Calculus (2): Applications of Differentiation

10

INCREASING AND DECREASING FUNCTIONS

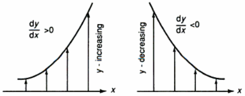


Fig. 10.1

On any stretch of the curve $y = f(x)$, where $\frac{dy}{dx} > 0$, the curve slopes upwards. Hence y *increases* as x *increases* and $f(x)$ is an **increasing** function.

Similarly if $\frac{dy}{dx} < 0$, y decreases as x increases and $f(x)$ is a **decreasing** function.

At any point where $\frac{dy}{dx} = 0$, $f(x)$ has a **stationary** value and is neither increasing nor decreasing. Such a point is a **stationary** point.

Example 1

For what range of values of x is the function $y = x^3 - 3x^2 - 9x + 4$ (a) decreasing, (b) increasing?

$$\frac{dy}{dx} = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x - 3)(x + 1)$$

(a) $\frac{dy}{dx} < 0$ for $-1 < x < 3$ and the function is decreasing in this interval (Fig.10.2).

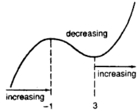


Fig. 10.2

(b) $\frac{dy}{dx} > 0$ for $x < -1$ or $x > 3$ and the function is increasing in these intervals.
The function has stationary values at $x = 3$ and at $x = -1$.

Example 2

(a) For what range of values of x is the function $y = x + \frac{1}{4x}$ increasing? (b) What are the coordinates of the stationary points?

(a) $\frac{dy}{dx} = 1 - \frac{1}{4x^2}$
$\frac{dy}{dx} > 0$ if $1 > \frac{1}{4x^2}$, i.e. $4x^2 > 1$ or $x^2 > \frac{1}{4}$.
Hence $x < -\frac{1}{2}$ or $x > \frac{1}{2}$.
This is illustrated in the sketch of the curve in Fig. 10.3.

(b) $\frac{dy}{dx} = 0$ where $1 - \frac{1}{4x^2} = 0$ i.e. $x^2 = \frac{1}{4}$ and $x = \pm\frac{1}{2}$.
So the coordinates of the stationary points A and B are $(\frac{1}{2},1)$ and $(-\frac{1}{2},-1)$.

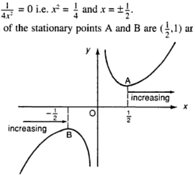


Fig. 10.3

TANGENTS AND NORMALS

As we have seen, if $y = f(x)$ is the equation of a curve, then $\frac{dy}{dx}$ gives the gradient of the tangent at any point (Fig. 10.4).

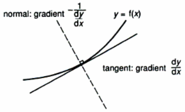


Fig. 10.4

Associated with the tangent is the **normal**, which is perpendicular to the tangent. The gradient of the tangent is $\frac{dy}{dx}$, so the gradient of the normal is $-\frac{1}{\frac{dy}{dx}}$.

Example 3

Find the equations of the tangent and the normal to the curve $y = x^2 - 2x - 3$ at the point where it meets the positive x-axis.

When $y = 0$, $x^2 - 2x - 3 = (x - 3)(x + 1) = 0$. So the curve meets the positive x-axis where $x = 3$.

$\frac{dy}{dx} = 2x - 2$ and when $x = 3$, $\frac{dy}{dx} = 4$.

We write this briefly as $\left(\frac{dy}{dx}\right)_{x=3} = 4$, meaning the *value* of $\frac{dy}{dx}$ when $x = 3$.

The equation of the tangent is then $y = 4(x - 3)$ i.e. $y = 4x - 12$.

The gradient of the normal $= -\frac{1}{4}$ so the equation of the normal is $y = -\frac{1}{4}(x - 3)$ i.e. $4y + x = 3$.

Example 4

(a) Find the equation of the normal to the curve $y = x + \frac{4}{x}$ at the point P where $x = 4$.

(b) If the normal meets the curve again at N, find the coordinates of N (Fig.10.5).

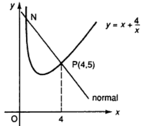


Fig.10.5

(a) The coordinates of P are $(4, 4 + \frac{4}{4})$ i.e. (4,5).

$\frac{dy}{dx} = 1 - \frac{4}{x^2}$ so $\left(\frac{dy}{dx}\right)_{x=4} = 1 - \frac{1}{4} = \frac{3}{4}$.

Hence the gradient of the normal $= -\frac{4}{3}$.

The equation of the normal is $y - 5 = -\frac{4}{3}(x - 4)$ i.e. $3y + 4x = 31$.

(b) To find where the normal meets the curve, we solve the simultaneous equations $3y + 4x = 31$ and $y = x + \frac{4}{x}$.

$x + \frac{4}{x} = \frac{31 - 4x}{3}$ i.e. $3x^2 + 12 = 31x - 4x^2$ or $7x^2 - 31x + 12 = 0$.

Then $(7x - 3)(x - 4) = 0$ which gives $x = 4$ or $\frac{3}{7}$.

$x = 4$ is the point P so the coordinates of N are $\left(\frac{3}{7}, \frac{3}{7} + \frac{4}{\frac{3}{7}}\right)$ i.e. $\left(\frac{3}{7}, \frac{205}{21}\right)$.

Example 5

(a) Find the x-coordinate of the point on the curve $y = 2x^3 + x^2 - 2x + 1$ where the curve is parallel to the line $y = 2x$.

(b) Is any part of the curve parallel to the line $y + 3x = 1$?

(a) $\frac{dy}{dx} = 6x^2 + 2x - 2$ and this must equal 2 (the gradient of $y = 2x$).

Then $6x^2 + 2x - 2 = 2$ which gives $3x^2 + x - 2 = 0$ or $(3x - 2)(x + 1) = 0$.

Hence $x = \frac{2}{3}$ or -1.

At these points the curve is parallel to $y = 2x$.

(b) If $\frac{dy}{dx} = 6x^2 + 2x - 2 = -3$ (the gradient of $y + 3x = 1$), then $6x^2 + 2x + 1 = 0$. But this equation has no real solutions. Hence the gradient of the curve is never equal to -3 and the curve is never parallel to $y + 3x = 1$.

Exercise 10.1 *(Answers on page 628.)*

1 Find the range of values of t for which V is increasing if $V = 4t^3 - 3t$.

2 In what interval must x lie if the function $y = x^4 - x^3$ is decreasing?

3 For what values of x is the function $y = \frac{x}{4} + \frac{1}{x}$ increasing? State the coordinates of the stationary points on the curve.

4 Find the range of values of x for which the function $y = 1 - x + 2x^2 - x^3$ is increasing.

5 Find the interval in which x lies if the function $y = 2x^3 + 3x^2 - 12x + 4$ is decreasing and the coordinates of the stationary points.

6 For what values of t is the function $s = 4 - 3t + 2t^2$ decreasing?

7 Find the equations of the tangent and the normal to the following curves at the given point:

(a) $y = x^2 - 2$; $x = -3$
(b) $y = 2x^3$; $x = 1$
(c) $y = 1 - x - 3x^2$; $x = -1$
(d) $y = 2x^3 - x - 1$; $x = -1$
(e) $y = \frac{4}{x}$; $x = -2$
(f) $y = \frac{3}{x+1}$; $y = 1$
(g) $y = \frac{1}{1-2x}$; $y = -1$
(h) $y = 2x + \frac{4}{x}$; $x = -2$
(i) $y = 2x^2 - 3$; $x = 2$
(j) $y = 1 - x - 3x^2 - x^3$; $x = -1$
(k) $y = 3 - \frac{2}{x}$; $y = 7$
(l) $y = \frac{2}{1-2x}$; $y = 1$

8 The tangent to the curve $y = x^2 - 2x + 3$ at a certain point is parallel to the line $y = x$. Find the equation of the tangent and the coordinates of the point where it meets the x-axis.

9 Find (a) the coordinates of the point on the curve $y = 3x^2 + 2x + 1$ where the tangent is parallel to the line $4x + y = 5$ and (b) the equation of the normal at that point.

10 (a) Find the equation of the tangent to the curve $y = x^3 - 2x^2 + x$ at the origin.
(b) At what point does it meet the curve again?

11 The normal to the curve $y = 2x - \frac{1}{1-x}$ where $x = 2$, meets the curve again at the point P. Find
(a) the equation of the normal,
(b) the coordinates of P,
(c) the equation of the tangent at P.

12 The normal to the curve $y = 2x - \frac{1}{x+1}$ at the point where $x = 1$ meets the curve again at a second point. Find the x-coordinate of this point.

13 A and B are points on the curve $y = 2x - \frac{6}{x}$ whose x-coordinates are 1 and 3 respectively. Find the equations of the tangents at A and B and the coordinates of the point where they intersect.

14 Show that the function $y = x^3 + x^2 + 5x + 6$ is always increasing.

15 If the function $y = x^3 + ax^2 + 3x - 1$ is always increasing, find the range of possible values of a.

16 (a) The normal at the point A(–1,2) on the curve $y = 3 - x^2$ meets the curve again at B. Find (i) the equation of the normal at A and (ii) the coordinates of B.

