More Equations and Inequalities

As you study Chapter 9 you will be able to recognize and solve a variety of equations and inequalities. As you work through the chapter, write the solution set for each inequality here. Then use the letter next to each answer to complete the puzzle.

Solve each inequality.

_____ 1. $x < 3$ and $x \geq -2$ i. $(-\infty, \infty)$

_____ 2. $x < 3$ or $x \geq -2$ s. $(-2, 3)$

_____ 3. $x > 3$ and $x \leq -2$ d. $[-2, 3)$

_____ 4. $x > 3$ or $x \leq -2$ v. $(-\infty, -2] \cup (3, \infty)$

_____ 5. $\dfrac{x - 3}{x + 2} \leq 0$ i. $\{\,\}$

_____ 6. $\dfrac{x - 3}{x + 2} \geq 0$ o. $(-\infty, -2) \cup (3, \infty)$

_____ 7. $(x - 3)(x + 2) < 0$ n. $(-2, 3]$

_____ 8. $(x - 3)(x + 2) > 0$ i. $(-\infty, -2) \cup [3, \infty)$

He wears glasses during his math class because it improves . . .

$$\overline{}\ \overline{}\ \overline{}\ \overline{}\ \overline{}\ \overline{}\ \overline{}\ \overline{}$$
1   6   4   2   7   3   8   5

Section 9.1   Compound Inequalities
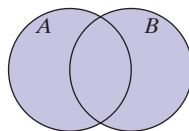
1. Union and Intersection

In Chapter 1 we graphed simple inequalities and expressed the solution set in interval notation and in set-builder notation. In this chapter, we will solve **compound inequalities** that involve the union or intersection of two or more inequalities.

> ### *A* Union *B* and *A* Intersection *B*
>
> The **union** of sets A and B, denoted $A \cup B$, is the set of elements that belong to set A or to set B or to both sets A and B.
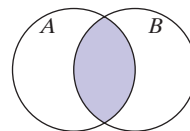>
> The **intersection** of two sets A and B, denoted $A \cap B$, is the set of elements common to both A and B.

The concepts of the union and intersection of two sets are illustrated in Figures 9-1 and 9-2.

$A \cup B$
A union *B*
The elements in *A or B or* both
Figure 9-1

$A \cap B$
A intersection *B*
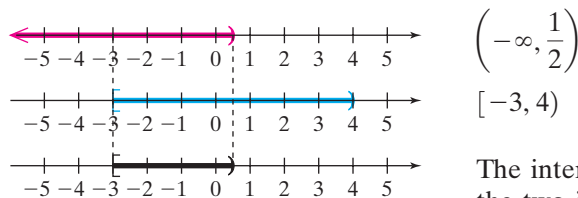The elements in *A and B*
Figure 9-2

Example 1   Finding the Union and Intersection of Two Intervals

Find the union or intersection as indicated.

a. $\left(-\infty, \frac{1}{2}\right) \cap [-3, 4)$ **b.** $(-\infty, -2) \cup [-4, 3)$

Solution:

a. $\left(-\infty, \frac{1}{2}\right) \cap [-3, 4)$ To find the intersection, graph each interval separately. Then find the real numbers common to both intervals.
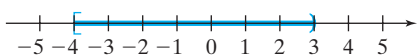

$\left(-\infty, \frac{1}{2}\right)$

$[-3, 4)$

The intersection is the "overlap" of the two intervals: $\left[-3, \frac{1}{2}\right)$

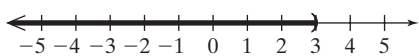The intersection is $\left[-3, \frac{1}{2}\right)$.

b. $(-\infty, -2) \cup [-4, 3)$ To find the union, graph each interval separately. The union is the collection of real numbers that lie in the first interval, the second interval, or both intervals.
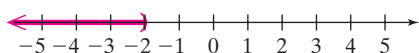
$(-\infty, -2)$

$[-4, 3)$

The union consists of all real numbers in the red interval along with the real numbers in the blue interval: $(-\infty, 3)$

The union is $(-\infty, 3)$.

Skill Practice

1. Find the intersection $(-\infty, 2) \cap [-5, \infty)$. Write the answer in interval notation.
2. Find the union $(-\infty, -3) \cup (-\infty, 0)$. Write the answer in interval notation.

2. Solving Compound Inequalities: And

The solution to two inequalities joined by the word *and* is the intersection of their solution sets. The solution to two inequalities joined by the word *or* is the union of their solution sets.

> **Steps to Solve a Compound Inequality**
>
> 1. Solve and graph each inequality separately.
>
> 2. • If the inequalities are joined by the word *and*, find the intersection of the two solution sets.
>   • If the inequalities are joined by the word *or*, find the union of the two solution sets.
>
> 3. Express the solution set in interval notation or in set-builder notation.

As you work through the examples in this section, remember that multiplying or dividing an inequality by a negative factor reverses the direction of the inequality sign.

Example 2 Solving Compound Inequalities: And

Solve the compound inequalities.

a. $-2x < 6$ and $x + 5 \le 7$

b. $4.4a + 3.1 < -12.3$ and $-2.8a + 9.1 < -6.3$

c. $-\dfrac{2}{3}x \le 6$ and $-\dfrac{1}{2}x < 1$

Solution:

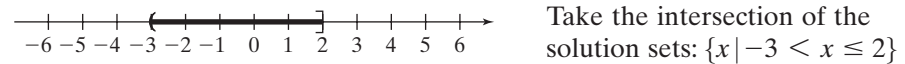**a.** $-2x < 6$ and $x + 5 \le 7$ Solve each inequality separately.

$\dfrac{-2x}{-2} > \dfrac{6}{-2}$ and $x \le 2$ Reverse the first inequality sign.

$x > -3$ and $x \le 2$

Skill Practice Answers

1. $[-5, 2)$ **2.** $(-\infty, 0)$

$$\{x\,|\,x > -3\}$$

$$\{x\,|\,x \le 2\}$$

Take the intersection of the solution sets: $\{x\,|-3 < x \le 2\}$
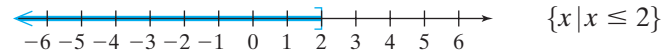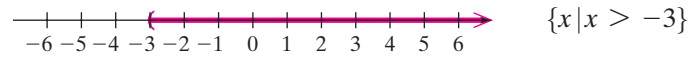
The solution is $\{x\,|-3 < x \le 2\}$ or equivalently, in interval notation, $(-3, 2]$.

b.

$$4.4a + 3.1 < -12.3 \quad \text{and} \quad -2.8a + 9.1 < -6.3$$

$$4.4a < -15.4 \quad \text{and} \quad -2.8a < -15.4$$

$$\frac{4.4a}{4.4} < \frac{-15.4}{4.4} \quad \text{and} \quad \frac{-2.8a}{-2.8} > \frac{-15.4}{-2.8} \quad \begin{array}{l}\text{Reverse the second} \\ \text{inequality sign.}\end{array}$$

$$a < -3.5 \quad \text{and} \quad a > 5.5$$

$$\{a\,|\,a < -3.5\}$$

$$\{a\,|\,a > 5.5\}$$

The intersection of the solution sets is the empty set: { }
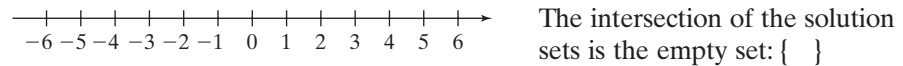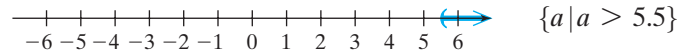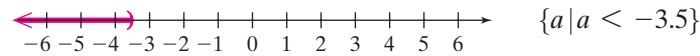
There are no real numbers that are simultaneously less than -3.5 and greater than 5.5. Hence, there is no solution.

c.

$$-\frac{2}{3}x \le 6 \quad \text{and} \quad -\frac{1}{2}x < 1$$

$$-\frac{3}{2}\left(-\frac{2}{3}x\right) \ge -\frac{3}{2}(6) \quad \text{and} \quad -2\left(-\frac{1}{2}x\right) > -2(1) \quad \begin{array}{l}\text{Solve each inequal-} \\ \text{ity separately.}\end{array}$$

$$x \ge -9 \quad \text{and} \quad x > -2$$

$$\{x\,|\,x \ge -9\}$$

$$\{x\,|\,x > -2\}$$

Take the intersection of the solution sets: $\{x\,|\,x > -2\}$
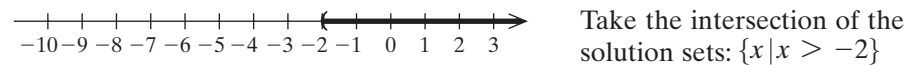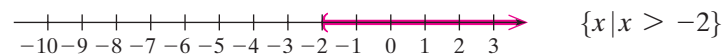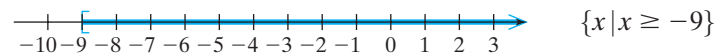
The solution is $\{x\,|\,x > -2\}$ or, in interval notation, $(-2, \infty)$.

Skill Practice Solve the compound inequalities.

3. $5x + 2 \ge -8$ and $-4x > -24$ **4.** $-\frac{1}{3}y - \frac{1}{2} > 2$ and $6y + 2 > 2$

5. $-2.1x > 4.2$ and $3.5x < -10.5$

In Section 1.7, we learned that the inequality $a < x < b$ is the intersection of two simultaneous conditions implied on x.

$$a < x < b \text{ is equivalent to } a < x \quad \text{and} \quad x < b$$

Skill Practice Answers

3. $\{x\,|-2 \le x < 6\}; [-2, 6)$
4. No solution
5. $\{x\,|\,x < -3\}; (-\infty, -3)$

Example 3	Solving Compound Inequalities: And

Solve the inequality $-2 \leq -3x + 1 < 5$.

Solution:

$$-2 \leq \boxed{-3x + 1} < 5$$

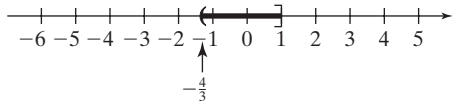|			
$-2 \leq -3x + 1$	and	$-3x + 1 < 5$	Set up the intersection of two inequalities.
$-3 \leq -3x$	and	$-3x < 4$	Solve each inequality.
$\dfrac{-3}{-3} \geq \dfrac{-3x}{-3}$	and	$\dfrac{-3x}{-3} > \dfrac{4}{-3}$	Reverse the direction of the inequality signs.
$1 \geq x$	and	$x > -\dfrac{4}{3}$	
$x \leq 1$	and	$x > -\dfrac{4}{3}$	Rewrite the inequalities.
	$-\dfrac{4}{3} < x \leq 1$		Take the intersection of the solution sets.

The solution is $\{x \mid -\frac{4}{3} < x \leq 1\}$ or, equivalently in interval notation, $(-\frac{4}{3}, 1]$.

Skill Practice	Solve the inequality.

6. $-6 < 5 - 2x \leq 1$

TIP: As an alternative approach to Example 3, we can isolate the variable x in the "middle" portion of the inequality. Recall that the operations performed on the middle part of the inequality must also be performed on the left- and right-hand sides.

$-2 \leq -3x + 1 < 5$	
$-2 - 1 \leq -3x + 1 - 1 < 5 - 1$	Subtract 1 from all three parts of the inequality.
$-3 \leq -3x < 4$	Simplify.
$\dfrac{-3}{-3} \geq \dfrac{-3x}{-3} > \dfrac{4}{-3}$	Divide by -3 in all three parts of the inequality. (Remember to reverse inequality signs.)
$1 \geq x > -\dfrac{4}{3}$	Simplify.
$-\dfrac{4}{3} < x \leq 1$	Rewrite the inequality.

Skill Practice Answers

6. $\left\{ x \mid 2 \leq x < \dfrac{11}{2} \right\}; \left[2, \dfrac{11}{2} \right)$

3. Solving Compound Inequalities: Or

Example 4   Solving Compound Inequalities: Or

Solve the compound inequalities.

a. $-3y - 5 > 4$ or $4 - y \leq 6$

b. $4x + 3 < 16$ or $-2x < 3$

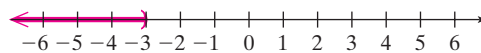**c.** $\dfrac{1}{3}x < 2$ or $-\dfrac{1}{2}x + 1 > 0$

Solution:

a. $-3y - 5 > 4$ or $4 - y \leq 6$

$\qquad\quad -3y > 9$ or $\qquad -y \leq 2$ Solve each inequality separately.

$\qquad\quad \dfrac{-3y}{-3} < \dfrac{9}{-3}$ or $\dfrac{-y}{-1} \geq \dfrac{2}{-1}$ Reverse the inequality signs.
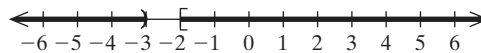
$\qquad\qquad y < -3$ or $\qquad y \geq -2$

    $\{y \mid y < -3\}$
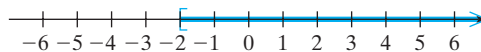
    $\{y \mid y \geq -2\}$

    Take the union of the solution sets $\{y \mid y < -3$ or $y \geq -2\}$.

The solution is $\{y \mid y < -3$ or $y \geq -2\}$ or, equivalently in interval notation, $(-\infty, -3) \cup [-2, \infty)$.

b. $4x + 3 < 16$ or $-2x < 3$

$\qquad\quad 4x < 13$ or $x > -\dfrac{3}{2}$ Solve each inequality separately.

$\qquad\quad x < \dfrac{13}{4}$ or $x > -\dfrac{3}{2}$
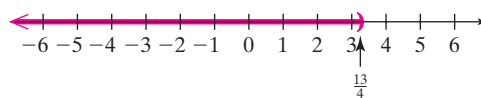
    $\{x \mid x < \tfrac{13}{4}\}$

    $\{x \mid x > -\tfrac{3}{2}\}$

    Take the union of the solution sets.

The union of the solution sets is $\{x \mid x$ is any real number$\}$ or equivalently $(-\infty, \infty)$.

c. $\frac{1}{3}x < 2$ or $-\frac{1}{2}x + 1 > 0$

$3\left(\frac{1}{3}x\right) < 3(2)$ or $-\frac{1}{2}x > -1$ Solve each inequality separately.
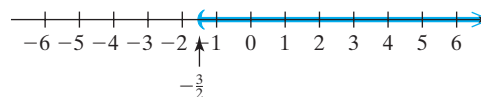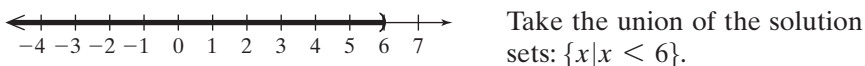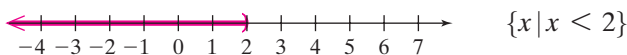
$x < 6$ or $-2\left(-\frac{1}{2}x\right) < -2(-1)$

$x < 6$ or $x < 2$

$\{x \mid x < 6\}$

$\{x \mid x < 2\}$

Take the union of the solution sets: $\{x \mid x < 6\}$.

The union of the solution sets is $\{x \mid x < 6\}$ or, in interval notation, $(-\infty, 6)$.

Skill Practice Solve the compound inequalities.

7. $-10t - 8 \geq 12$ or $3t - 6 > 3$ **8.** $x - 7 > -2$ or $-6x > -48$

9. $2.5x > 10$ or $-0.75x < 3$

---

4. Applications of Compound Inequalities

Compound inequalities are used in many applications, as shown in Examples 5 and 6.

Example 5 Translating Compound Inequalities

The normal level of thyroid-stimulating hormone (TSH) for adults ranges from 0.4 to 4.8 microunits per milliliter (μU/mL). Let x represent the amount of TSH measured in microunits per milliliter.

a. Write an inequality representing the normal range of TSH.

b. Write a compound inequality representing abnormal TSH levels.

Solution:

a. $0.4 \leq x \leq 4.8$ **b.** $x < 0.4$ or $x > 4.8$

Skill Practice

10. The length of a normal human pregnancy is from 37 to 41 weeks, inclusive.

a. Write an inequality representing the normal length of a pregnancy.

b. Write a compound inequality representing an abnormal length for a pregnancy.

Skill Practice Answers

7. $\{t \mid t \leq -2$ or $t > 3\}$;
$(-\infty, -2] \cup (3, \infty)$
8. All real numbers; $(-\infty, \infty)$
9. $\{x \mid x > -4\}$; $(-4, \infty)$
10a. $37 \leq w \leq 41$
b. $w < 37$ or $w > 41$

Example 6	Translating and Solving a Compound Inequality

The sum of a number and 4 is between -5 and 12. Find all such numbers.

Solution:

Let x represent a number.

$$-5 < x + 4 < 12 \qquad \text{Translate the inequality.}$$

$$-5 - 4 < x + 4 - 4 < 12 - 4 \qquad \text{Subtract 4 from all three parts of the inequality.}$$

$$-9 < x < 8$$

The number may be any real number between -9 and 8: $\{x \mid -9 < x < 8\}$.

Skill Practice

11. The sum of twice a number and 11 is between 21 and 31. Find all such numbers.

Skill Practice Answers

11. Any real number between 5 and 10: $\{n \mid 5 < n < 10\}$

Section 9.1	**Practice Exercises**

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Study Skills Exercise

1. Define the key terms.

   a. Compound inequality   **b. Intersection**   **c. Union**

Review Exercises

For Exercises 2–8, review solving linear inequalities from Section 1.7. Write the answers in interval notation.

2. $6u + 5 > 2$ **3.** $-2 + 3z \le 4$ **4.** $-\dfrac{3}{4}p \le 12$ **5.** $-6q > -\dfrac{1}{3}$

6. $-1.5 < 0.1x \le 8.1$ **7.** $-0.2 < 2.6 + 7t < 4$ **8.** $0 \le 4x - 1 < 5$

Concept 1: Union and Intersection

For Exercises 9–14, find the intersection and union of sets as indicated. Write the answers in interval notation.

9. a. $(-2, 5) \cap [-1, \infty)$ **10. a.** $(-\infty, 4) \cap [-1, 5)$ **11. a.** $\left(-\dfrac{5}{2}, 3\right) \cap \left(-1, \dfrac{9}{2}\right)$

    b. $(-2, 5) \cup [-1, \infty)$ **b.** $(-\infty, 4) \cup [-1, 5)$ **b.** $\left(-\dfrac{5}{2}, 3\right) \cup \left(-1, \dfrac{9}{2}\right)$

12. a. $(-3.4, 1.6) \cap (-2.2, 4.1)$ **13. a.** $(-4, 5] \cap (0, 2]$ **14. a.** $[-1, 5) \cap (0, 3)$

    b. $(-3.4, 1.6) \cup (-2.2, 4.1)$ **b.** $(-4, 5] \cup (0, 2]$ **b.** $[-1, 5) \cup (0, 3)$

Concept 2: Solving Compound Inequalities: And

For Exercises 15–24, solve the inequality and graph the solution. Write the answer in interval notation.

