y$.

Then $xy = b^M b^N = b^{M+N}$.

Writing the expression $xy = b^{M+N}$ in logarithmic form, we have,

$$\log_b (xy) = M + N$$

$$\log_b (xy) = \log_b x + \log_b y \checkmark$$

To demonstrate the product property for logarithms, simplify the following expressions by using the order of operations.

$$\log_3 (3 \cdot 9) \stackrel{?}{=} \log_3 3 + \log_3 9$$

$$\log_3 27 \stackrel{?}{=} 1 + 2$$

$$3 = 3 \checkmark$$

Quotient Property for Logarithms

Let b, x, and y be positive real numbers where $b \neq 1$. Then

$$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$$

The logarithm of a quotient equals the difference of the logarithms of the numerator and denominator.

Skill Practice Answers

1. 1 **2.** 7 **3.** c

The proof of the quotient property for logarithms is similar to the proof of the product property and is omitted here. To demonstrate the quotient property for logarithms, simplify the following expressions by using the order of operations.

$$\log\left(\frac{1{,}000{,}000}{100}\right) \stackrel{?}{=} \log 1{,}000{,}000 - \log 100$$

$$\log 10{,}000 \stackrel{?}{=} 6 - 2$$

$$4 = 4 \checkmark$$

Power Property for Logarithms

Let b and x be positive real numbers where $b \neq 1$. Let p be any real number. Then

$$\log_b x^p = p \log_b x$$

Proof:

Let $M = \log_b x$, which implies $b^M = x$.

Raise both sides to the power p: $(b^M)^p = x^p$ or equivalently $b^{Mp} = x^p$

Write the expression $b^{Mp} = x^p$ in logarithmic form: $\log_b x^p = Mp = pM$ or equivalently $\log_b x^p = \text{p} \log_b x$. ✓

To demonstrate the power property for logarithms, simplify the following expressions by using the order of operations.

$$\log_4 4^2 \stackrel{?}{=} 2 \log_4 4$$

$$2 \stackrel{?}{=} 2 \cdot 1$$

$$2 = 2 \checkmark$$

The properties of logarithms are summarized in the box.

Properties of Logarithms

Let b, x, and y be positive real numbers where $b \neq 1$, and let p be a real number. Then the following properties of logarithms are true.

1. $\log_b 1 = 0$	**5.** $\log_b (xy) = \log_b x + \log_b y$	**Product property for logarithms**
2. $\log_b b = 1$	**6.** $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$	**Quotient property for logarithms**
3. $\log_b b^p = p$	**7.** $\log_b x^p = p \log_b x$	**Power property for logarithms**
4. $b^{\log_b x} = x$		

2. Expanded Logarithmic Expressions

In many applications it is advantageous to expand a logarithm into a sum or difference of simpler logarithms.

Example 2 Writing a Logarithmic Expression in Expanded Form

Write the expressions as the sum or difference of logarithms of x, y, and z. Assume all variables represent positive real numbers.

a. $\log_3\left(\frac{xy^3}{z^2}\right)$ **b.** $\log\left(\frac{\sqrt{x+y}}{10}\right)$ **c.** $\log_b\sqrt[5]{\frac{x^4}{yz^3}}$

Solution:

a. $\log_3\left(\frac{xy^3}{z^2}\right)$

$= \log_3 xy^3 - \log_3 z^2$ Quotient property for logarithms (property 6)

$= [\log_3 x + \log_3 y^3] - \log_3 z^2$ Product property for logarithms (property 5)

$= \log_3 x + 3\log_3 y - 2\log_3 z$ Power property for logarithms (property 7)

b. $\log\left(\frac{\sqrt{x+y}}{10}\right)$

$= \log(\sqrt{x+y}) - \log(10)$ Quotient property for logarithms (property 6)

$= \log(x+y)^{1/2} - 1$ Write $\sqrt{x+y}$ as $(x+y)^{1/2}$ and simplify $\log 10 = 1$.

$= \frac{1}{2}\log(x+y) - 1$ Power property for logarithms (property 7)

c. $\log_b\sqrt[5]{\frac{x^4}{yz^3}}$

$= \log_b\left(\frac{x^4}{yz^3}\right)^{1/5}$ Write $\sqrt[5]{\frac{x^4}{yz^3}}$ as $\left(\frac{x^4}{yz^3}\right)^{1/5}$.

$= \frac{1}{5}\log_b\left(\frac{x^4}{yz^3}\right)$ Power property for logarithms (property 7)

$= \frac{1}{5}[\log_b x^4 - \log_b(yz^3)]$ Quotient property for logarithms (property 6)

$= \frac{1}{5}[\log_b x^4 - (\log_b y + \log_b z^3)]$ Product property for logarithms (property 5)

$= \frac{1}{5}[\log_b x^4 - \log_b y - \log_b z^3]$ Distributive property

$$= \frac{1}{5}[4 \log_b x - \log_b y - 3 \log_b z]$$ Power property for logarithms (property 7)

or $$\frac{4}{5}\log_b x - \frac{1}{5}\log_b y - \frac{3}{5}\log_b z$$

Skill Practice Write the expression as the sum or difference of logarithms of a, b, and c. Assume all variables represent positive real numbers.

4. $\log_5\left(\frac{a^2b^3}{c}\right)$ **5.** $\log\frac{(a-b)^2}{\sqrt{10}}$ **6.** $\log_3\sqrt[4]{\frac{a}{bc^5}}$

3. Single Logarithmic Expressions

In some applications it is necessary to write a sum or difference of logarithms as a single logarithm.

Example 3 Writing a Sum or Difference of Logarithms as a Single Logarithm

Rewrite the expressions as a single logarithm, and simplify the result, if possible. Assume all variable expressions within the logarithms represent positive real numbers.

a. $\log_2 560 - \log_2 7 - \log_2 5$ **b.** $2\log x - \frac{1}{2}\log y + 3\log z$

c. $\frac{1}{2}[\log_5(x^2 - y^2) - \log_5(x + y)]$

Solution:

a. $\log_2 560 - \log_2 7 - \log_2 5$

$= \log_2 560 - (\log_2 7 + \log_2 5)$ Factor out -1 from the last two terms.

$= \log_2 560 - \log_2(7 \cdot 5)$ Product property for logarithms (property 5)

$= \log_2\left(\frac{560}{7 \cdot 5}\right)$ Quotient property for logarithms (property 6)

$= \log_2 16$ Simplify inside parentheses.

$= 4$

b. $2\log x - \frac{1}{2}\log y + 3\log z$

$= \log x^2 - \log y^{1/2} + \log z^3$ Power property for logarithms (property 7)

$= \log x^2 + \log z^3 - \log y^{1/2}$ Group terms with positive coefficients.

$= \log(x^2z^3) - \log y^{1/2}$ Product property for logarithms (property 5)

$= \log\left(\frac{x^2z^3}{y^{1/2}}\right)$ or $\log\left(\frac{x^2z^3}{\sqrt{y}}\right)$ Quotient property for logarithms (property 6)

Skill Practice Answers

4. $2\log_5 a + 3\log_5 b - \log_5 c$

5. $2\log(a - b) - \frac{1}{2}$

6. $\frac{1}{4}\log_3 a - \frac{1}{4}\log_3 b - \frac{5}{4}\log_3 c$

c. $\frac{1}{2}[\log_5(x^2 - y^2) - \log_5(x + y)]$

$= \frac{1}{2}\log_5\left(\frac{x^2 - y^2}{x + y}\right)$ Quotient property for logarithms (property 6)

$= \frac{1}{2}\log_5\left[\frac{(x + y)(x - y)}{x + y}\right]$ Factor and simplify within the parentheses.

$= \frac{1}{2}\log_5(x - y)$

$= \log_5(x - y)^{1/2}$ or $\log_5\sqrt{x - y}$ Power property for logarithms (property 7)

Skill Practice Write the expression as a single logarithm, and simplify the result, if possible.

7. $\log_3 54 + \log_3 10 - \log_3 20$

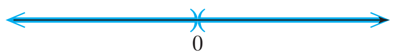
8. $3\log x + \frac{1}{3}\log y - 2\log z$

9. $\frac{1}{2}[\log_2(a^2 + ab) - \log_2 a]$

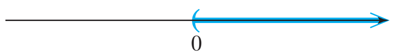
It is important to note that the properties of logarithms may be used to write a single logarithm as a sum or difference of logarithms. Furthermore, the properties may be used to write a sum or difference of logarithms as a single logarithm. In either case, these operations may change the domain.

For example, consider the function $y = \log_b x^2$. Using the power property for logarithms, we have $y = 2\log_b x$. Consider the domain of each function:

$y = \log_b x^2$ Domain: $(-\infty, 0) \cup (0, \infty)$



$y = 2\log_b x$ Domain: $(0, \infty)$



These two functions are equivalent only for values of x in the intersection of the two domains, that is, for $(0, \infty)$.

Skill Practice Answers

7. 3 **8.** $\log\left(\frac{x^3\sqrt[3]{y}}{z^2}\right)$

9. $\log_2\sqrt{a + b}$

Section 10.5 Practice Exercises

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Study Skills Exercise

1. Define the key terms.

a. Product property of logarithms **b. Quotient property of logarithms**

c. Power property of logarithms

Review Exercises

For Exercises 2–5, find the values of the logarithmic and exponential expressions without using a calculator.

2. 8^{-2} **3.** $\log 10{,}000$ **4.** $\log_2 32$ **5.** 6^{-1}

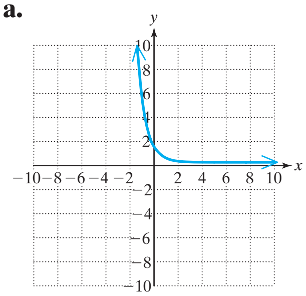
For Exercises 6–9, approximate the values of the logarithmic and exponential expressions by using a calculator.

6. $(\sqrt{2})^{\pi}$ **7.** $\log 8$ **8.** $\log 27$ **9.** $\pi^{\sqrt{2}}$

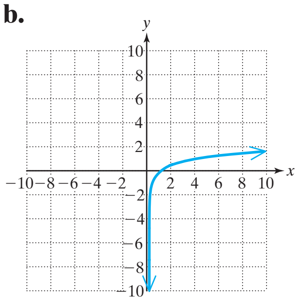
For Exercises 10–13, match the function with the appropriate graph.

10. $f(x) = 4^x$ **11.** $q(x) = \left(\frac{1}{5}\right)^x$ **12.** $h(x) = \log_5 x$ **13.** $k(x) = \log_{1/3} x$

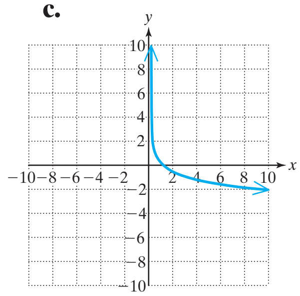
a.



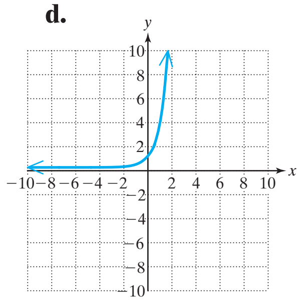
b.



c.



d.



Concept 1: Properties of Logarithms

14. Property 1 of logarithms states that $\log_b 1 = 0$. Write an example of this property.

15. Property 2 of logarithms states that $\log_b b = 1$. Write an example of this property.

16. Property 3 of logarithms states that $\log_b b^p = p$. An example is $\log_6 6^3 = 3$. Write another example of this property.

17. Property 4 of logarithms states that $b^{\log_b x} = x$. An example is $2^{\log_2 5} = 5$. Write another example of this property.

For Exercises 18–39, evaluate each expression.

18. $\log_3 3$ **19.** $\log 10$ **20.** $\log_5 5^4$ **21.** $\log_4 4^5$

22. $6^{\log_6 11}$ **23.** $7^{\log_7 2}$ **24.** $\log 10^3$ **25.** $\log_6 6^3$

26. $\log_3 1$ **27.** $\log_8 1$ **28.** $10^{\log 9}$ **29.** $8^{\log_8 5}$

30. $\log_{1/2} 1$ **31.** $\log_{1/3}\left(\frac{1}{3}\right)$ **32.** $\log_2 1 + \log_2 2^3$ **33.** $\log 10^4 + \log 10$

34. $\log_4 4 + \log_2 1$ **35.** $\log_7 7 + \log_4 4^2$ **36.** $\log_{1/4}\left(\frac{1}{4}\right)^{2x}$ **37.** $\log_{2/3}\left(\frac{2}{3}\right)^p$

38. $\log_a a^4$ **39.** $\log_y y^2$

40. Compare the expressions by approximating their values on a calculator. Which two expressions appear to be equivalent?

a. $\log(3 \cdot 5)$ **b.** $\log 3 \cdot \log 5$ **c.** $\log 3 + \log 5$

41. Compare the expressions by approximating their values on a calculator. Which two expressions appear to be equivalent?

a. $\log\left(\frac{6}{5}\right)$ **b.** $\frac{\log 6}{\log 5}$ **c.** $\log 6 - \log 5$

42. Compare the expressions by approximating their values on a calculator. Which two expressions appear to be equivalent?

a. $\log 20^2$ **b.** $[\log 20]^2$ **c.** $2 \log 20$

43. Compare the expressions by approximating their values on a calculator. Which two expressions appear to be equivalent?

a. $\log \sqrt{4}$ **b.** $\frac{1}{2}\log 4$ **c.** $\sqrt{\log 4}$

Concept 2: Expanded Logarithmic Expressions

For Exercises 44–61, expand into sums and/or differences of logarithms. Assume all variables represent positive real numbers.

44. $\log_3\left(\frac{x}{5}\right)$ **45.** $\log_2\left(\frac{y}{z}\right)$ **46.** $\log(2x)$

47. $\log_6(xyz)$ **48.** $\log_5 x^4$ **49.** $\log_7 z^{1/3}$

50. $\log_4\left(\frac{ab}{c}\right)$ **51.** $\log_2\left(\frac{x}{yz}\right)$ **52.** $\log_b\left(\frac{\sqrt{xy}}{z^3 w}\right)$

53. $\log\left(\frac{a\sqrt[3]{b}}{cd^2}\right)$ **54.** $\log_2\left(\frac{x+1}{y^2\sqrt{z}}\right)$ **55.** $\log\left(\frac{a+1}{b\sqrt[3]{c}}\right)$

56. $\log\left(\sqrt[3]{\frac{ab^2}{c}}\right)$ **57.** $\log_5\left(\sqrt[4]{\frac{w^3 z}{x^2}}\right)$ **58.** $\log\left(\frac{1}{w^5}\right)$

59. $\log_3\left(\frac{1}{z^4}\right)$ **60.** $\log_b\left(\frac{\sqrt{a}}{b^3 c}\right)$ **61.** $\log_x\left(\frac{x}{y\sqrt{z}}\right)$

Concept 3: Single Logarithmic Expressions

For Exercises 62–75, write the expressions as a single logarithm. Assume all variables represent positive real numbers.

62. $\log x + \log y - \log z$ **63.** $\log C + \log A + \log B + \log I + \log N$

64. $2\log_3 x - 3\log_3 y + \log_3 z$ **65.** $\frac{1}{3}\log_2 x - 5\log_2 y - 3\log_2 z$

66. $2\log_3 a - \frac{1}{4}\log_3 b + \log_3 c$ **67.** $\log_5 a - \frac{1}{2}\log_5 b - 3\log_5 c$

68. $\log_b x - 3\log_b x + 4\log_b x$ **69.** $2\log_3 z + \log_3 z - \frac{1}{2}\log_3 z$

70. $5\log_8 a - \log_8 1 + \log_8 8$ **71.** $\log_2 2 + 2\log_2 b - \log_2 1$

72. $2 \log (x + 6) + \frac{1}{3} \log y - 5 \log z$

73. $\frac{1}{4} \log (a + 1) - 2 \log b - 4 \log c$

74. $\log_b (x + 1) - \log_b (x^2 - 1)$

75. $\log_x (p^2 - 4) - \log_x (p - 2)$

Expanding Your Skills

For Exercises 76–85, find the values of the logarithms given that $\log_b 2 \approx 0.693$, $\log_b 3 \approx 1.099$, and $\log_b 5 \approx 1.609$.