(b) Find the coordinates of the point C on the curve where the curve is parallel to the normal at A.

STATIONARY POINTS: MAXIMA AND MINIMA

As we have seen, a curve $y = f(x)$ has a **stationary** point where $\frac{dy}{dx} = 0$. There are three types of stationary point: **maximum**, **minimum** and **point of inflexion** (Fig.10.6).

Maximum and minimum points are also called **turning** points as the tangent 'turns round' at these points.

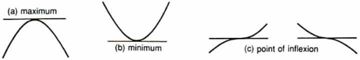


Fig. 10.6

The essential condition for a stationary point is $\frac{dy}{dx} = 0$.

If this equation has solutions, they are the x-coordinates of the stationary points. We then test for the type of point. A curve may have one or more or none of these points.

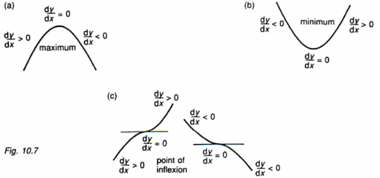


Fig. 10.7

Maximum point

$\frac{dy}{dx}$ passes from positive values through 0 to negative values (Fig. 10.7(a)).

Minimum point

$\frac{dy}{dx}$ passes from negative values through 0 to positive values (Fig. 10.7(b)).

Point of inflexion

$\frac{dy}{dx}$ has the same sign on each side of the zero value (Fig.10.7(c)).

Note that maximum and minimum apply only in the neighbourhood of the stationary point. The values of the function at this point are not necessarily the greatest and least values of the function overall.

Quadratic Function

We have already seen that the quadratic function $y = ax^2 + bx + c$ *always* has either a maximum (when $a < 0$) or a minimum point ($a > 0$).

As $\frac{dy}{dx} = 2ax + b$, this equation always has a solution.

Example 6

Which type of stationary point does $y = 1 - 2x - 2x^2$ have and what is the value of y at that point?

As $a = -2 < 0$, the curve has a maximum point.

$\frac{dy}{dx} = -2 - 4x$ and $\frac{dy}{dx} = 0$ gives $x = -\frac{1}{2}$.

So the curve has a maximum at $(-\frac{1}{2}, 1\frac{1}{2})$.

Example 7

Find the nature of the stationary points on the curve $y = 4x^3 - 3x^2 - 6x + 2$.

$\frac{dy}{dx} = 12x^2 - 6x - 6 = 6(2x^2 - x - 1) = 6(2x + 1)(x - 1)$

$\frac{dy}{dx} = 0$ when $x = -\frac{1}{2}$ or 1.

A simple test to decide on the nature of the stationary point is to examine the **sign** of $\frac{dy}{dx}$ on each side of that point.

Consider the signs of the factors $(2x + 1)$ and $(x - 1)$.

x	slightly $< -\frac{1}{2}$	$-\frac{1}{2}$	slightly $> -\frac{1}{2}$
sign of $(2x + 1)(x - 1)$	$(-)(-) = +$	0	$(+)(-) = -$
sketch of tangent	/	—	\

The sketch of the curve around $x = -\frac{1}{2}$ is $\frown$ i.e. there is a maximum point at $x = -\frac{1}{2}$.

x	slightly < 1	1	slightly > 1
sign of $(2x + 1)(x - 1)$	$(+)(-) = -$	0	$(-)(-) = +$
sketch of tangent	$\searrow$	—	$\nearrow$

The sketch of the curve around $x = 1$ is $\smile$ i.e. there is a minimum point at $x = 1$. When $x = -\frac{1}{2}$, $y = 4(-\frac{1}{8}) - 3(\frac{1}{4}) - 6(-\frac{1}{2}) + 2 = 3\frac{3}{4}$ (a maximum value), and when $x = 1$, $y = -3$ (a minimum value).

Example 8

Examine the nature of the stationary point(s) on the curve $y = x^3 - x^2 + 5x - 1$.

$\frac{dy}{dx} = 3x^2 - 2x + 5$. For stationary points, $3x^2 - 2x + 5 = 0$. This equation has no solutions, so the curve has no stationary points.

Example 9

What type of stationary point(s) does the curve $y = x^3 - 3x^2 + 3x - 1$ have?

$\frac{dy}{dx} = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x - 1)^2$

$\frac{dy}{dx} = 0$ gives $x = 1$. There is only one stationary point on the curve.

x	slightly < 1	1	slightly > 1
sign of $(x - 1)^2$	+	0	+
sketch of tangent	$\nearrow$	—	$\nearrow$

The sketch of the curve is $\nearrow$ which is a point of inflexion.

The $\frac{d^2y}{dx^2}$ Test for Maxima and Minima

The sign test is adequate for simple functions but $\frac{d^2y}{dx^2}$ can be used to test for maxima and minima.

Around a maximum point, $\frac{dy}{dx}$ passes from positive to negative so it is a *decreasing* function (Fig.10.8). Hence the gradient of $\frac{dy}{dx}$, i.e. $\frac{d^2y}{dx^2}$, is negative at that point.

Around a minimum point, $\frac{dy}{dx}$ passes from negative to positive so it is an *increasing* function (Fig.10.9). Hence the gradient of $\frac{dy}{dx}$ i.e. $\frac{d^2y}{dx^2}$, is positive at that point.

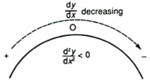


Fig. 10.8

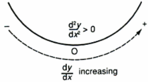


Fig. 10.9

Maximum point	Minimum point
$\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$	$\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$

If $\frac{d^2y}{dx^2} = 0$, this test is indecisive. For such cases the sign test should be used (see Example 11).

Example 10

Find the type of stationary points on the curve $y = 4x + \frac{1}{x}$ and the coordinates of these points.

$\frac{dy}{dx} = 4 - \frac{1}{x^2}$

If $\frac{dy}{dx} = 0$, then $x^2 = \frac{1}{4}$ and $x = \pm\frac{1}{2}$.

$\frac{d^2y}{dx^2} = \frac{2}{x^3}$

When $x = +\frac{1}{2}$, $\frac{d^2y}{dx^2} > 0$ so this is a minimum point at $(\frac{1}{2}, 4)$.

When $x = -\frac{1}{2}$, $\frac{d^2y}{dx^2} < 0$ so this is a maximum point at $(-\frac{1}{2}, -4)$.

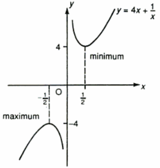


Fig. 10.10

Example 11

Find the nature of the stationary points on (a) $y = x^3$, (b) $y = x^4$.

(a) $\frac{dy}{dx} = 3x^2$ so $\frac{dy}{dx} = 0$ gives $x = 0$.

$\frac{d^2y}{dx^2} = 6x = 0$ when $x = 0$.

x	slightly < 0	0	slightly > 0
sign of $3x^2$	+	0	+
sketch of tangent	/	—	/

The sign test shows that this is a point of inflexion (Fig.10.11(a)).

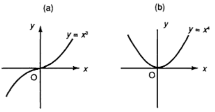


Fig. 10.11

(b) $\frac{dy}{dx} = 4x^3$ and again $\frac{dy}{dx} = 0$ gives $x = 0$.

$\frac{d^2y}{dx^2} = 12x^2 = 0$ when $x = 0$.

x	slightly < 0	0	slightly > 0
sign of $4x^3$	–	0	+
sketch of tangent	\	—	/

The sign test shows that this is a minimum (Fig.10.11(b)).

Exercise 10.2 *(Answers on page 628.)*

1 For each of the following functions, find (i) the x-coordinates and the nature of the stationary points (if any) and (ii) the value of the function at these points.

(a) $7 - 6x - x^2$ (b) $x^2 - 3x$

(c) $x^3 - 3x - 2$ (d) $x^5 - 2$

(e) $2x^3 + 3x^2 - 36x + 4$ (f) $2x^3 - x^2 + 1$

(g) $3x^4 - 4x^3 + 1$

(h) $x + \frac{25}{x}$

(i) $3x - \frac{1}{x}$

(j) $x^3 - 6x^2 + 12x + 2$

(k) $x^3 + 3x - 2$

(l) $x^3 + 3x^2 + 4x + 4$

(m) $x^2 + \frac{16}{x^2}$

(n) $3x^2 - 5x - 2$

(o) x^6

(p) $x^3 + x^2 + 1$

(q) $x^3 - x^2 - 5x - 1$

(r) $x^3 + 3x^2 + 3x - 4$

(s) $x - \frac{1}{9x}$

(t) $x + \frac{1}{x - 1}$

(u) $x^2 + \frac{16}{x}$

(v) $2x - \frac{6}{(5 - 3x)}$

2 The function $y = ax^3 - 12x + 2$ has a turning point where $x = 2$. Find (a) the value of a, (b) the nature of this turning point.

3 The function $y = 2x^3 + ax^2 - 12x - 4$ has a minimum point where $x = 1$. Find (a) the value of a, (b) the position of the maximum point.