15. $y - 7 \geq -9$ and $y + 2 \leq 5$

16. $a + 6 > -2$ and $5a < 30$

17. $2t + 7 < 19$ and $5t + 13 > 28$

18. $5p + 2p \geq -21$ and $-9p + 3p \geq -24$

19. $21k - 11 \leq 6k + 19$ and
$3k - 11 < -k + 7$

20. $6w - 1 > 3w - 11$ and
$-3w + 7 \leq 8w - 13$

21. $\frac{2}{3}(2p - 1) \geq 10$ and $\frac{4}{5}(3p + 4) \geq 20$

22. $5(a + 3) + 9 < 2$ and $3(a - 2) + 6 < 10$

23. $-2 < -x - 12$ and $-14 < 5(x - 3) + 6x$

24. $-8 \geq -3y - 2$ and $3(y - 7) + 16 > 4y$

25. Write $-4 \leq t < \frac{3}{4}$ as two separate inequalities.

26. Write $-2.8 < y \leq 15$ as two separate inequalities.

27. Explain why $6 < x < 2$ has no solution.

28. Explain why $4 < t < 1$ has no solution.

29. Explain why $-5 > y > -2$ has no solution.

30. Explain why $-3 > w > -1$ has no solution.

For Exercises 31–40, solve the inequality and graph the solution set. Write the answer in interval notation.

31. $0 \leq 2b - 5 < 9$

32. $-6 < 3k - 9 \leq 0$

33. $-1 < \frac{a}{6} \leq 1$

34. $-3 \leq \frac{1}{2}x < 0$

35. $-\frac{2}{3} < \frac{y - 4}{-6} < \frac{1}{3}$

36. $\frac{1}{3} > \frac{t - 4}{-3} > -2$

37. $5 \leq -3x - 2 \leq 8$

38. $-1 < -2x + 4 \leq 5$

39. $12 > 6x + 3 \geq 0$

40. $-4 \geq 2x - 5 > -7$

Concept 3: Solving Compound Inequalities: Or

For Exercises 41–54, solve the inequality and graph the solution set. Write the answer in interval notation.

41. $h + 4 < 0$ or $6h > -12$

42. $5y > 12$ or $y - 3 < -2$

43. $2y - 1 \geq 3$ or $y < -2$

44. $x < 0$ or $3x + 1 \geq 7$

45. $1.2 > 7.2z - 9.6$ or $3.1 \leq 6.3 - 1.6z$

46. $9.5 > 3.1 + 0.8z$ or $-2.8 > 6.1 + 0.89z$

47. $5(x - 1) \geq -5$ or $5 - x \leq 11$

48. $-p + 7 \geq 10$ or $3(p - 1) \leq 12$

49. $\frac{5}{3}v \leq 5$ or $-v - 6 < 1$

50. $\frac{3}{8}u + 1 > 0$ or $-2u \geq -4$

51. $\frac{3t - 1}{10} > \frac{1}{2}$ or $\frac{3t - 1}{10} < -\frac{1}{2}$

52. $\frac{6 - x}{12} > \frac{1}{4}$ or $\frac{6 - x}{12} < -\frac{1}{6}$

53. $0.5w + 5 < 2.5w - 4$ or $0.3w \leq -0.1w - 1.6$

**54.** $1.25a + 3 \leq 0.5a - 6$ or $2.5a - 1 \geq 9 - 1.5a$

Mixed Exercises

For Exercises 55–66, solve the inequality. Write the answer in interval notation.

55. a. $3x - 5 < 19$ and $-2x + 3 < 23$
 b. $3x - 5 < 19$ or $-2x + 3 < 23$

56. a. $0.5(6x + 8) > 0.8x - 7$ and $4(x + 1) < 7.2$
 b. $0.5(6x + 8) > 0.8x - 7$ or $4(x + 1) < 7.2$

57. a. $8x - 4 \geq 6.4$ or $0.3(x + 6) \leq -0.6$
 b. $8x - 4 \geq 6.4$ and $0.3(x + 6) \leq -0.6$

58. a. $-2r + 4 \leq -8$ or $3r + 5 \leq 8$
 b. $-2r + 4 \leq -8$ and $3r + 5 \leq 8$

59. $-4 \leq \frac{2 - 4x}{3} < 8$

60. $-1 < \frac{3 - x}{2} \leq 0$

61. $5 \geq -4(t - 3) + 3t$ or
 $6 < 12t + 8(4 - t)$

62. $3 > -(w - 3) + 4w$ or
 $-5 \geq -3(w - 5) + 6w$

63. $-7(3 - x) < 9[x - 3(x + 1)]$ and
 $6x - (4 - x) + 3 < -3[8 - 2(x + 1)]$

64. $-5[x + 3(2 - x)] \leq 4(3 - 5x) + 7$ and
 $4 - [-4 - (2x - 3)] \geq 9 - (10 - 6x)$

65. $\frac{-x + 3}{2} > \frac{4 + x}{5}$ or $\frac{1 - x}{4} > \frac{2 - x}{3}$

66. $\frac{y - 7}{-3} < \frac{1}{4}$ or $\frac{y + 1}{-2} > -\frac{1}{3}$

Concept 4: Applications of Compound Inequalities

67. The normal number of white blood cells for human blood is between 4800 and 10,800 cells per cubic millimeter, inclusive. Let x represent the number of white blood cells per cubic millimeter.

  a. Write an inequality representing the normal range of white blood cells per cubic millimeter.

  b. Write a compound inequality representing abnormal levels of white blood cells per cubic millimeter.
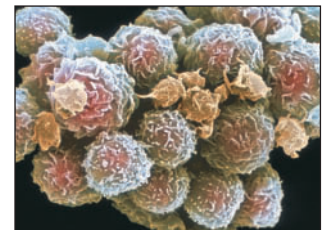
68. Normal hemoglobin levels in human blood for adult males are between 13 and 16 grams per deciliter (g/dL), inclusive. Let x represent the level of hemoglobin measured in grams per deciliter.

   a. Write an inequality representing normal hemoglobin levels for adult males.

   b. Write a compound inequality representing abnormal levels of hemoglobin for adult males.

69. The normal number of platelets in human blood is between 2.0×10^5 and 3.5×10^5 platelets per cubic millimeter, inclusive. Let x represent the number of platelets per cubic millimeter.

   a. Write an inequality representing a normal platelet count per cubic millimeter.

   b. Write a compound inequality representing abnormal platelet counts per cubic millimeter.

70. Normal hemoglobin levels in human blood for adult females are between 12 and 15 g/dL, inclusive. Let x represent the level of hemoglobin measured in grams per deciliter.

   a. Write an inequality representing normal hemoglobin levels for adult females.

   b. Write a compound inequality representing abnormal levels of hemoglobin for adult females.

71. Twice a number is between -3 and 12. Find all such numbers.

72. The difference of a number and 6 is between 0 and 8. Find all such numbers.

73. One plus twice a number is either greater than 5 or less than -1. Find all such numbers.

74. One-third of a number is either less than -2 or greater than 5. Find all such numbers.

Polynomial and Rational Inequalities

Section 9.2

1. Solving Inequalities Graphically

In Sections 1.7 and 9.1, we solved simple and compound linear inequalities. In this section we will solve polynomial and rational inequalities. We begin by defining a quadratic inequality.

   Quadratic inequalities are inequalities that can be written in any of the following forms:

$$ax^2 + bx + c \geq 0 \qquad ax^2 + bx + c \leq 0$$

$$ax^2 + bx + c > 0 \qquad ax^2 + bx + c < 0 \qquad \text{where } a \neq 0$$

Recall from Section 8.4 that the graph of a quadratic function defined by $f(x) = ax^2 + bx + c \, (a \neq 0)$ is a parabola that opens upward or downward. The quadratic inequality $f(x) > 0$ or equivalently $ax^2 + bx + c > 0$ is asking the question, "For what values of x is the value of the function positive (above the x-axis)?" The inequality $f(x) < 0$ or equivalently $ax^2 + bx + c < 0$ is asking, "For what values of x is the value of the function negative (below the x-axis)?" The graph of a quadratic function can be used to answer these questions.

Concepts

1. Solving Inequalities Graphically

2. Solving Polynomial Inequalities by Using the Test Point Method

3. Solving Rational Inequalities by Using the Test Point Method

4. Inequalities with "Special Case" Solution Sets

Example 1   Using a Graph to Solve a Quadratic Inequality

Use the graph of $f(x) = x^2 - 6x + 8$ in Figure 9-3 to solve the inequalities.

   a. $x^2 - 6x + 8 < 0$ **b.** $x^2 - 6x + 8 > 0$

Solution:

From Figure 9-3, we see that the graph of $f(x) = x^2 - 6x + 8$ is a parabola opening upward. The function factors as $f(x) = (x - 2)(x - 4)$. The x-intercepts are at $x = 2$ and $x = 4$, and the y-intercept is $(0, 8)$.
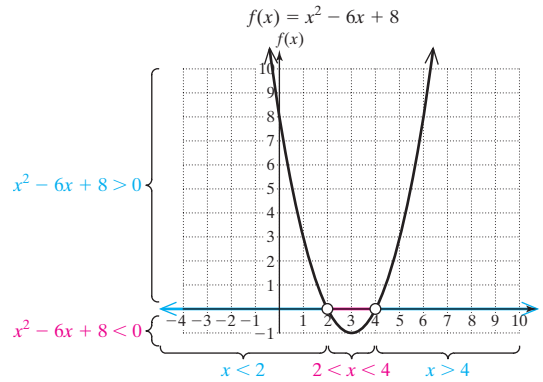

$$f(x) = x^2 - 6x + 8$$

Figure 9-3

a. The solution to $x^2 - 6x + 8 < 0$ is the set of real numbers x for which $f(x) < 0$. Graphically, this is the set of all x-values corresponding to the points where the parabola is below the x-axis (shown in red). Hence

$$x^2 - 6x + 8 < 0 \qquad \text{for } \{x \mid 2 < x < 4\} \text{ or equivalently, } (2, 4)$$

b. The solution to $x^2 - 6x + 8 > 0$ is the set of x-values for which $f(x) > 0$. This is the set of x-values where the parabola is above the x-axis (shown in blue). Hence

$$x^2 - 6x + 8 > 0 \qquad \text{for } \{x \mid x < 2 \quad \text{or} \quad x > 4\} \text{ or } (-\infty, 2) \cup (4, \infty)$$

Skill Practice

1. Refer to the graph of $f(x) = x^2 + 3x - 4$ to solve the inequalities.

a. $x^2 + 3x - 4 > 0$

b. $x^2 + 3x - 4 < 0$
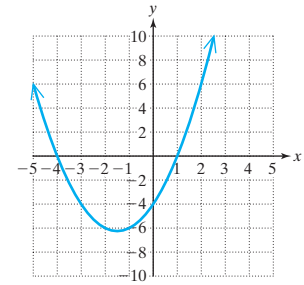
Notice that $x = 2$ and $x = 4$ define the boundaries of the solution sets to the inequalities in Example 1. These values are the solutions to the related equation $x^2 - 6x + 8 = 0$.

TIP: The inequalities in Example 1 are strict inequalities. Therefore, $x = 2$ and $x = 4$ (where $f(x) = 0$) are not included in the solution set. However, the corresponding inequalities using the symbols $\leq$ and $\geq$ do include the values where $f(x) = 0$. Hence,

The solution to $x^2 - 6x + 8 \leq 0$ is $\{x \mid 2 \leq x \leq 4\}$ or $[2, 4]$

The solution to $x^2 - 6x + 8 \geq 0$ is $\{x \mid x \leq 2 \text{ or } x \geq 4\}$ or $(-\infty, -2] \cup [4, \infty)$

Skill Practice Answers

1a. $\{x \mid x < -4 \text{ or } x > 1\}$;
$(-\infty, -4) \cup (1, \infty)$
b. $\{x \mid -4 < x < 1\}$; $(-4, 1)$

Example 2 Using a Graph to Solve a Rational Inequality

Use the graph of $g(x) = \dfrac{1}{x+1}$ in Figure 9-4 to solve the inequalities.

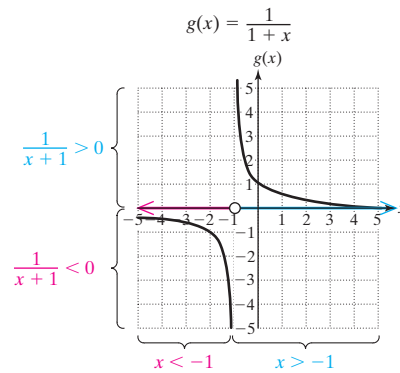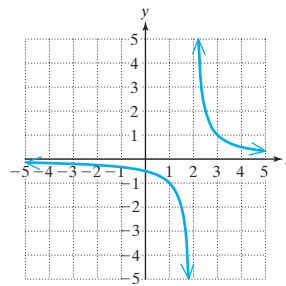**a.** $\dfrac{1}{x+1} < 0$ **b.** $\dfrac{1}{x+1} > 0$

Solution:

$g(x) = \dfrac{1}{1+x}$

Figure 9-4

a. Figure 9-4 indicates that $g(x)$ is below the x-axis for $x < -1$ (shown in red). Therefore, the solution to $\dfrac{1}{x+1} < 0$ is $\{x \mid x < -1\}$ or, equivalently, $(-\infty, -1)$.

b. Figure 9-4 indicates that $g(x)$ is above the x-axis for $x > -1$ (shown in blue). Therefore, the solution to $\dfrac{1}{x+1} > 0$ is $\{x \mid x > -1\}$ or, equivalently, $(-1, \infty)$.

TIP: Notice that $x = -1$ defines the boundary of the solution sets to the inequalities in Example 2. At $x = -1$ the inequality is undefined.

Skill Practice

2. Refer to the graph of $g(x) = \dfrac{1}{x-2}$ to solve the inequalities.

a. $\dfrac{1}{x-2} > 0$ **b.** $\dfrac{1}{x-2} < 0$

2. Solving Polynomial Inequalities by Using the Test Point Method

Examples 1 and 2 demonstrate that the boundary points of an inequality provide the boundaries of the solution set.

Boundary Points

The **boundary points** of an inequality consist of the real solutions to the related equation and the points where the inequality is undefined.

Testing points in regions bounded by these points is the basis of the **test point method** to solve inequalities.

Solving Inequalities by Using the Test Point Method

1. Find the boundary points of the inequality.

2. Plot the boundary points on the number line. This divides the number line into regions.

3. Select a test point from each region and substitute it into the original inequality.
   - If a test point makes the original inequality true, then that region is part of the solution set.

4. Test the boundary points in the original inequality.
   - If a boundary point makes the original inequality true, then that point is part of the solution set.

Example 3   Solving Polynomial Inequalities by Using the Test Point Method

Solve the inequalities by using the test point method.

a. $2x^2 + 5x < 12$ **b.** $x(x - 2)(x + 4)^2(x - 4) > 0$

Solution:

a. $2x^2 + 5x < 12$ **Step 1:** Find the boundary points. Because polynomials are defined for all values of x, the only boundary points are the real solutions to the related equation.

$2x^2 + 5x = 12$ Solve the related equation.

$2x^2 + 5x - 12 = 0$

$(2x - 3)(x + 4) = 0$

$x = \frac{3}{2}$ $x = -4$ The boundary points are $\frac{3}{2}$ and -4.

Region I | Region II | Region III

$$-6 \; -5 \; -4 \; -3 \; -2 \; -1 \;\; 0 \;\; 1 \;\; 2 \;\; 3 \;\; 4 \;\; 5 \;\; 6$$

Step 2: Plot the boundary points.

Step 3: Select a test point from each region.

Test $x = -5$

$$2x^2 + 5x < 12$$

$$2(-5)^2 + 5(-5) \overset{?}{<} 12$$

$$50 - 25 \overset{?}{<} 12$$

$$25 \overset{?}{<} 12 \quad \text{False}$$

Test $x = -4$

$$2x^2 + 5x < 12$$

$$2(-4)^2 + 5(-4) \overset{?}{<} 12$$

$$32 - 20 \overset{?}{<} 12$$

$$12 \overset{?}{<} 12 \quad \text{False}$$

Test $x = 0$

$$2x^2 + 5x < 12$$

$$2(0)^2 + 5(0) \overset{?}{<} 12$$

$$0 \overset{?}{<} 12$$

$$0 \overset{?}{<} 12 \quad \text{True}$$

Test $x = \frac{3}{2}$

$$2x^2 + 5x < 12$$

$$2\left(\frac{3}{2}\right)^2 + 5\left(\frac{3}{2}\right) \overset{?}{<} 12$$

$$2\left(\frac{9}{4}\right) + \frac{15}{2} \overset{?}{<} 12$$

$$\frac{9}{2} + \frac{15}{2} \overset{?}{<} 12$$

$$\frac{24}{2} \overset{?}{<} 12 \quad \text{False}$$

Test $x = 2$

$$2x^2 + 5x < 12$$

$$2(2)^2 + 5(2) \overset{?}{<} 12$$

$$8 + 10 \overset{?}{<} 12$$

$$18 \overset{?}{<} 12 \quad \text{False}$$

Step 4: Test the boundary points.

Neither boundary point makes the inequality true. Therefore, the boundary points are not included in the solution set.

False   True   False

$$-6 \; -5 \; -4 \; -3 \; -2 \; -1 \;\; 0 \;\; 1 \;\; 2 \;\; 3 \;\; 4 \;\; 5 \;\; 6$$
$$\frac{3}{2}$$

> **TIP:** The strict inequality, $<$, excludes values of x for which $2x^2 + 5x = 12$. This implies that the boundary points are not included in the solution set.

The solution is $\{x \mid -4 < x < \frac{3}{2}\}$ or equivalently in interval notation $(-4, \frac{3}{2})$.

Calculator Connections

Graph $Y_1 = 2x^2 + 5x$ and $Y_2 = 12$.
Notice that $Y_1 < Y_2$ for $\{x \mid -4 < x < \frac{3}{2}\}$.

$Y_1 = 2x^2 + 5x$
20
$Y_2 = 12$
$(-4, 12)$ $(\frac{3}{2}, 12)$
-8   8
-4 $\frac{3}{2}$
-5

b. $x(x - 2)(x + 4)^2(x - 4) > 0$

$$x(x - 2)(x + 4)^2(x - 4) = 0$$

$$x = 0 \quad x = 2 \quad x = -4 \quad x = 4$$

Step 1: Find the boundary points.
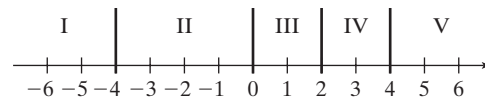
Step 2: Plot the boundary points.
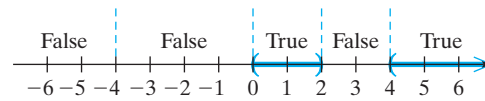
Step 3: Select a test point from each region.