76. $\log_b 6$

77. $\log_b 4$

78. $\log_b 12$

79. $\log_b 25$

80. $\log_b 81$

81. $\log_b 30$

82. $\log_b \left(\frac{5}{2}\right)$

83. $\log_b \left(\frac{25}{3}\right)$

84. $\log_b 10^6$

85. $\log_b 2^{12}$

86. The intensity of sound waves is measured in decibels and is calculated by the formula

$$B = 10 \log \left(\frac{I}{I_0}\right)$$

where I_0 is the minimum detectable decibel level.

a. Expand this formula by using the properties of logarithms.

b. Let $I_0 = 10^{-16}$ W/cm^2 and simplify.

87. The Richter scale is used to measure the intensity of an earthquake and is calculated by the formula

$$R = \log \left(\frac{I}{I_0}\right)$$

where I_0 is the minimum level detectable by a seismograph.

a. Expand this formula by using the properties of logarithms.

b. Let $I_0 = 1$ and simplify.

Graphing Calculator Exercises

88. **a.** Graph $Y_1 = \log x^2$ and state its domain.

b. Graph $Y_2 = 2 \log x$ and state its domain.

c. For what values of x are the expressions $\log x^2$ and $2 \log x$ equivalent?

89. **a.** Graph $Y_1 = \log (x - 1)^2$ and state its domain.

b. Graph $Y_2 = 2 \log (x - 1)$ and state its domain.

c. For what values of x are the expressions $\log (x - 1)^2$ and $2 \log (x - 1)$ equivalent?

Section 10.6 The Irrational Number e

Concepts

1. The Irrational Number e
2. Computing Compound Interest
3. The Natural Logarithmic Function
4. Change-of-Base Formula
5. Applications of the Natural Logarithmic Function

1. The Irrational Number e

The exponential function base 10 is particularly easy to work with because integral powers of 10 represent different place positions in the base-10 numbering system. In this section, we introduce another important exponential function whose base is an irrational number called e.

Consider the expression $(1 + \frac{1}{x})^x$. The value of the expression for increasingly large values of x approaches a constant (Table 10-7).

Table 10-7

x	$\left(1 + \frac{1}{x}\right)^x$
100	2.70481382942
1000	2.71692393224
10,000	2.71814592683
100,000	2.71826823717
1,000,000	2.71828046932
1,000,000,000	2.71828182710

As x approaches infinity, the expression $(1 + \frac{1}{x})^x$ approaches a constant value that we call $\boldsymbol{e}$. From Table 10-7, this value is approximately 2.718281828.

$$e \approx 2.718281828$$

The value of e is an irrational number (a nonterminating, nonrepeating decimal) and is a universal constant, as is π.

Example 1 Graphing $f(x) = e^x$

Graph the function defined by $f(x) = e^x$.

Solution:

Because the base of the function is greater than 1 ($e \approx 2.718281828$), the graph is an increasing exponential function. We can use a calculator to evaluate $f(x) = e^x$ at several values of x.

If you are using your calculator correctly, your answers should match those found in Table 10-8. Values are rounded to 3 decimal places. The corresponding graph of $f(x) = e^x$ is shown in Figure 10-16.

Table 10-8

x	$f(x) = e^x$
−3	0.050
−2	0.135
−1	0.368
0	1.000
1	2.718
2	7.389
3	20.086

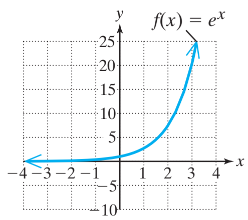
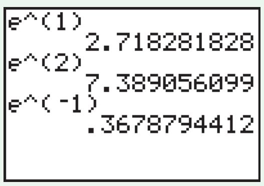


Figure 10-16

Calculator Connections

Practice using your calculator by evaluating e^x for $x = 1, x = 2$, and $x = -1$.



Skill Practice

1. Graph $f(x) = e^x + 1$.

2. Computing Compound Interest

One particularly interesting application of exponential functions is in computing compound interest.

1. If the number of compounding periods per year is finite, then the amount in an account is given by

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

where P is the initial principal, r is the annual interest rate, n is the number of times compounded per year, and t is the time in years that the money is invested.

2. If the number of compound periods per year is infinite, then interest is said to be **compounded continuously**. In such a case, the amount in an account is given by

$$A(t) = Pe^{rt}$$

where P is the initial principal, r is the annual interest rate, and t is the time in years that the money is invested.

Example 2 Computing the Balance on an Account

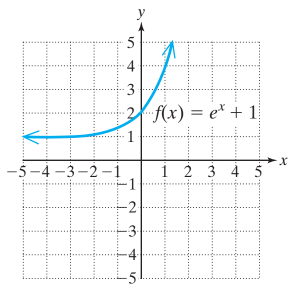
Suppose \$5000 is invested in an account earning 6.5% interest. Find the balance in the account after 10 years under the following compounding options.

a. Compounded annually
b. Compounded quarterly
c. Compounded monthly
d. Compounded daily
e. Compounded continuously

Solution:

Compounding Option	n Value	Formula	Result
Annually	$n = 1$	$A(10) = 5000\left(1 + \frac{0.065}{1}\right)^{(1)(10)}$	\$9385.69
Quarterly	$n = 4$	$A(10) = 5000\left(1 + \frac{0.065}{4}\right)^{(4)(10)}$	\$9527.79
Monthly	$n = 12$	$A(10) = 5000\left(1 + \frac{0.065}{12}\right)^{(12)(10)}$	\$9560.92
Daily	$n = 365$	$A(10) = 5000\left(1 + \frac{0.065}{365}\right)^{(365)(10)}$	\$9577.15
Continuously	Not applicable	$A(10) = 5000e^{(0.065)(10)}$	\$9577.70

Skill Practice Answers

1.


Notice that there is a \$191.46 difference in the account balance between annual compounding and daily compounding. However, the difference between compounding daily and compounding continuously is small—\$0.55. As n gets infinitely large, the function defined by

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

converges to $A(t) = Pe^{rt}$.

Skill Practice

2. Suppose \$1000 is invested at 5%. Find the balance after 8 years under the following options.
- **a.** Compounded annually
- **b.** Compounded quarterly
- **c.** Compounded monthly
- **d.** Compounded daily
- **e.** Compounded continuously

3. The Natural Logarithmic Function

Recall that the common logarithmic function $y = \log x$ has a base of 10. Another important logarithmic function is called the **natural logarithmic function**. The natural logarithmic function has a base of e and is written as $y = \ln x$. That is,

$$y = \ln x = \log_e x$$

Calculator Connections

```
ln(1)
                0
ln(2)
     .6931471806
ln(3)
     1.098612289
```

Example 3 Graphing $y = \ln x$

Graph $y = \ln x$.

Solution:

Because the base of the function $y = \ln x$ is e and $e > 1$, the graph is an increasing logarithmic function. We can use a calculator to find specific points on the graph of $y = \ln x$ by pressing the ln key.

Practice using your calculator by evaluating $\ln x$ for the following values of x. If you are using your calculator correctly, your answers should match those found in Table 10-9. Values are rounded to 3 decimal places. The corresponding graph of $y = \ln x$ is shown in Figure 10-17.

Table 10-9

x	$\ln x$
1	0.000
2	0.693
3	1.099
4	1.386
5	1.609
6	1.792
7	1.946

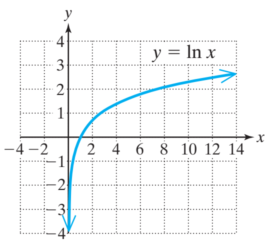


Figure 10-17

Note that the natural logarithmic function $f(x) = \ln x$ is the inverse of $g(x) = e^x$. That means that the graphs are symetric to each other about the line $y = x$.

Skill Practice Answers

2a. \$1477.46 **b.** \$1488.13
c. \$1490.59 **d.** \$1491.78
e. \$1491.82

Skill Practice

3. Graph $y = \ln x + 1$.

The properties of logarithms stated in Section 10.5 are also true for natural logarithms.

Properties of the Natural Logarithmic Function

Let x and y be positive real numbers, and let p be a real number. Then the following properties are true.

1. $\ln 1 = 0$	**5.** $\ln (xy) = \ln x + \ln y$	Product property for logarithms
2. $\ln e = 1$	**6.** $\ln \left(\frac{x}{y}\right) = \ln x - \ln y$	Quotient property for logarithms
3. $\ln e^p = p$	**7.** $\ln x^p = p \ln x$	Power property for logarithms
4. $e^{\ln x} = x$		

Example 4 Simplifying Expressions with Natural Logarithms

Simplify the expressions. Assume that all variable expressions within the logarithms represent positive real numbers.

a. $\ln e$ **b.** $\ln 1$ **c.** $\ln (e^{x+1})$ **d.** $e^{\ln (x+1)}$

Solution:

a. $\ln e = 1$ Property 2

b. $\ln 1 = 0$ Property 1

c. $\ln (e^{x+1}) = x + 1$ Property 3

d. $e^{\ln (x+1)} = x + 1$ Property 4

Skill Practice Simplify.

4. $\ln e^2$ **5.** $-3 \ln 1$ **6.** $\ln e^{(x+y)}$ **7.** $e^{\ln (3x)}$

Example 5 Writing a Sum or Difference of Natural Logarithms as a Single Logarithm

Write the expression as a single logarithm. Assume that all variable expressions within the logarithms represent positive real numbers.

$$\ln (x^2 - 9) - \ln (x - 3) - 2 \ln x$$

Solution:

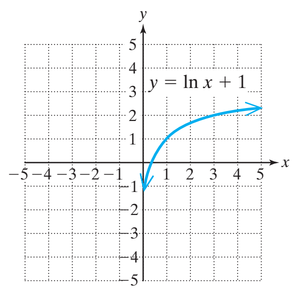
$\ln (x^2 - 9) - \ln (x - 3) - 2 \ln x$

$= \ln (x^2 - 9) - \ln (x - 3) - \ln x^2$ Power property for logarithms (property 7)

$= \ln (x^2 - 9) - [\ln (x - 3) + \ln x^2]$ Factor out -1 from the last two terms.

Skill Practice Answers

3.



4. 2 **5.** 0 **6.** $x + y$

7. $3x$

$= \ln (x^2 - 9) - \ln [(x - 3)x^2]$ Product property for logarithms (property 5)

$= \ln \left[\dfrac{x^2 - 9}{(x - 3)x^2}\right]$ Quotient property for logarithms (property 6)

$= \ln \left[\dfrac{(x - 3)(x + 3)}{(x - 3)x^2}\right]$ Factor.

$= \ln \left(\dfrac{x + 3}{x^2}\right)$ Simplify.

Skill Practice

8. Write as a single logarithm.

$$\ln (x + 4) - \ln (x^2 - 16) + \ln x$$

Example 6 Writing a Logarithmic Expression in Expanded Form

Write the expression

$$\ln \left(\frac{e}{x^2\sqrt{y}}\right)$$

as a sum or difference of logarithms of x and y. Assume all variable expressions within the logarithm represent positive real numbers.

Solution:

$\ln \left(\dfrac{e}{x^2\sqrt{y}}\right)$

$= \ln e - \ln (x^2\sqrt{y})$ Quotient property for logarithms (property 6)

$= \ln e - [\ln x^2 + \ln \sqrt{y}]$ Product property for logarithms (property 5)

$= 1 - \ln x^2 - \ln y^{1/2}$ Distributive property. Also simplify $\ln e = 1$.

$= 1 - 2 \ln x - \dfrac{1}{2} \ln y$ Power property for logarithms (property 7)

Skill Practice

9. Write as a sum or difference of logarithms of x and y. Assume all variables represent positive real numbers.

$$\ln \left(\frac{x^3\sqrt{y}}{e^2}\right)$$

4. Change-of-Base Formula

A calculator can be used to approximate the value of a logarithm with a base of 10 or a base of e by using the [log] key or the [ln] key, respectively. However, to use a calculator to evaluate a logarithmic expression with a base other than 10 or e, we must use the **change-of-base formula**.

Skill Practice Answers

8. $\ln \left(\dfrac{x}{x - 4}\right)$

9. $3 \ln x + \dfrac{1}{2} \ln y - 2$

Change-of-Base Formula

Let a and b be positive real numbers such that $a \neq 1$ and $b \neq 1$. Then for any positive real number x,

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Proof:

Let $M = \log_b x$, which implies that $b^M = x$.

Take the logarithm, base a, on both sides: $\log_a b^M = \log_a x$

Apply the power property for logarithms: $M \cdot \log_a b = \log_a x$

Divide both sides by $\log_a b$: $\frac{M \cdot \cancel{\log_a b}}{\cancel{\log_a b}} = \frac{\log_a x}{\log_a b}$

$$M = \frac{\log_a x}{\log_a b}$$

Because $M = \log_b x$, we have $\log_b x = \frac{\log_a x}{\log_a b}$ ✓

The change-of-base formula converts a logarithm of one base to a ratio of logarithms of a different base. For the sake of using a calculator, we often apply the change-of-base formula with base 10 or base *e*.

Example 7 Using the Change-of-Base Formula

a. Use the change-of-base formula to evaluate $\log_4 80$ by using base 10. (Round to three decimal places.)

b. Use the change-of-base formula to evaluate $\log_4 80$ by using base e. (Round to three decimal places.)

Solution:

a. $\log_4 80 = \frac{\log_{10} 80}{\log_{10} 4} = \frac{\log 80}{\log 4} \approx \frac{1.903089987}{0.6020599913} \approx 3.161$

b. $\log_4 80 = \frac{\log_e 80}{\log_e 4} = \frac{\ln 80}{\ln 4} \approx \frac{4.382026635}{1.386294361} \approx 3.161$

To check the result, we see that $4^{3.161} \approx 80$.

Skill Practice

10. Use the change-of-base formula to evaluate $\log_5 95$ by using base 10. Round to three decimal places.

11. Use the change-of-base formula to evaluate $\log_5 95$ by using base e. Round to three decimal places.

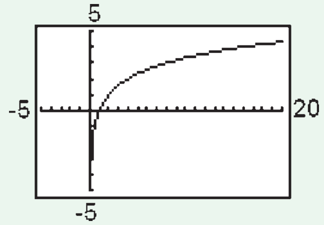
Skill Practice Answers

10. 2.829 **11.** 2.829

Calculator Connections

The change-of-base formula can be used to graph logarithmic functions with bases other than 10 or e. For example, to graph $Y_1 = \log_2 x$, we can enter the function as either

$$Y_1 = \frac{\log x}{\log 2} \quad \text{or} \quad Y_1 = \frac{\ln x}{\ln 2}$$



5. Applications of the Natural Logarithmic Function

Plant and animal tissue contains both carbon 12 and carbon 14. Carbon 12 is a stable form of carbon, whereas carbon 14 is a radioactive isotope with a half-life of approximately 5730 years. While a plant or animal is living, it takes in carbon from the atmosphere either through photosynthesis or through its food. The ratio of carbon 14 to carbon 12 in a living organism is constant and is the same as the ratio found in the atmosphere.

When a plant or animal dies, it no longer ingests carbon from the atmosphere. The amount of stable carbon 12 remains unchanged from the time of death, but the carbon 14 begins to decay. Because the rate of decay is constant, a tissue sample can be dated by comparing the percentage of carbon 14 still present to the percentage of carbon 14 assumed to be in its original living state.

The age of a tissue sample is a function of the percent of carbon 14 still present in the organism according to the following model:

$$A(p) = \frac{\ln p}{-0.000121}$$

where $A(p)$ is the age in years and p is the percentage (in decimal form) of carbon 14 still present.