4 The function $y = x^3 + ax^2 - 7x - 1$ has a stationary value where $x = 1$. Find (a) the value of a and (b) the type and position of the stationary points.

5 Find (a) the positions and nature of the stationary points on the curve $y = x^3 - 2x^2 + 1$ and (b) the coordinates of the point where the gradient on the curve is a minimum.

6 For what value of t is $s = t^3 - 9t^2 + 15t - 10$

(a) a maximum,

(b) a minimum?

For what value of t is $\frac{ds}{dt}$ a minimum?

7 Given that $v = 1 - t + 2t^2 - t^3$, find the value of t for which $\frac{dv}{dt}$ is a maximum and explain why it is a maximum.

8 The function $y = ax^3 + bx^2 - 12x + 13$ passes through the point (1,0) and has a stationary point where $x = -1$. Find

(a) the value of a and of b,

(b) the type and position of the stationary points.

9 Find the value of x for which $y = 4x^3 - x^2 - 2x + 1$ has

(a) a maximum,

(b) a minimum value.

Hence find the values of θ for the function $T = 4\cos^3\theta - \cos^2\theta - 2\cos\theta + 1$ at its maximum and minimum values.

10 For the function $A = \pi r^3 - 6r^2 + 3$, find, in terms of π, the values of r at the stationary points, and find which type each point is.

11 If $y = 4x^3 + 3ax^2 + 48x - 3$, in what interval must a *not* lie if y has stationary points? If $a = 10$, find the x-coordinates and the nature of the stationary points.

12 Find the type and position of the stationary point(s) on the curve $y = \frac{1}{x - 1} + \frac{1}{2 - x}$.

MAXIMUM AND MINIMUM PROBLEMS

The methods we have learnt can be used to find the maximum and minimum values of a quantity which varies under certain conditions.

Example 12

Two numbers x and y are connected by the relation $x + y = 6$. Find the values of x and y which give a stationary point of the function $T = 2x^2 + 3y^2$ and determine whether they make T a maximum or minimum.

We must express T in terms of *one* of the variables x or y.
Choosing x, $y = 6 - x$ and $T = 2x^2 + 3(6 - x)^2$.
For a stationary point, we put $\frac{dT}{dx} = 0$.
Then $\frac{dT}{dx} = 4x - 6(6 - x) = 10x - 36 = 0$ and so $x = 3.6$ and $y = 2.4$.
To decide whether this gives a maximum or minimum we find $\frac{d^2T}{dx^2}$.
$\frac{d^2T}{dx^2} = 10$ which is positive.
Hence T will have a minimum value when $x = 3.6$, $y = 2.4$.

Example 13

A cylindrical can (with lid) of radius r cm is made from 300 cm² of thin sheet metal.

(a) Show that its height, h cm, is given by $h = \frac{150 - \pi r^2}{\pi r}$.

(b) Find r and h so that the can will contain the maximum possible volume and find this volume.

(a) The surface area A of a cylinder radius r, height h is given by
$A = 2\pi r^2 + 2\pi rh = 300$.
Hence $2\pi rh = 300 - 2\pi r^2$ and $h = \frac{150 - \pi r^2}{\pi r}$

(b) The volume $V = \pi r^2 h$ and V is to be maximized. We must express V in terms of one variable and so we substitute for h from (a).
Then $V = \pi r^2 \frac{150 - \pi r^2}{\pi r} = r(150 - \pi r^2) = 150r - \pi r^3$.
To find the maximum value of V, we set $\frac{dV}{dr} = 0$.
$\frac{dV}{dr} = 150 - 3\pi r^2 = 0$ and so $3\pi r^2 = 150$ giving $r = \sqrt{\frac{50}{\pi}} \approx 4$ cm.
Checking that this is a maximum, $\frac{d^2V}{dr^2} = -6\pi r$ which is < 0.

From (a), when $r = 4$ cm, $h = \frac{150 - 50}{\pi \times 4} \approx 8$ cm.

Hence to obtain the maximum volume, the radius = 4 cm and the height = 8 cm.
The maximum volume is then $\pi r^2 h = \pi 4^2 \times 8 = 402$ cm³.

[Note that the height = the diameter. A can of this shape will give maximum volume for a given surface area.]

Example 14

The length of a closed rectangular box is 3 times its width (Fig.10.12). If its volume is 972 cm³, find the dimensions of the box if the surface area is to be a minimum.

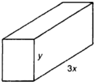


Fig. 10.12

Take the width as x cm, length $3x$ cm and let the height be y cm for the moment.

The volume $V = 3x^2y = 972$ i.e. $x^2y = 324$ (i)

The surface area $A = 6x^2 + 6xy + 2xy = 6x^2 + 8xy$ (ii)

We must now express A in terms of one variable.

From (i), $y = \frac{324}{x^2}$

and so $A = 6x^2 + 8x\frac{324}{x^2} = 6x^2 + \frac{2592}{x}$, and $\frac{dA}{dx} = 12x - \frac{2592}{x^2}$.

To minimize A, we set $\frac{dA}{dx} = 0$.

Then $12x - \frac{2592}{x^2} = 0$ giving $12x^3 = 2592$ or $x^3 = 216$. Hence $x = 6$.

To verify that this is a minimum, $\frac{d^2A}{dx^2} = 12 + \frac{2 \times 2592}{x^3}$ which will be positive.

From (i), when $x = 6$, $y = 324 \div 36 = 9$.

Hence the dimensions are 18 cm by 6 cm by 9 cm for the minimum surface area.

Example 15

Triangle ABC is isosceles with AB = AC = 20 cm and BC = 24 cm (Fig.10.13). A rectangle PQRS is drawn inside the triangle with PQ on BC, and S and R on AB and AC respectively.

(a) If PQ = 2x cm, show that the area A cm² of the rectangle is given by $A = \frac{8x(12 - x)}{3}$.

(b) Hence find the value of x for which A is a maximum.

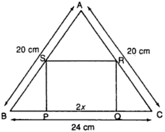


Fig. 10.13

(a) We need to know the height of the rectangle.
Let SP = y cm (Fig.10.14).
If M is the midpoint of BC, then BM = 12, BP = $12 - x$ and, by Pythagoras' Theorem, AM = 16.

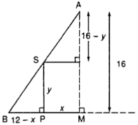


Fig. 10.14

From △SBP, $\tan \angle SBP = \frac{y}{12-x}$ and from △ABM, $\tan \angle SBP = \frac{16}{12} = \frac{4}{3}$.

Then $\frac{y}{12-x} = \frac{4}{3}$ so $y = \frac{4(12-x)}{3}$.

Hence $A = 2xy = \frac{8x(12-x)}{3}$.

(b) $A = \frac{96x - 8x^2}{3} = 32x - \frac{8x^2}{3}$ and $\frac{dA}{dx} = 32 - \frac{16x}{3}$.

When $\frac{dA}{dx} = 0, x = 6$.

$\frac{d^2A}{dx^2} = -\frac{16}{3}$ which is negative thus confirming that this gives a maximum.

Exercise 10.3 *(Answers on page 629.)*

1 Given that $x + y = 8$, find the minimum value of $x + y^2$.

2 Find the minimum value of $x^2 - xy + y^2$ given that $x + y = 10$.

3 x and y are numbers such that $x + y = 4$. Find the minimum value of $x^2 + xy + 2y^2$.

4 Given that $u = 3 + 4t^2 - 2t^3$, find the maximum value of u for the domain $0 \le t \le 2$, showing that it is a maximum.

5 If $s = 7 + 8t + 5t^2 - t^3$, find the value of t which gives a minimum value of s, showing that it is a minimum.

6 What is the minimum value of $x + \frac{1}{x}$ if $x > 0$?

7 If $R = \frac{V^2}{4} + \frac{500}{V}$, find the value of V for which R is a minimum.

8 A rectangular box, with a lid, is made from thin metal. Its length = $2x$ cm and its width = x cm. If the box must have a volume of 72 cm^3,
(a) show that the area A cm^2 of metal used is given by $A = 4x^2 + \frac{216}{x}$,
(b) find the value of x so that A is a minimum.

9 The cost $\$C$ of running a boat on a trip is given by $C = 4v^2 + \frac{1000}{v}$ where v is the average speed in km h^{-1}. Find the value of v for which the cost is a minimum.

10 It is estimated that the load L which can safely be placed on a beam of width x, length y and height h is given by $L = \frac{4xy^2}{h}$. If $h = 30$ and $x + y = 15$, find the greatest load that the beam can bear.

11 A piece of wire of length 20 cm is formed into the shape of a sector of a circle of radius r cm and angle θ radians.

(a) Show that $\theta = \frac{20 - 2r}{r}$ and that the area of the sector is $r(10 - r)$ cm^2.

(b) Hence find the values of r and θ to give the maximum area.

12 A cylinder is placed inside a circular cone of radius 18 cm and height 12 cm so that its base is level with the base of the cone, as shown in Fig.10.15.