Test $x = -5$: $-5(-5-2)(-5+4)^2(-5-4) \overset{?}{>} 0$ $-315 \overset{?}{>} 0$ False

Test $x = -1$: $-1(-1-2)(-1+4)^2(-1-4) \overset{?}{>} 0$ $-135 \overset{?}{>} 0$ False

Test $x = 1$: $1(1-2)(1+4)^2(1-4) \overset{?}{>} 0$ $75 \overset{?}{>} 0$ True

Test $x = 3$: $3(3-2)(3+4)^2(3-4) \overset{?}{>} 0$ $-147 \overset{?}{>} 0$ False

Test $x = 5$: $5(5-2)(5+4)^2(5-4) \overset{?}{>} 0$ $1215 \overset{?}{>} 0$ True

Step 4: The boundary points are not included because the inequality, $>$, is strict.

The solution is $\{x \mid 0 < x < 2 \text{ or } x > 4\}$ or, equivalently in interval notation, $(0, 2) \cup (4, \infty)$.

Calculator Connections

Graph $Y_1 = x(x - 2)(x + 4)^2(x - 4)$. Y_1 is positive (above the x-axis) for $\{x \mid 0 < x < 2 \text{ or } x > 4\}$ or equivalently $(0, 2) \cup (4, \infty)$.
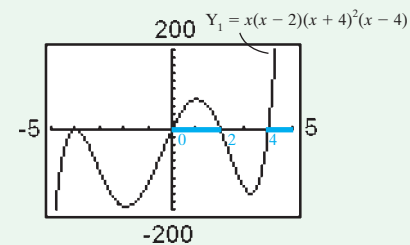
Skill Practice   Solve the inequalities by using the test point method. Write the answers in interval notation.

3. $x^2 + x > 6$ **4.** $t(t - 5)(t + 2)^2 > 0$

Example 4   **Solving a Polynomial Inequality Using the Test Point Method**

Solve the inequality by using the test point method. $x^2 + x - 4 \geq 0$

Solution:

$$x^2 + x - 4 \geq 0$$

$$x^2 + x - 4 = 0$$

$$x = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(-4)}}{2(1)}$$

Step 1: Find the boundary points in the related equation. Since this equation is not factorable, use the quadratic formula to find the solutions.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Skill Practice Answers

3. $(-\infty, -3) \cup (2, \infty)$
4. $(-\infty, -2) \cup (-2, 0) \cup (5, \infty)$

$$x = \frac{-1 \pm \sqrt{17}}{2}$$

$$x = \frac{-1 - \sqrt{17}}{2} \approx -2.56 \text{ and } x = \frac{-1 + \sqrt{17}}{2} \approx 1.56$$

Region I | Region II | Region III

$-6\ -5\ -4\ -3\ -2\ -1\ \ 0\ \ 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6$
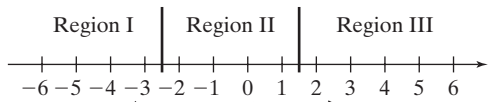
Step 2: Plot the boundary points.

Step 3: Select a test point from each region.

Test $x = -3$

$$x^2 + x - 4 \geq 0$$
$$(-3)^2 + (-3) - 4 \overset{?}{\geq} 0$$
$$2 \overset{?}{\geq} 0 \text{ True}$$

Test $x = 0$

$$x^2 + x - 4 \geq 0$$
$$(0)^2 + (0) - 4 \overset{?}{\geq} 0$$
$$-4 \overset{?}{\geq} 0 \text{ False}$$

Test $x = 2$

$$x^2 + x - 4 \geq 0$$
$$(2)^2 + (2) - 4 \overset{?}{\geq} 0$$
$$2 \overset{?}{\geq} 0 \text{ True}$$

True　　　False　　　True

$-6\ -5\ -4\ -3\ -2\ -1\ \ 0\ \ 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6$
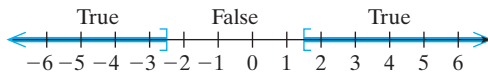
Step 4: Test the boundary points. Both boundary points make the inequality true. Therefore, both boundary points are included in the solution set.

The solution is $\left\{ x \mid x \leq \dfrac{-1 - \sqrt{17}}{2} \text{ or } x \geq \dfrac{-1 + \sqrt{17}}{2} \right\}$ or equivalently

in interval notation: $\left(-\infty, \dfrac{-1 - \sqrt{17}}{2} \right] \cup \left[\dfrac{-1 + \sqrt{17}}{2}, \infty \right)$.

Skill Practice Solve the inequality using the test point method. Write the answer in interval notation.

5. $x^2 - 3x - 1 < 0$

3. Solving Rational Inequalities by Using the Test Point Method

The test point method can be used to solve rational inequalities. A **rational inequality** is an inequality in which one or more terms is a rational expression. The solution set to a rational inequality must exclude all values of the variable that make the inequality undefined. That is, exclude all values that make the denominator equal to zero for any rational expression in the inequality.

Example 5 Solving a Rational Inequality by Using the Test Point Method

Solve the inequality by using the test point method. $\quad \dfrac{x + 2}{x - 4} \leq 3$

Skill Practice Answers

5. $\left(\dfrac{3 - \sqrt{13}}{2}, \dfrac{3 + \sqrt{13}}{2} \right)$

Solution:

$$\frac{x + 2}{x - 4} \le 3$$

Step 1: Find the boundary points. Note that the inequality is undefined for $x = 4$. Hence $x = 4$ is automatically a boundary point. To find any other boundary points, solve the related equation.

$$\frac{x + 2}{x - 4} = 3$$

$$(x - 4)\left(\frac{x + 2}{x - 4}\right) = (x - 4)(3) \qquad \text{Clear fractions.}$$

$$x + 2 = 3(x - 4) \qquad\qquad \text{Solve for } x.$$

$$x + 2 = 3x - 12$$

$$-2x = -14$$

$$x = 7$$

The solution to the related equation is $x = 7$, and the inequality is undefined for $x = 4$. Therefore, the boundary points are $x = 4$ and $x = 7$.

Region I | Region II | Region III

```
  +  +  +  +  +  +  +  +  +  +  +  +
 -3 -2 -1  0  1  2  3  4  5  6  7  8
```

Step 2: Plot boundary points.

Step 3: Select test points.

Test $x = 0$

$$\frac{x + 2}{x - 4} \le 3$$

$$\frac{0 + 2}{0 - 4} \overset{?}{\le} 3$$

$$-\frac{1}{2} \overset{?}{\le} 3 \quad \text{True}$$

Test $x = 5$

$$\frac{x + 2}{x - 4} \le 3$$

$$\frac{5 + 2}{5 - 4} \overset{?}{\le} 3$$

$$\frac{7}{1} \overset{?}{\le} 3 \quad \text{False}$$

Test $x = 8$

$$\frac{x + 2}{x - 4} \le 3$$

$$\frac{8 + 2}{8 - 4} \overset{?}{\le} 3$$

$$\frac{10}{4} \overset{?}{\le} 3$$

$$\frac{5}{2} \overset{?}{\le} 3 \quad \text{True}$$

Test $x = 4$:

$$\frac{x + 2}{x - 4} \le 3$$

$$\frac{4 + 2}{4 - 4} \overset{?}{\le} 3$$

$$\frac{6}{0} \overset{?}{\le} 3 \quad \text{Undefined}$$

Test $x = 7$:

$$\frac{x + 2}{x - 4} \le 3$$

$$\frac{7 + 2}{7 - 4} \overset{?}{\le} 3$$

$$\frac{9}{3} \overset{?}{\le} 3 \quad \text{True}$$

Step 4: Test the boundary points.

The boundary point $x = 4$ cannot be included in the solution set, because it is undefined in the inequality. The boundary point $x = 7$ makes the original inequality true and must be included in the solution set.

```
  +  +  +  +  +  +  +  )  +  +  [  +
 -3 -2 -1  0  1  2  3  4  5  6  7  8
```
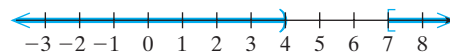
The solution is $\{x \mid x < 4 \text{ or } x \ge 7\}$ or, equivalently in interval notation, $(-\infty, 4) \cup [7, \infty)$.

Calculator Connections

Graph $Y_1 = \dfrac{x+2}{x-4}$ and $Y_2 = 3$.

Y_1 has a vertical asymptote at $x = 4$. Furthermore, $Y_1 = Y_2$ at $x = 7$. $Y_1 \le Y_2$ (that is, Y_1 is below Y_2) for $x < 4$ and for $x \ge 7$.
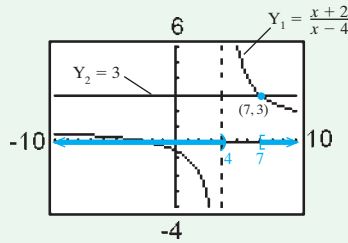
Skill Practice Solve the inequality by using the test point method. Write the answer in interval notation.

6. $\dfrac{x-5}{x+4} \le -1$

Example 6 Solving a Rational Inequality by Using the Test Point Method

Solve the inequality. $\quad \dfrac{3}{x-1} > 0$

Solution:

$$\frac{3}{x-1} > 0$$

$$\frac{3}{x-1} = 0$$

$$(\cancel{x-1}) \cdot \left(\frac{3}{\cancel{x-1}}\right) = (x-1) \cdot 0$$

$$3 = 0$$

Step 1: Find the boundary points. Note that the inequality is undefined for $x = 1$, so $x = 1$ is a boundary point. To find any other boundary points, solve the related equation.

Clear fractions.

There is no solution to the related equation.

The only boundary point is $x = 1$.

Region I | Region II

−5 −4 −3 −2 −1 0 1 2 3 4 5
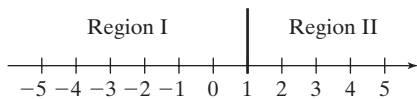
Step 2: Plot boundary points.

Test $x = 0$:

$$\frac{3}{(0)-1} \overset{?}{>} 0$$

$$\frac{3}{-1} \overset{?}{>} 0 \text{ False}$$

Test $x = 2$:

$$\frac{3}{(2)-1} \overset{?}{>} 0$$

$$\frac{3}{1} > 0 \text{ True}$$

Step 3: Select test points.

Step 4: The boundary point $x = 1$ cannot be included in the solution set because it is undefined in the original inequality.

False | True

−5 −4 −3 −2 −1 0 1 2 3 4 5
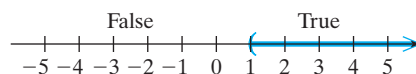
The solution is $\{x \mid x > 1\}$ or equivalently in interval notation, $(1, \infty)$.

Skill Practice Answers

6. $\left(-4, \dfrac{1}{2}\right]$

Skill Practice   Solve the inequality.

7. $\dfrac{-5}{y + 2} < 0$

---

4. Inequalities with "Special Case" Solution Sets

The solution to an inequality is often one or more regions on the real number line. Sometimes, however, the solution to an inequality may be a single point on the number line, the empty set, or the set of all real numbers.

Example 7   **Solving Inequalities**

Solve the inequalities.

    a. $x^2 + 6x + 9 \geq 0$ **b.** $x^2 + 6x + 9 > 0$

    c. $x^2 + 6x + 9 \leq 0$ **d.** $x^2 + 6x + 9 < 0$

Solution:

a. $x^2 + 6x + 9 \geq 0$ Notice that $x^2 + 6x + 9$ is a perfect square trinomial.

      $(x + 3)^2 \geq 0$ Factor $x^2 + 6x + 9 = (x + 3)^2$.

The quantity $(x + 3)^2$ is a perfect square and is greater than or equal to zero for all real numbers, x. The solution is all real numbers, $(-\infty, \infty)$.
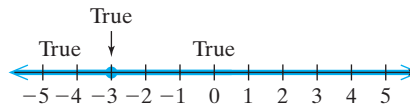
b. $x^2 + 6x + 9 > 0$

      $(x + 3)^2 > 0$ This is the same inequality as in part (a) with the exception that the inequality is strict. The solution set does not include the point where $x^2 + 6x + 9 = 0$. Therefore, the boundary point $x = -3$ is *not* included in the solution set.
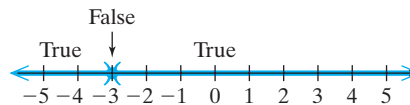
The solution set is $\{x \mid x < -3 \text{ or } x > -3\}$ or equivalently $(-\infty, -3) \cup (-3, \infty)$.

c. $x^2 + 6x + 9 \leq 0$

      $(x + 3)^2 \leq 0$

A perfect square cannot be less than zero. However, $(x + 3)^2$ is equal to zero at $x = -3$. Therefore, the solution set is $\{-3\}$.
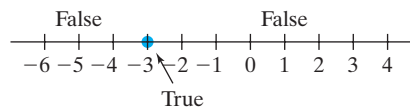
TIP: The graph of $f(x) = x^2 + 6x + 9$, or equivalently $f(x) = (x + 3)^2$, is equal to zero at $x = -3$ and positive (above the x-axis) for all other values of x in its domain.
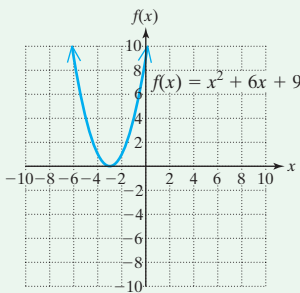
Skill Practice Answers

7. $(-2, \infty)$

d. $x^2 + 6x + 9 < 0$

$(x + 3)^2 < 0$

A perfect square cannot be negative; therefore, there are no real numbers x such that $(x + 3)^2 < 0$. There is no solution.

Skill Practice Solve the inequalities.

8. $x^2 - 4x + 4 \geq 0$ **9.** $x^2 - 4x + 4 > 0$

10. $x^2 - 4x + 4 \leq 0$ **11.** $x^2 - 4x + 4 < 0$

Skill Practice Answers

8. All real numbers; $(-\infty, \infty)$
9. $(-\infty, 2) \cup (2, \infty)$
10. $\{2\}$
11. No solution

Section 9.2   Practice Exercises

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Study Skills Exercise

1. Define the key terms.

    a. Quadratic inequality   **b. Boundary points**

    c. Test point method   **d. Rational inequality**

Review Exercises

For Exercises 2–8, solve the compound inequalities. Write the solutions in interval notation.

2. $6x - 10 > 8$ or $8x + 2 < 5$ **3.** $3(a - 1) + 2 > 0$ or $2a > 5a + 12$

4. $5(k - 2) > -25$ and $7(1 - k) > 7$ **5.** $2y + 4 \geq 10$ and $5y - 3 \leq 13$

6. $0 < 3(x + 1) \leq 4$ **7.** $6 \geq 4 - 2x \geq -2$

8. $-4 > 5 - x > -6$

Concept 1: Solving Inequalities Graphically

For Exercises 9–12, estimate from the graph the intervals for which the inequality is true.

9.

    a. $f(x) > 0$ **b.** $f(x) < 0$

    c. $f(x) \leq 0$ **d.** $f(x) \geq 0$

10.

    a. $g(x) < 0$ **b.** $g(x) > 0$
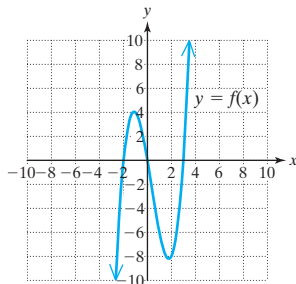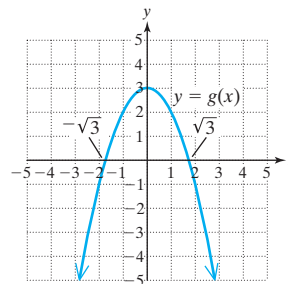
    c. $g(x) \geq 0$ **d.** $g(x) \leq 0$

11.
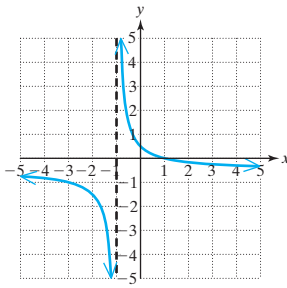
     a. $h(x) \geq 0$ **b.** $h(x) \leq 0$

     c. $h(x) < 0$ **d.** $h(x) > 0$

12.
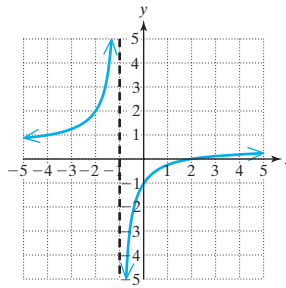
     a. $k(x) \leq 0$ **b.** $k(x) \geq 0$

     c. $k(x) > 0$ **d.** $k(x) < 0$

Concept 2: Solving Polynomial Inequalities by Using the Test Point Method

For Exercises 13–18, solve the equation and related inequalities.

13. **a.** $3(4 - x)(2x + 1) = 0$

     b. $3(4 - x)(2x + 1) < 0$

     c. $3(4 - x)(2x + 1) > 0$

14. **a.** $5(y + 6)(3 - 5y) = 0$

     b. $5(y + 6)(3 - 5y) < 0$

     c. $5(y + 6)(3 - 5y) > 0$

15. **a.** $x^2 + 7x = 30$

     b. $x^2 + 7x < 30$

     c. $x^2 + 7x > 30$

16. **a.** $q^2 - 4q = 5$

     b. $q^2 - 4q \leq 5$

     c. $q^2 - 4q \geq 5$

17. **a.** $2p(p - 2) = p + 3$

     b. $2p(p - 2) \leq p + 3$

     c. $2p(p - 2) \geq p + 3$

18. **a.** $3w(w + 4) = 10 - w$

     b. $3w(w + 4) < 10 - w$

     c. $3w(w + 4) > 10 - w$

For Exercises 19–38, solve the polynomial inequality. Write the answer in interval notation.

19. $(t - 7)(t + 1) < 0$

20. $(p - 4)(p + 2) > 0$

21. $-6(4 + 2x)(5 - x) > 0$

22. $-8(2t + 5)(6 - t) < 0$

23. $m(m + 1)^2(m + 5) \leq 0$

24. $w^2(3 - w)(w + 2) \geq 0$

25. $a^2 - 12a \leq -32$

26. $w^2 + 20w \geq -64$

27. $5x^2 - 2x - 1 > 0$

28. $3x^2 + 2x - 4 < 0$

29. $x^2 + 3x \leq 6$

30. $x^2 - 8 \leq 3x$

31. $b^2 - 121 < 0$

32. $c^2 - 25 < 0$

33. $3p(p - 2) - 3 \geq 2p$

34. $2t(t + 3) - t \leq 12$

35. $x^3 - x^2 \leq 12x$

36. $x^3 + 36 > 4x^2 + 9x$

37. $w^3 + w^2 > 4w + 4$

38. $2p^3 - 5p^2 \leq 3p$

Concept 3: Solving Rational Inequalities by Using the Test Point Method

For Exercises 39–42, solve the equation and related inequalities.