Example 8 Applying the Natural Logarithmic Function to Radioactive Decay

Using the formula

$$A(p) = \frac{\ln p}{-0.000121}$$

a. Find the age of a bone that has 72% of its initial carbon 14.

b. Find the age of the Iceman, a body uncovered in the mountains of northern Italy in 1991. Samples of his hair revealed that 51.4% of the original carbon 14 was present after his death.

Solution:

a. $$A(p) = \frac{\ln p}{-0.000121}$$

$$A(0.72) = \frac{\ln (0.72)}{-0.000121} \qquad \text{Substitute 0.72 for } p.$$

$$\approx 2715 \text{ years}$$

b. $A(p) = \dfrac{\ln p}{-0.000121}$

$A(0.514) = \dfrac{\ln(0.514)}{-0.000121}$ Substitute 0.514 for p.

≈ 5500 years   The body of the Iceman is approximately 5500 years old.

Skill Practice

12. Use the formula $A(p) = \dfrac{\ln p}{-0.000121}$ (where $A(p)$ is the age in years and p is the percent of carbon 14 still present) to determine the age of a human skull that has 90% of its initial carbon 14.

Skill Practice Answers

12. ≈ 871 years old

Section 10.6 Practice Exercises

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Study Skills Exercise

1. Define the key terms.

a. e

b. Continuously compounded interest

c. Natural logarithmic function

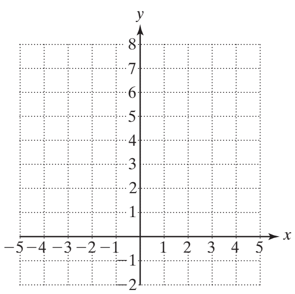
d. Change-of-base formula

Review Exercises

For Exercises 2–5, fill out the tables and graph the functions. For Exercises 4 and 5 round to two decimal places.

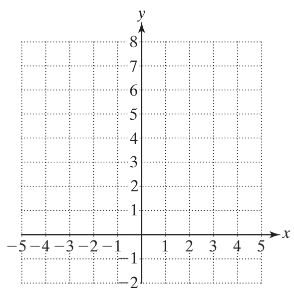
2. $f(x) = \left(\dfrac{3}{2}\right)^x$

x	$f(x)$
−3	
−2	
−1	
0	
1	
2	
3	



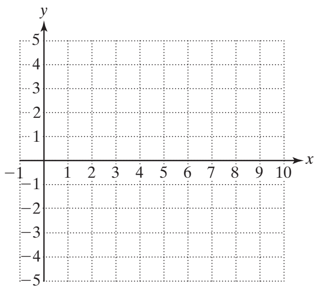
3. $g(x) = \left(\dfrac{1}{5}\right)^x$

x	$g(x)$
−3	
−2	
−1	
0	
1	
2	
3	



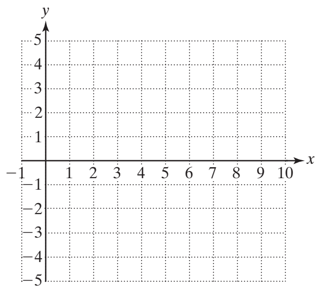
4. $q(x) = \log(x + 1)$

x	$q(x)$
-0.5	
0	
4	
9	



5. $r(x) = \log x$

x	$r(x)$
0.5	
1	
5	
10	

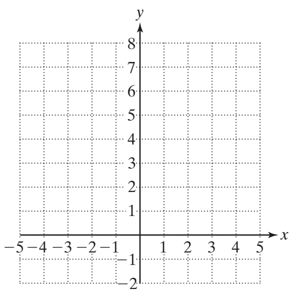


Concept 1: The Irrational Number *e*

For Exercises 6–9, graph the equation by completing the table and plotting points. Identify the domain. Round to two decimal places when necessary.

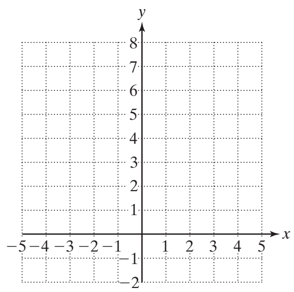
6. $y = e^{x+1}$

x	y
-4	
-3	
-2	
-1	
0	
1	



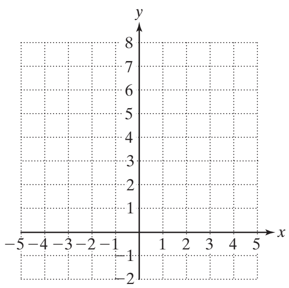
7. $y = e^{x+2}$

x	y
-5	
-4	
-3	
-2	
-1	
0	



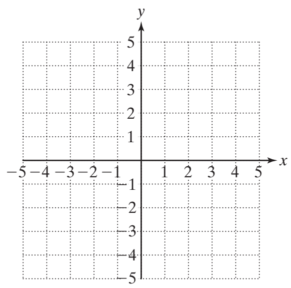
8. $y = e^x + 2$

x	y
-2	
-1	
0	
1	
2	
3	



9. $y = e^x - 1$

x	y
-4	
-3	
-2	
-1	
0	
1	



Concept 2: Computing Compound Interest

In Exercises 10–15, use the model

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

for interest compounded n times per year. Use the model $A(t) = Pe^{rt}$ for interest compounded continuously.

10. Suppose an investor deposits \$10,000 in a certificate of deposit for 5 years for which the interest is compounded monthly. Find the total amount of money in the account for the following interest rates. Compare your answers and comment on the effect of interest rate on an investment.

a. $r = 4.0\%$ **b.** $r = 6.0\%$ **c.** $r = 8.0\%$ **d.** $r = 9.5\%$

11. Suppose an investor deposits \$5000 in a certificate of deposit for 8 years for which the interest is compounded quarterly. Find the total amount of money in the account for the following interest rates. Compare your answers and comment on the effect of interest rate on an investment.

a. $r = 4.5\%$ **b.** $r = 5.5\%$ **c.** $r = 7.0\%$ **d.** $r = 9.0\%$

12. Suppose an investor deposits \$8000 in a savings account for 10 years at 4.5% interest. Find the total amount of money in the account for the following compounding options. Compare your answers. How does the number of compound periods per year affect the total investment?

a. Compounded annually
b. Compounded quarterly
c. Compounded monthly
d. Compounded daily
e. Compounded continuously

13. Suppose an investor deposits \$15,000 in a savings account for 8 years at 5.0% interest. Find the total amount of money in the account for the following compounding options. Compare your answers. How does the number of compound periods per year affect the total investment?

a. Compounded annually
b. Compounded quarterly
c. Compounded monthly
d. Compounded daily
e. Compounded continuously

14. Suppose an investor deposits \$5000 in an account bearing 6.5% interest compounded continuously. Find the total amount in the account for the following time periods. How does the length of time affect the amount of interest earned?

a. 5 years **b.** 10 years **c.** 15 years **d.** 20 years **e.** 30 years

15. Suppose an investor deposits \$10,000 in an account bearing 6.0% interest compounded continuously. Find the total amount in the account for the following time periods. How does the length of time affect the amount of interest earned?

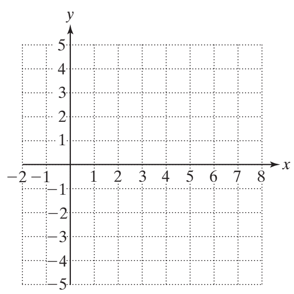
a. 5 years **b.** 10 years **c.** 15 years **d.** 20 years **e.** 30 years

Concept 3: The Natural Logarithmic Function

For Exercises 16–19, graph the equation by completing the table and plotting the points. Identify the domain. Round to two decimal places when necessary.

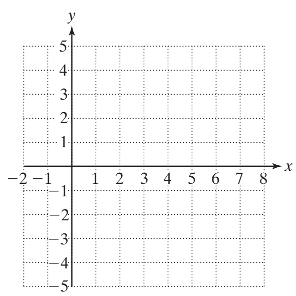
16. $y = \ln(x - 2)$

x	y
2.25	
2.50	
2.75	
3	
4	
5	
6	



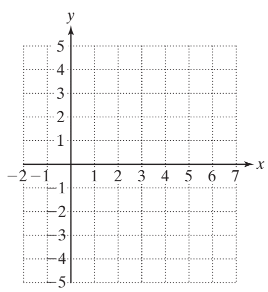
17. $y = \ln(x - 1)$

x	y
1.25	
1.50	
1.75	
2	
3	
4	
5	



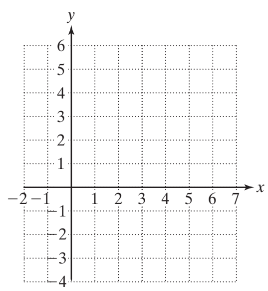
18. $y = \ln x - 1$

x	y
0.25	
0.5	
0.75	
1	
2	
3	
4	

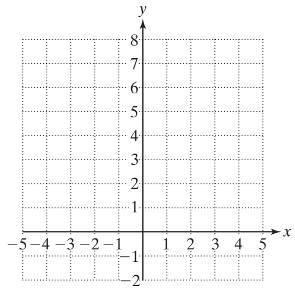


19. $y = \ln x + 2$

x	y
0.25	
0.5	
0.75	
1	
2	
3	
4	

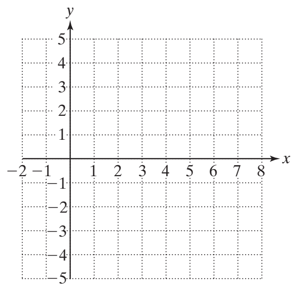


20. **a.** Graph $f(x) = 10^x$



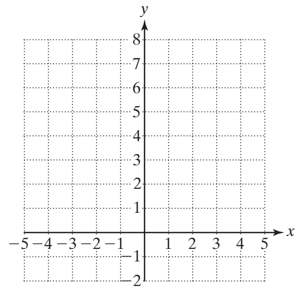
b. Identify the domain and range of *f.*

c. Graph $g(x) = \log x$.



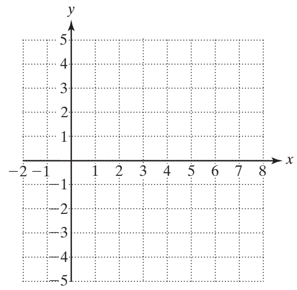
d. Identify the domain and range of *g.*

21. **a.** Graph $f(x) = e^x$.



b. Identify the domain and range of *f.*

c. Graph $g(x) = \ln x$.



d. Identify the domain and range of *g.*

For Exercises 22–29, simplify the expressions. Assume all variables represent positive real numbers.

22. $\ln e$

23. $\ln e^2$

24. $\ln 1$

25. $e^{\ln x}$

26. $\ln e^{-6}$

27. $\ln e^{(5-3x)}$

28. $e^{\ln(2x+3)}$

29. $e^{\ln 4}$

For Exercises 30–37, write the expression as a single logarithm. Assume all variables represent positive real numbers.

30. $6 \ln p + \frac{1}{3} \ln q$

31. $2 \ln w + \ln z$

32. $\frac{1}{2}(\ln x - 3 \ln y)$

33. $\frac{1}{3}(4 \ln a - \ln b)$

34. $2 \ln a - \ln b - \frac{1}{3} \ln c$

35. $-\ln x + 3 \ln y - \ln z$

36. $4 \ln x - 3 \ln y - \ln z$

37. $\frac{1}{2} \ln c + \ln a - 2 \ln b$

For Exercises 38–45, write the expression as a sum and/or difference of $\ln a$, $\ln b$, and $\ln c$. Assume all variables represent positive real numbers.

38. $\ln\left(\frac{a}{b}\right)^2$

39. $\ln \sqrt[3]{\frac{a}{b}}$

40. $\ln(b^2 \cdot e)$

41. $\ln(\sqrt{c} \cdot e)$

42. $\ln\left(\frac{a^4\sqrt{b}}{c}\right)$

43. $\ln\left(\frac{\sqrt{ab}}{c^3}\right)$

44. $\ln\left(\frac{ab}{c^2}\right)^{1/5}$

45. $\ln \sqrt{2ab}$

Concept 4: Change-of-Base Formula

46. a. Evaluate $\log_6 200$ by computing $\dfrac{\log 200}{\log 6}$ to four decimal places.

b. Evaluate $\log_6 200$ by computing $\dfrac{\ln 200}{\ln 6}$ to four decimal places.

c. How do your answers to parts (a) and (b) compare?

47. a. Evaluate $\log_8 120$ by computing $\dfrac{\log 120}{\log 8}$ to four decimal places.

b. Evaluate $\log_8 120$ by computing $\dfrac{\ln 120}{\ln 8}$ to four decimal places.

c. How do your answers to parts (a) and (b) compare?

For Exercises 48–59, use the change-of-base formula to approximate the logarithms to 4 decimal places. Check your answers by using the exponential key on your calculator.

48. $\log_2 7$ **49.** $\log_3 5$ **50.** $\log_8 24$ **51.** $\log_4 17$

52. $\log_8 (0.012)$ **53.** $\log_7 (0.251)$ **54.** $\log_9 1$ **55.** $\log_2 \left(\dfrac{1}{5}\right)$

56. $\log_4 \left(\dfrac{1}{100}\right)$ **57.** $\log_5 (0.0025)$ **58.** $\log_7 (0.0006)$ **59.** $\log_2 (0.24)$

Concept 5: Applications of the Natural Logarithmic Function

Under continuous compounding, the amount of time t in years required for an investment to double is a function of the interest rate r:

$$t = \frac{\ln 2}{r}$$

Use the formula for Exercises 60–62.

60. a. If you invest \$5000, how long will it take the investment to reach \$10,000 if the interest rate is 4.5%? Round to one decimal place.

b. If you invest \$5000, how long will it take the investment to reach \$10,000 if the interest rate is 10%? Round to one decimal place.

c. Using the doubling time found in part (b), how long would it take a \$5000 investment to reach \$20,000 if the interest rate is 10%?

61. a. If you invest \$3000, how long will it take the investment to reach \$6000 if the interest rate is 5.5%? Round to one decimal place.

b. If you invest \$3000, how long will it take the investment to reach \$6000 if the interest rate is 8%? Round to one decimal place.

c. Using the doubling time found in part (b), how long would it take a \$3000 investment to reach \$12,000 if the interest rate is 8%?

62. **a.** If you invest \$4000, how long will it take the investment to reach \$8000 if the interest rate is 3.5%? Round to one decimal place.

b. If you invest \$4000, how long will it take the investment to reach \$8000 if the interest rate is 5%? Round to one decimal place.

c. Using the doubling time found in part (b), how long would it take a \$4000 investment to reach \$16,000 if the interest rate is 5%?

63. On August 31, 1854, an epidemic of cholera was discovered in London, England, resulting from a contaminated community water pump at Broad Street. By the end of September more than 600 citizens who drank water from the pump had died.

The cumulative number of deaths from cholera in the 1854 London epidemic can be approximated by

$$D(t) = 91 + 160 \ln (t + 1)$$

where t is the number of days after the start of the epidemic ($t = 0$ corresponds to September 1, 1854).

a. Approximate the total number of deaths as of September 1 ($t = 0$).

b. Approximate the total number of deaths as of September 5, September 10, and September 20.

Graphing Calculator Connections

64. **a.** Graph the function defined by $f(x) = \log_3 x$ by graphing $Y_1 = \dfrac{\log x}{\log 3}$.

b. Graph the function defined by $f(x) = \log_3 x$ by graphing $Y_2 = \dfrac{\ln x}{\ln 3}$.

c. Does it appear that $Y_1 = Y_2$ on the standard viewing window?

65. **a.** Graph the function defined by $f(x) = \log_7 x$ by graphing $Y_1 = \dfrac{\log x}{\log 7}$.

b. Graph the function defined by $f(x) = \log_7 x$ by graphing $Y_2 = \dfrac{\ln x}{\ln 7}$.

c. Does it appear that $Y_1 = Y_2$ on the standard viewing window?