(a) If the radius of the cylinder is r cm, show that its height h cm is given by $h = \frac{2}{3}(18 - r)$.

(b) Hence find the value of r to give the maximum possible volume of the cylinder and find this volume in terms of π.

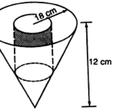


Fig. 10.15

13 A straight line passes through the point (2,3) and its gradient is m. It meets the positive x– and y– axes at A and B respectively.

(a) State the equation of the line in terms of m.

(b) Show that $OA = 2 - \frac{3}{m}$ and find a similar expression for OB.

(c) Show that the area of $\triangle OAB = 6 - \frac{9}{2m} - 2m$.

(d) Hence find the value of m for which this area is a minimum, showing that it is a minimum.

14 From a rectangular piece of thin cardboard 16 cm by 10 cm, the shaded squares each of side x cm are removed (Fig.10.16). The remainder is folded up to form a tray.

(a) Show that the volume V cm^3 of this tray is given by $V = 4(x^3 - 13x^2 + 40x)$.

(b) Hence find a possible value of x which will give the maximum value of V.

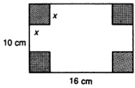


Fig. 10.16

15 The cost of making x articles per day is $\$(\frac{1}{2}x^2 + 50x + 50)$ and the selling price of each one is $\$(80 - \frac{1}{4}x)$. Find

(a) the daily profit in terms of x,

(b) the value of x to give the maximum profit.

16 Ship A is at O at noon and is sailing due East at 10 km h^{-1} (Fig.10.17). At that time, ship B is 100 km due South of O and is sailing at 20 km h^{-1} due North.

(a) State the distances in km of A and B from O after t hours.

(b) Show that the distance S km between A and B is then given by $S^2 = 500t^2 - 4000t + 10\ 000$.

(c) Find the value of t for which S^2 is a minimum and hence find the minimum distance between the ships.

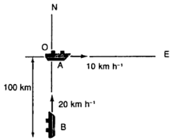


Fig. 10.17

17 The dimensions of a cylinder of radius r are such that the sum of its length and its circumference is 8π cm.

(a) Show that its length is $\pi(8 - 2r)$ cm.

(b) Hence state its volume in terms of r and find the value of r which gives the maximum volume.

18 In Fig.10.18, ABCD is a rectangle which fits inside the semicircle of radius 10 cm and centre O.

(a) If AB = $2x$ cm, show that the area A cm^2 of the rectangle is given by $A^2 = 4x^2(100 - x^2)$.

(b) Find the value of x which makes A^2 a maximum.

(c) Hence find the maximum area of the rectangle.

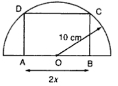


Fig. 10.18

19 In Fig.10.19, ABCD is a rectangle where AB = 9 m and AD = 6 m. CE = 4 m and FE is parallel to AD. X is a point on FE where XF = x m and M is the midpoint of BC. Find

(a) AX^2 and XM^2 in terms of x,

(b) the value of x for which $AX^2 + XM^2$ is a minimum.

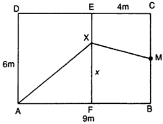


Fig 10.19

20 In $\triangle ABC$, $\angle BAC = 60°$, AB = 4 cm and AC = 2 cm. P lies on AB extended where BP = x cm, while Q lies on AC extended where CQ = y cm. Given that $x + y = 10$, show that $PQ^2 = 3x^2 - 24x + 112$ and find the value of x which will make PQ^2 a minimum. State the ratio of BC:PQ in that case.

21 The position vectors $\mathbf{r}_A$ and $\mathbf{r}_B$ of two points A and B are given by $\mathbf{r}_A = 2t\mathbf{i} + (1 + t)\mathbf{j}$ and $\mathbf{r}_B = (t + 1)\mathbf{i} - (t + 2)\mathbf{j}$.

(a) Find the values of t for which OA is perpendicular to OB where O is the origin.

(b) Find the vector $\overrightarrow{AB}$ in terms of t.

(c) Find the value of t for which $|\overrightarrow{AB}|^2$ is a minimum.

(d) Hence find the shortest distance between A and B.

22 A can is in the shape of a closed cylinder with a hemisphere at one end (Fig.10.20). Its volume is 45π cm³. Taking r cm as the radius of the cylinder and h cm as its height, show that

(a) $r^2h + \frac{2r^3}{3} = 45$,

(b) the external surface area A of the can is given by $A = \frac{5\pi r^2}{3} + \frac{90\pi}{r}$

(c) Hence find the value of r for which A is a minimum and find the minimum value of A.

(Volume of a sphere = $\frac{4\pi r^3}{3}$, surface area of a sphere = $4\pi r^2$).

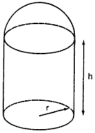


Fig. 10.20

23 ABC is an isosceles triangle with AB = AC = 10 cm and $\angle BAC = 60°$. A particle P starts from B and moves along BA at a speed of 2 cm s^{-1}. Another particle Q starts from A at the same time and moves along AC at a speed of 4 cm s^{-1}.

(a) Write down the distances of P and Q from A at time t seconds after the start. Find

(b) an expression for PQ^2 in terms of t and

(c) the value of t for which PQ^2 is a minimum.

(d) Hence find the minimum length of PQ.

24 Fig.10.21 shows a framework in the shape of a rectangular box made from straight pieces of wire. The total length of these pieces is 60 cm.

(a) Show that $y = (15 - 5x)$ cm.

(b) Find an expression for the volume enclosed by the framework in terms of x and hence find (c) the value of x which makes this volume a maximum and (d) the maximum volume.

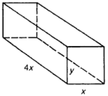


Fig. 10.21

25 A piece of wire 48 cm long is divided into two parts. One part is formed into the shape of a circle of radius r cm while the other part is formed into a square of side x cm.

(a) Show that $r = \frac{24 - 2x}{\pi}$.

(b) Find an expression in terms of x for the total area A of the two shapes and hence calculate (correct to 3 significant figures) the value of x for which A is a minimum.

26 In $\triangle ABC$, $\angle A = 60°$ and AB = x cm, AC = y cm where $x + 2y = k$ (a constant). Find an expression for BC^2 in terms of x and k and hence find the ratio $x{:}y$ for which BC^2 is a minimum.

27 ABCD is a square of side 10 cm. P lies on BC where BP = x cm and Q lies on CD where CQ = $\frac{3x}{2}$ cm. (a) Find an expression in terms of x for the area of $\triangle APQ$ and hence (b) find the value of x which makes this area a minimum.

28 A rectangular box has a square cross–section and the sum of its length and the perimeter of this cross–section is 2 m. If the length of the box is x m, show that its volume V m^3 is given by $V = \frac{x(2 - x)^2}{16}$.

Hence find the maximum volume of the box.

29 Fig. 10.22 shows part of the parabola $y = 8x - x^2$ with a rectangle ABCD which fits between the curve and the x-axis. Taking AB = $2x$ show that (a) OB = $x + 4$ and (b) the area of ABCD = $32x - 2x^3$ units^2. Hence find the value of x which makes this area a maximum and state the maximum area.

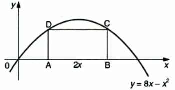


Fig. 10.22

VELOCITY AND ACCELERATION

A common rate of change is the **speed** of a moving body. This is the rate of change of distance travelled with respect to time. The average speed is

$$\frac{\text{distance travelled}}{\text{time taken}}$$

Speed is usually measured in m s^{-1} but also in cm s^{-1} or km h^{-1}.

If the *direction* is to be taken into account, then we speak of the **velocity** of the body. The magnitude of the vector velocity is the speed.

Now if the time $\longrightarrow$ 0, we shall have the limiting value of the average speed, i.e. the speed at a particular instant or the **instantaneous** speed. So if s is the distance travelled in time t and s is a function of t, then $\frac{ds}{dt}$ will give the speed v at a given instant.

$$v = \frac{ds}{dt} \text{ where } s \text{ is a function of } t$$

If v itself is changing, then we have the rate of change of speed v with respect to t, called the **acceleration (*a*)**.

Now $$a = \frac{dv}{dt} = \frac{d(\frac{ds}{dt})}{dt} = \frac{d^2s}{dt^2} \text{ where } v \text{ is a function of } t$$

Acceleration is the rate of *increase* of the velocity with respect to time and hence its standard unit is metres per second *per second*, written m s^{-2}.

A positive acceleration means that the speed is increasing, while a negative acceleration (or a **deceleration** or **retardation**) means that the speed is decreasing.

If the distance s is measured from a fixed point O, its value at any time t is also called the **displacement** of the particle from O. This is its actual distance *from* O at time t which is **not** necessarily the same as the distance *travelled* up to time t. This is illustrated in Example 16.

Example 16

A particle starts from a point O and moves in a straight line so that its distance s cm from O after time t seconds is given by $s = 2t^2 - \frac{t^3}{6}$. *Find*

(a) its initial velocity and acceleration,

(b) the time after the start when it comes to a momentary halt,

(c) its distance from O at this time.