39. a. $\dfrac{10}{x-5} = 5$ **40. a.** $\dfrac{8}{a+1} = 4$ **41. a.** $\dfrac{z+2}{z-6} = -3$ **42. a.** $\dfrac{w-8}{w+6} = 2$

b. $\dfrac{10}{x-5} < 5$ **b.** $\dfrac{8}{a+1} > 4$ **b.** $\dfrac{z+2}{z-6} \leq -3$ **b.** $\dfrac{w-8}{w+6} \leq 2$

c. $\dfrac{10}{x-5} > 5$ **c.** $\dfrac{8}{a+1} < 4$ **c.** $\dfrac{z+2}{z-6} \geq -3$ **c.** $\dfrac{w-8}{w+6} \geq 2$

For Exercises 43–54, solve the rational inequalities. Write the answer in interval notation.

43. $\dfrac{2}{x-1} \geq 0$ **44.** $\dfrac{-3}{x+2} \leq 0$ **45.** $\dfrac{b+4}{b-4} > 0$ **46.** $\dfrac{a+1}{a-3} < 0$

47. $\dfrac{3}{2x-7} < -1$ **48.** $\dfrac{8}{4x+9} > 1$ **49.** $\dfrac{x+1}{x-5} \geq 4$ **50.** $\dfrac{x-2}{x+6} \leq 5$

51. $\dfrac{1}{x} \leq 2$ **52.** $\dfrac{1}{x} \geq 3$ **53.** $\dfrac{(x+2)^2}{x} > 0$ **54.** $\dfrac{(x-3)^2}{x} < 0$

Concept 4: Inequalities with "Special Case" Solution Sets

For Exercises 55–70, solve the inequalities.

55. $x^2 + 10x + 25 \geq 0$ **56.** $x^2 + 6x + 9 < 0$ **57.** $x^2 + 2x + 1 < 0$ **58.** $x^2 + 8x + 16 \geq 0$

59. $\dfrac{x^2}{x^2+4} < 0$ **60.** $\dfrac{x^2}{x^2+4} \geq 0$ **61.** $x^4 + 3x^2 \leq 0$ **62.** $x^4 + 2x^2 \leq 0$

63. $x^2 + 24x + 144 > 0$ **64.** $x^2 + 12x + 36 < 0$ **65.** $x^2 + 24x + 144 \leq 0$ **66.** $x^2 + 12x + 36 \geq 0$

67. $x^2 + 3x + 5 < 0$ **68.** $2x^2 + 3x + 3 > 0$ **69.** $-5x^2 + x < 1$ **70.** $-3x^2 - x > 6$

Mixed Exercises

For Exercises 71–94, identify the inequality as one of the following types: linear, quadratic, rational, or polynomial (degree > 2). Then solve the inequality and write the answer in interval notation.

71. $2y^2 - 8 \leq 24$ **72.** $8p^2 - 18 > 0$ **73.** $(5x+2)^2 > -4$

74. $(3 - 7x)^2 < -1$ **75.** $4(x-2) < 6x - 3$ **76.** $-7(3-y) > 4 + 2y$

77. $\dfrac{2x+3}{x+1} \leq 2$ **78.** $\dfrac{5x-1}{x+3} \geq 5$ **79.** $4x^3 - 40x^2 + 100x > 0$

80. $2y^3 - 12y^2 + 18y < 0$ **81.** $2p^3 > 4p^2$ **82.** $w^3 \leq 5w^2$

83. $\dfrac{1}{x^2+3} < -4$ **84.** $\dfrac{1}{4t^2+5} > -2$ **85.** $x^2 - 2 < 0$

86. $y^2 - 3 > 0$ **87.** $x^2 + 5x - 2 \geq 0$ **88.** $t^2 + 7t + 3 \leq 0$

89. $\dfrac{a + 2}{a - 5} \geq 0$ **90.** $\dfrac{t + 1}{t - 2} \leq 0$ **91.** $2 \geq t - 3$

92. $-5p + 8 < p$ **93.** $4x^2 + 2 \geq -x$ **94.** $5x - 4 \geq 2x^2$

Graphing Calculator Exercises

95. To solve the inequality $\dfrac{x}{x - 2} > 0$
enter Y_1 as $x/(x - 2)$ and determine where the graph is above the x-axis. Write the solution in interval notation.

96. To solve the inequality $\dfrac{x}{x - 2} < 0$
enter Y_1 as $x/(x - 2)$ and determine where the graph is below the x-axis. Write the solution in interval notation.

97. To solve the inequality $x^2 - 1 < 0$, enter Y_1 as $x^2 - 1$ and determine where the graph is below the x-axis. Write the solution in interval notation.

98. To solve the inequality $x^2 - 1 > 0$, enter Y_1 as $x^2 - 1$ and determine where the graph is above the x-axis. Write the solution in interval notation.

For Exercises 99–102, determine the solution by graphing the inequalities.

99. $x^2 + 10x + 25 \leq 0$ **100.** $-x^2 + 10x - 25 \geq 0$

101. $\dfrac{8}{x^2 + 2} < 0$ **102.** $\dfrac{-6}{x^2 + 3} > 0$

Absolute Value Equations

1. Solving Absolute Value Equations

An equation of the form $|x| = a$ is called an **absolute value equation**. The solution includes all real numbers whose absolute value equals a. For example, the solutions to the equation $|x| = 4$ are $x = 4$ as well as $x = -4$, because $|4| = 4$ and $|-4| = 4$. In Chapter 1, we introduced a geometric interpretation of $|x|$. The absolute value of a number is its distance from zero on the number line (Figure 9-5). Therefore, the solutions to the equation $|x| = 4$ are the values of x that are 4 units away from zero.

Concepts

1. Solving Absolute Value Equations
2. Solving Equations Having Two Absolute Values

$$|x| = 4$$

$$x = -4 \quad \text{or} \quad x = 4$$
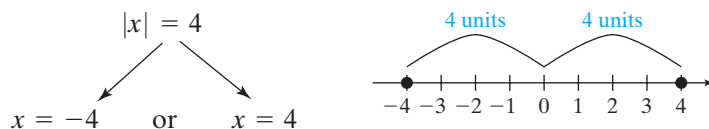
Figure 9-5

Absolute Value Equations of the Form $|x| = a$

If a is a real number, then

1. If $a \geq 0$, the equation $|x| = a$ is equivalent to $x = a$ or $x = -a$.

2. If $a < 0$, there is no solution to the equation $|x| = a$.

Example 1 Solving Absolute Value Equations

Solve the absolute value equations.

a. $|x| = 5$ **b.** $|w| - 2 = 12$ **c.** $|p| = 0$ **d.** $|x| = -6$

Solution:

a. $|x| = 5$ The equation is in the form $|x| = a$, where $a = 5$.

$x = 5 \quad \text{or} \quad x = -5$ Rewrite the equation as $x = a$ or $x = -a$.

b. $|w| - 2 = 12$

$|w| = 14$ Isolate the absolute value to write the equation in the form $|w| = a$.

$w = 14 \quad \text{or} \quad w = -14$ Rewrite the equation as $w = a$ or $w = -a$.

TIP: The absolute value must be isolated on one side of the equal sign before $|x| = a$ can be rewritten as $x = a$ or $x = -a$.

c. $|p| = 0$

$p = 0 \quad \text{or} \quad p = -0$ Rewrite as two equations. Notice that the second equation $p = -0$ is the same as the first equation. Intuitively, $p = 0$ is the only number whose absolute value equals 0.

d. $|x| = -6$ This equation is of the form $|x| = a$, but a is negative. There is no number whose absolute value is negative.

No solution

Skill Practice Solve the absolute value equations.

1. $|y| = 7$ **2.** $|v| + 6 = 10$ **3.** $|w| = 0$ **4.** $|z| = -12$

Skill Practice Answers

1. $y = 7$ or $y = -7$
2. $v = 4$ or $v = -4$
3. $w = 0$ **4.** No solution

We have solved absolute value equations of the form $|x| = a$. Notice that x can represent any algebraic quantity. For example, to solve the equation $|2w - 3| = 5$, we still rewrite the absolute value equation as two equations. In this case, we set the quantity $2w - 3$ equal to 5 and to -5, respectively.

$$|2w - 3| = 5$$

$$2w - 3 = 5 \quad \text{or} \quad 2w - 3 = -5$$

Steps to Solve an Absolute Value Equation

1. Isolate the absolute value. That is, write the equation in the form $|x| = a$, where a is a constant real number.

2. If $a < 0$, there is no solution.

3. Otherwise, if $a \geq 0$, rewrite the absolute value equation as $x = a$ or $x = -a$.

4. Solve the individual equations from step 3.

5. Check the answers in the original absolute value equation.

Example 2   Solving Absolute Value Equations

Solve the absolute value equations.

a. $|2w - 3| = 5$ **b.** $|2c - 5| + 6 = 2$

Solution:

a. $|2w - 3| = 5$ $\qquad\qquad\qquad$ The equation is already in the form $|x| = a$, where $x = 2w - 3$.

$2w - 3 = 5 \quad$ or $\quad 2w - 3 = -5 \qquad$ Rewrite as two equations.

$\qquad 2w = 8 \quad$ or $\qquad 2w = -2 \qquad$ Solve each equation.

$\qquad\quad w = 4 \quad$ or $\qquad\quad w = -1$

Check: $w = 4 \qquad$ Check: $w = -1 \qquad$ Check the solutions in the original equation.

$|2w - 3| = 5 \qquad\quad |2w - 3| = 5$

$|2(4) - 3| \stackrel{?}{=} 5 \qquad |2(-1) - 3| \stackrel{?}{=} 5$

$|8 - 3| \stackrel{?}{=} 5 \qquad\quad |-2 - 3| \stackrel{?}{=} 5$

$|5| \stackrel{?}{=} 5 \checkmark \qquad\qquad |-5| \stackrel{?}{=} 5 \checkmark$

Calculator Connections

To confirm the answers to Example 2(a), graph $Y_1 = \text{abs}(2x - 3)$ and $Y_2 = 5$. The solutions to the equation $|2w - 3| = 5$ are the x-coordinates of the points of intersection $(4, 5)$ and $(-1, 5)$.
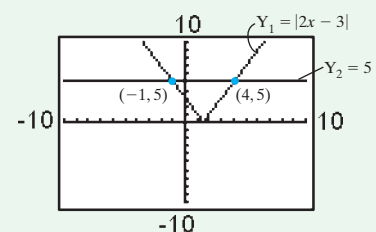
b. $|2c - 5| + 6 = 2$

$\qquad |2c - 5| = -4$ Isolate the absolute value. The equation is in the form $|x| = a$, where $x = 2c - 5$ and $a = -4$. Because $a < 0$, there is no solution.

$\qquad$ No solution   There are no numbers c that will make an absolute value equal to a negative number.

> **Avoiding Mistakes:**
>
> Always isolate the absolute value first. Otherwise you will get answers that do not check.

Calculator Connections

The graphs of $Y_1 = \text{abs}(2x - 5) + 6$ and $Y_2 = 2$ do not intersect.
$\qquad$ Therefore, there is no solution to the equation $|2c - 5| + 6 = 2$.

Skill Practice   Solve the absolute value equations.

5. $|4x + 1| = 9$ **6.** $|3z + 10| + 3 = 1$

Example 3   Solving Absolute Value Equations

Solve the absolute value equations.

a. $-2\left|\frac{2}{5}p + 3\right| - 7 = -19$ **b.** $|4.1 - p| + 6.9 = 6.9$

Solution:

a. $-2\left|\frac{2}{5}p + 3\right| - 7 = -19$

$\qquad -2\left|\frac{2}{5}p + 3\right| = -12$ Isolate the absolute value.

$\qquad \dfrac{-2\left|\frac{2}{5}p + 3\right|}{-2} = \dfrac{-12}{-2}$

$\qquad \left|\frac{2}{5}p + 3\right| = 6$

$\dfrac{2}{5}p + 3 = 6$ or $\dfrac{2}{5}p + 3 = -6$ Rewrite as two equations.

$2p + 15 = 30$ or $2p + 15 = -30$ Multiply all terms by 5 to clear fractions.

$2p = 15$ or $2p = -45$

$p = \dfrac{15}{2}$ or $p = -\dfrac{45}{2}$ Both solutions check in the original equation.

Skill Practice Answers

5. $x = 2$ or $x = -\dfrac{5}{2}$

6. No solution

b. $|4.1 - p| + 6.9 = 6.9$

$\qquad |4.1 - p| = 0$ Isolate the absolute value.

$4.1 - p = 0$ or $4.1 - p = -0$ Rewrite as two equations. Notice that the equations are the same.

$\qquad -p = -4.1$ Subtract 4.1 from both sides.

$\qquad p = 4.1$ Check: $p = 4.1$

$\qquad\qquad |4.1 - p| + 6.9 = 6.9$

$\qquad\qquad |4.1 - 4.1| + 6.9 \overset{?}{=} 6.9$

$\qquad\qquad |0| + 6.9 \overset{?}{=} 6.9$

The solution is 4.1. $6.9 = 6.9 \checkmark$

Skill Practice   Solve the absolute value equations.

7. $3\left|\dfrac{3}{2}a + 1\right| + 2 = 14$ **8.** $|1.2 + x| - 3.5 = -3.5$

2. Solving Equations Having Two Absolute Values

Some equations have two absolute values. The solutions to the equation $|x| = |y|$ are $x = y$ or $x = -y$. That is, if two quantities have the same absolute value, then the quantities are equal or the quantities are opposites.

> **Equality of Absolute Values**
>
> $$|x| = |y| \text{ implies that } x = y \text{ or } x = -y.$$

Example 4 Solving an Equation Having Two Absolute Values

Solve the equations.

a. $|2w - 3| = |5w + 1|$ **b.** $|x - 4| = |x + 8|$

Solution:

a. $|2w - 3| = |5w + 1|$

Avoiding Mistakes:
To take the opposite of the quantity $5w + 1$, use parentheses and apply the distributive property.

$2w - 3 = 5w + 1$ or $2w - 3 = -(5w + 1)$ Rewrite as two equations, $x = y$ or $x = -y$.

$2w - 3 = 5w + 1$ or $2w - 3 = -5w - 1$ Solve for w.

$-3w - 3 = 1$ or $7w - 3 = -1$

$-3w = 4$ or $7w = 2$

$w = -\dfrac{4}{3}$ or $w = \dfrac{2}{7}$

The solutions are $-\dfrac{4}{3}$ and $\dfrac{2}{7}$.   Both values check in

Skill Practice Answers

7. $a = 2$ or $a = -\dfrac{10}{3}$

8. $x = -1.2$

b. $|x - 4| = |x + 8|$

$x - 4 = x + 8$ or $x - 4 = -(x + 8)$ Rewrite as two equations, $x = y$ or $x = -y$.

$-4 = 8$ or $x - 4 = -x - 8$ Solve for x.

↑
contradiction

$2x - 4 = -8$

$2x = -4$

$x = -2$

The only solution is -2.

$x = -2$ checks in the original equation.

Skill Practice Answers

Skill Practice   Solve the equations.

9. $|3 - 2x| = |3x - 1|$ **10.** $|4t + 3| = |4t - 5|$

9. $x = \dfrac{4}{5}$ or $x = -2$

10. $t = \dfrac{1}{4}$

Section 9.3   Practice Exercises

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Study Skills Exercise

1. Define the key term **absolute value equation**.

Review Exercises

For Exercises 2–7, solve the inequalities. Write the answers in interval notation.

2. $3(a + 2) - 6 > 2$ and $-2(a - 3) + 14 > -3$ **3.** $3x - 5 \geq 7x + 3$ or $2x - 1 \leq 4x - 5$

4. $\dfrac{4}{y - 4} \geq 3$ **5.** $\dfrac{3}{t + 1} \leq 2$

6. $3(x - 2)(x + 4)(2x - 1) < 0$ **7.** $x^3 - 7x^2 - 8x > 0$

Concept 1: Solving Absolute Value Equations

For Exercises 8–39, solve the absolute value equations.

8. $|p| = 7$ **9.** $|q| = 10$ **10.** $|x| + 5 = 11$ **11.** $|x| - 3 = 20$

12. $|y| = \sqrt{2}$ **13.** $|y| = \dfrac{5}{8}$ **14.** $|w| - 3 = -5$ **15.** $|w| + 4 = -8$

16. $|3q| = 0$ **17.** $|4p| = 0$ **18.** $\left|3x - \dfrac{1}{2}\right| = \dfrac{1}{2}$ **19.** $|4x + 1| = 6$

20. $\left|\dfrac{7z}{3} - \dfrac{1}{3}\right| + 3 = 6$ **21.** $\left|\dfrac{w}{2} + \dfrac{3}{2}\right| - 2 = 7$ **22.** $\left|\dfrac{5y + 2}{2}\right| = 6$ **23.** $\left|\dfrac{2t - 1}{3}\right| = 5$

24. $|0.2x - 3.5| = -5.6$ **25.** $|1.81 + 2x| = -2.2$ **26.** $1 = -4 + \left|2 - \dfrac{1}{4}w\right|$ **27.** $-12 = -6 - |6 - 2x|$

28. $10 = 4 + |2y + 1|$ **29.** $-1 = -|5x + 7|$ **30.** $-2|3b - 7| - 9 = -9$

31. $-3|5x + 1| + 4 = 4$ **32.** $-2|x + 3| = 5$ **33.** $-3|x - 5| = 7$

34. $0 = |6x - 9|$ **35.** $7 = |4k - 6| + 7$ **36.** $\left|-\dfrac{1}{5} - \dfrac{1}{2}k\right| = \dfrac{9}{5}$

37. $\left|-\dfrac{1}{6} - \dfrac{2}{9}h\right| = \dfrac{1}{2}$ **38.** $-3|2 - 6x| + 5 = -10$ **39.** $5|1 - 2x| - 7 = 3$

Concept 2: Solving Equations Having Two Absolute Values

For Exercises 40–53, solve the absolute value equations.