66. **a.** Graph the function defined by $f(x) = \log_{1/3} x$ by graphing $Y_1 = \dfrac{\log x}{\log \left(\frac{1}{3}\right)}$.

b. Graph the function defined by $f(x) = \log_{1/3} x$ by graphing $Y_2 = \dfrac{\ln x}{\ln \left(\frac{1}{3}\right)}$.

c. Does it appear that $Y_1 = Y_2$ on the standard viewing window?

67. **a.** Graph the function defined by $f(x) = \log_{1/7} x$ by graphing $Y_1 = \dfrac{\log x}{\log \left(\frac{1}{7}\right)}$.

b. Graph the function defined by $f(x) = \log_{1/7} x$ by graphing $Y_2 = \dfrac{\ln x}{\ln \left(\frac{1}{7}\right)}$.

c. Does it appear that $Y_1 = Y_2$ on the standard viewing window?

For Exercises 68–70, graph the functions on your calculator.

68. Graph $s(x) = \log_{1/2} x$

69. Graph $y = e^{x-2}$

70. Graph $y = e^{x-1}$

Chapter 10 Problem Recognition Exercises—Logarithmic and Exponential Forms

Fill out the table by writing the exponential expressions in logarithmic form, and the logarithmic expressions in exponential form. Use the fact that:

$$y = \log_b x \quad \text{is equivalent to} \quad b^y = x$$

	Exponential Form	Logarithmic Form
1.	$2^5 = 32$	
2.		$\log_3 81 = 4$
3.	$z^y = x$	
4.		$\log_b a = c$
5.	$10^3 = 1000$	
6.		$\log 10 = 1$
7.	$e^a = b$	
8.		$\ln p = q$
9.	$\left(\frac{1}{2}\right)^2 = \frac{1}{4}$	
10.		$\log_{1/3} 9 = -2$
11.	$10^{-2} = 0.01$	
12.		$\log 4 = x$
13.	$e^0 = 1$	
14.		$\ln e = 1$
15.	$25^{1/2} = 5$	
16.		$\log_{16} 2 = \frac{1}{4}$
17.	$e^t = s$	
18.		$\ln w = r$
19.	$15^{-2} = \frac{1}{225}$	
20.		$\log_3 p = -1$

Section 10.7 Logarithmic and Exponential Equations

Concepts

1. Solving Logarithmic Equations

Equations containing one or more logarithms are called **logarithmic equations**. For example,

$$\ln(2x + 5) = 1 \quad \text{and} \quad \log_4 = 1 - \log_4(x - 3)$$

are logarithmic equations. To solve a basic logarithmic equation such as $\ln(2x + 5) = 1$, write the expression in exponential form:

$$\ln(2x + 5) = 1 \Rightarrow e^1 = 2x + 5. \text{ Now solve for } x.$$

$$e - 5 = 2x$$

$$\frac{e - 5}{2} = x \quad \text{or} \quad x \approx -1.14$$

To solve equations containing more than one logarithm of first degree, use the following guidelines.

Guidelines to Solve Logarithmic Equations

1. Isolate the logarithms on one side of the equation.
2. Write a sum or difference of logarithms as a single logarithm.
3. Rewrite the equation in step 2 in exponential form.
4. Solve the resulting equation from step 3.
5. Check all solutions to verify that they are within the domain of the logarithmic expressions in the original equation.

Example 1 Solving a Logarithmic Equation

Solve the equation.

$$\log_4 x = 1 - \log_4(x - 3)$$

Solution:

$\log_4 x = 1 - \log_4(x - 3)$	
$\log_4 x + \log_4(x - 3) = 1$	Isolate the logarithms on one side of the equation.
$\log_4[x(x - 3)] = 1$	Write as a single logarithm.
$\log_4(x^2 - 3x) = 1$	Simplify inside the parentheses.
$x^2 - 3x = 4^1$	Write the equation in exponential form.
$x^2 - 3x - 4 = 0$	The resulting equation is quadratic.
$(x - 4)(x + 1) = 0$	Factor.
$x = 4 \quad \text{or} \quad x = -1$	Apply the zero product rule.

Notice that $x = -1$ is *not* a solution because $\log_4 x$ is not defined at $x = -1$. However, $x = 4$ *is* defined in both expressions $\log_4 x$ and $\log_4(x - 3)$. We can substitute $x = 4$ into the original equation to show that it checks.

Check: $x = 4$

$$\log_4 x = 1 - \log_4(x - 3)$$

$$\log_4 4 \stackrel{?}{=} 1 - \log_4(4 - 3)$$

$$1 \stackrel{?}{=} 1 - \log_4 1$$

$$1 \stackrel{?}{=} 1 - 0 \checkmark \text{ True}$$

The solution is $x = 4$.

Skill Practice

1. Solve the equation.

$$\log_3(x - 8) = 2 - \log_3 x$$

TIP: The equation from Example 1 involved the logarithmic functions $y = \log_4 x$ and $y = \log_4(x - 3)$. The domains of these functions are $\{x \mid x > 0\}$ and $\{x \mid x > 3\}$, respectively. Therefore, the solutions to the equation are restricted to x-values in the intersection of these two sets, that is, $\{x \mid x > 3\}$. The solution $x = 4$ satisfies this requirement, whereas $x = -1$ does not.

Example 2 Solving Logarithmic Equations

Solve the equations.

a. $\log(x + 300) = 3.7$

b. $\ln(x + 2) + \ln(x - 1) = \ln(9x - 17)$

Solution:

a. $\log(x + 300) = 3.7$ — The equation has a single logarithm that is already isolated.

$10^{3.7} = x + 300$ — Write the equation in exponential form.

$10^{3.7} - 300 = x$ — Solve for x.

$x = 10^{3.7} - 300 \approx 4711.87$

Check: $x = 10^{3.7} - 300$ — Check the exact value of x in the original equation.

$$\log(x + 300) = 3.7$$

$$\log[(10^{3.7} - 300) + 300] \stackrel{?}{=} 3.7$$

$$\log(10^{3.7} - 300 + 300) \stackrel{?}{=} 3.7$$

$\log 10^{3.7} \stackrel{?}{=} 3.7$ — Property 3 of logarithms: $\log_b b^p = p$

$$3.7 = 3.7 \checkmark \text{ True}$$

The solution $x = 10^{3.7} - 300$ checks.

Skill Practice Answers

1. $x = 9$ ($x = -1$ does not check)

b. $\ln(x + 2) + \ln(x - 1) = \ln(9x - 17)$

$\ln(x + 2) + \ln(x - 1) - \ln(9x - 17) = 0$ — Isolate the logarithms on one side.

$\ln\left[\frac{(x + 2)(x - 1)}{9x - 17}\right] = 0$ — Write as a single logarithm.

$e^0 = \frac{(x + 2)(x - 1)}{9x - 17}$ — Write the equation in exponential form.

$1 = \frac{(x + 2)(x - 1)}{9x - 17}$ — Simplify.

$(1) \cdot (9x - 17) = \left[\frac{(x + 2)(x - 1)}{\cancel{9x - 17}}\right] \cdot \cancel{(9x - 17)}$ — Multiply by the LCD.

$9x - 17 = (x + 2)(x - 1)$ — The equation is quadratic.

$9x - 17 = x^2 + x - 2$

$0 = x^2 - 8x + 15$

$0 = (x - 5)(x - 3)$

$x = 5 \quad \text{or} \quad x = 3$

The solutions $x = 5$ and $x = 3$ are both within the domain of the logarithmic functions in the original equation.

Check: $x = 5$

$\ln(x + 2) + \ln(x - 1) = \ln(9x - 17)$

$\ln(5 + 2) + \ln(5 - 1) \stackrel{?}{=} \ln[9(5) - 17]$

$\ln 7 + \ln 4 \stackrel{?}{=} \ln(45 - 17)$

$\ln(7 \cdot 4) \stackrel{?}{=} \ln(28)$

$\ln 28 = \ln 28$ ✓ True

Check: $x = 3$

$\ln(x + 2) + \ln(x - 1) = \ln(9x - 17)$

$\ln(3 + 2) + \ln(3 - 1) \stackrel{?}{=} \ln[9(3) - 17]$

$\ln 5 + \ln 2 \stackrel{?}{=} \ln(27 - 17)$

$\ln(5 \cdot 2) \stackrel{?}{=} \ln(10)$

$\ln 10 = \ln 10$ ✓ True

Both solutions check.

Skill Practice Solve the equations.

2. $\log(2p + 6) = 1$ **3.** $\ln(t - 3) + \ln(t - 1) = \ln(2t - 5)$

Skill Practice Answers

2. $p = 2$
3. $t = 4$ ($t = 2$ does not check)

2. Applications of Logarithmic Equations

Example 3 Applying a Logarithmic Equation to Earthquake Intensity

The magnitude of an earthquake (the amount of seismic energy released at the hypocenter of the earthquake) is measured on the Richter scale. The Richter scale value R is determined by the formula

$$R = \log\left(\frac{I}{I_0}\right)$$

where I is the intensity of the earthquake and I_0 is the minimum measurable intensity of an earthquake. (I_0 is a "zero-level" quake—one that is barely detected by a seismograph.)

a. Compare the Richter scale values of earthquakes that are (i) 100,000 times (10^5 times) more intense than I_0 and (ii) 1,000,000 times (10^6 times) more intense than I_0.

b. On October 17, 1989, an earthquake measuring 7.1 on the Richter scale occurred in the Loma Prieta area in the Santa Cruz Mountains. The quake devastated parts of San Francisco and Oakland, California, bringing 63 deaths and over 3700 injuries. Determine how many times more intense this earthquake was than a zero-level quake.

Solution:

a. $R = \log\left(\frac{I}{I_0}\right)$

i. Earthquake 100,000 times I_0

$R = \log\left(\frac{10^5 \cdot I_0}{I_0}\right)$ Substitute $I = 10^5 I_0$.

$= \log 10^5$

$= 5$

ii. Earthquake 1,000,000 times I_0

$R = \log\left(\frac{10^6 \cdot I_0}{I_0}\right)$ Substitute $I = 10^6 I_0$.

$= \log 10^6$

$= 6$

Notice that the value on the Richter scale corresponds to the magnitude (power of 10) of the energy released. That is, a 1-unit increase on the Richter scale represents a 10-fold increase in the intensity of an earthquake.

b. $R = \log\left(\frac{I}{I_0}\right)$

$7.1 = \log\left(\frac{I}{I_0}\right)$ Substitute $R = 7.1$.

$\frac{I}{I_0} = 10^{7.1}$ Write the equation in exponential form.

$I = 10^{7.1} \cdot I_0$ Solve for I.

The Loma Prieta earthquake in 1989 was $10^{7.1}$ times ($\approx$12,590,000 times) more intense than a zero-level earthquake.

Skill Practice

4. In December 2004, an earthquake in the Indian Ocean measuring 9.0 on the Richter scale caused a tsunami that killed more than 300,000 people. Determine how many times more intense this earthquake was than a zero-level quake. Use the formula

$$R = \log\left(\frac{I}{I_0}\right)$$

where I is the intensity of the quake, I_0 is the measure of a zero-level quake, and R is the Richter scale value.

Skill Practice Answers

4. The earthquake was 10^9 times more intense than a zero-level quake.

3. Solving Exponential Equations

An equation with one or more exponential expressions is called an **exponential equation**. The following property is often useful in solving exponential equations.

Equivalence of Exponential Expressions

Let x, y, and b be real numbers such that $b > 0$ and $b \neq 1$. Then

$$b^x = b^y \quad \text{implies} \quad x = y$$

The equivalence of exponential expressions indicates that if two exponential expressions of the same base are equal, their exponents must be equal.

Example 4 Solving Exponential Equations

Solve the equations.

a. $4^{2x-9} = 64$ **b.** $(2^x)^{x+3} = \frac{1}{4}$

Solution:

a.

$4^{2x-9} = 64$

$4^{2x-9} = 4^3$ Write both sides with a common base.

$2x - 9 = 3$ If $b^x = b^y$, then $x = y$.

$2x = 12$ Solve for x.

$x = 6$

To check, substitute $x = 6$ into the original equation.

$$4^{2(6)-9} \stackrel{?}{=} 64$$

$$4^{12-9} \stackrel{?}{=} 64$$

$$4^3 = 64 \checkmark \text{ True}$$

b.

$(2^x)^{x+3} = \frac{1}{4}$

$2^{x^2+3x} = 2^{-2}$ Apply the multiplication property of exponents. Write both sides of the equation with a common base.

$x^2 + 3x = -2$ If $b^x = b^y$, then $x = y$.

$x^2 + 3x + 2 = 0$ The resulting equation is quadratic.

$(x + 2)(x + 1) = 0$ Solve for x.

$x = -2$ or $x = -1$

The check is left to the reader.

Skill Practice Solve the equations.

5. $2^{3x+1} = 8$ **6.** $(3^x)^{x-5} = \frac{1}{81}$

Skill Practice Answers

5. $x = \frac{2}{3}$ **6.** $x = 1$ or $x = 4$

Example 5 Solving an Exponential Equation

Solve the equation. $4^x = 25$

Solution:

Because 25 cannot be written as an integral power of 4, we cannot use the property that if $b^x = b^y$, then $x = y$. Instead we can rewrite the equation in its corresponding logarithmic form to solve for x.

$4^x = 25$

$x = \log_4 25$ Write the equation in logarithmic form

$= \dfrac{\ln 25}{\ln 4} \approx 2.322$ Change-of-base formula

The same result can be reached by taking a logarithm of any base on both sides of the equation. Then by applying the power property of logarithms, the unknown exponent can be written as a factor.

$4^x = 25$

$\log 4^x = \log 25$ Take the common logarithm of both sides.

$x \log 4 = \log 25$ Apply the power property of logarithms to express the exponent as a factor. This is now a linear equation in x.

$\dfrac{x \log 4}{\log 4} = \dfrac{\log 25}{\log 4}$ Solve for x.

$x = \dfrac{\log 25}{\log 4} \approx 2.322$

Skill Practice

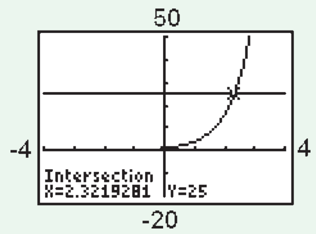
7. Solve the equation.

$5^x = 32$

Calculator Connections

Graph $Y_1 = 4^x$ and $Y_2 = 25$.

An *Intersect* feature can be used to find the x-coordinate where $Y_1 = Y_2$.



Guidelines to Solve Exponential Equations

1. Isolate one of the exponential expressions in the equation.
2. Take a logarithm on both sides of the equation. (The natural logarithmic function or the common logarithmic function is often used so that the final answer can be approximated with a calculator.)
3. Use the power property of logarithms (property 7) to write exponents as factors. Recall: $\log_b M^p = p \log_b M$.
4. Solve the resulting equation from step 3.

Example 6 Solving Exponential Equations by Taking a Logarithm on Both Sides

Solve the equations.

a. $2^{x+3} = 7^x$ **b.** $e^{-3.6x} = 9.74$

Skill Practice Answers

7. $x = \dfrac{\ln 32}{\ln 5} \approx 2.153$

Solution:

a.

$2^{x+3} = 7^x$

$\ln 2^{(x+3)} = \ln 7^x$ Take the natural logarithm of both sides.

$(x + 3) \ln 2 = x \ln 7$ Use the power property of logarithms.

$x (\ln 2) + 3 (\ln 2) = x \ln 7$ Apply the distributive property.

$x (\ln 2) - x (\ln 7) = -3 \ln 2$ Collect x-terms on one side.

$x(\ln 2 - \ln 7) = -3 \ln 2$ Factor out x.

$\frac{x(\ln 2 - \ln 7)}{(\ln 2 - \ln 7)} = \frac{-3 \ln 2}{(\ln 2 - \ln 7)}$ Solve for x.

$x = \frac{-3 \ln 2}{(\ln 2 - \ln 7)} \approx 1.660$

TIP: The exponential equation $2^{x+3} = 7^x$ could have been solved by taking a logarithm of *any* base on both sides of the equation. For example, using base 10 yields

$\log 2^{x+3} = \log 7^x$

$(x + 3) \log 2 = x \log 7$ Apply the power property for logarithms.