(d) What maximum velocity does it reach before that time?

(e) After what time does the particle pass through O again?

(A 'particle' means a body small enough for its dimensions to be ignored.)

If $s = 2t^2 - \frac{t^3}{6}$, then the velocity $v = \frac{ds}{dt} = 4t - \frac{t^2}{2}$ (i)

and the acceleration $a = \frac{dv}{dt} = 4 - t$ (ii)

(a) When $t = 0$ (the start), $v = 0$ and $a = 4$. The particle starts from rest (motionless) with an acceleration of 4 cm s^{-2}.
From (ii), note that the acceleration decreases to 0 in the first 4 seconds and then becomes negative.

(b) From (i), $v = 0$ when $4t - \frac{t^2}{2} = 0$ i.e. $t(4 - \frac{t}{2}) = 0$ which gives $t = 0$ (the start) or $t = 8$.
At $t = 8$, $a = 4 - 8 = -4$ so the particle stops and instantly reverses direction, moving back towards O. Such a position, where $v = 0$ but $a \neq 0$, is called 'instantaneous rest'.

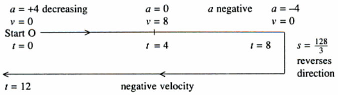
(c) When $t = 8$, $s = 2(8)^2 - \frac{8^3}{6} = \frac{128}{3}$ cm.

(d) The maximum velocity occurs when $\frac{dv}{dt} = 0$.
From (ii), this occurs when $t = 4$ and v is then $4(4) - \frac{16}{2} = 8$ cm s^{-1}.

(e) $s = 0$ when $2t^2 - \frac{t^3}{6} = 0$ i.e. $t^2(2 - \frac{t}{6}) = 0$ which gives $t = 0$ or or $t = 12$.

Hence the particle passes through O again after 12 seconds, now moving in the reverse direction.

The following diagram shows the features of the motion.



At time $t = 8$, the displacement = distance travelled = $\frac{128}{3}$.

At time $t = 12$, the displacement = 0 but the distance travelled was $\frac{256}{3}$. The particle reversed during that time.

Example 17

The distance s m of a particle moving in a straight line measured from a fixed point O on the line is given by $s = t^2 - 3t + 2$ where t is the time in seconds from the start. Find

(a) its initial distance from O,
(b) its initial velocity and in which direction,
(c) its initial acceleration,
(d) the times when it passes through O and with what velocity,
(e) when and where it is at instantaneous rest.

(a) At the start, $t = 0$. Then $s = 2$ m. The particle starts 2 m from O.

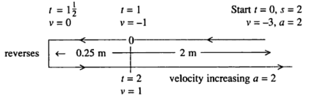
(b) $v = \frac{ds}{dt} = 2t - 3$
When $t = 0$, $v = -3$, i.e. in the direction towards O.

(c) $a = \frac{dv}{dt} = 2$
The acceleration is constant i.e. 2 m s^{-2}.

(d) $s = 0$ when $t^2 - 3t + 2 = (t - 2)(t - 1) = 0$ i.e. when $t = 2$ or 1.
When $t = 1$, $v = -1$ and when $t = 2$, $v = 1$.

(e) The particle is at instantaneous rest when $v = 0$, i.e. when $t = 1\frac{1}{2}$ seconds.
Then $s = (1\frac{1}{2})^2 - 3(1\frac{1}{2}) + 2 = -0.25$ m.

Putting these facts together, the following diagrammatic representation of the motion can be made:



Exercise 10.4 *(Answers on page 629.)*

1 A particle, moving in a straight line, starts from rest and its displacement s m from a fixed point of the line is given by $s = t^2 - kt$ where k is a constant and t is the time (in seconds) after the start. If it comes to instantaneous rest after 2 seconds, find
(a) the value of k,
(b) the initial velocity of the particle.

2 The distance s m of a particle moving in a straight line measured from a fixed point O on the line is given by $s = t^2 - 2t$ where t is the time in seconds after the start.
(a) What is the initial velocity of the particle?
(b) When is the particle at instantaneous rest?
(c) When does it pass through O for the second time?
(d) What is the acceleration of the particle?

3 For a particle moving in a straight line, its displacement s m from a point O on the line is given by $s = t^2 - 5t + 6$, where t is the time in seconds from the start. Find
(a) the initial distance of the particle from O,
(b) its initial velocity,
(c) when it is at instantaneous rest,
(d) at what time(s) after the start it passes through O.
(e) the distance travelled in the first 3 seconds.

4 A small body moves along the x-axis so that its distance x from the origin at time t s is given by $x = 2t^3 - 15t^2 + 24t + 20$. Find
(a) the velocity with which it starts,
(b) when it is at instantaneous rest,
(c) the minimum distance of the body from the origin.
(d) Between what times is the particle moving towards the origin?
(e) What is its acceleration at the times in (d)?

5 A particle moves in a straight line. Its displacement s m from a fixed point on the line is given by $s = t^2 - 4t - 5$, at a time t after the start, where $t \geq 0$. Find
(a) where the particle starts and its initial velocity,
(b) when and where it comes to instantaneous rest,
(c) when it passes through the fixed point,
(d) its acceleration.

6 A particle moves along the x-axis and its x-coordinate at time t s after the start is given by $x = 2t^3 - 9t^2 + 12t - 1$ for $t \geq 0$.
(a) Find its x-coordinate and velocity at the start.
(b) At what times does the particle come to instantaneous rest?
(c) What is its maximum velocity in the direction of the negative x-axis?
(e) When is its acceleration zero?

7 The velocity v cm s^{-1} of a particle moving in a straight line is given by $v = 6t - kt^2$, where k is a constant and t s is the time from the start. If its acceleration is 0 when $t = 1$, find
(a) the value of k,
(b) the time when the particle comes to instantaneous rest,
(c) the maximum velocity of the particle.

SMALL INCREMENTS: APPROXIMATE CHANGES

Given a function $y = f(x)$, suppose x is changed by an increment δx to become $x + \delta x$. Then y changes by an increment δy. We can find an approximate value for δy in a simple way using $\frac{dy}{dx}$, provided **δx is small.**

In Fig. 10.23, A is the point on $y = f(x)$ where $x = k$. AB = δx and BC = δy. AT is the tangent at A and the gradient of this tangent = $\left(\frac{dy}{dx}\right)_{x=k}$

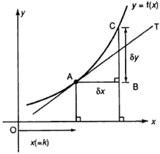


Fig. 10.23

Now if δx is small, we can take $\frac{BC}{AB} = \frac{\delta y}{\delta x}$ to be approximately equal to the gradient of the tangent at A.

Then $\frac{\delta y}{\delta x} \approx \left(\frac{dy}{dx}\right)_{x=k}$ and so

$$\delta y \approx \left(\frac{dy}{dx}\right)_{x=k} \times \delta x$$

Example 18

If $y = \frac{27}{x^2}$, *find the approximate change in y if x is increased from 3 to 3.01.*

Here $k = 3$.

$\frac{dy}{dx} = -\frac{54}{x^3}$ and so $\left(\frac{dy}{dx}\right)_{x=3} = -2.$

Then $\delta y \approx \left(\frac{dy}{dx}\right)_{x=3} \times \delta x = -2 \times 0.01 = -0.02.$

Note that the negative value indicates a *decrease* in the value of y.

Example 19

Given that $T = x^3 - 2x^2 + 1$ *and x is decreased from 2 to 1.985, find the new value of T approximately.*

$\frac{dT}{dx} = 3x^2 - 4x$ so $\left(\frac{dT}{dx}\right)_{x=2} = 4.$

Then $\delta T \approx \left(\frac{dT}{dx}\right)_{x=2} \times \delta x$

$= 4 \times (-0.015)$ (as x was decreased)

$= -0.06$

Hence the new value of $T = 2^3 - 2(2^2) + 1 - 0.06 = 0.94.$

Example 20

The volume V of a sphere is given by $V = \frac{4\pi r^3}{3}$ where r is the radius.

(a) State an expression for the approximate change in V if r is changed by a small amount δr.

(b) Hence find the approximate percentage change in V if r is increased by 1%.

(a) $\frac{dV}{dr} = 4\pi r^2$

$\delta V \approx \left(\frac{dV}{dr}\right) \times \delta r = 4\pi r^2 \times \delta r.$

(b) If r is increased by 1% then $\delta r = 0.01r$.

The percentage increase in $V \approx \frac{\delta V}{V} \times 100\% = \frac{4\pi r^2 \delta r}{\frac{4}{3}\pi r^3} \times 100\%$

$= \frac{3}{r} \times 0.01r \times 100\% = 3\%$

Example 21

If $y = 3 - x + 2x^2$ and x is increased from 4 by an amount $\frac{r}{100}$ where r is small, find in terms of r

(a) the approximate change in y,

(b) the percentage change in y.