40. $|4x - 2| = |-8|$ **41.** $|3x + 5| = |-5|$ **42.** $|4w + 3| = |2w - 5|$

43. $|3y + 1| = |2y - 7|$ **44.** $|2y + 5| = |7 - 2y|$ **45.** $|9a + 5| = |9a - 1|$

46. $\left|\dfrac{4w - 1}{6}\right| = \left|\dfrac{2w}{3} + \dfrac{1}{4}\right|$ **47.** $\left|\dfrac{3p + 2}{4}\right| = \left|\dfrac{1}{2}p - 2\right|$ **48.** $|2h - 6| = |2h + 5|$

49. $|6n - 7| = |4 - 6n|$ **50.** $|3.5m - 1.2| = |8.5m + 6|$ **51.** $|11.2n + 9| = |7.2n - 2.1|$

52. $|4x - 3| = -|2x - 1|$ **53.** $-|3 - 6y| = |8 - 2y|$

Expanding Your Skills

54. Write an absolute value equation whose solution is the set of real numbers 6 units from zero on the number line.

55. Write an absolute value equation whose solution is the set of real numbers $\frac{7}{2}$ units from zero on the number line.

56. Write an absolute value equation whose solution is the set of real numbers $\frac{4}{3}$ units from zero on the number line.

57. Write an absolute value equation whose solution is the set of real numbers 9 units from zero on the number line.

For Exercises 58–63, solve the absolute value equations.

58. $|5y - 3| + \sqrt{5} = 1 + \sqrt{5}$ **59.** $|2x - \sqrt{3}| + 4 = 4 + \sqrt{3}$ **60.** $|\sqrt{3} + x| = 7$

61. $\sqrt{2} + |w - 8| = 3 + 4\sqrt{2}$ **62.** $|w - \sqrt{6}| = |3w + \sqrt{6}|$ **63.** $\left|\dfrac{\sqrt{5}}{2}x - 4\right| = 6$

Graphing Calculator Exercises

For Exercises 64–71, enter the left side of the equation as Y_1 and enter the right side of the equation as Y_2. Then use the *Intersect* feature or *Zoom* and *Trace* to approximate the *x*-values where the two graphs intersect (if they intersect).

64. $|4x - 3| = 5$

65. $|x - 4| = 3$

66. $|8x + 1| + 8 = 1$

67. $|3x - 2| + 4 = 2$

68. $|x - 3| = |x + 2|$

69. $|x + 4| = |x - 2|$

70. $|2x - 1| = |-x + 3|$

71. $|3x| = |2x - 5|$

Absolute Value Inequalities

Section 9.4

1. Solving Absolute Value Inequalities by Definition

Concepts

In Section 9.3, we studied absolute value equations in the form $|x| = a$. In this section we will solve absolute value *inequalities*. An inequality in any of the forms $|x| < a$, $|x| \leq a$, $|x| > a$, or $|x| \geq a$ is called an **absolute value inequality**.

Recall that an absolute value represents distance from zero on the real number line. Consider the following absolute value equation and inequalities.

1. Solving Absolute Value Inequalities by Definition

2. Solving Absolute Value Inequalities by the Test Point

3. Translating to an Absolute Value Expression

1. $|x| = 3$

$x = 3$ or $x = -3$

Solution:

The set of all points 3 units from zero on the number line

2. $|x| > 3$ **Solution:**

$x < -3$ or $x > 3$ The set of all points more than 3 units from zero

3. $|x| < 3$ **Solution:**

$-3 < x < 3$ The set of all points less than 3 units from zero

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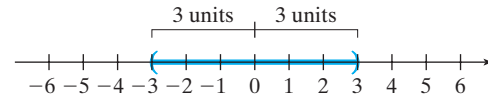
Absolute Value Equations and Inequalities

Let a be a real number such that $a > 0$. Then

Equation/ Inequality	Solution (Equivalent Form)	Graph
$\lvert x \rvert = a$	$x = -a$ or $x = a$	
$\lvert x \rvert > a$	$x < -a$ or $x > a$	
$\lvert x \rvert < a$	$-a < x < a$	

To solve an absolute value inequality, first isolate the absolute value and then rewrite the absolute value inequality in its equivalent form.

Example 1   Solving Absolute Value Inequalities

Solve the inequalities.

a. $|3w + 1| - 4 < 7$ **b.** $3 \le 1 + \left| \dfrac{1}{2}t - 5 \right|$

Solution:

a. $|3w + 1| - 4 < 7$

$|3w + 1| < 11$ ⟵ Isolate the absolute value first.

The inequality is in the form $|x| < a$, where $x = 3w + 1$.

$-11 < 3w + 1 < 11$ Rewrite in the equivalent form $-a < x < a$.

$-12 < 3w < 10$ Solve for w.

$-4 < w < \dfrac{10}{3}$
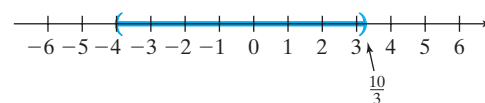
The solution is $\{w \mid -4 < w < \frac{10}{3}\}$ or, equivalently in interval notation, $\left(-4, \frac{10}{3}\right)$.

Calculator Connections

Graph $Y_1 = \text{abs}(3x + 1) - 4$ and $Y_2 = 7$.
On the given display window, $Y_1 < Y_2$
(Y_1 is below Y_2) for $-4 < x < \frac{10}{3}$.
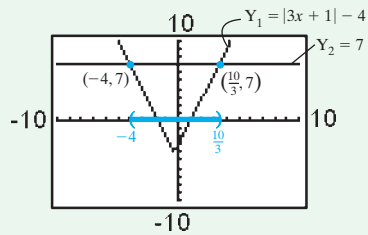
b. $3 \leq 1 + \left|\dfrac{1}{2}t - 5\right|$

$1 + \left|\dfrac{1}{2}t - 5\right| \geq 3$ Write the inequality with the absolute value on the left.

$\left|\dfrac{1}{2}t - 5\right| \geq 2$ Isolate the absolute value.

The inequality is in the form $|x| \geq a$, where $x = \frac{1}{2}t - 5$.

$\dfrac{1}{2}t - 5 \leq -2$ or $\dfrac{1}{2}t - 5 \geq 2$ Rewrite in the equivalent form $x \leq -a$ or $x \geq a$.

$\dfrac{1}{2}t \leq 3$ or $\dfrac{1}{2}t \geq 7$ Solve the compound inequality.

$2\left(\dfrac{1}{2}t\right) \leq 2(3)$ or $2\left(\dfrac{1}{2}t\right) \geq 2(7)$ Clear fractions.
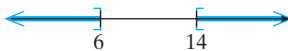
$t \leq 6$ or $t \geq 14$

The solution is $\{t \mid t \leq 6 \text{ or } t \geq 14\}$ or, equivalently in interval notation, $(-\infty, 6] \cup [14, \infty)$.

Calculator Connections

Graph $Y_1 = \text{abs}((1/2)x - 5) + 1$ and $Y_2 = 3$.
On the given display window, $Y_1 \geq Y_2$ for
$x \leq 6$ or $x \geq 14$.
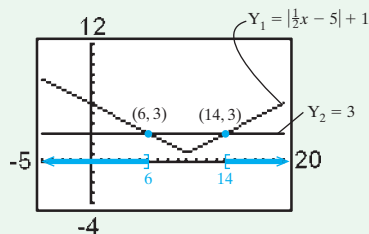
Skill Practice   Solve the inequalities. Write the solutions in interval notation.

1. $|2t + 5| + 2 \leq 11$ **2.** $5 < 1 + \left|\dfrac{1}{3}c - 1\right|$

By definition, the absolute value of a real number will always be nonnegative. Therefore, the absolute value of any expression will always be greater than

Skill Practice Answers
1. $[-7, 2]$
2. $(-\infty, -9) \cup (15, \infty)$

a negative number. Similarly, an absolute value can never be less than a negative number. Let a represent a positive real number. Then

- The solution to the inequality $|x| > -a$ is all real numbers, $(-\infty, \infty)$.
- There is no solution to the inequality $|x| < -a$.

Example 2   Solving Absolute Value Inequalities

Solve the inequalities.

a. $|3d - 5| + 7 < 4$ **b.** $|3d - 5| + 7 > 4$

Solution:

a. $|3d - 5| + 7 < 4$ Isolate the absolute value. An absolute value

   $|3d - 5| < -3$ expression cannot be less than a negative number. Therefore, there is no solution.

   No solution

b. $|3d - 5| + 7 > 4$ Isolate the absolute value. The inequality is in

   $|3d - 5| > -3$ the form $|x| > a$, where a is negative. An absolute value of any real number is greater than a negative number. Therefore, the

   All real numbers, $(-\infty, \infty)$ solution is all real numbers.

Calculator Connections

By graphing $Y_1 = \text{abs}(3x - 5) + 7$ and $Y_2 = 4$, we see that $Y_1 > Y_2$ (Y_1 is above Y_2) for all real numbers x on the given display window.
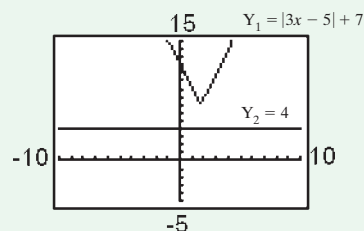
Skill Practice   Solve the inequalities.

3. $|4p + 2| + 6 < 2$ **4.** $|4p + 2| + 6 > 2$

Example 3   Solving Absolute Value Inequalities

Solve the inequalities.

a. $|4x + 2| \geq 0$ **b.** $|4x + 2| > 0$

Solution:

a. $|4x + 2| \geq 0 \longleftarrow$ The absolute value is already isolated.

   The absolute value of any real number is nonnegative. Therefore, the solution is all real numbers, $(-\infty, \infty)$.

b. $|4x + 2| > 0$

   An absolute value will be greater than zero at all points *except where it is equal to zero*. That is, the point(s) for which $|4x + 2| = 0$ must be excluded from the solution set.

$|4x + 2| = 0$

$4x + 2 = 0$ or $4x + 2 = -0$ The second equation is the same as the first.

$4x = -2$

$x = -\dfrac{1}{2}$ Therefore, exclude $x = -\frac{1}{2}$ from the solution.
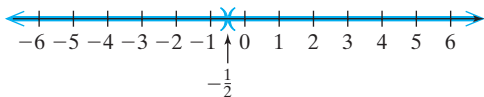
The solution is $\{x \mid x \neq -\frac{1}{2}\}$ or equivalently in interval notation, $(-\infty, -\frac{1}{2}) \cup (-\frac{1}{2}, \infty)$.

Calculator Connections

Graph $Y_1 = \text{abs}(4x + 2)$. From the graph, $Y_1 = 0$ at $x = -\frac{1}{2}$ (the x-intercept). On the given display window, $Y_1 > 0$ for $x < -\frac{1}{2}$ or $x > -\frac{1}{2}$.
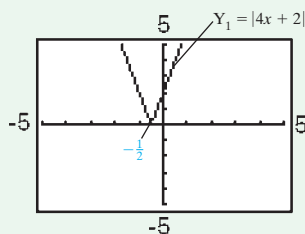
Skill Practice   Solve the inequalities.

5. $|3x - 1| \geq 0$ **6.** $|3x - 1| > 0$

2. Solving Absolute Value Inequalities by the Test Point Method

For each problem in Example 1, the absolute value inequality was converted to an equivalent compound inequality. However, sometimes students have difficulty setting up the appropriate compound inequality. To avoid this problem, you may want to use the test point method to solve absolute value inequalities.

Solving Inequalities by Using the Test Point Method

1. Find the boundary points of the inequality. (Boundary points are the real solutions to the related equation and points where the inequality is undefined.)

2. Plot the boundary points on the number line. This divides the number line into regions.

3. Select a test point from each region and substitute it into the original inequality.
   - If a test point makes the original inequality true, then that region is part of the solution set.

4. Test the boundary points in the original inequality.
   - If a boundary point makes the original inequality true, then that point is part of the solution set.

Skill Practice Answers

5. $(-\infty, \infty)$

6. $\left\{x \mid x \neq \dfrac{1}{3}\right\}; \left(-\infty, \dfrac{1}{3}\right) \cup \left(\dfrac{1}{3}, \infty\right)$

To demonstrate the use of the test point method, we will repeat the absolute value inequalities from Example 1. Notice that regardless of the method used, the absolute value is always isolated *first* before any further action is taken.

Example 4   **Solving Absolute Value Inequalities by the Test Point Method**

Solve the inequalities by using the test point method.

a. $|3w + 1| - 4 < 7$ **b.** $3 \le 1 + \left|\frac{1}{2}t - 5\right|$

Solution:

a. $|3w + 1| - 4 < 7$

$|3w + 1| < 11$ ← ————————————— Isolate the absolute value.

$|3w + 1| = 11$ **Step 1:** Solve the related equation.

$3w + 1 = 11$ or $3w + 1 = -11$ Write as an equivalent system of two equations.

$3w = 10$ or $3w = -12$

$w = \dfrac{10}{3}$ or $w = -4$ These are the only boundary points.

Step 2: Plot the boundary points.

Region I │   Region II   │ Region III

$-6\ -5\ -4\ -3\ -2\ -1\quad 0\quad 1\quad 2\quad 3\quad 4\quad 5\quad 6$

Step 3: Select a test point from each region.
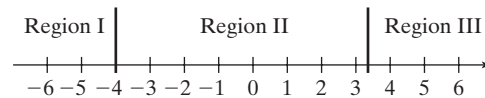
Test $w = -5$:

$|3(-5) + 1| - 4 \overset{?}{<} 7$

$|-14| - 4 \overset{?}{<} 7$

$14 - 4 \overset{?}{<} 7$

$10 \overset{?}{<} 7$ False

Test $w = 0$:

$|3(0) + 1| - 4 \overset{?}{<} 7$

$|1| - 4 \overset{?}{<} 7$

$-3 \overset{?}{<} 7$ True

Test $w = 4$:

$|3(4) + 1| - 4 \overset{?}{<} 7$

$|13| - 4 \overset{?}{<} 7$

$13 - 4 \overset{?}{<} 7$

$9 \overset{?}{<} 7$ False

False   True   False

$-6\ -5\ -4\ -3\ -2\ -1\quad 0\quad 1\quad 2\quad 3\quad 4\quad 5\quad 6$
$\frac{10}{3}$
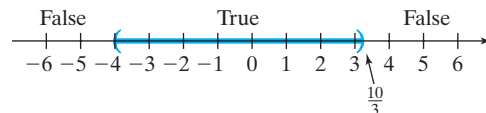
Step 4: Because the original inequality is a strict inequality, the boundary points (where equality occurs) are not included.

The solution is $\{w | -4 < w < \frac{10}{3}\}$ or, equivalently in interval notation, $(-4, \frac{10}{3})$.

b. $3 \leq 1 + \left| \frac{1}{2}t - 5 \right|$

$1 + \left| \frac{1}{2}t - 5 \right| \geq 3$ Write the inequality with the absolute value on the left.

$\left| \frac{1}{2}t - 5 \right| \geq 2 \longleftarrow$ Isolate the absolute value.

$\left| \frac{1}{2}t - 5 \right| = 2$ **Step 1:** Solve the related equation.

$\frac{1}{2}t - 5 = 2$ or $\frac{1}{2}t - 5 = -2$ Write the union of two equations.

$\frac{1}{2}t = 7$ or $\frac{1}{2}t = 3$

$t = 14$ or $t = 6$ These are the boundary points.

Step 2: Plot the boundary points.

Region I | Region II | Region III

————————|——————————|——————————→
           6   14
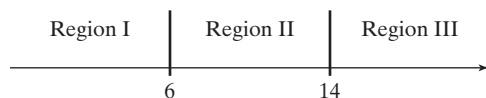
Step 3: Select a test point from each region.

Test $t = 0$:   **Test $t = 10$:**   **Test $t = 16$:**

$3 \overset{?}{\leq} 1 + \left| \frac{1}{2}(0) - 5 \right|$ $3 \overset{?}{\leq} 1 + \left| \frac{1}{2}(10) - 5 \right|$ $3 \overset{?}{\leq} 1 + \left| \frac{1}{2}(16) - 5 \right|$

$3 \overset{?}{\leq} 1 + |0 - 5|$ $3 \overset{?}{\leq} 1 + |5 - 5|$ $3 \overset{?}{\leq} 1 + |8 - 5|$

$3 \overset{?}{\leq} 1 + |-5|$ $3 \overset{?}{\leq} 1 + |0|$ $3 \overset{?}{\leq} 1 + |3|$

$3 \leq 1 + 5$ True $3 \overset{?}{\leq} 1$ False $3 \leq 4$ True

Step 4: The original inequality uses the sign $\geq$. Therefore, the boundary points (where equality occurs) must be part of the solution set.

      True   False   True

←————————]———————————[————————→
          6   14
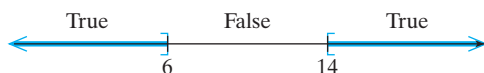
The solution is $\{x \mid x \leq 6 \text{ or } x \geq 14\}$ or, equivalently in interval notation, $(-\infty, 6] \cup [14, \infty)$.

Skill Practice   Solve the inequalities by using the test point method.

7. $10 \geq 6 + |3t - 4|$ **8.** $\left| \frac{1}{2}c + 4 \right| + 1 > 6$

Skill Practice Answers

7. $\left[0, \frac{8}{3} \right]$

8. $(-\infty, -18) \cup (2, \infty)$

> **Example 5**   Solving Absolute Value Inequalities

Solve the inequalities.

a. $\left| \dfrac{1}{3}x + 4 \right| < 0$ **b.** $\left| \dfrac{1}{3}x + 4 \right| \leq 0$

Solution:

a. $\left| \dfrac{1}{3}x + 4 \right| < 0$ ⟵ The absolute value is already isolated.

     No solution   Because the absolute value of any real number is nonnegative, an absolute value cannot be strictly less than zero. Therefore, there is no solution to this inequality.

b. $\left| \dfrac{1}{3}x + 4 \right| \leq 0$ ⟵ The absolute value is already isolated.

An absolute value will never be less than zero. However, an absolute value may be equal to zero. Therefore, the only solutions to this inequality are the solutions to the related equation.

$\left| \dfrac{1}{3}x + 4 \right| = 0$ Set up the related equation.

$\dfrac{1}{3}x + 4 = 0$

$\dfrac{1}{3}x = -4$

$x = -12$ This is the only boundary point.