$x \log 2 + 3 \log 2 = x \log 7$ Apply the distributive property.

$x \log 2 - x \log 7 = -3 \log 2$ Collect x-terms on one side of the equation.

$x(\log 2 - \log 7) = -3 \log 2$ Factor out x.

$x = \frac{-3 \log 2}{\log 2 - \log 7} \approx 1.660$

b.

$e^{-3.6x} = 9.74$

$\ln e^{-3.6x} = \ln 9.74$ The exponential expression has a base of e, so it is convenient to take the natural logarithm of both sides.

$(-3.6x) \ln e = \ln 9.74$ Use the power property of logarithms.

$-3.6x = \ln 9.74$ Simplify (recall that $\ln e = 1$).

$x = \frac{\ln 9.74}{-3.6} \approx -0.632$

Skill Practice Solve the equations.

8. $3^x = 8^{x+2}$ **9.** $e^{-0.2t} = 7.52$

Skill Practice Answers

8. $x = \frac{2 \ln 8}{\ln 3 - \ln 8} \approx -4.240$

9. $t = \frac{\ln 7.52}{-0.2} \approx -10.088$

4. Applications of Exponential Equations

Example 7 Applying an Exponential Function to World Population

The population of the world was estimated to have reached 6.5 billion in April 2006. The population growth rate for the world is estimated to be 1.4%. (Source: U.S. Census Bureau)

$$P(t) = 6.5(1.014)^t$$

represents the world population in billions as a function of the number of years after April 2006. ($t = 0$ represents April 2006).

a. Use the function to estimate the world population in April 2010.

b. Use the function to estimate the amount of time after April 2006 required for the world population to reach 13 billion.

Solution:

a.

$$P(t) = 6.5(1.014)^t$$

$$P(4) = 6.5(1.014)^4$$ The year 2010 corresponds to $t = 4$.

$$\approx 6.87$$ In 2010, the world's population will be approximately 6.87 billion.

b.

$$P(t) = 6.5(1.014)^t$$

$$13 = 6.5(1.014)^t$$ Substitute $P(t) = 13$ and solve for t.

$$\frac{13}{6.5} = \frac{6.5(1.014)^t}{6.5}$$ Isolate the exponential expression on one side of the equation.

$$2 = 1.014^t$$

$$\ln 2 = \ln 1.014^t$$ Take the natural logarithm of both sides.

$$\ln 2 = t \ln 1.014$$ Use the power property of logarithms.

$$\frac{\ln 2}{\ln 1.014} = \frac{t \ln 1.014}{\ln 1.014}$$ Solve for t.

$$t = \frac{\ln 2}{\ln 1.014} \approx 50$$ The population will reach 13 billion (double the April 2006 value) approximately 50 years after 2006.

Note: It has taken thousands of years for the world's population to reach 6.5 billion. However, with a growth rate of 1.4%, it will take only 50 years to gain an additional 6.5 billion.

Skill Practice Use the population function from Example 7.

10. Estimate the world population in April 2020.

11. Estimate the year in which the world population will reach 9 billion.

Skill Practice Answers

10. Approximately 7.9 billion

11. The year 2029

On Friday, April 25, 1986, a nuclear accident occurred at the Chernobyl nuclear reactor, resulting in radioactive contaminates being released into the atmosphere. The most hazardous isotopes released in this accident were ^{137}Cs (cesium 137), ^{131}I (iodine 131), and ^{90}Sr (strontium 90). People living close to Chernobyl (in Ukraine) were at risk of radiation exposure from inhalation, from absorption through the skin, and from food contamination. Years after the incident, scientists have seen an increase in the incidence of thyroid disease among children living in the contaminated areas. Because iodine is readily absorbed in the thyroid gland, scientists suspect that radiation from iodine 131 is the cause.

Example 8 Applying an Exponential Equation to Radioactive Decay

The half-life of radioactive iodine ^{131}I is 8.04 days. If 10 g of iodine 131 is initially present, then the amount of radioactive iodine still present after t days is approximated by

$$A(t) = 10e^{-0.0862t}$$

where t is the time in days.

a. Use the model to approximate the amount of ^{131}I still present after 2 weeks. Round to the nearest 0.1 g.

b. How long will it take for the amount of ^{131}I to decay to 0.5 g? Round to the nearest 0.1 year.

Solution:

a. $A(t) = 10e^{-0.0862t}$

$A(14) = 10e^{-0.0862(14)}$ Substitute $t = 14$ (2 weeks).

≈ 3.0 g

b. $A(t) = 10e^{-0.0862t}$

$0.5 = 10e^{-0.0862t}$ Substitute $A = 0.5$.

$\dfrac{0.5}{10} = \dfrac{10e^{-0.0862t}}{10}$ Isolate the exponential expression.

$0.05 = e^{-0.0862t}$

$\ln(0.05) = \ln(e^{-0.0862t})$ Take the natural logarithm of both sides.

$\ln(0.05) = -0.0862t$ The resulting equation is linear.

$\dfrac{\ln(0.05)}{-0.0862} = \dfrac{-0.0862t}{-0.0862}$ Solve for t.

$t = \dfrac{\ln(0.05)}{-0.0862} \approx 34.8$ years

Note: Radioactive iodine (^{131}I) is used in medicine in appropriate dosages to treat patients with hyperactive (overactive) thyroids. Because iodine is readily absorbed in the thyroid gland, the radiation is localized and will reduce the size of the thyroid while minimizing damage to surrounding tissues.

Skill Practice Radioactive strontium 90 (^{90}Sr) has a half-life of 28 years. If 100 g of strontium 90 is initially present, the amount left after t years is approximated by

$$A(t) = 100e^{-0.0248t}$$

12. Find the amount of ^{90}Sr present after 85 years.
13. How long will it take for the amount of ^{90}Sr to decay to 40 g?

Skill Practice Answers
12. 12.1 g **13.** 36.9 years

Section 10.7 Practice Exercises

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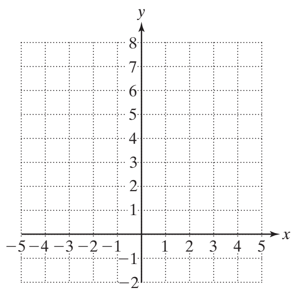
Study Skills Exercise

1. Define the key terms.

a. Logarithmic equation **b. Exponential equation**

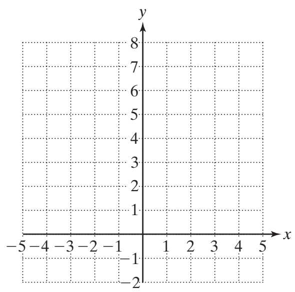
Review Exercises

2. a. Graph $f(x) = e^x$.



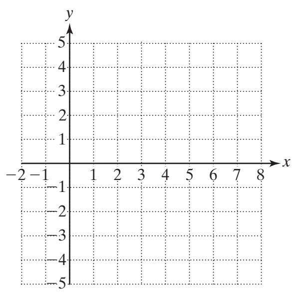
b. Write the domain and range in interval notation.

3. a. Graph $g(x) = 3^x$.



b. Write the domain and range in interval notation.

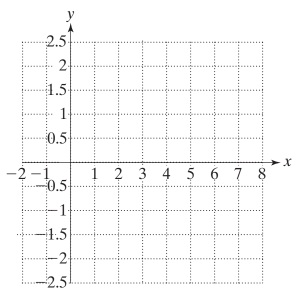
4. a. Graph $h(x) = \ln x$.



b. Write the equation of the vertical asymptote.

c. Write the domain and range in interval notation.

5. a. Graph $k(x) = \log x$.



b. Write the equation of the vertical asymptote.

c. Write the domain and range in interval notation.

For Exercises 6–9, write the expression as a single logarithm. Assume all variables represent positive real numbers.

6. $\log_b(x - 1) + \log_b(x + 2)$

7. $\log_b x + \log_b(2x + 3)$

8. $\log_b x - \log_b(1 - x)$

9. $\log_b(x + 2) - \log_b(3x - 5)$

Concept 1: Solving Logarithmic Equations

Solve the logarithmic equations in Exercises 10–41.

10. $\log_3 x = 2$

11. $\log_4 x = 9$

12. $\log p = 42$

13. $\log q = \frac{1}{2}$

14. $\ln x = 0.08$

15. $\ln x = 19$

16. $\log_2 x = -4$

17. $\log x = -1$

18. $\log_x 25 = 2 \quad (x > 0)$

19. $\log_x 100 = 2 \quad (x > 0)$

20. $\log_b 10{,}000 = 4 \quad (b > 0)$

21. $\log_b e^3 = 3 \quad (b > 0)$

22. $\log_y 5 = \frac{1}{2} \quad (y > 0)$

23. $\log_b 8 = \frac{1}{2} \quad (b > 0)$

24. $\log_4(c + 5) = 3$

25. $\log_5(a - 4) = 2$

26. $\log_5(4y + 1) = 1$

27. $\log_6(5t - 2) = 1$

28. $\ln(1 - x) = 0$

29. $\log_4(2 - x) = 1$

30. $\log_3 8 - \log_3(x + 5) = 2$

31. $\log_2(x + 3) - \log_2(x + 2) = 1$

32. $\log_3 k + \log_3(2k + 3) = 2$

33. $\log_2(h - 1) + \log_2(h + 1) = 3$

34. $\log(x + 2) = \log(3x - 6)$

35. $\log x = \log(1 - x)$

36. $\ln x - \ln(4x - 9) = 0$

37. $\ln(x + 5) - \ln x = \ln(4x)$

38. $\log_5(3t + 2) - \log_5 t = \log_5 4$

39. $\log(6y - 7) + \log y = \log 5$

40. $\log(4m) = \log 2 + \log(m - 3)$

41. $\log(-h) + \log 3 = \log(2h - 15)$

Concept 2: Applications of Logarithmic Equations

42. On May 28, 2004, an earthquake that measured 6.3 on the Richter scale occurred in northern Iran. Use the formula $R = \log\left(\frac{I}{I_0}\right)$ to determine how many times more intense this earthquake was than a zero-level quake.

43. Papua, Indonesia, had a 7.0 earthquake on February 4, 2004. Determine how many times more intense this earthquake was than a zero-level quake.

The decibel level of sound can be found by the equation $D = 10\log\left(\frac{I}{I_0}\right)$, where I is the intensity of the sound and I_0 is the intensity of the least audible sound that an average healthy person can hear. Generally I_0 is found as 10^{-12} watt per square meter (W/m^2). Use this information to answer Exercises 44–45.

44. Given that heavy traffic has a decibel level of 89.3 and that $I_0 = 10^{-12}$, find the intensity of the sound of heavy traffic.

45. Given that normal conversation has a decibel level of 65 and $I_0 = 10^{-12}$, find the intensity of the sound of normal conversation.

Concept 3: Solving Exponential Equations

For Exercises 46–61, solve the exponential equation by using the property that $b^x = b^y$ implies $x = y$, for $b > 0$ and $b \neq 1$.

46. $5^x = 625$

47. $3^x = 81$

48. $2^{-x} = 64$

49. $6^{-x} = 216$

50. $36^x = 6$

51. $343^x = 7$

52. $4^{2x-1} = 64$

53. $5^{3x-1} = 125$

54. $81^{3x-4} = \frac{1}{243}$

55. $4^{2x-7} = \frac{1}{128}$

56. $\left(\frac{2}{3}\right)^{-x+4} = \frac{8}{27}$

57. $\left(\frac{1}{4}\right)^{3x+2} = \frac{1}{64}$

58. $16^{-x+1} = 8^{5x}$

59. $27^{(1/3)x} = \left(\frac{1}{9}\right)^{2x-1}$

60. $(4^x)^{x+1} = 16$

61. $(3^x)^{x+2} = \frac{1}{3}$

For Exercises 62–73, solve the exponential equations by taking the logarithm of both sides. (Round the answers to three decimal places.)

62. $8^a = 21$

63. $6^y = 39$

64. $e^x = 8.1254$

65. $e^x = 0.3151$

66. $10^t = 0.0138$

67. $10^p = 16.8125$

68. $e^{0.07h} = 15$

69. $e^{0.03k} = 4$

70. $32e^{0.04m} = 128$

71. $8e^{0.05n} = 160$

72. $3^{x+1} = 5^x$

73. $2^{x-1} = 7^x$

Concept 4: Applications of Exponential Equations

74. The population of China can be modeled by the function

$$P(t) = 1237(1.0095)^t$$

where $P(t)$ is in millions and t is the number of years since 1998.

a. Using this model, what was the population in the year 2002?

b. Predict the population in the year 2012.

c. If this growth rate continues, in what year will the population reach 2 billion people (2 billion is 2000 million)?

75. The population of Delhi, India, can be modeled by the function

$$P(t) = 9817(1.031)^t$$

where $P(t)$ is in thousands and t is the number of years since 2001.

a. Using this model, predict the population in the year 2010.

b. If this growth rate continues, in what year will the population reach 15 million (15 million is 15,000 thousand)?

76. The growth of a certain bacteria in a culture is given by the model

$$A(t) = 500e^{0.0277t}$$

where $A(t)$ is the number of bacteria and t is time in minutes.

a. What is the initial number of bacteria?

b. What is the population after 10 min?

c. How long will it take for the population to double (that is, reach 1000)?

77. The population of the bacteria *Salmonella typhimurium* is given by the model

$$A(t) = 300e^{0.01733t}$$

where $A(t)$ is the number of bacteria and t is time in minutes.

a. What is the initial number of bacteria?

b. What is the population after 10 min?

c. How long will it take for the population to double?

78. Suppose \$5000 is invested at 7% interest compounded continuously. How long will it take for the investment to grow to \$10,000? Use the model $A(t) = Pe^{rt}$ and round to the nearest tenth of a year.

79. Suppose \$2000 is invested at 10% interest compounded continuously. How long will it take for the investment to triple? Use the model $A(t) = Pe^{rt}$ and round to the nearest year.

80. Phosphorus 32 (^{32}P) has a half-life of approximately 14 days. If 10 g of ^{32}P is present initially, then the amount of phosphorus 32 still present after t days is given by $A(t) = 10(0.5)^{t/14}$.

a. Find the amount of phosphorus 32 still present after 5 days. Round to the nearest tenth of a gram.

b. Find the amount of time necessary for the amount of ^{32}P to decay to 4 g. Round to the nearest tenth of a day.

81. Polonium 210 (^{210}Po) has a half-life of approximately 138.6 days. If 4 g of ^{210}Po is present initially, then the amount of polonium 210 still present after t days is given by $A(t) = 4e^{-0.005t}$.

a. Find the amount of polonium 138 still present after 50 days. Round to the nearest tenth of a gram.

b. Find the amount of time necessary for the amount of ^{32}P to decay to 0.5 g. Round to the nearest tenth of a day.

82. Suppose you save \$10,000 from working an extra job. Rather than spending the money, you decide to save the money for retirement by investing in a mutual fund that averages 12% per year. How long will it take for this money to grow to \$1,000,000? Use the model $A(t) = Pe^{rt}$ and round to the nearest tenth of a year.

83. The model $A = Pe^{rt}$ is used to compute the total amount of money in an account after t years at an interest rate r, compounded continuously. The value P is the initial principal. Find the amount of time required for the investment to double as a function of the interest rate. (*Hint:* Substitute $A = 2P$ and solve for t.)

Expanding Your Skills

84. The isotope of plutonium of mass 238 (written ^{238}Pu) is used to make thermoelectric power sources for spacecraft. The heat and electric power derived from such units have made the Voyager, Gallileo, and Cassini missions to the outer reaches of our solar system possible. The half-life of ^{238}Pu is 87.7 years.