(a) $\frac{dy}{dx} = -1 + 4x$ and $\left(\frac{dy}{dx}\right)_{x=4} = 15.$

Then $\delta y \approx 15 \times \frac{r}{100} = \frac{3r}{20}.$

(b) The original value of y was $3 - 4 + 32 = 31$ and the percentage change in

$y = \frac{\delta y}{y} \times 100\% \approx \frac{\frac{3r}{20}}{31} \times 100\% = \frac{15r}{31}\%.$

Example 22

If $y = 2x^2 - 3x + 1$, find the positive value of x for which y = 3. Hence find the approximate increase in x which will change y from 3 to 3.015.

When $y = 3$, $2x^2 - 3x + 1 = 3$ so $2x^2 - 3x - 2 = 0$ or $(2x + 1)(x - 2) = 0$ giving $x = 2$ (positive value).

$\frac{dy}{dx} = 4x - 3$ and $\left(\frac{dy}{dx}\right)_{x=2} = 5$; $\delta y = 0.015.$

Then substituting in $\delta y \approx \left(\frac{dy}{dx}\right)_{x=2} \times \delta x,$

$0.015 = 5\delta x$ giving $\delta x \approx 0.003.$

Exercise 10.5 *(Answers on page 629.)*

1 If $y = x^2 - x + 1$, find the approximate change in y when x is increased from 4 to 4.025.

2 Given the function $y = x^3 + x^2 - 4$, x is increased from 4 to 4.05. What is the approximate change in y?

3 If $z = 2x^2 - 7$, find the approximate change in z when x is decreased from 4 to 3.99.

4 Given that $y = (x + 2)^5$, find the approximate change in y when x is increased from 2 to 2.005.

5 Given that $y = (x^2 - x - 1)^4$, find the change in y following an increase in x from 2 to 2.01.

6 For the function $T = \frac{5}{s+1}$, find the new value of T approximately due to an increase in s from 9 to 9.1.

7 $P = (1 - \frac{1}{x})^3$. When $x = 2$, it is decreased by 3%. Find the approximate percentage change in P.

8 Find the approximate change in T for the function $T = 4 + 3u - 2u^2$ when u is increased by 5% from the value of 2.

9 The radius of a circle is increased by 5%. Calculate the approximate percentage increase in
(a) the circumference,
(b) the area of the circle.

10 A piece of wire of length 20 cm is shaped into the form of a sector of a circle of radius r cm and angle θ radians.
(a) Show that the area A cm^2 of the sector is given by $A = r(10 - r)$.
(b) If r is increased by 2% when $r = 2.5$ cm, find the approximate percentage change in A.

11 The height of a cone is 20 cm but the radius of its circular base is increased from 10 cm to 10.01 cm. Find the approximate change in the volume of the cone in terms of π.

12 If $y = x^3 - 3x^2$, find, in terms of k, (a) the approximate increase in y if x is increased from 4 to $4 + k$, where k is small and (b) the approximate percentage change in y.

13 Each side of a cube is increased by p% where p is small. What is the approximate percentage increase in the volume of the cube in terms of p?

14 $y = x^2 - \frac{1}{1-x}$. If x is increased from 3 to 3.001, find the approximate change in y.

15 If x is decreased from 5 to 4.98 in the function $y = \frac{2}{x-1}$, what is the approximate percentage change in y?

16 Find the positive value of x when $y = 4$ for the function $y = x^2 - 5x - 2$.
Hence find the approximate change in x when y changes from 4 to 4.02.

17 The y-coordinate of a point in the first quadrant on the curve $y = 3x^2 - 8x - 1$ is 2. Find its x-coordinate. What is the approximate change in x if the point is moved to a position on the curve where $y = 2.04$?

18 For the function $y = 3x^2 + ax + b$, where a and b are constants, when x changes from 2 to 2.02, y changes from 2 to 2.12 approximately. Find the values of a and b.

19 In an experiment to find the values of T from the formula $T = \frac{2}{x^2 + 4}$, values of x are read from a measuring device. A value of x is read as 2.04 but should be 2. What is the approximate error in the value of T?

20 U is calculated from the formula $U = \frac{2}{x - 1}$. Measurements of x are taken but they are liable to an error of $\pm 1.5\%$. When x is measured as 3, what are the greatest and least values of U?

21 Given that $v = \frac{1}{u} + \frac{1}{1 - u}$, find the approximate change in v when u is increased from 2 to 2.04.

22 For the function $A = \frac{1}{(r - 1)^2}$, a small change in r when $r = 2$ $(r - 1)^2$ produces a 2% reduction in the value of A. Find the change in r approximately.

Connected Rates of Change

Example 23

Some oil is spilt onto a level surface and spreads out in the shape of a circle. The radius r cm of the circle is increasing at the rate of 0.5 cm s^{-1}. At what rate is the area of the circle increasing when the radius is 5 cm?

The rate of change of the radius wrt time (t) = $\frac{dr}{dt} = 0.5$.

We wish to find the rate of change of the area A i.e. $\frac{dA}{dt}$.

We can find a link between these two rates by using the rule for the differential coefficient of combined functions i.e. $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$.

We know that $A = \pi r^2$ so $\frac{dA}{dr} = 2\pi r = 2\pi \times 5$.

Then $\frac{dA}{dt} = 2\pi \times 5 \times 0.5 = 15.7\ \text{cm}^2\ \text{s}^{-1}$.

This method can always be used to compare the rates of change of two connected quantities x and y with respect to a third quantity. The relation between x and y gives $\frac{dy}{dx}$.

Example 24

Water is emptied from a cylindrical tank of radius 20 cm at the rate of 2.5 litres s^{-1} and fresh water is added at the rate of 2 litres s^{-1} (Fig. 10.24). At what rate is the water level in the tank changing?

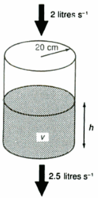


Fig. 10.24

Let the depth of water be h cm at time t s.
The rate of the change of the volume V cm^3 of the water
$= \frac{dV}{dt} = +2000 - 2500 = -500$ cm^3 s^{-1}
(1 litre = 1000 cm^3; so 2000 cm^3 of water flowing in and 2500 cm^3 of water flowing out, per second).
The rate of change of the water level $= \frac{dh}{dt}$ which we have to find.
Then $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$.
Since $V = \pi r^2 h$ and $\frac{dV}{dh} = 400\pi$,
therefore $-500 = 400\pi \times \frac{dh}{dt}$ giving $\frac{dh}{dt} = \frac{-500}{400\pi} = -0.40$ cm s^{-1}
The water level is *falling* at the rate of 0.4 cm s^{-1}.

Example 25

A hollow circular cone is held upside down with its axis vertical (Fig. 10.25). Liquid is added at the constant rate of 20 cm^3 s^{-1} but leaks away through a small hole in the vertex at the constant rate of 15 cm^3 s^{-1}. At what rate is the depth of the liquid in the cone changing when it is 12 cm?

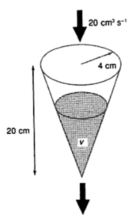


Fig. 10.25

If V cm³ is the volume of the liquid in the cone at time t s, then
$\frac{dV}{dt} = +20 - 15 = 5$ cm³ s⁻¹.

Let h cm be the height of the liquid at time t s. We have to find $\frac{dh}{dt}$.
$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$ so we find a relation between V and h.

Taking r as the radius of the water surface, from Fig. 10.26 we have:
$\tan\theta = \frac{4}{20} = \frac{r}{h}$ so $r = \frac{h}{5}$ and $V = \frac{1}{3}\pi r^2 h = \frac{\pi h^3}{75}$.

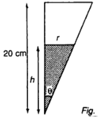


Fig. 10.26

Then $\frac{dV}{dh} = \frac{\pi h^2}{25}$.

So $\frac{dV}{dt} = 5 = \frac{\pi \times 12^2}{25} \times \frac{dh}{dt}$

and $\frac{dh}{dt} = \frac{25 \times 5}{\pi \times 144} \approx 0.28$ cm s⁻¹.

The depth of liquid is rising at the rate of 0.28 cm s⁻¹.

Note that in this example the rate of change of the depth depends on how much liquid is already in the cone as the cross-section is not constant.

Example 26

The pressure P units and the volume V m³ of a quantity of gas stored at a constant temperature in a cylinder are related by Boyle's Law PV = k (a constant). At a certain time, the volume of gas in the cylinder is 30 m³ and its pressure is 20 units. If the gas is being compressed at the rate of 6 m³ s⁻¹, at what rate is the pressure changing?

$PV = k$ so $k = 20 \times 30 = 600$ units m^3.

The relation between P and V is $PV = 600$ or $P = \frac{600}{V}$.

Now $\frac{dP}{dt} = \frac{dP}{dV} \times \frac{dV}{dt}$ and $\frac{dP}{dV} = -\frac{600}{V^2}$

We are given that $\frac{dV}{dt} = -6$ (decreasing).

So $\frac{dP}{dt} = -\frac{600}{30^2} \times (-6) = 4$ units per second (increasing).