The solution set is $\{-12\}$.

```
        False        True        False
                      ↓
  ←————————————————————●————————————————————→
                     −12
```
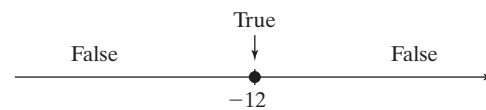
> ### Calculator Connections
>
> Graph $Y_1 = \text{abs}((1/3)x + 4)$. Notice that on the given viewing window the graph of Y_1 does not extend below the x-axis. Therefore, there is no solution to the inequality $Y_1 < 0$.
>   Because $Y_1 = 0$ at $x = -12$, the inequality $Y_1 \leq 0$ has a solution at $x = -12$.
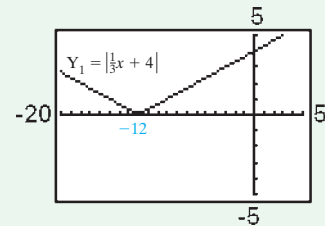
> **Skill Practice**   Solve the inequalities.
>
> **9.** $\left| \dfrac{3}{5}x + 3 \right| < 0$ **10.** $\left| \dfrac{3}{5}x + 3 \right| \leq 0$

Skill Practice Answers
9. No solution   **10.** $\{-5\}$

3. Translating to an Absolute Value Expression

Absolute value expressions can be used to describe distances. The distance between c and d is given by $|c - d|$. For example, the distance between -2 and 3 on the number line is $|(-2) - 3| = 5$ as expected.

Example 6 Expressing Distances with Absolute Value

Write an absolute value inequality to represent the following phrases.

a. All real numbers x, whose distance from zero is greater than 5 units

b. All real numbers x, whose distance from -7 is less than 3 units

Solution:

a. All real numbers x, whose distance from zero is greater than 5 units

$|x - 0| > 5$ or simply $|x| > 5$
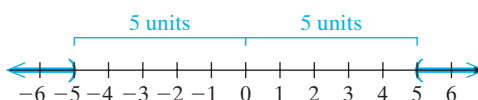
b. All real numbers x, whose distance from -7 is less than 3 units

$|x - (-7)| < 3$ or simply $|x + 7| < 3$
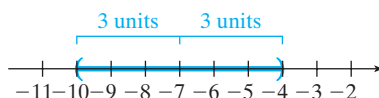
Skill Practice Write an absolute value inequality to represent the following phrases.

11. All real numbers whose distance from zero is greater than 10 units

12. All real numbers whose distance from 4 is less than 6

Absolute value expressions can also be used to describe boundaries for measurement error.

Example 7 Expressing Measurement Error with Absolute Value

Latoya measured a certain compound on a scale in the chemistry lab at school. She measured 8 g of the compound, but the scale is only accurate to ± 0.1 g. Write an absolute value inequality to express an interval for the true mass, x, of the compound she measured.

Solution:

Because the scale is only accurate to ± 0.1 g, the true mass, x, of the compound may deviate by as much as 0.1 g above or below 8 g. This may be expressed as an absolute value inequality:

$|x - 8.0| \leq 0.1$ or equivalently $7.9 \leq x \leq 8.1$

Skill Practice

13. Vonzell molded a piece of metal in her machine shop. She measured the thickness at 12 mm. Her machine is accurate to ± 0.05 mm. Write an absolute value inequality to express an interval for the true measurement of the thickness, t, of the metal.

Skill Practice Answers

13. $|t - 12| \leq 0.05$

Section 9.4   Practice Exercises

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Study Skills Exercise

1. Define the key term **absolute value inequality**.

Review Exercises

For Exercises 2–3, solve the equations.

2. $|10x - 6| = -5$

3. $2 = |5 - 7x| + 1$

For Exercises 4–7, solve the inequalities and graph the solution set. Write the solution in interval notation.

4. $-15 < 3w - 6 \leq -9$

5. $5 - 2y \leq 1$ and $3y + 2 \geq 14$

6. $m - 7 \leq -5$ or $m - 7 \geq -10$

7. $3b - 2 < 7$ or $b - 2 > 4$

Concepts 1 and 2: Solving Absolute Value Inequalities

For Exercises 8–18, solve the equations and inequalities. For each inequality, graph the solution set and express the solution in interval notation.

8. a. $|x| = 5$

    b. $|x| > 5$

    c. $|x| < 5$

9. a. $|a| = 4$

    b. $|a| > 4$

    c. $|a| < 4$

10. a. $|x - 3| = 7$

    b. $|x - 3| > 7$

    c. $|x - 3| < 7$

11. a. $|w + 2| = 6$

    b. $|w + 2| > 6$

    c. $|w + 2| < 6$

12. a. $|p| = -2$

    b. $|p| > -2$

    c. $|p| < -2$

13. a. $|x| = -14$

    b. $|x| > -14$

    c. $|x| < -14$

14. a. $|y + 1| = -6$

    b. $|y + 1| > -6$

    c. $|y + 1| < -6$

15. a. $|z - 4| = -3$

    b. $|z - 4| > -3$

    c. $|z - 4| < -3$

16. a. $|x| = 0$

    b. $|x| > 0$

    c. $|x| < 0$

17. a. $|p + 3| = 0$

    b. $|p + 3| > 0$

    c. $|p + 3| < 0$

18. a. $|k - 7| = 0$

    b. $|k - 7| > 0$

    c. $|k - 7| < 0$

For Exercises 19–48, solve the absolute value inequalities by using either the definition or the test point method. Graph the solution set and write the solution in interval notation.

19. $|x| > 6$

20. $|x| \le 6$

21. $|t| \le 3$

22. $|p| > 3$

23. $|y + 2| \ge 0$

24. $0 \le |7n + 2|$

25. $5 \le |2x - 1|$

26. $|x - 2| \ge 7$

27. $|k - 7| < -3$

28. $|h + 2| < -9$

29. $\left|\dfrac{w - 2}{3}\right| - 3 \le 1$

30. $\left|\dfrac{x + 3}{2}\right| - 2 \ge 4$

31. $14 \le |9 - 4y|$

32. $1 > |2m - 7|$

33. $\left|\dfrac{2x + 1}{4}\right| < 5$

34. $\left|\dfrac{x - 4}{5}\right| \le 7$

35. $8 < |4 - 3x| + 12$

36. $-16 < |5x - 1| - 1$

37. $5 - |2m + 1| > 5$

38. $3 - |5x + 3| > 3$

39. $|p + 5| \le 0$

40. $|y + 1| - 4 \le -4$

41. $|z - 6| + 5 > 5$

42. $|2c - 1| - 4 > -4$

43. $5|2y - 6| + 3 \ge 13$

44. $7|y + 1| - 3 \ge 11$

45. $-3|6 - t| + 1 > -5$

46. $-4|8 - x| + 2 > -14$

47. $|0.02x + 0.06| - 0.1 < 0.05$

48. $|0.05x - 0.04| - 0.01 < 0.11$

Concept 3: Translating to an Absolute Value Expression

For Exercises 49–52, write an absolute value inequality equivalent to the expression given.

49. All real numbers whose distance from 0 is greater than 7

50. All real numbers whose distance from -3 is less than 4

51. All real numbers whose distance from 2 is at most 13

52. All real numbers whose distance from 0 is at least 6

53. A 32-oz jug of orange juice may not contain exactly 32 oz of juice. The possibility of measurement error exists when the jug is filled in the factory. If the maximum measurement error is ± 0.05 oz, write an absolute value inequality representing the range of volumes, x, in which the orange juice jug may be filled.

54. The length of a board is measured to be 32.3 in. The maximum measurement error is ± 0.2 in. Write an absolute value inequality that represents the range for the length of the board, x.

55. A bag of potato chips states that its weight is $6\frac{3}{4}$ oz. The maximum measurement error is $\pm\frac{1}{8}$ oz. Write an absolute value inequality that represents the range for the weight, x, of the bag of chips.

56. A $\frac{7}{8}$-in. bolt varies in length by at most $\pm\frac{1}{16}$ in. Write an absolute value inequality that represents the range for the length, x, of the bolt.

57. The width, w, of a bolt is supposed to be 2 cm but may have a 0.01-cm margin of error. Solve $|w - 2| \le 0.01$, and interpret the solution to the inequality in the context of this problem.

58. In the 2004 election, Senator Barak Obama was projected to receive 70% of the votes with a margin of error of 3%. Solve $|p - 0.70| \le 0.03$, and interpret the solution to the inequality in the context of this problem.

Expanding Your Skills

For Exercises 59–62, match the graph with the inequality:

59.

60.

61.

62.
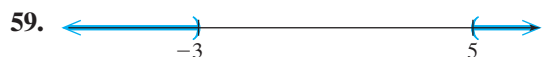
a. $|x - 2| < 4$ **b.** $|x - 1| > 4$ **c.** $|x - 3| < 2$ **d.** $|x - 5| > 1$

Graphing Calculator Exercises

To solve an absolute value inequality by using a graphing calculator, let Y_1 equal the left side of the inequality and let Y_2 equal the right side of the inequality. Graph both Y_1 and Y_2 on a standard viewing window and use an *Intersect* feature or *Zoom* and *Trace* to approximate the intersection of the graphs. To solve $Y_1 > Y_2$, determine all x-values where the graph of Y_1 is above the graph of Y_2. To solve $Y_1 < Y_2$, determine all x-values where the graph of Y_1 is below the graph of Y_2.

For Exercises 63–72, solve the inequalities using a graphing calculator.

63. $|x + 2| > 4$ **64.** $|3 - x| > 6$

65. $\left|\dfrac{x+1}{3}\right| < 2$

66. $\left|\dfrac{x-1}{4}\right| < 1$

67. $|x - 5| < -3$

68. $|x + 2| < -2$

69. $|2x + 5| > -4$

70. $|1 - 2x| > -4$

71. $|6x + 1| \leq 0$

72. $|3x - 4| \leq 0$

Chapter 9   Problem Recognition Exercises— Equations and Inequalities

For Exercises 1–24, identify the category for each equation or inequality (choose from the list here). Then solve the equation or inequality.

- Linear
- Quadratic
- Polynomial (degree greater than 2)
- Rational
- Absolute value

1. $z^2 + 10z + 9 > 0$

2. $5a - 2 = 6(a + 4)$

3. $\dfrac{x - 4}{2x + 4} \geq 1$

4. $4x^2 - 7x = 2$

5. $|3x - 1| + 4 < 6$

6. $2t^2 - t + 1 < 0$

7. $\dfrac{1}{2}p - \dfrac{2}{3} < \dfrac{1}{6}p - 4$

8. $p^2 + 3p \leq 4$

9. $3y^2 - 2y - 2 \geq 0$

10. $\dfrac{x + 6}{x + 4} = 7$

11. $3(2x - 4) \geq 1 - (x - 3)$

12. $|6x + 5| + 3 = 2$

13. $(x - 3)(2x + 1)(x + 5) \geq 0$

14. $-x^2 - x - 3 \leq 0$

15. $\dfrac{-6}{y - 2} < 2$

16. $x^2 - 1 = 5x$

17. $\left|\dfrac{x}{4} + 2\right| + 6 > 6$

18. $x^2 - 36 < 0$

19. $3y^3 + 5y^2 - 12y - 20 = 0$

20. $|8 - 2x| = 16$

21. $5b - 10 - 3b = 3(b - 2) - 4$

22. $\dfrac{x}{x - 5} \geq 0$

23. $|6x + 1| = |5 + 6x|$

24. $4x(x - 5)^2(2x + 3) < 0$

Section 9.5	Linear Inequalities in Two Variables

Concepts

1. Graphing Linear Inequalities in Two Variables

2. Compound Linear Inequalities in Two Variables

3. Graphing a Feasible Region

1. Graphing Linear Inequalities in Two Variables

A **linear inequality in two variables** x and y is an inequality that can be written in one of the following forms: $ax + by < c$, $ax + by > c$, $ax + by \leq c$, or $ax + by \geq c$, provided a and b are not both zero.

A solution to a linear inequality in two variables is an ordered pair that makes the inequality true. For example, solutions to the inequality $x + y < 6$ are ordered pairs (x, y) such that the sum of the x- and y-coordinates is less than 6. This inequality has an infinite number of solutions, and therefore it is convenient to express the solution set as a graph.

To graph a linear inequality in two variables, we will follow these steps.

> ### Graphing a Linear Inequality in Two Variables
>
> **1.** Solve for y, if possible.
>
> **2.** Graph the related equation. Draw a dashed line if the inequality is strict, $<$ or $>$. Otherwise, draw a solid line.
>
> **3.** Shade above or below the line as follows:
>
> - Shade *above* the line if the inequality is of the form $y > ax + b$ or $y \geq ax + b$.
>
> - Shade *below* the line if the inequality is of the form $y < ax + b$ or $y \leq ax + b$.

This process is demonstrated in Example 1.

Example 1	Graphing a Linear Inequality in Two Variables

Graph the solution set. $-3x + y \leq 1$

Solution:

$-3x + y \leq 1$

$\qquad y \leq 3x + 1$ Solve for y.

Next graph the line defined by the related equation $y = 3x + 1$.

Because the inequality is of the form $y \leq ax + b$, the solution to the inequality is the region *below* the line $y = 3x + 1$. See Figure 9-6.
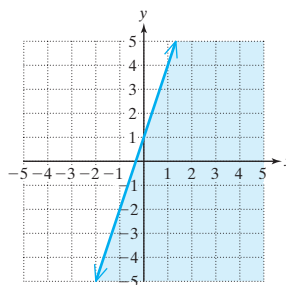
Figure 9-6

Skill Practice   Graph the solution set.

1. $2x + y \geq -4$

After graphing the solution to a linear inequality, we can verify that we have shaded the correct side of the line by using test points. In Example 1, we can pick an arbitrary ordered pair within the shaded region. Then substitute the x- and y-coordinates in the original inequality. If the result is a true statement, then that ordered pair is a solution to the inequality and suggests that other points from the same region are also solutions.

For example, the point $(0, 0)$ lies within the shaded region (Figure 9-7).

$\qquad -3x + y \leq 1$ Substitute $(0, 0)$ in the original inequality.

$\qquad -3(0) + (0) \overset{?}{\leq} 1$

$\qquad\qquad 0 + 0 \leq 1$ ✔ True   The point $(0, 0)$ from the shaded region is a solution.

In Example 2, we will graph the solution set to a strict inequality. A strict inequality uses the symbol $<$ or $>$. In such a case, the boundary line will be drawn as a dashed line. This indicates that the boundary itself is *not* part of the solution set.
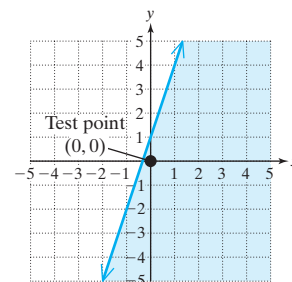
Figure 9-7

Example 2   Graphing a Linear Inequality in Two Variables

Graph the solution set. $-4y < 5x$

Solution:

$-4y < 5x$

$\dfrac{-4y}{-4} > \dfrac{5x}{-4}$ Solve for y. Reverse the inequality sign.

$y > -\dfrac{5}{4}x$

Skill Practice Answers

1.
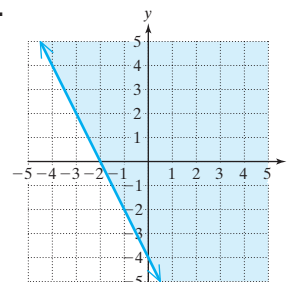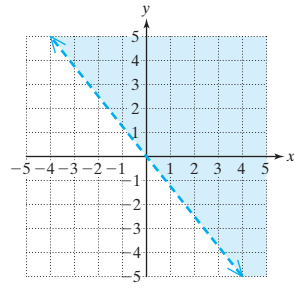
Figure 9-8

Graph the line defined by the related equation, $y = -\frac{5}{4}x$. The boundary line is drawn as a dashed line because the inequality is strict. Also note that the line passes through the origin.

Because the inequality is of the form $y > ax + b$, the solution to the inequality is the region *above* the line. See Figure 9-8.

Skill Practice　Graph the solution set.

2. $-3y < x$
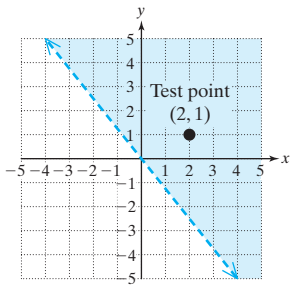
Figure 9-9

In Example 2, we cannot use the origin as a test point, because the point $(0, 0)$ is on the boundary line. Be sure to select a test point strictly within the shaded region. In this case, we choose $(2, 1)$. See Figure 9-9.

$$-4y < 5x$$
$$-4(1) \overset{?}{<} 5(2) \qquad \text{Substitute } (2, 1) \text{ in the original inequality.}$$
$$-4 < 10 \;\; ✔ \;\; \text{True} \qquad \text{The point } (2, 1) \text{ from the shaded region is a solution to the original inequality.}$$

In Example 3, we encounter a situation in which we cannot solve for the *y*-variable.

Example 3　**Graphing a Linear Inequality in Two Variables**

Graph the solution set.　$4x \geq -12$

Solution:

$$4x \geq -12$$
$$x \geq -3$$

In this inequality, there is no *y*-variable. However, we can simplify the inequality by solving for *x*.

Graph the related equation $x = -3$. This is a vertical line. The boundary is drawn as a solid line, because the inequality is not strict, $\geq$.

Figure 9-10

To shade the appropriate region, refer to the inequality, $x \geq -3$. The points for which x is greater than -3 are to the right of $x = -3$. Therefore, shade the region to the *right* of the line (Figure 9-10).

Selecting a test point such as $(0, 0)$ from the shaded region indicates that we have shaded the correct side of the line.

$$4x \geq -12 \qquad \text{Substitute } x = 0.$$
$$4(0) \geq -12 \;\; ✔ \qquad \text{True}$$

Skill Practice　Graph the solution set.

3. $-2x \geq 2$

Skill Practice Answers

2.

3.
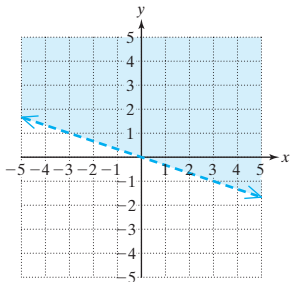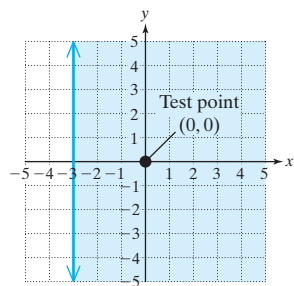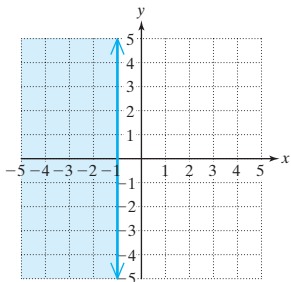
2. Compound Linear Inequalities in Two Variables

Some applications require us to find the union or intersection of two or more linear inequalities.

Example 4 | Graphing a Compound Linear Inequality

Graph the solution set of the compound inequality.

$$y > \tfrac{1}{2}x + 1 \quad \text{and} \quad x + y < 1$$

Solution:

Solve each inequality for y.