Suppose a hypothetical space probe is launched in the year 2002 with 2.0 kg of ^{238}Pu. Then the amount of ^{238}Pu available to power the spacecraft decays over time according to

$$P(t) = 2e^{-0.0079t}$$

where $t \geq 0$ is the time in years and $P(t)$ is the amount of plutonium still present (in kilograms).

a. Suppose the space probe is due to arrive at Pluto in the year 2045. How much plutonium will remain when the spacecraft reaches Pluto? Round to 2 decimal places.

b. If 1.5 kg of ^{238}Pu is required to power the spacecraft's data transmitter, will there be enough power in the year 2045 for us to receive close-up images of Pluto?

85. ^{99m}Tc is a radionuclide of technetium that is widely used in nuclear medicine. Although its half-life is only 6 hr, the isotope is continuously produced via the decay of its longer-lived parent ^{99}Mo (molybdenum 99) whose half-life is approximately 3 days. The ^{99}Mo generators (or "cows") are sold to hospitals in which the ^{99m}Tc can be "milked" as needed over a period of a few weeks. Once separated from its parent, the ^{99m}Tc may be chemically incorporated into a variety of imaging agents, each of which is designed to be taken up by a specific target organ within the body. Special cameras, sensitive to the gamma rays emitted by the technetium, are then used to record a "picture" (similar in appearance to an X-ray film) of the selected organ.

Suppose a technician prepares a sample of ^{99m}Tc-pyrophosphate to image the heart of a patient suspected of having had a mild heart attack. If the injection contains 10 millicuries (mCi) of ^{99m}Tc at 1:00 P.M., then the amount of technetium still present is given by

$$T(t) = 10e^{-0.1155t}$$

where $t > 0$ represents the time in hours after 1:00 P.M. and $T(t)$ represents the amount of ^{99m}Tc (in millicuries) still present.

a. How many millicuries of ^{99m}Tc will remain at 4:20 P.M. when the image is recorded? Round to the nearest tenth of a millicurie.

b. How long will it take for the radioactive level of the ^{99m}Tc to reach 2 mCi? Round to the nearest tenth of an hour.

For Exercises 86–89, solve the equations.

86. $(\log x)^2 - 2\log x - 15 = 0$
(*Hint:* Let $u = \log x$.)

87. $(\log_2 z)^2 - 3\log_2 z - 4 = 0$

88. $(\log_3 w)^2 + 5\log_3 w + 6 = 0$

89. $(\ln x)^2 - 2\ln x = 0$

Graphing Calculator Exercises

90. The amount of money a company receives from sales is related to the money spent on advertising according to

$$S(x) = 400 + 250\log x, \quad x \geq 1$$

where $S(x)$ is the amount in sales (in \$1000s) and x is the amount spent on advertising (in \$1000s).

a. The value of $S(1) = 400$ means that if \$1000 is spent on advertising, the total sales will be \$400,000.

i. Find the total sales for this company if \$11,000 is spent on advertising.

ii. Find the total sales for this company if \$21,000 is spent on advertising.

iii. Find the total sales for this company if \$31,000 is spent on advertising.

b. Graph the function $y = S(x)$ on a window where $0 \leq x \leq 40$ and $0 \leq y \leq 1000$. Using the graph and your answers from part (a), describe the relationship between total sales and the money spent on advertising. As the money spent on advertising is increased, what happens to the rate of increase of total sales?

c. How many advertising dollars are required for the total sales to reach \$1,000,000? That is, for what value of x will $S(x) = 1000$?

91. Graph $Y_1 = 8\text{^}x$ and $Y_2 = 21$ on a window where $0 \le x \le 5$ and $0 \le y \le 40$. Use the graph and an *Intersect* feature or *Zoom* and *Trace* to support your answer to Exercise 62.

92. Graph $Y_1 = 6\text{^}x$ and $Y_2 = 39$ on a window where $0 \le x \le 5$ and $0 \le y \le 50$. Use the graph and an *Intersect* feature or *Zoom* and *Trace* to support your answer to Exercise 63.

Chapter 10 SUMMARY

Section 10.1 Algebra and Composition of Functions

Key Concepts

The Algebra of Functions

Given two functions f and g, the functions $f + g, f - g, f \cdot g$, and $\dfrac{f}{g}$ are defined as

$(f + g)(x) = f(x) + g(x)$

$(f - g)(x) = f(x) - g(x)$

$(f \cdot g)(x) = f(x) \cdot g(x)$

$\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$ provided $g(x) \neq 0$

Composition of Functions

The **composition** of f and g, denoted $f \circ g$, is defined by the rule

$(f \circ g)(x) = f(g(x))$ provided that $g(x)$ is in the domain of f

The **composition** of g and f, denoted $g \circ f$, is defined by the rule

$(g \circ f)(x) = g(f(x))$ provided that $f(x)$ is in the domain of g

Examples

Example 1

Let $g(x) = 5x + 1$ and $h(x) = x^3$. Find:

1. $(g + h)(3) = g(3) + h(3) = 16 + 27 = 43$
2. $(g \cdot h)(-1) = g(-1) \cdot h(-1) = (-4) \cdot (-1) = 4$
3. $(g - h)(x) = 5x + 1 - x^3$
4. $\left(\dfrac{g}{h}\right)(x) = \dfrac{5x + 1}{x^3}$

Example 2

Find $(f \circ g)(x)$ and $(g \circ f)(x)$ given the functions defined by $f(x) = 4x + 3$ and $g(x) = 7x$.

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(7x) \\ &= 4(7x) + 3 \\ &= 28x + 3\end{aligned}$$

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(4x + 3) \\ &= 7(4x + 3) \\ &= 28x + 21\end{aligned}$$

Section 10.2 Inverse Functions

Key Concepts

Horizontal Line Test

Consider a function defined by a set of points (x, y) in a rectangular coordinate system. Then the graph defines y as a one-to-one function of x if no horizontal line intersects the graph in more than one point.

Finding an Equation of the Inverse of a Function

For a one-to-one function defined by $y = f(x)$, the equation of the inverse can be found as follows:

1. Replace $f(x)$ by y.
2. Interchange x and y.
3. Solve for y.
4. Replace y by $f^{-1}(x)$.

The graphs defined by $y = f(x)$ and $y = f^{-1}(x)$ are symmetric with respect to the line $y = x$.

Definition of an Inverse Function

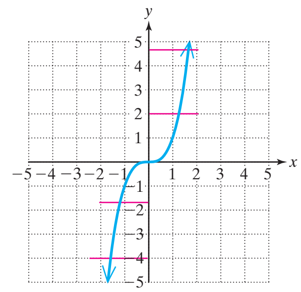
If f is a one-to-one function represented by ordered pairs of the form (x, y), then the inverse function, denoted f^{-1}, is the set of ordered pairs defined by ordered pairs of the form (y, x).

Inverse Function Property

If f is a one-to-one function, then g is the inverse of f if and only if $(f \circ g)(x) = x$ for all x in the domain of g, and $(g \circ f)(x) = x$ for all x in the domain of f.

Examples

Example 1



The function is one-to-one because it passes the horizontal line test.

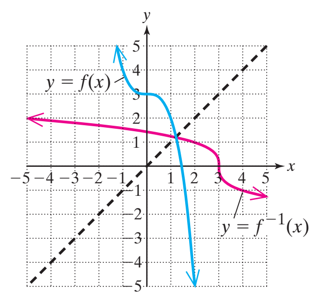
Example 2

Find the inverse of the one-to-one function defined by $f(x) = 3 - x^3$.

1. $y = 3 - x^3$
2. $x = 3 - y^3$
3. $x - 3 = -y^3$

   $-x + 3 = y^3$

   $\sqrt[3]{-x + 3} = y$
4. $f^{-1}(x) = \sqrt[3]{-x + 3}$



Example 3

Verify that the functions defined by $f(x) = x - 1$ and $g(x) = x + 1$ are inverses.

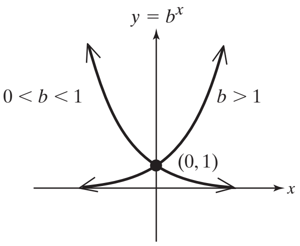
$(f \circ g)(x) = f(x + 1) = (x + 1) - 1 = x$

$(g \circ f)(x) = g(x - 1) = (x - 1) + 1 = x$

Section 10.3 Exponential Functions

Key Concepts

A function $y = b^x$ $(b > 0, b \neq 1)$ is an **exponential function**.



The domain is $(-\infty, \infty)$.

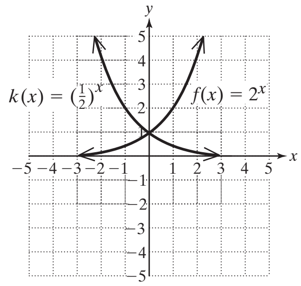
The range is $(0, \infty)$

The line $y = 0$ is a horizontal asymptote.

The y-intercept is $(0, 1)$.

Examples

Example 1

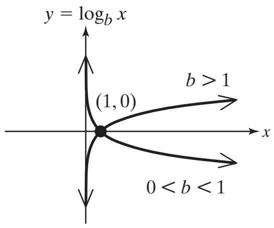


Section 10.4 Logarithmic Functions

Key Concepts

The function $y = \log_b x$ is a **logarithmic function**.

$y = \log_b x \quad \Leftrightarrow \quad b^y = x \quad (x > 0, b > 0, b \neq 1)$



For $y = \log_b x$, the domain is $(0, \infty)$.

The range is $(-\infty, \infty)$.

The line $x = 0$ is a vertical asymptote.

The x-intercept is $(1, 0)$.

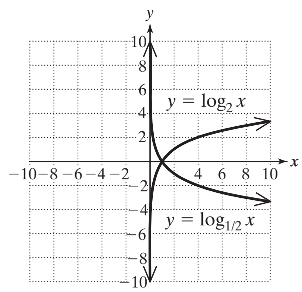
The function $y = \log x$ is the **common logarithmic function** (base 10).

Examples

Example 1

$\log_4 64 = 3$ because $4^3 = 64$

Example 2



Example 3

$\log 10{,}000 = 4$ because $10^4 = 10{,}000$

Section 10.5 Properties of Logarithms

Key Concepts

Let b, x, and y be positive real numbers where $b \neq 1$, and let p be a real number. Then the following properties are true.

1. $\log_b 1 = 0$
2. $\log_b b = 1$
3. $\log_b b^p = p$
4. $b^{\log_b(x)} = x$
5. $\log_b(xy) = \log_b x + \log_b y$
6. $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$
7. $\log_b x^p = p\log_b x$

The **properties of logarithms** can be used to write multiple logarithms as a single logarithm.

The **properties of logarithms** can be used to write a single logarithm as a sum or difference of logarithms.

Examples

Example 1

1. $\log_5 1 = 0$
2. $\log_6 6 = 1$
3. $\log_4 4^7 = 7$
4. $2^{\log_2(5)} = 5$
5. $\log(5x) = \log 5 + \log x$
6. $\log_7\left(\frac{z}{10}\right) = \log_7 z - \log_7 10$
7. $\log x^5 = 5\log x$

Example 2

$$\log x - \frac{1}{2}\log y - 3\log z$$
$$= \log x - (\log y^{1/2} + \log z^3)$$
$$= \log x - \log(\sqrt{y}z^3)$$
$$= \log\left(\frac{x}{\sqrt{y}z^3}\right)$$

Example 3

$$\log\sqrt[3]{\frac{x}{y^2}}$$
$$= \frac{1}{3}\log\left(\frac{x}{y^2}\right)$$
$$= \frac{1}{3}(\log x - \log y^2)$$
$$= \frac{1}{3}(\log x - 2\log y)$$
$$= \frac{1}{3}\log x - \frac{2}{3}\log y$$

Section 10.6 The Irrational Number *e*

Key Concepts

As x becomes infinitely large, the expression $\left(1 + \frac{1}{x}\right)^x$ approaches the irrational number ***e***, where $e \approx 2.718281$.

The balance of an account earning compound interest n times per year is given by

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

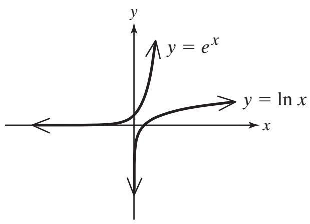
where P = principal, r = interest rate, t = time in years, and n = number of compound periods per year.

The balance of an account earning interest continuously is given by

$$A(t) = Pe^{rt}$$

The function $y = e^x$ is the exponential function with base e.

The **natural logarithm function** $y = \ln x$ is the logarithm function with base e.



Change-of-Base Formula

$$\log_b x = \frac{\log_a x}{\log_a b} \qquad a > 0, a \neq 1, b > 0, b \neq 1$$

Examples

Example 1

Find the account balance for $8000 invested for 10 years at 7% compounded quarterly.

$P = 8000 \qquad t = 10 \qquad r = 0.07 \qquad n = 4$

$$A(10) = 8000\left(1 + \frac{0.07}{4}\right)^{(4)(10)}$$

$$= \$16{,}012.78$$

Example 2

Find the account balance for the same investment compounded continuously.

$P = 8000 \qquad t = 10 \qquad r = 0.07$

$$A(t) = 8000e^{(0.07)(10)}$$

$$= \$16{,}110.02$$

Example 3

Use a calculator to approximate the value of the expressions.

$e^{7.5} \approx 1808.04$

$e^{-\pi} \approx 0.0432$

$\ln 107 \approx 4.6728$

$\ln\left(\frac{1}{\sqrt{2}}\right) \approx -0.3466$

Example 4

$$\log_3 59 = \frac{\log 59}{\log 3} \approx 3.7115$$

Section 10.7 Logarithmic and Exponential Equations

Key Concepts

Guidelines to Solve Logarithmic Equations

1. Isolate the logarithms on one side of the equation.
2. Write a sum or difference of logarithms as a single logarithm.
3. Rewrite the equation in step 2 in exponential form.
4. Solve the resulting equation from step 3.
5. Check all solutions to verify that they are within the domain of the logarithmic expressions in the equation.

Examples

Example 1

$\log(3x - 1) + 1 = \log(2x + 1)$

Step 1: $\log(3x - 1) - \log(2x + 1) = -1$

Step 2: $\log\left(\dfrac{3x - 1}{2x + 1}\right) = -1$

Step 3: $10^{-1} = \dfrac{3x - 1}{2x + 1}$

Step 4: $\dfrac{1}{10} = \dfrac{3x - 1}{2x + 1}$

$$2x + 1 = 10(3x - 1)$$

$$2x + 1 = 30x - 10$$

$$-28x = -11$$

$$x = \frac{11}{28}$$

Step 5: $x = \dfrac{11}{28}$ Checks in original equation

The equivalence of exponential expressions can be used to solve **exponential equations**.

If $b^x = b^y$ then $x = y$

Example 2

$5^{2x} = 125$

$5^{2x} = 5^3$ implies that $2x = 3$

$$x = \frac{3}{2}$$

Guidelines to Solve Exponential Equations

1. Isolate one of the exponential expressions in the equation.
2. Take a logarithm of both sides of the equation.
3. Use the power property of logarithms to write exponents as factors.
4. Solve the resulting equation from step 3.

Example 3

$$4^{x+1} - 2 = 1055$$

Step 1: $4^{x+1} = 1057$

Step 2: $\ln(4^{x+1}) = \ln 1057$

Step 3: $(x + 1)\ln 4 = \ln 1057$

Step 4: $x + 1 = \dfrac{\ln 1057}{\ln 4}$

$$x = \frac{\ln 1057}{\ln 4} - 1 \approx 4.023$$

Chapter 10 Review Exercises

Section 10.1

For Exercises 1–8, refer to the functions defined here.

$$f(x) = x - 7 \qquad g(x) = -2x^3 - 8x$$
$$m(x) = \sqrt{x} \qquad n(x) = \frac{1}{x - 2}$$

Find the indicated function values. Write the domain in interval notation.

1. $(f - g)(x)$ **2.** $(f + g)(x)$

3. $(f \cdot n)(x)$ **4.** $(f \cdot m)(x)$

5. $\left(\frac{f}{g}\right)(x)$ **6.** $\left(\frac{g}{f}\right)(x)$

7. $(m \circ f)(x)$ **8.** $(n \circ f)(x)$

For Exercises 9–12, refer to the functions defined for Exercises 1–8. Find the function values, if possible.