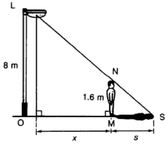
Example 27

A street lamp is 8 m high. A man of height 1.6 m walks along the street away from the lamp at a steady rate of 1 m s^{-1}. At what rate is the length of his shadow changing?

In Fig. 10.27, L is the lamp and OL = 8 m.

MN = 1.6 m is the man and MS = s m his shadow.

Fig. 10.27



Let OM = x m.

Now $\frac{dx}{dt}$ = rate at which the man is walking = 1 m s^{-1}. We require $\frac{ds}{dt}$.

$\frac{ds}{dt} = \frac{ds}{dx} \times \frac{dx}{dt}$ so we find the relation between s and x.

$\tan \angle NSM = \frac{1.6}{s} = \frac{8}{x+s}$ which gives $s = \frac{x}{4}$.

Then $\frac{ds}{dx} = \frac{1}{4}$.

Hence $\frac{ds}{dx} = \frac{1}{4} \times 1 = 0.25$ m s^{-1} which is the rate at which the length of his shadow is increasing.

Exercise 10.6 *(Answers on page 630.)*

1 At what rate is the area of a circle decreasing when its radius is 8 cm and decreasing at 0.4 cm s^{-1}?

2 The area of a circle is decreasing at the rate of 2 cm^2 s^{-1}. How fast is the radius decreasing when the area is 9π cm^2?

3 The radius r cm of a sphere is 10 cm and it is increasing at the rate of 0.25 cm s^{-1}. At what rate is (a) the volume, (b) the surface area, increasing?
(For a sphere, volume = $\frac{4\pi r^3}{3}$ and surface area = $4\pi r^2$).

4 A spherical balloon is being inflated by blowing in 2×10^3 cm^3 of air per second. At what rate is its radius increasing when its diameter is 20 cm?

5 ABC is a triangle with $\angle B = 90°$ and AB has a fixed length of 8 cm.
The length of BC is increasing at 0.5 cm s^{-1}. At what rate is the area of the triangle increasing?

6 A closed cylinder is of fixed length 10 cm but its radius is increasing at the rate of 1.5 cm s^{-1}. Find the rate of increase of its total surface area when the radius is 4 cm. (Leave the answer in terms of π).

7 A circular cylinder has a diameter of 40 cm and is being filled with water at the rate of 1.5 litres s^{-1}. At what rate is the water level rising?

8 The length of each side of a cubical framework of straight wires is expanding at the rate of 0.02 m s^{-1}. At what rate in cm^3 s^{-1} is the volume of the framework changing when each side is 0.2 m long?

9 x and y are connected by the equation $y = \frac{x^2 - 3}{x}$. If x is changing at a rate of 0.3 units per second, find the rate of change of y when $x = 3$.

10 $y = (2r^2 - r + 1)^3$ and $x = 4r$. At what rate is y changing with respect to x when $r = 0.5$?

11 The height of a cone remains constant at 20 cm. The radius of the base is 5 cm and is increasing at 0.2 cm s^{-1}. At what rate is the volume of the cone changing?

12 The volume V cm^3 of liquid in a container is given by $V = 2x^3 - 4x^2 + 5$ where x cm is the depth of the liquid. At what rate is the volume increasing when $x = 4$ and is increasing at the rate of 1.5 cm s^{-1}?

13 Liquid escapes from a circular cylinder of radius 5 cm at a rate of 50 cm^3 s^{-1}. How fast is the level of the liquid in the cylinder falling?

14 A hollow cone of radius 15 cm and height 25 cm, is held vertex down with its axis vertical. Liquid is poured into the cone at the rate of 500 cm^3 s^{-1}. How fast is the level of the liquid rising when the radius of its surface is 10 cm?

15 In an electrical circuit the resistance $R = \frac{10}{I}$ where I is the current flowing in the circuit. If I is increasing at 0.05 units per second, what is the rate of change of R when $R = 5$ units?

16 Two quantities p and q are related by the equation $(p - 1)(q + 2) = k$ where k is a constant. When $p = 5$ units, q is 7 units and q is changing at the rate of 0.04 units per second. Find the rate at which p is changing.

17 Water is being poured into a cylinder of radius 10 cm at a rate of 360 cm^3 s^{-1} but leaks out at a rate of 40 cm^3 s^{-1}. At what speed is the water level changing?

18 In Fig. 10.28, the sides of the rectangle ABCD are 18 cm and 10 cm. The rectangle KLMN lies inside ABCD and the shaded area has a width of x cm at each side.

(a) Express the shaded area in terms of x.

(b) If the shaded area is $\frac{8}{15}$ of the area of ABCD, find the value of x.

(c) The area of KLMN varies as x decreases at a constant rate of 0.25 cm s^{-1}. Find the rate at which the shaded area is decreasing when it is $\frac{8}{15}$ of the area of ABCD.

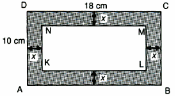


Fig. 10.28

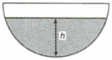


Fig. 10.29

19 A hemispherical bowl contains liquid as shown in Fig. 10.29. The volume V cm^3 of liquid is given by $V = \frac{1}{3}\pi h^2(24 - h)$ where h is the greatest depth of the liquid in cm. If liquid is poured into the bowl at the rate of 100 cm^3 s^{-1}, at what rate is the greatest depth of the liquid increasing when it is 2 cm? (Leave the answer in terms of π).

20 Sand falls on to level ground at a rate of 1000 cm^3 s^{-1} and piles up in the form of a circular cone whose vertical angle is 60°.

(a) Given that $\tan 30° = \frac{1}{\sqrt{3}}$, show that the radius r of the base is given by $r = \frac{h}{\sqrt{3}}$ where h is the height.

(b) Show that the volume V of the pile is $\frac{\pi h^3}{9}$.

(c) Hence find the rate at which the height of the pile is increasing when $h = 20$ cm.

21 In Fig. 10.30, ABC is an isosceles triangle where AC = BC = 13 cm and AB = 10 cm. PQ moves towards AB at a steady rate of 0.5 cm s^{-1} keeping parallel to AB. If PQ is x cm from C, show that

(a) PQ = $\frac{5x}{6}$ cm,

(b) the shaded area = $\frac{5}{12}(144 - x^2)$ cm^2.

(c) Hence find the rate at which the shaded area is decreasing when PQ is half way towards AB from C.

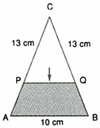


Fig. 10.30

22 (a) If $\frac{dL}{dt} = k$ where k is a number, show that $\frac{dL^2}{dt} = 2kL$.

(b) ABC is a triangle in which $\angle CAB = 60°$ and AB is of fixed length 5 cm. If AC = 8 cm, show that BC = 7 cm.

(c) Taking AC = x cm and L = length of BC, find an expression for L^2 in terms of x.

(d) Find $\frac{dL^2}{dt}$ when $x = 8$ and is increasing at 1 cm s^{-1}.

(e) Hence, using (a) find the rate at which the length of BC is changing.

SUMMARY

- As x increases,

  $y = f(x)$ is increasing for $\frac{dy}{dx} > 0$

  $y = f(x)$ is decreasing for $\frac{dy}{dx} < 0$

- Gradient of tangent to $y = f(x)$ is $\frac{dy}{dx}$.

  Gradient of normal is $-\frac{1}{\frac{dy}{dx}}$.

- For a stationary point (maximum, minimum or point of inflexion), $\frac{dy}{dx} = 0$.
  The stationary point is maximum if $\frac{d^2y}{dx^2} < 0$, minimum if $\frac{d^2y}{dx^2} > 0$.
  If $\frac{d^2y}{dx^2} = 0$, use the sign test.

- If distance s is a function of time t, then velocity $v = \frac{ds}{dt}$,
  acceleration $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$.

- If $y = f(x)$ and x is changed from a value k by a small increment δx,

  $\delta y \approx \left(\frac{dy}{dx}\right)_{x=k} \times \delta x$

- If $y = f(x)$, $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$.

REVISION EXERCISE 10 *(Answers on page 630.)*

A

1 Find the range of values of x for which the function $y = x^3 - 6x^2 - 15x + 3$ is increasing.

2 For what value of x does the function $y = 4x^3 - 6x^2 - 9x + 5$ have a minimum stationary point?

3 The area of a circle increases from 25π to 25.5π. Calculate the approximate increase in the radius.

4 Variables x and y are related by the equation $y = \frac{2x - 6}{x}$.

(i) Obtain an expression for $\frac{dy}{dx}$ and hence find an expression for the approximate increase in y as x increases from 4 to $4 + p$, where p is small.

(ii) Given that x and y are functions of t and that $\frac{dy}{dt} = 0.4$, find the corresponding rate of change of x when $y = 1$. (C)

5 The area, A cm^2, of the image of a rocket on a radar screen is given by the formula $A = \frac{12}{r^2}$, where r km is the distance of the rocket from the screen. The rocket is approaching at 0.5 km s^{-1}. When the rocket is 10 km away, at what rate is the area of the image changing? When A is changing at 0.096 cm^2 s^{-1}, how far away is the rocket? (C)

6 A piece of wire, 60 cm long, is bent to form the shape shown in Fig. 10.31. This shape consists of a semicircular arc, radius r cm, and three sides of a rectangle of height x cm.