First inequality

$y > \tfrac{1}{2}x + 1$

The inequality is of the form $y > ax + b$. Graph *above* the boundary line. (See Figure 9-11).
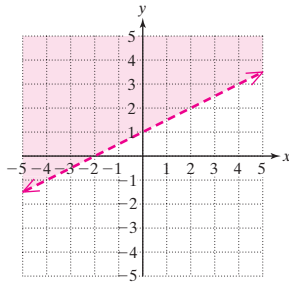
Second inequality

$x + y < 1$

$y < -x + 1$

The inequality is of the form $y < ax + b$. Graph *below* the boundary line. (See Figure 9-12).
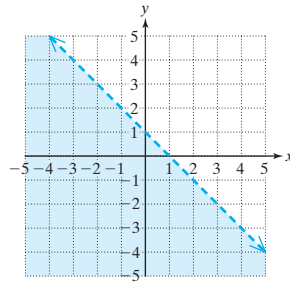
Figure 9-11

Figure 9-12

The region bounded by the inequalities is the region above the line $y = \tfrac{1}{2}x + 1$ and below the line $y = -x + 1$. This is the intersection or "overlap" of the two regions (shown in purple in Figure 9-13).
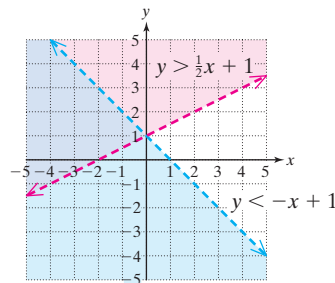
Figure 9-13

The intersection is the solution set to the system of inequalities. See Figure 9-14.
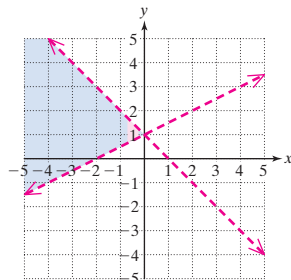
Figure 9-14

Skill Practice   Graph the solution set.

4. $x - 3y > 3$

   $y < -2x + 4$

Example 5 demonstrates the union of the solution sets of two linear inequalities.

Example 5   **Graphing a Compound Linear Inequality**

Graph the solution set of the compound inequality.

$$3y \leq 6 \quad \text{or} \quad y - x \leq 0$$

Solution:

First inequality	Second inequality
$3y \leq 6$	$y - x \leq 0$
$y \leq 2$	$y \leq x$

The graph of $y \leq 2$ is the region on and below the horizontal line $y = 2$. (See Figure 9-15.)

The inequality is of the form $y \leq ax + b$. Graph a solid line and the region below the line. (See Figure 9-16.)
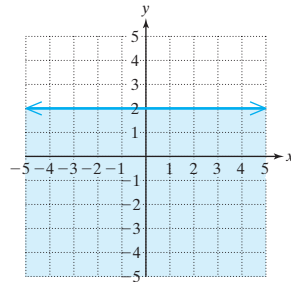
Figure 9-15
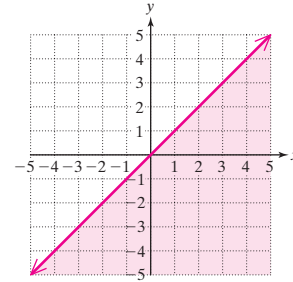
Figure 9-16
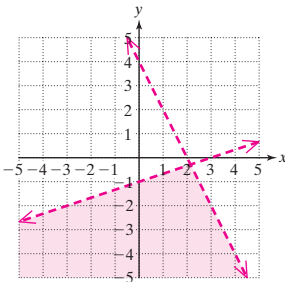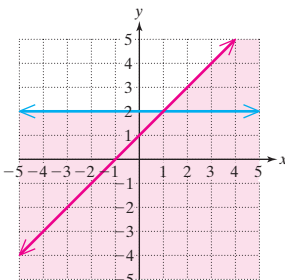
The solution to the compound inequality $3y \leq 6$ or $y - x \leq 0$ is the union of these regions, Figure 9-17.
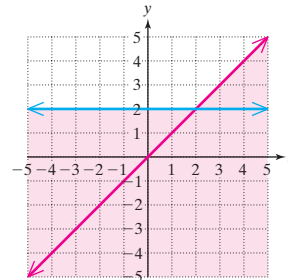
Figure 9-17

Skill Practice   Graph the solution set.

5. $2y \leq 4 \quad \text{or} \quad y \leq x + 1$

Skill Practice Answers

4.

5.

| Example 6 | Graphing Compound Linear Inequalities

Describe the region of the plane defined by the following systems of inequalities.

$$x \leq 0 \quad \text{and} \quad y \geq 0$$

Solution:

$x \leq 0$ $x \leq 0$ on the y-axis and in the second and third quadrants.

$y \geq 0$ $y \geq 0$ on the x-axis and in the first and second quadrants.

The intersection of these regions is the set of points in the second quadrant (with the boundary included).
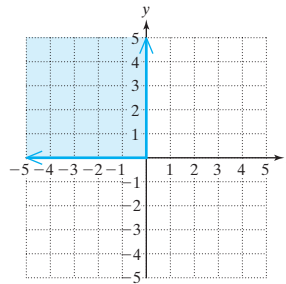
| **Skill Practice** | Graph the region defined by the system of inequalities.

6. $x \leq 0$ and $y \leq 0$

3. Graphing a Feasible Region

When two variables are related under certain constraints, a system of linear inequalities can be used to show a region of feasible values for the variables.

| Example 7 | Graphing a Feasible Region

Susan has two tests on Friday: one in chemistry and one in psychology. Because the two classes meet in consecutive hours, she has no study time between tests. Susan estimates that she has a maximum of 12 hr of study time before the tests, and she must divide her time between chemistry and psychology.

Let x represent the number of hours Susan spends studying chemistry.

Let y represent the number of hours Susan spends studying psychology.

a. Find a set of inequalities to describe the constraints on Susan's study time.

b. Graph the constraints to find the feasible region defining Susan's study time.

Solution:

a. Because Susan cannot study chemistry or psychology for a negative period of time, we have $x \geq 0$ and $y \geq 0$.

Furthermore, her total time studying cannot exceed 12 hr: $x + y \leq 12$.

A system of inequalities that defines the constraints on Susan's study time is

$$x \geq 0$$

$$y \geq 0$$

$$x + y \leq 12$$

Skill Practice Answers

6.
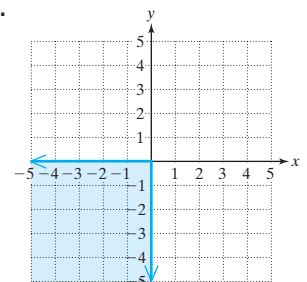
b. The first two conditions $x \geq 0$ and $y \geq 0$ represent the set of points in the first quadrant. The third condition $x + y \leq 12$ represents the set of points below and including the line $x + y = 12$ (Figure 9-18).
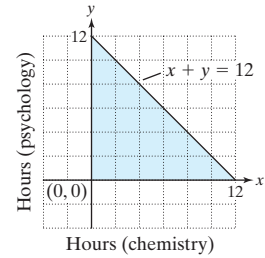
Figure 9-18

Discussion:

1. Refer to the feasible region drawn in Example 7(b). Is the ordered pair $(8, 5)$ part of the feasible region?

No. The ordered pair $(8, 5)$ indicates that Susan spent 8 hr studying chemistry and 5 hr studying psychology. This is a total of 13 hr, which exceeds the constraint that Susan only had 12 hr to study. The point $(8, 5)$ lies outside the feasible region, above the line $x + y = 12$ (Figure 9-19).

2. Is the ordered pair $(7, 3)$ part of the feasible region?

Yes. The ordered pair $(7, 3)$ indicates that Susan spent 7 hr studying chemistry and 3 hr studying psychology.

This point lies within the feasible region and satisfies all three constraints.

$$x \geq 0 \longrightarrow \qquad 7 \geq 0 \qquad \text{True}$$

$$y \geq 0 \longrightarrow \qquad 3 \geq 0 \qquad \text{True}$$

$$x + y \leq 12 \longrightarrow (7) + (3) \leq 12 \qquad \text{True}$$

Notice that the ordered pair $(7, 3)$ corresponds to a point where Susan is not making full use of the 12 hr of study time.
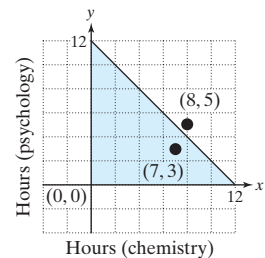
Figure 9-19

3. Suppose there was one additional constraint imposed on Susan's study time. She knows she needs to spend at least twice as much time studying chemistry as she does studying psychology. Graph the feasible region with this additional constraint.

Because the time studying chemistry must be at least twice the time studying psychology, we have $x \geq 2y$.

This inequality may also be written as $y \leq \dfrac{x}{2}$

Figure 9-20 shows the first quadrant with the constraint $y \leq \dfrac{x}{2}$.
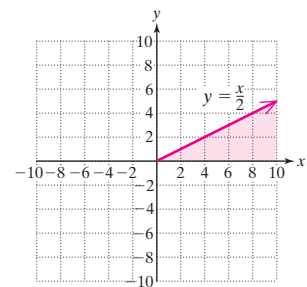
Figure 9-20

4. At what point in the feasible region is Susan making the most efficient use of her time for both classes?

First and foremost, Susan must make use of *all* 12 hr. This occurs for points along the line $x + y = 12$. Susan will also want to study for both classes with approximately twice as much time devoted to chemistry. Therefore, Susan will be deriving the maximum benefit at the point of intersection of the line $x + y = 12$ and the line $y = \dfrac{x}{2}$.

Using the substitution method, replace $y = \dfrac{x}{2}$ into the equation $x + y = 12$.

$x + \dfrac{x}{2} = 12$

$2x + x = 24$ Clear fractions.

$3x = 24$

$x = 8$ Solve for x.

$y = \dfrac{(8)}{2}$ To solve for y, substitute $x = 8$
into the equation $y = \dfrac{x}{2}$.

$y = 4$
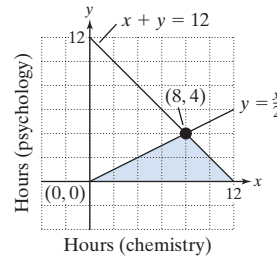
Therefore Susan should spend 8 hr studying chemistry and 4 hr studying psychology.

Skill Practice

7. A local pet rescue group has a total of 30 cages that can be used to hold cats and dogs. Let x represent the number of cages used for cats, and let y represent the number used for dogs.
   a. Write a set of inequalities to express the fact that the number of cat and dog cages cannot be negative.
   b. Write an inequality to describe the constraint on the total number of cages for cats and dogs.
   c. Graph the system of inequalities to find the feasible region describing the available cages.

Skill Practice Answers

7a. $x \geq 0$ and $y \geq 0$
b. $x + y \leq 30$
c.
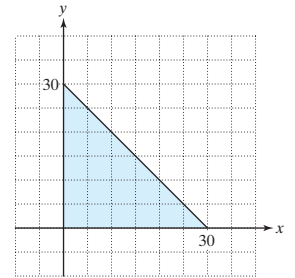
Section 9.5   Practice Exercises

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Study Skills Exercise

1. Define the key term **linear inequality in two variables**.

Review Exercises

For Exercises 2–5, solve the inequalities.

2. $5 < x + 1$ and $-2x + 6 \geq -6$

3. $5 - x \leq 4$ and $6 > 3x - 3$

4. $4 - y < 3y + 12$ or $-2(y + 3) \geq 12$

5. $-2x < 4$ or $3x - 1 \leq -13$

Concept 1: Graphing Linear Inequalities in Two Variables

For Exercises 6–9, decide if the following points are solutions to the inequality.

6. $2x - y > 8$

   a. $(3, -5)$
   b. $(-1, -10)$
   c. $(4, -2)$
   d. $(0, 0)$

7. $3y + x < 5$

   a. $(-1, 7)$
   b. $(5, 0)$
   c. $(0, 0)$
   d. $(2, -3)$

8. $y \leq -2$

   a. $(5, -3)$ **c.** $(0, 0)$

   b. $(-4, -2)$ **d.** $(3, 2)$

9. $x \geq 5$

   a. $(4, 5)$ **c.** $(8, 8)$

   b. $(5, -1)$ **d.** $(0, 0)$

For Exercises 10–15, decide which inequality symbol should be used ($<, >, \geq, \leq$) by looking at the graph.

10.

$x - y$ ____ 2

11.

y ____ $-2x + 3$

12.

y ____ -4

13.

x ____ 3

14.

x ____ 0 and y ____ 0

15.
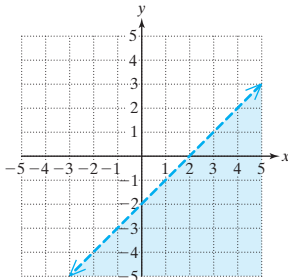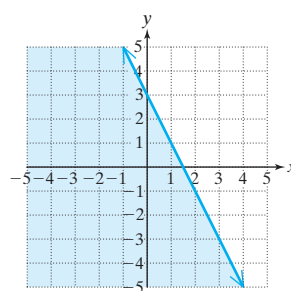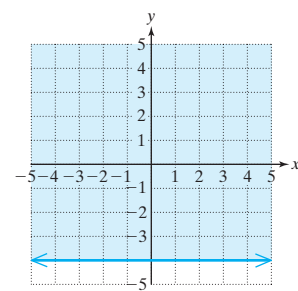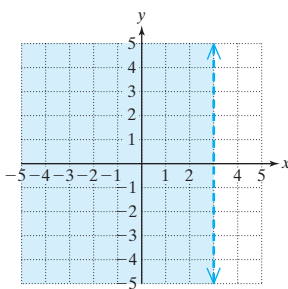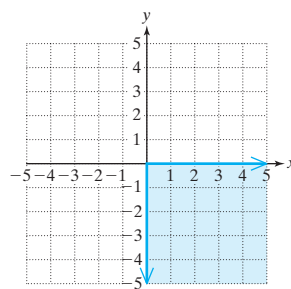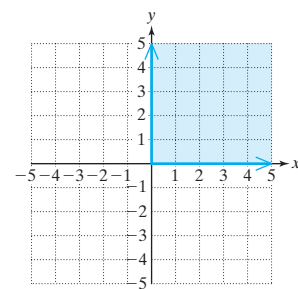
x ____ 0 and y ____ 0

For Exercises 16–39, graph the solution set.

16. $x - 2y > 4$
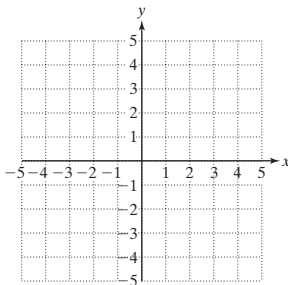
17. $x - 3y > 6$
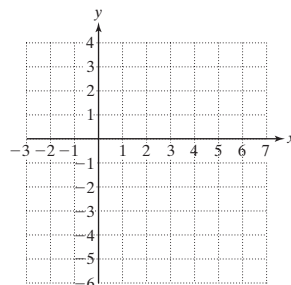
18. $5x - 2y < 10$
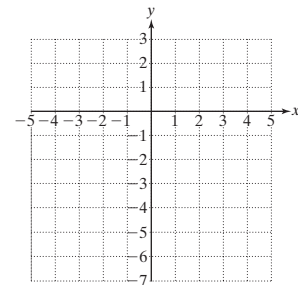
19. $x - 3y < 8$
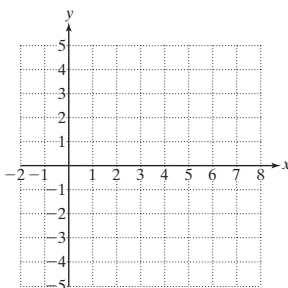
20. $2x \leq -6y + 12$
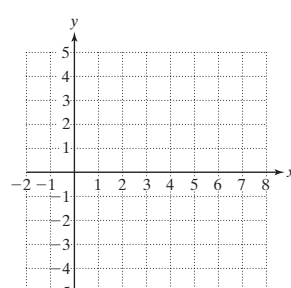
21. $4x < 3y + 12$
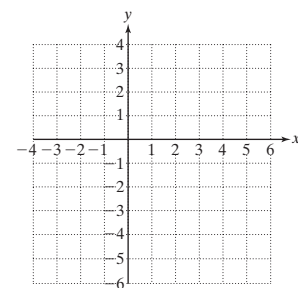
22. $2y \leq 4x$
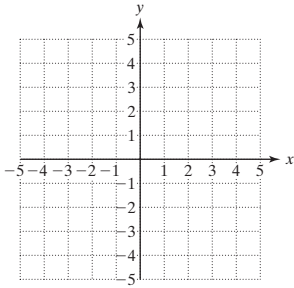
23. $-6x < 2y$
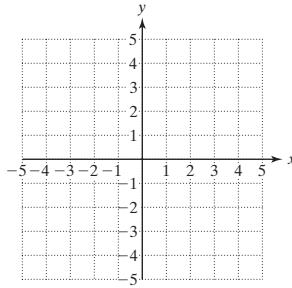
24. $y \geq -2$
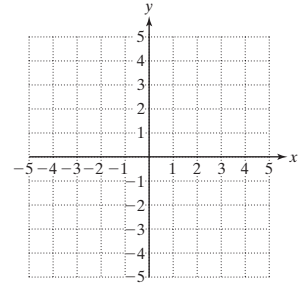
25. $y \geq 5$
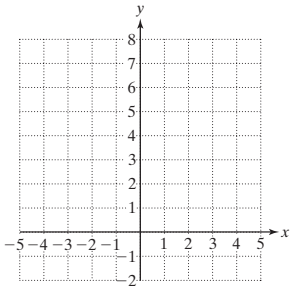
26. $4x < 5$
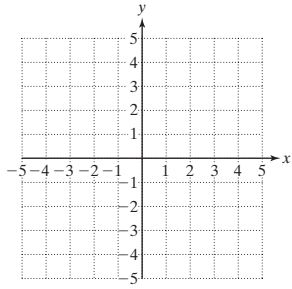
27. $x + 6 < 7$
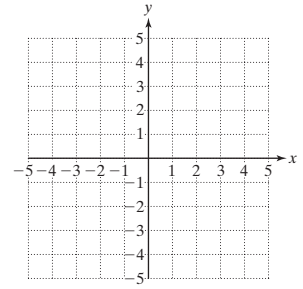
28. $y \geq \dfrac{2}{5}x - 4$
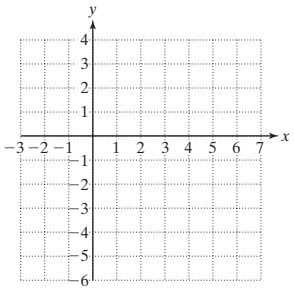
29. $y \geq -\dfrac{5}{2}x - 4$
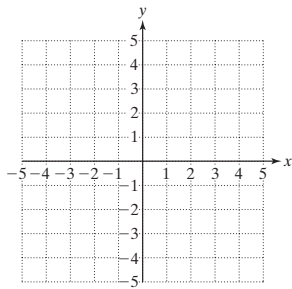
30. $y \leq \dfrac{1}{3}x + 6$
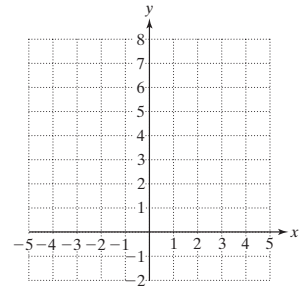
31. $y \leq -\dfrac{1}{4}x + 2$
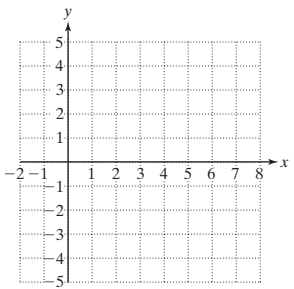
32. $y - 5x > 0$
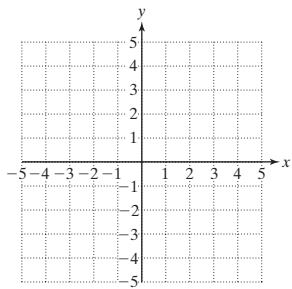
33. $y - \dfrac{1}{2}x > 0$
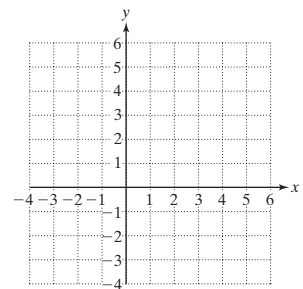
34. $\dfrac{x}{5} + \dfrac{y}{4} < 1$
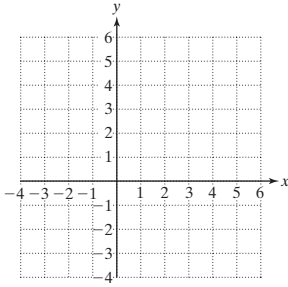
35. $x + \dfrac{y}{2} \geq 2$
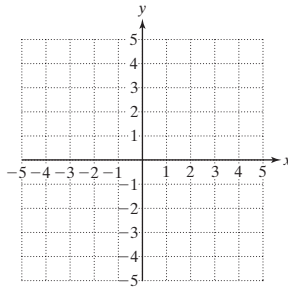
36. $0.1x + 0.2y \leq 0.6$
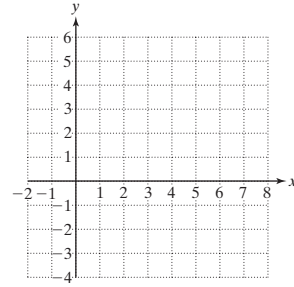
37. $0.3x - 0.2y < 0.6$
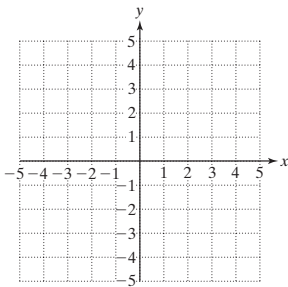
38. $x \leq -\dfrac{2}{3}y$
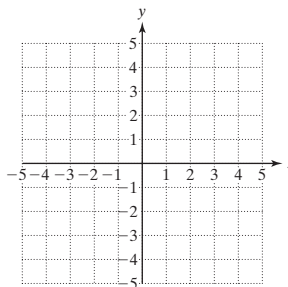
39. $x \geq -\dfrac{5}{4}y$
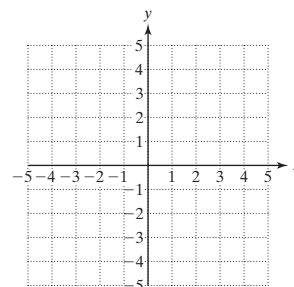
Concept 2: Compound Linear Inequalities in Two Variables