9. $(m \circ g)(-2)$ **10.** $(n \circ g)(-1)$

11. $(f \circ g)(4)$ **12.** $(g \circ f)(8)$

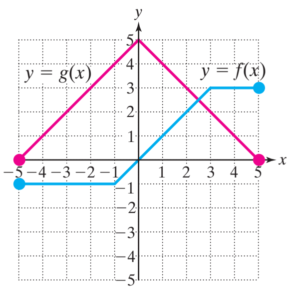
13. Given $f(x) = 2x + 1$ and $g(x) = x^2$

a. Find $(g \circ f)(x)$.

b. Find $(f \circ g)(x)$.

c. Based on your answers to parts (a) and (b), is $f \circ g$ equal to $g \circ f$?

For Exercises 14–19, refer to the graph. Approximate the function values, if possible.



14. $\left(\frac{f}{g}\right)(1)$ **15.** $(f \cdot g)(-2)$

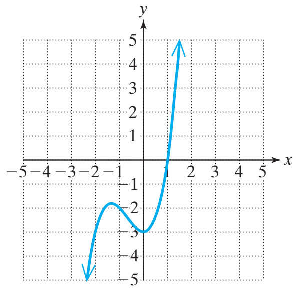
16. $(f + g)(-4)$ **17.** $(f - g)(2)$

18. $(g \circ f)(-3)$ **19.** $(f \circ g)(4)$

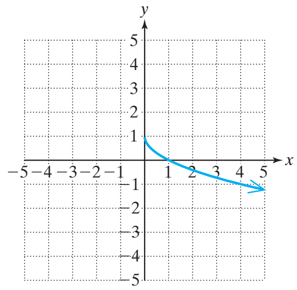
Section 10.2

For Exercises 20–21, determine if the function is one-to-one by using the horizontal line test.

20.



21.



For Exercises 22–26, write the inverse for each one-to-one function.

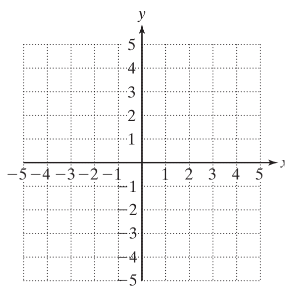
22. $\{(3, 5), (2, 9), (0, -1), (4, 1)\}$

23. $q(x) = \frac{3}{4}x - 2$ **24.** $g(x) = \sqrt[5]{x} + 3$

25. $f(x) = (x - 1)^3$ **26.** $n(x) = \frac{4}{x - 2}$

27. Verify that the functions defined by $f(x) = 5x - 2$ and $g(x) = \frac{1}{5}x + \frac{2}{5}$ are inverses by showing that $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$.

28. Graph the functions q and q^{-1} from Exercise 23 on the same coordinate axes. What can you say about the relationship between these two graphs?



29. a. Find the domain and range of the function defined by $h(x) = \sqrt{x + 1}$.

b. Find the domain and range of the function defined by $k(x) = x^2 - 1, x \geq 0$.

30. Determine the inverse of the function $p(x) = \sqrt{x} + 2$.

Section 10.3

For Exercises 31–38, evaluate the exponential expressions. Use a calculator and round to 3 decimal places, if necessary.

31. 4^5

32. 6^{-2}

33. $8^{1/3}$

34. $\left(\frac{1}{100}\right)^{-1/2}$

35. 2^{π}

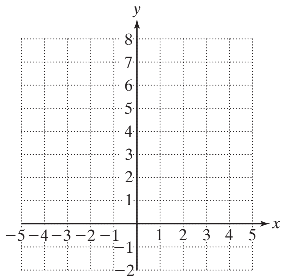
36. $5^{\sqrt{3}}$

37. $(\sqrt{7})^{1/2}$

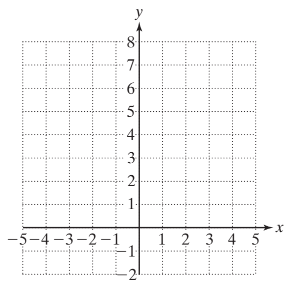
38. $\left(\frac{3}{4}\right)^{4/3}$

For Exercises 39–42, graph the functions.

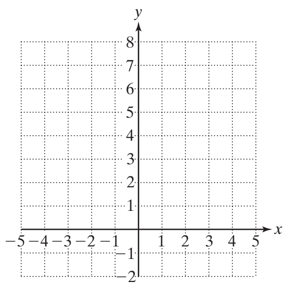
39. $f(x) = 3^x$



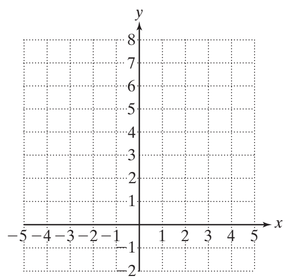
40. $g(x) = \left(\frac{1}{4}\right)^x$



41. $h(x) = 5^{-x}$



42. $k(x) = \left(\frac{2}{5}\right)^{-x}$



43. a. Does the graph of $y = b^x, b > 0, b \neq 1$, have a vertical or a horizontal asymptote?

b. Write an equation of the asymptote.

44. Background radiation is radiation that we are exposed to from naturally occurring sources including the soil, the foods we eat, and the sun. Background radiation varies depending on where we live. A typical background radiation level is 150 millirems (mrem) per year. (A rem is a measure of energy produced from radiation.) Suppose a substance emits 30,000 mrem per year and has a half-life of 5 years. The function defined by

$$A(t) = 30{,}000\left(\frac{1}{2}\right)^{t/5}$$

gives the radiation level (in millirems) of this substance after t years.

a. What is the radiation level after 5 years?

b. What is the radiation level after 15 years?

c. Will the radiation level of this substance be below the background level of 150 mrem after 50 years?

Section 10.4

For Exercises 45–52, evaluate the logarithms without using a calculator.

45. $\log_3\left(\frac{1}{27}\right)$

46. $\log_5 1$

47. $\log_7 7$

48. $\log_2 2^8$

49. $\log_2 16$

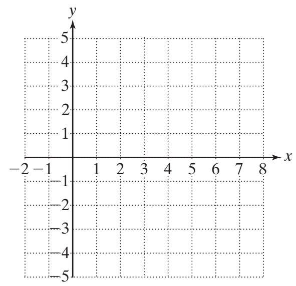
50. $\log_3 81$

51. $\log 100{,}000$

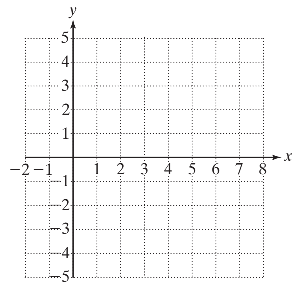
52. $\log_8\left(\frac{1}{8}\right)$

For Exercises 53–54, graph the logarithmic functions.

53. $q(x) = \log_3 x$



54. $r(x) = \log_{1/2} x$



55. a. Does the graph of $y = \log_b x$ have a vertical or a horizontal asymptote?

b. Write an equation of the asymptote.

56. Acidity of a substance is measured by its pH. The pH can be calculated by the formula $\text{pH} = -\log\,[\text{H}^+]$, where $[\text{H}^+]$ is the hydrogen ion concentration.

a. What is the pH of a fruit with a hydrogen ion concentration of 0.00316 mol/L? Round to one decimal place.

b. What is the pH of an antacid tablet with $[\text{H}^+] = 3.16 \times 10^{-10}$? Round to one decimal place.

Section 10.5

For Exercises 57–60, evaluate the logarithms without using a calculator.

57. $\log_8 8$

58. $\log_{11} 11^6$

59. $\log_{1/2} 1$

60. $12^{\log_{12} 7}$

61. Complete the properties. Assume x, y, and b are positive real numbers such that $b \neq 1$.

a. $\log_b (xy) =$

b. $\log_b x - \log_b y =$

c. $\log_b x^p =$

For Exercises 62–65, write the logarithmic expressions as a single logarithm.

62. $\frac{1}{4}(\log_b y - 4\log_b z + 3\log_b x)$

63. $\frac{1}{2}\log_3 a + \frac{1}{2}\log_3 b - 2\log_3 c - 4\log_3 d$

64. $\log 540 - 3\log 3 - 2\log 2$

65. $-\log_4 18 + \log_4 6 + \log_4 3 - \log_4 1$

66. Which of the following is equivalent to $\frac{2\log 7}{\log 7 + \log 6}$?

a. $\frac{\log 7}{\log 6}$ **b.** $\frac{\log 49}{\log 42}$ **c.** $\frac{2}{\log 6}$

67. Which of the following is equivalent to $\frac{\log 8^{-3}}{\log 2 + \log 4}$?

a. -3 **b.** $-3\log\left(\frac{4}{3}\right)$ **c.** $\frac{-3\log 4}{\log 3}$

Section 10.6

For Exercises 68–75, use a calculator to approximate the expressions to 4 decimal places.

68. e^5

69. $e^{\sqrt{7}}$

70. $32e^{0.008}$

71. $58e^{-0.0125}$

72. $\ln 6$

73. $\ln\left(\frac{1}{9}\right)$

74. $\log 22$

75. $\log e^3$

For Exercises 76–79, use the change-of-base formula to approximate the logarithms to 4 decimal places.

76. $\log_2 10$

77. $\log_9 80$

78. $\log_5 (0.26)$

79. $\log_4 (0.0062)$

80. An investor wants to deposit \$20,000 in an account for 10 years at 5.25% interest. Compare the amount she would have if her money were invested with the following different compounding options. Use

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

for interest compounded n times per year and $A(t) = Pe^{rt}$ for interest compounded continuously.

a. Compounded annually

b. Compounded quarterly

c. Compounded monthly

d. Compounded continuously

81. To measure a student's retention of material at the end of a course, researchers give the student a test on the material every month for 24 months after the course is over. The student's average score t months after completing the course is given by

$$S(t) = 75e^{-0.5t} + 20$$

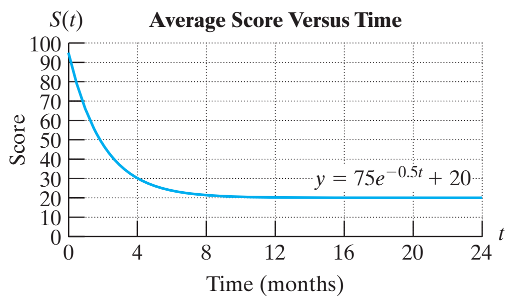
where t is the time in months and $S(t)$ is the test score.

a. Find $S(0)$ and interpret the result.

b. Find $S(6)$ and interpret the result.

c. Find $S(12)$ and interpret the result.

d. The graph of $y = S(t)$ is shown here. Does it appear that the student's average score is approaching a limiting value? Explain.



Section 10.7

For Exercises 82–89, identify the domain. Write the answer in interval notation.

82. $f(x) = e^x$

83. $g(x) = e^{x+6}$

84. $h(x) = e^{x-3}$

85. $k(x) = \ln x$

86. $q(x) = \ln(x + 5)$

87. $p(x) = \ln(x - 7)$

88. $r(x) = \ln(4 - 3x)$

89. $w(x) = \ln(5 - x)$

Solve the logarithmic equations in Exercises 90–97. If necessary, round to two decimal places.

90. $\log_5 x = 3$

91. $\log_7 x = -2$

92. $\log_6 y = 3$

93. $\log_3 y = \frac{1}{12}$

94. $\log(2w - 1) = 3$

95. $\log_2(3w + 5) = 5$

96. $\log p - 1 = -\log(p - 3)$

97. $\log_4(2 + t) - 3 = \log_4(3 - 5t)$

Solve the exponential equations in Exercises 98–105. If necessary, round to four decimal places.

98. $4^{3x+5} = 16$

99. $5^{7x} = 625$

100. $4^a = 21$

101. $5^a = 18$

102. $e^{-x} = 0.1$

103. $e^{-2x} = 0.06$

104. $10^{2n} = 1512$

105. $10^{-3m} = \frac{1}{821}$

106. $2^{x+3} = 7^x$

107. $14^{x-5} = 6^x$

108. Radioactive iodine (^{131}I) is used to treat patients with a hyperactive (overactive) thyroid. Patients with this condition may have symptoms that include rapid weight loss, heart palpitations, and high blood pressure. The half-life of radioactive iodine is 8.04 days. If a patient is given an initial dose of 2 μg, then the amount of iodine in the body after t days is approximated by

$$A(t) = 2e^{-0.0862t}$$

where t is the time in days and $A(t)$ is the amount (in micrograms) of ^{131}I remaining.

a. How much radioactive iodine is present after a week? Round to two decimal places.

b. How much radioactive iodine is present after 30 days? Round to two decimal places.

c. How long will it take for the level of radioactive iodine to reach 0.5 μg?

109. The growth of a certain bacterium in a culture is given by the model $A(t) = 150e^{0.007t}$, where $A(t)$ is the number of bacteria and t is time in minutes. Let $t = 0$ correspond to the initial number of bacteria.

a. What is the initial number of bacteria?

b. What is the population after $\frac{1}{2}$ hr?

c. How long will it take for the population to double?

110. The value of a car is depreciated with time according to

$$V(t) = 15{,}000e^{-0.15t}$$

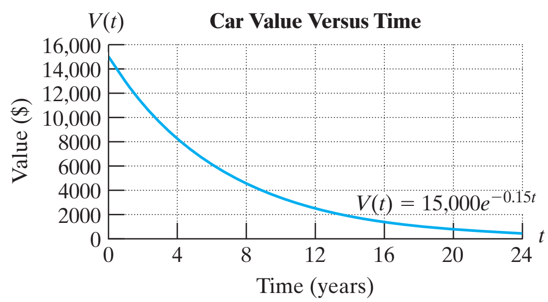
where $V(t)$ is the value in dollars and t is the time in years.

a. Find $V(0)$ and interpret the result in the context of this problem.

b. Find $V(10)$ and interpret the result in the context of this problem. Round to the nearest dollar.

c. Find the time required for the value of the car to drop to \$5000. Round to the nearest tenth of a year.

d. The graph of $y = V(t)$ is shown here. Does it appear that the value of the car is approaching a limiting value? If so what does the limiting value appear to be?



Chapter 10 Test

For Exercises 1–8, refer to these functions.

$f(x) = x - 4 \qquad g(x) = \sqrt{x + 2} \qquad h(x) = \dfrac{1}{x}$

Find the function values if possible.

1. $\left(\dfrac{f}{g}\right)(x)$

2. $(h \cdot g)(x)$

3. $(g \circ f)(x)$

4. $(h \circ f)(x)$

5. $(f - g)(7)$

6. $(h + f)(2)$

7. $(h \circ g)(14)$

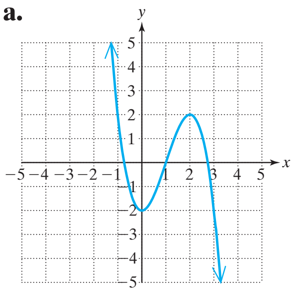
8. $(g \circ f)(0)$

9. If $f(x) = x - 4$ and $g(x) = \sqrt{x + 2}$, write the domain of the function $\frac{g}{f}$.

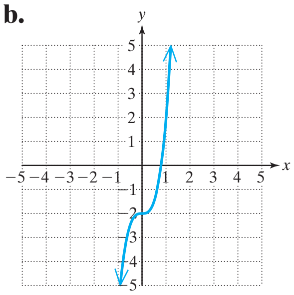
10. Explain how to determine graphically if a function is one-to-one.

11. Which of the functions is one-to-one?

a.



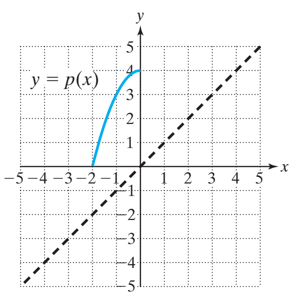
b.