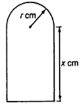


Fig. 10.31

Express x in terms of r and hence show that the area enclosed, A cm^2, is given by $A = 60r - 2r^2 - \frac{\pi r^2}{2}$.

Hence determine, to 3 significant figures, the value of r for which A is either a maximum or a minimum. Determine whether this value of r makes A a maximum or a minimum. (C)

7 If $y = 10 - x + 5x^2$, find the approximate percentage change in y when x is increased by p% (p small) when $x = 4$.

8 Under a heating process, the length, x cm, of each side of a metal cube increases from an initial value of 9.9 cm at a constant rate of 0.005 cm s^{-1}. Express the volume, V cm^3, and the surface area, A cm^2, of the cube in terms of x.

Write down expressions for $\frac{dV}{dx}$ and $\frac{dA}{dx}$.

Hence find (i) the rate at which V is increasing when the cube has been heated for 20 s, (ii) the approximate increase in A as x increases from 10 to 10.001 cm. (C)

9 $R = \frac{V^3}{25} + \frac{10\,800}{V}$. Find the value of V for which R is least.

10 A piece of wire, 100 cm in length, is divided into two parts. One part is bent to form an equilateral triangle of side x cm and the other is bent to form a square of side y cm. Express y in terms of x and hence show that A cm^2, the total area enclosed by the two shapes, is such that

$A = \frac{\sqrt{3}x^2}{4} + \frac{(100 - 3x)^2}{16}$.

Calculate the value of x for which A has a stationary value.
Determine whether this value of x makes A a maximum or a minimum. (C)

11 Show that the equation of the normal to the curve $y = 2x + \frac{6}{x}$ at the point (2,7) is $y + 2x = 11$. Given that this normal meets the curve again at P, find the x-coordinate of P. (C)

12 The diagram shows a solid body which consists of a right circular cylinder fixed, with no overlap, to a rectangular block. The block has a square base of side $2x$ cm and a height of x cm. The cylinder has a radius of x cm and a height of y cm. Given that the total volume of the solid is 27 cm³, express y in terms of x.
Hence show that the total surface area, A cm², of the solid is given by

$$A = \frac{54}{x} + 8x^2.$$

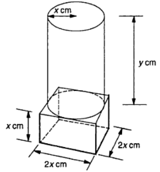


Fig. 10.32

Find
(i) the value of x for which A has a stationary value,
(ii) the value of A and of y corresponding to this value of x.
Determine whether the stationary value of A is a maximum or a minimum. (C)

13 Fig.10.31 shows part of the curve $y = 4 + 3x - x^2$ and the line $2y - 2 = x$. OB = b and BCD is parallel to the y-axis.
(a) Express the length of CD in terms of b.
(b) Hence find the value of b for which the length of CD is a maximum.

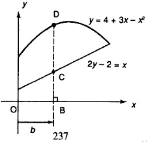


Fig. 10.33

14 A circular cylinder of height $2h$ cm is fitted inside a sphere of radius 10 cm. Find an expression for the radius of the cylinder in terms of h and hence find the maximum volume of the cylinder.

15 A point moves on the x-axis and its position at time t is given by $x = t(t^2 - 6t + 12)$. Show that its velocity at the origin is 12 and find its position when it comes to instantaneous rest. If v is its velocity and a its acceleration at time t, show that $a^2 = 12v$.

16 A piece of wire, of fixed length L cm, is bent to form the boundary OPQO of a sector of a circle (Fig. 10.34). The circle has centre O and radius r cm. The angle of the sector is θ radians.

Show that the area A cm², of the sector is given by $A = \frac{1}{2}rL - r^2$.

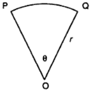


Fig. 10.34

(a) Find a relationship between r and L for which A has a stationary value and find the corresponding value of θ. Determine the nature of this stationary value.
(b) Show that, for this value of θ, the area of the triangle OPQ is approximately 45.5% of the area of the sector OPQ. (C)

17 A line of gradient m ($m < 0$) passes through the point (3,2) and meets the axes at P and Q. Find the coordinates of P and Q in terms of m and show that the area of $\triangle$POQ is $6 - \frac{2}{m} - \frac{9m}{2}$. Hence find the minimum area of $\triangle$POQ.

18 A particle is travelling in a straight line and its distance s cm from a fixed point on the line after t seconds is given by $s = 12t - 15t^2 + 4t^3$. Find.
(a) the velocity and acceleration after 3 seconds,
(b) the distance between the two points where it is at instantaneous rest.

19 A rectangular box without a lid is made from thin cardboard. The sides of the base are $2x$ cm and $3x$ cm and the height of the box is h cm. If the total surface area is 200 cm², show that

$$h = \frac{20}{x} - \frac{3x}{5}$$

and hence find the dimensions of the box to give the maximum volume.

20 Show that the height of a circular cone of volume V and radius r is given by $\frac{3V}{\pi r^2}$. If V remains constant but r is increased by 2%, find the approximate percentage change in h.

21 A particle P travels in a straight line so that its distance, s metres, from a fixed point O is given by $s = 11 + 6t^2 - t^3$ where t is the time in seconds measured from the start of the motion. Calculate

(i) the velocity of P after 3 seconds,

(ii) the velocity of P when its acceleration is instantaneously zero,

(iii) the average velocity of P over the first two seconds.

22 In Fig. 10.35, ABCD is a rectangle with AB = 6 cm and AD = 8 cm. DE = x cm. EC meets AB produced at F. Find the value of x which gives the minimum area of $\triangle$AFE and show that it is a minimum.

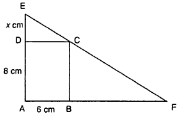


Fig. 10.35

B

23 Given the function $y = ax^3 + bx^2 + cx + d$, find the values of a, b, c and d if the curve

(i) passes through the point (0,–3),

(ii) has a stationary point at (–1,1),

(iii) the value of $\frac{d^2y}{dx^2} = 2$ when $x = 1$.

24 Find the nature of the stationary points on the curve $y = 3x^4 + 4x^3 + 2$.

25 A cylinder of radius r cm is placed upright inside a cone so that the top of the cylinder is 4 cm above the top of the cone as in Fig. 10.36. The cone has a radius of 6 cm and a height of 18 cm. The part of the cylinder inside the cone is h cm deep.

(a) Show that $h + 3r = 18$.

(b) Find an expression in terms of r for the volume of the cylinder.

(c) Hence find the value of h for which the volume of the cylinder is a maximum.

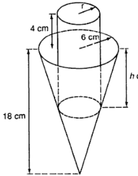


Fig. 10.36

26 (a) If $\frac{1}{u} + \frac{1}{v} = 2$, show that $u = \frac{v}{2v-1}$ and that this equals $\frac{1}{2}(1 + \frac{1}{2v-1})$.

(b) If v is increased by 2% when it is 2, find the percentage change in u.

27 A water trough 100 cm long has a cross section in the shape of a vertical trapezium ABCD as shown in Fig. 10.37. AB = 30 cm and AD and BC are each inclined at 60° to the horizontal. The trough is placed on level ground and is being filled at the rate of 10 litres s^{-1}.

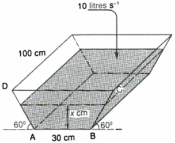


Fig. 10.37

(a) Given that $\tan 60° = \sqrt{3}$, show that the volume V cm^3 of water in the trough when it is x cm deep is given by $V = 100x(30 + \frac{x}{\sqrt{3}})$.

(b) Hence calculate the rate at which the water level is rising when $x = 15$ cm.

28 A point A moves along the positive x-axis away from the origin O at a speed of 4 cm s^{-1} where OA > 5 cm. B is a fixed point on the positive y-axis where OB = 20 cm. P is a fixed point on the positive x-axis where OP = 5 cm and Q lies on the line joining B and A with PQ parallel to the y-axis.

(a) Show that when OA = x cm, PQ = $20(1 - \frac{5}{x})$ cm.

(b) Hence find the speed of Q along PQ as A moves when (i) $x = 12$ cm, (ii) $x = 20$ cm.

(c) Obtain an expression in terms of x for the acceleration of Q along PQ.

29 In △OAB, ∠AOB = 60°, OA = 10 cm and OB = 4 cm. P lies on OA where OP = x cm and Q lies on OB. Given that the area of △OPQ is twice that of △OAB, find in terms of x, (a) OQ, (b) PQ^2. Hence find the value of x which will make PQ^2 a minimum and the corresponding length of OQ.

30 Find the point of intersection P of the curves $y^2 = 4x$ and $4y = x^2$ and sketch the parts of these curves which lie between the origin O and P. A lies on $y^2 = 4x$ with x-coordinate 2. B is a variable point (x,y) on the curve $4y = x^2$, lying between O and P. Find an expression for the area of △OAB and hence find the maximum area of this triangle.