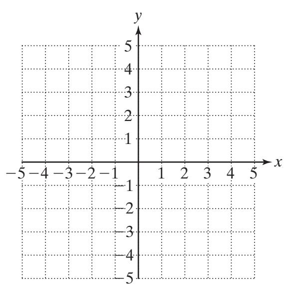


12. Write an equation of the inverse of the one-to-one function defined by $f(x) = \frac{1}{4}x + 3$.

13. Write an equation of the inverse of the function defined by $g(x) = (x - 1)^2, \quad x \geq 1$.

14. Given the graph of the function $y = p(x)$, graph its inverse $p^{-1}(x)$.

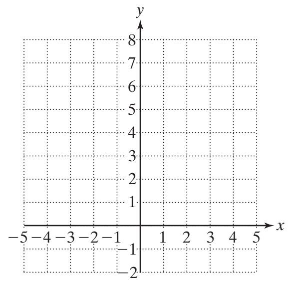




15. Use a calculator to approximate the expression to 4 decimal places.

a. $10^{2/3}$ **b.** $3^{\sqrt{10}}$ **c.** 8^{π}

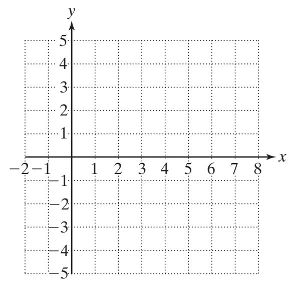
16. Graph $f(x) = 4^{x-1}$.



17. a. Write in logarithmic form. $16^{3/4} = 8$

b. Write in exponential form. $\log_x 31 = 5$

18. Graph $g(x) = \log_3 x$.



19. Complete the change-of-base formula: $\log_b n =$ _____

20. Use a calculator to approximate the expression to four decimal places:

a. $\log 21$ **b.** $\log_4 13$ **c.** $\log_{1/2} 6$

21. Using the properties of logarithms, expand and simplify. Assume all variables represent positive real numbers.

a. $-\log_3\left(\dfrac{3}{9x}\right)$ **b.** $\log\left(\dfrac{1}{10^5}\right)$

22. Write as a single logarithm. Assume all variables represent positive real numbers.

a. $\dfrac{1}{2}\log_b x + 3\log_b y$ **b.** $\log a - 4\log a$

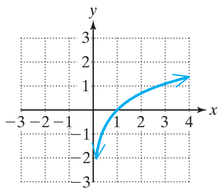
23. Use a calculator to approximate the expression to four decimal places, if necessary.

a. $e^{1/2}$ **b.** e^{-3}

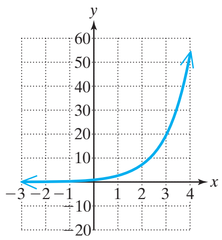
c. $\ln\left(\frac{1}{3}\right)$ **d.** $\ln e$

24. Identify the graphs as $y = e^x$ or $y = \ln(x)$.

a.



b.



25. Researchers found that t months after taking a course, students remembered p% of the material according to

$$p(t) = 92 - 20\ln(t + 1)$$

where $0 \le t \le 24$ is the time in months.

a. Find $p(4)$ and interpret the results.

b. Find $p(12)$ and interpret the results.

c. Find $p(0)$ and interpret the results.

26. The population of New York City has a 2% growth rate and can be modeled by the function $P(t) = 8008(1.02)^t$, where $P(t)$ is in thousands and t is in years ($t = 0$ corresponds to the year 2000).

a. Using this model, predict the population in the year 2010.

b. If this growth rate continues, in what year will the population reach 12 million (12 million is 12,000 thousand)?

27. A certain bacterial culture grows according to

$$P(t) = \frac{1{,}500{,}000}{1 + 5000e^{-0.8t}}$$

where P is the population of the bacteria and t is the time in hours.

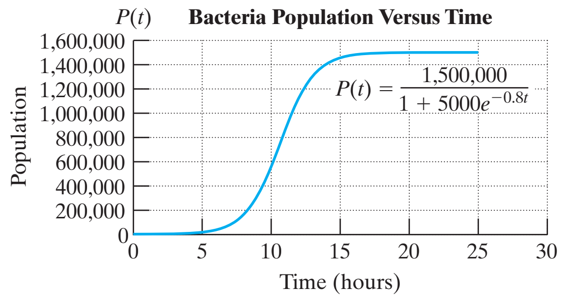
a. Find $P(0)$ and interpret the result. Round to the nearest whole number.

b. How many bacteria will be present after 6 hr?

c. How many bacteria will be present after 12 hr?

d. How many bacteria will be present after 18 hr?

e. From the graph does it appear that the population of bacteria is reaching a limiting value? If so, what does the limiting value appear to be?



For Exercises 28–33, solve the exponential and logarithmic equations. If necessary, round to 3 decimal places.

28. $\log x + \log(x - 21) = 2$

29. $\log_{1/2} x = -5$

30. $\ln(x + 7) = 2.4$

31. $3^{x+4} = \frac{1}{27}$

32. $4^x = 50$

33. $e^{2.4x} = 250$

34. Atmospheric pressure P decreases exponentially with altitude x according to

$$P(x) = 760e^{-0.000122x}$$

where $P(x)$ is the pressure measured in millimeters of mercury (mm Hg) and x is the altitude measured in meters.

a. Find $P(2500)$ and interpret the result. Round to one decimal place.

b. Find the pressure at sea level.

c. Find the altitude at which the pressure is 633 mm Hg.

35. Use the formula $A(t) = Pe^{rt}$ to compute the value of an investment under continuous compounding.

a. If \$2000 is invested at 7.5% compounded continuously, find the value of the investment after 5 years.

b. How long will it take the investment to double? Round to two decimal places.

Chapters 1–10 Cumulative Review Exercises

1. Simplify completely.

$$\frac{8 - 4 \cdot 2^2 + 15 \div 5}{|-3 + 7|}$$

2. Divide.

$$\frac{-8p^2 + 4p^3 + 6p^5}{8p^2}$$

3. Divide $(t^4 - 13t^2 + 36) \div (t - 2)$. Identify the quotient and remainder.

4. Simplify. $\sqrt{x^2 - 6x + 9}$

5. Simplify. $\dfrac{4}{\sqrt[3]{40}}$

6. Simplify. Write the answer with positive exponents only.

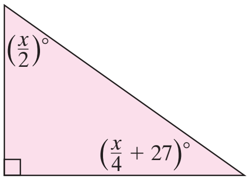
$$\frac{2^{2/5}c^{-1/4}d^{1/5}}{2^{-8/5}c^{3/4}d^{1/10}}$$

7. Find the area of the rectangle.

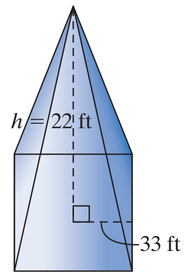
8. Perform the indicated operation.

$$\frac{4 - 3i}{2 + 5i}$$

9. Find the measure of each angle in the right triangle.



10. Find the positive slope of the sides of a pyramid with a square base 66 ft on a side and height of 22 ft.



11. How many liters of pure alcohol must be mixed with 8 L of 20% alcohol to bring the concentration up to 50% alcohol?

12. Bank robbers leave the scene of a crime and head north through winding mountain roads to their hideaway. Their average rate of speed is 40 mph. The police leave 6 min later in pursuit. If the police car averages 50 mph traveling the same route, how long will it take the police to catch the bank robbers?

13. Solve the system by using the Gauss-Jordan method.

$$5x + 10y = 25$$
$$-2x + 6y = -20$$

14. Solve for w. $-2[w - 3(w + 1)] = 4 - 7(w + 3)$

15. Solve for x. $ax - c = bx + d$

16. Solve for t. $s = \frac{1}{2}gt^2 \quad t \geq 0$

17. Solve for T. $\sqrt{1 - kT} = \dfrac{V_0}{V}$

18. Find the x-intercepts of the function defined by $f(x) = |x - 5| - 2$.

19. Let $f(t) = 6$, $g(t) = -5t$, and $h(t) = 2t^2$. Find

a. $(f \cdot g)(t)$ **b.** $(g \circ h)(t)$ **c.** $(h - g)(t)$

20. Solve for q. $|2q - 5| = |2q + 5|$

21. **a.** Find an equation of the line parallel to the y-axis and passing through the point (2, 6).

b. Find an equation of the line perpendicular to the y-axis and passing through the point (2, 6).

c. Find an equation of the line perpendicular to the line $2x + y = 4$ and passing through the point (2, 6). Write the answer in slope-intercept form.

22. The smallest angle in a triangle measures one-half the largest angle. The smallest angle measures 20° less than the middle angle. Find the measures of all three angles.

23. Solve the system.

$$\frac{1}{2}x - \frac{1}{4}y = 1$$

$$-2x + y = -4$$

24. Match the function with the appropriate graph.

i. $f(x) = \ln(x)$ ii. $g(x) = 3^x$

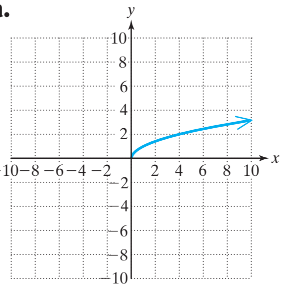
iii. $h(x) = x^2$ iv. $k(x) = -2x - 3$

v. $L(x) = |x|$ vi. $m(x) = \sqrt{x}$

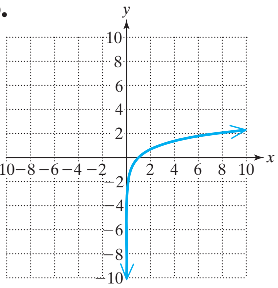
vii. $n(x) = \sqrt[3]{x}$ viii. $p(x) = x^3$

ix. $q(x) = \dfrac{1}{x}$ x. $r(x) = x$

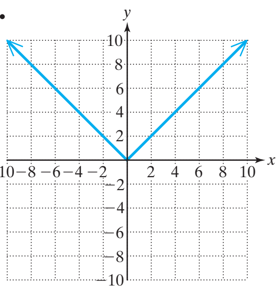
a.



b.



c.



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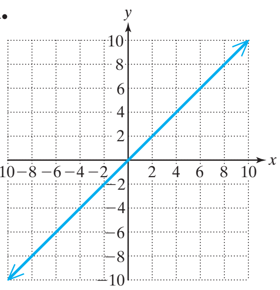

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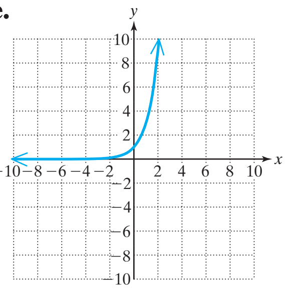

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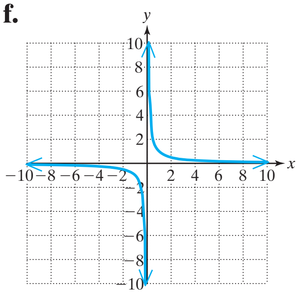

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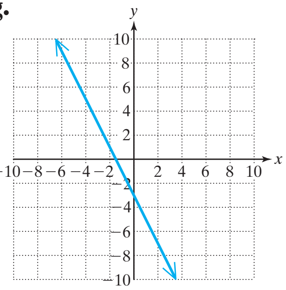

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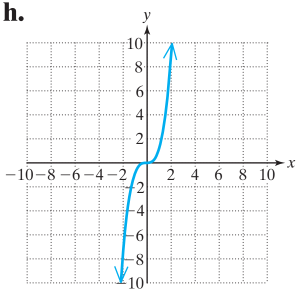
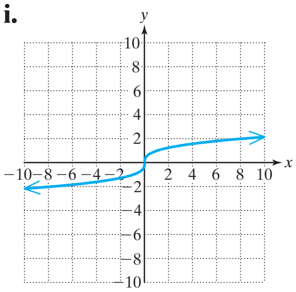
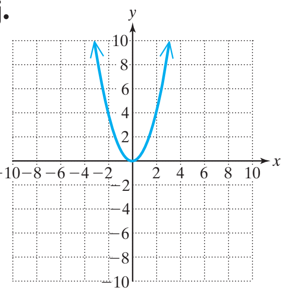


i.



j.



25. Find the domain. $f(x) = \sqrt{2x - 1}$.

26. Find $f^{-1}(x)$, given $f(x) = 5x - \frac{2}{3}$.

27. The volume of a gas varies directly as its temperature and inversely with pressure. At a temperature of 100 kelvins (K) and a pressure of 10 newtons per square meter (N/m^2), the gas occupies a volume of 30 m^3. Find the volume at a temperature of 200 K and pressure of 15 N/m^2.

28. Perform the indicated operations.

$$\frac{5x - 10}{x^2 - 4x + 4} \div \frac{5x^2 - 125}{25 - 5x} \cdot \frac{x^3 + 125}{10x + 5}$$

29. Perform the indicated operations.

$$\frac{x}{x - y} + \frac{y}{y - x} + x$$

30. Given the equation

$$\frac{2}{x-4} = \frac{5}{x+2}$$

a. Are there any restrictions on x for which the rational expressions are undefined?

b. Solve the equation.

c. Solve the related inequality.

$$\frac{2}{x-4} \geq \frac{5}{x+2}$$

Write the answer in interval notation.

31. Two more than 3 times the reciprocal of a number is $\frac{5}{4}$ less than the number. Find all such numbers.

32. Solve the equation. $\sqrt{-x} = x + 6$

33. Solve the inequality $2|x - 3| + 1 > -7$. Write the answer in interval notation.

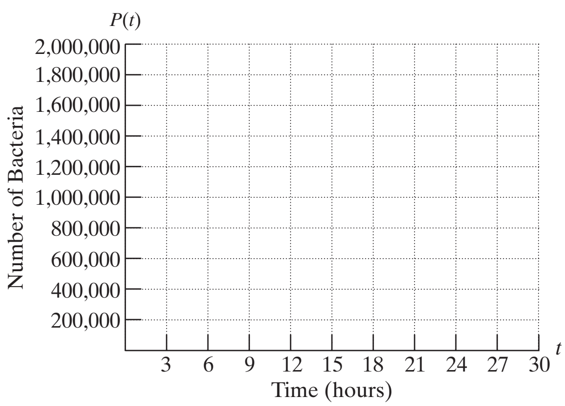
34. Four million *Escherichia coli (E. coli)* bacteria are present in a laboratory culture. An antibacterial agent is introduced and the population of bacteria decreases by one-half every 6 hr according to

$$P(t) = 4{,}000{,}000\left(\frac{1}{2}\right)^{t/6} \qquad t \geq 0$$

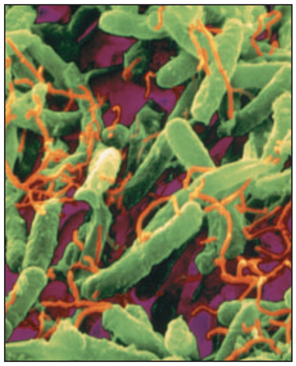
where t is the time in hours.

a. Find the population of bacteria after 6, 12, 18, 24, and 30 hr.

b. Sketch a graph of $y = P(t)$ based on the function values found in part (a).



c. Predict the time required for the population to decrease to 15,625 bacteria.



35. Evaluate the expressions without a calculator.

a. $\log_7 49$ **b.** $\log_4\left(\frac{1}{64}\right)$

c. $\log (1{,}000{,}000)$ **d.** $\ln e^3$

36. Use a calculator to approximate the expressions to four decimal places.

a. $\pi^{4.7}$ **b.** e^{π}

c. $(\sqrt{2})^{-5}$ **d.** $\log 5362$

e. $\ln (0.67)$ **f.** $\log_4 37$

37. Solve the equation. $e^x = 100$

38. Solve the equation. $\log_3(x + 6) - 3 = -\log_3 x$

39. Write the following expression as a single logarithm.

$$\frac{1}{2}\log z - 2\log x - 3\log y$$

40. Write the following expression as a sum or difference of logarithms

$$\ln\left(\sqrt[3]{\frac{x^2}{y}}\right)$$