For Exercises 40–55, graph the solution set of each compound inequality.

40. $y < 4$ and $y > -x + 2$
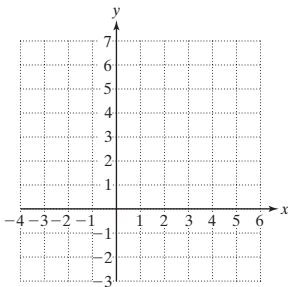
41. $y < 3$ and $x + 2y < 6$
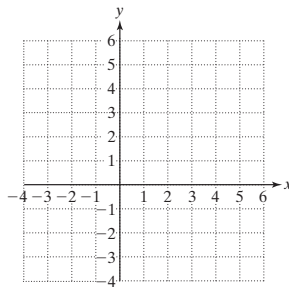
42. $2x + y \leq 5$ or $x \geq 3$
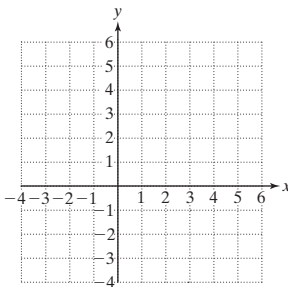
43. $x + 3y \geq 3$ or $x \leq -2$
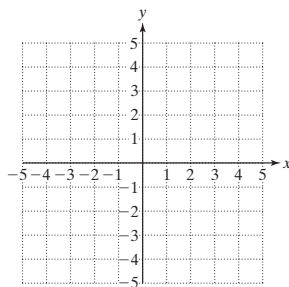
44. $x + y < 3$ and $4x + y < 6$
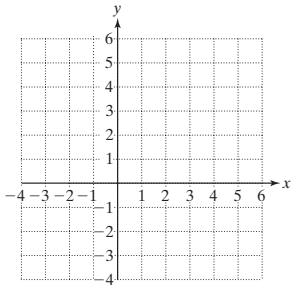
45. $x + y < 4$ and $3x + y < 9$
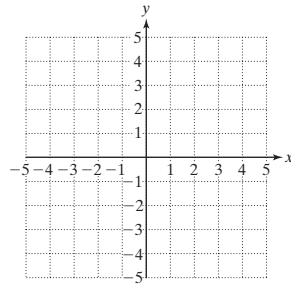
46. $2x - y \le 2$ or $2x + 3y \ge 6$
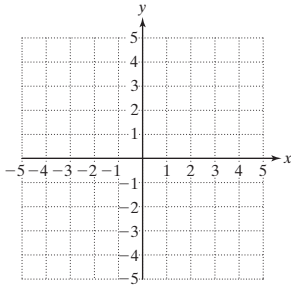
47. $3x + 2y \ge 4$ or $x - y \le 3$
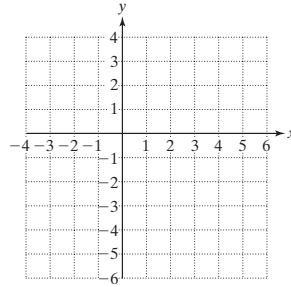
48. $x > 4$ and $y < 2$
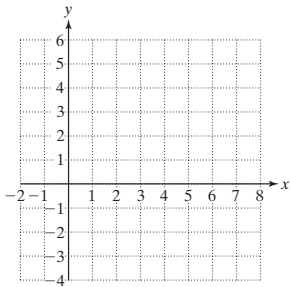
49. $x < 3$ and $y > 4$
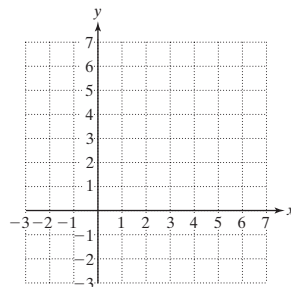
50. $x \le -2$ or $y \le 0$
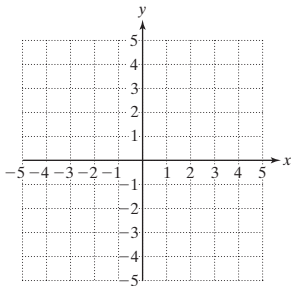
51. $x \ge 0$ or $y \ge -3$
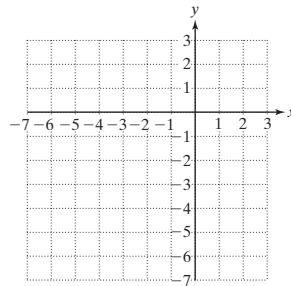
52. $x > 0$ and $x + y < 6$
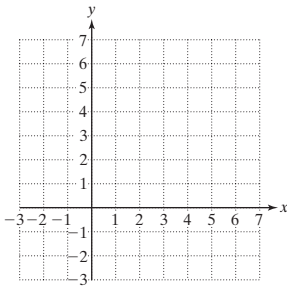
53. $x < 0$ and $x + y < 2$
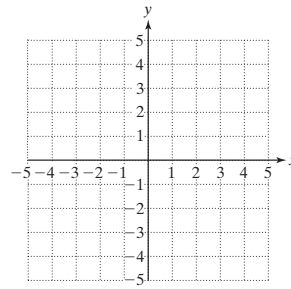
54. $y \le 0$ or $x - y \le -4$
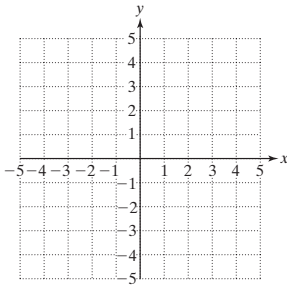
55. $y \ge 0$ or $x - y \ge -3$
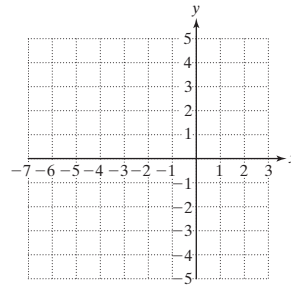
Concept 3: Graphing a Feasible Region

For Exercises 56–61, graph the feasible regions.

56. $x + y \le 3$ and

$x \ge 0, y \ge 0$
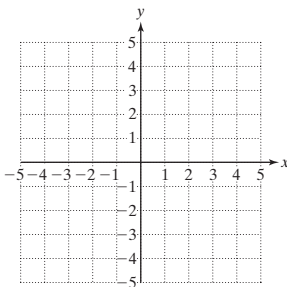
57. $x - y \le 2$ and

$x \ge 0, y \ge 0$
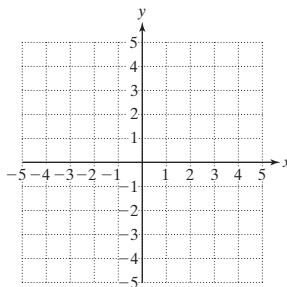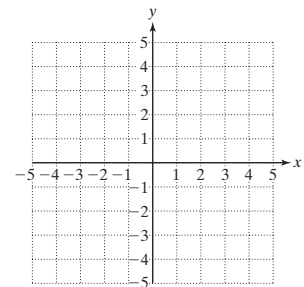

58. $y < \dfrac{1}{2}x - 3$ and

$x \le 0, y \ge -5$
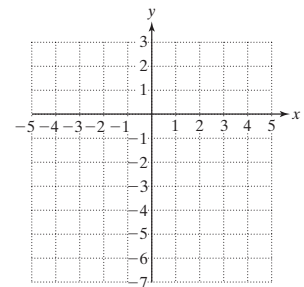
59. $y > \dfrac{1}{2}x - 3$ and

$x \ge -2, y \le 0$
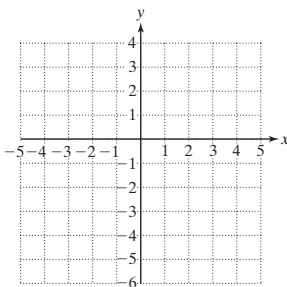
60. $x \ge 0, y \ge 0$

$x + y \le 8$ and

$3x + 5y \le 30$
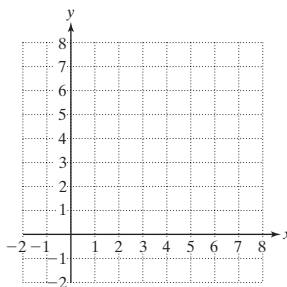
**61.** $x \ge 0, y \ge 0$

$x + y \le 5$ and

$x + 2y \le 6$

62. A manufacturer produces two models of desks. Model A requires $1\frac{1}{2}$ hr to stain and finish and $1\frac{1}{4}$ hr to assemble. Model B requires 2 hr to stain and finish and $\frac{3}{4}$ hr to assemble. The total amount of time available for staining and finishing is 12 hr and for assembling is 6 hr. Let x represent the number of Model A desks, and let y represent the number of Model B desks.

a. Write two inequalities that express the fact that the number of desks to be produced cannot be negative.

b. Write an inequality in terms of the number of Model A and Model B desks that can be produced if the total time for staining and finishing is at most 12 hr.

c. Write an inequality in terms of the number of Model A and Model B desks that can be produced if the total time for assembly is no more than 6 hr.

d. Identify the feasible region formed by graphing the preceding inequalities.

e. Is the point (4, 1) in the feasible region? What does the point (4, 1) represent in the context of this problem?

f. Is the point (5, 4) in the feasible region? What does the point (5, 4) represent in the context of this problem?
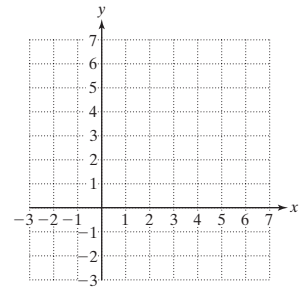
63. In scheduling two drivers for delivering pizza, James needs to have at least 65 hr scheduled this week. His two drivers, Karen and Todd, are not allowed to get overtime, so each one can work at most 40 hr. Let x represent the number of hours that Karen can be scheduled, and let y represent the number of hours Todd can be scheduled.

a. Write two inequalities that express the fact that Karen and Todd cannot work a negative number of hours.

b. Write two inequalities that express the fact that neither Karen nor Todd is allowed overtime (i.e., each driver can have at most 40 hr).

c. Write an inequality that expresses the fact that the total number of hours from both Karen and Todd needs to be at least 65 hr.

d. Graph the feasible region formed by graphing the inequalities.

e. Is the point (35, 40) in the feasible region? What does the point (35, 40) represent in the context of this problem?

f. Is the point (20, 40) in the feasible region? What does the point (20, 40) represent in the context of this problem?
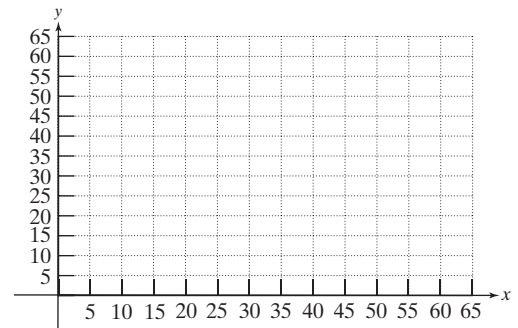
Chapter 9   SUMMARY

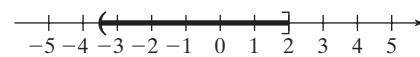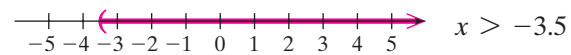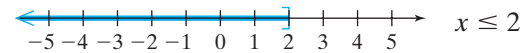## Section 9.1   Compound Inequalities

Key Concepts

Solve two or more inequalities joined by *and* by finding the intersection of their solution sets. Solve two or more inequalities joined by *or* by finding the union of the solution sets.
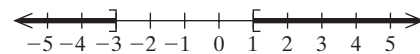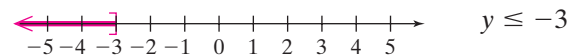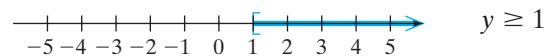
Examples

Example 1

$$-7x + 3 \geq -11 \qquad \text{and} \qquad 1 - x < 4.5$$
$$-7x \geq -14 \qquad \text{and} \qquad -x < 3.5$$
$$x \leq 2 \qquad \text{and} \qquad x > -3.5$$

$x \leq 2$

$x > -3.5$

The solution is $\{x \mid -3.5 < x \leq 2\}$ or equivalently $(-3.5, 2]$.

Example 2

$$5y + 1 \geq 6 \qquad \text{or} \qquad 2y - 5 \leq -11$$
$$5y \geq 5 \qquad \text{or} \qquad 2y \leq -6$$
$$y \geq 1 \qquad \text{or} \qquad y \leq -3$$

$y \geq 1$

$y \leq -3$

The solution is $\{y \mid y \geq 1 \text{ or } y \leq -3\}$ or equivalently $(-\infty, -3] \cup [1, \infty)$.

Section 9.2 Polynomial and Rational Inequalities

Key Concepts

The Test Point Method to Solve Polynomial and Rational Inequalities

1. Find the boundary points of the inequality. (Boundary points are the real solutions to the related equation and points where the inequality is undefined.)
2. Plot the boundary points on the number line. This divides the number line into regions.
3. Select a test point from each region and substitute it into the original inequality.
   - If a test point makes the original inequality true, then that region is part of the solution set.
4. Test the boundary points in the original inequality.
   - If a boundary point makes the original inequality true, then that point is part of the solution set.

Examples

Example 1

$$\frac{28}{2x - 3} \leq 4 \qquad \text{The inequality is undefined for } x = \frac{3}{2}. \text{ Find other possible boundary points by solving the related equation.}$$

$$\frac{28}{2x - 3} = 4 \qquad \text{Related equation}$$
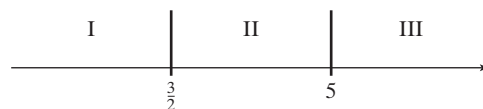
$$(2x - 3) \cdot \frac{28}{2x - 3} = (2x - 3) \cdot 4$$

$$28 = 8x - 12$$

$$40 = 8x$$

$$x = 5$$
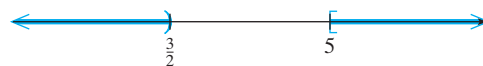
The boundaries are $x = \dfrac{3}{2}$ and $x = 5$

$$\begin{array}{c|c|c}
\text{I} & \text{II} & \text{III} \\
\hline
& \frac{3}{2} \quad 5 &
\end{array}$$

Region I: Test $x = 1$: $\dfrac{28}{2(1) - 3} \overset{?}{\leq} 4$ True

Region II: Test $x = 2$: $\dfrac{28}{2(2) - 3} \overset{?}{\leq} 4$ False

Region III: Test $x = 6$: $\dfrac{28}{2(6) - 3} \overset{?}{\leq} 4$ True

The boundary point $x = \frac{3}{2}$ is not included because $\dfrac{28}{2x - 3}$ is undefined there. The boundary $x = 5$ does check in the original inequality.

The solution is $\left(-\infty, \frac{3}{2}\right) \cup [5, \infty)$.

Section 9.3   Absolute Value Equations

Key Concepts

The equation $|x| = a$ is an absolute value equation. For $a \geq 0$, the solution to the equation $|x| = a$ is $x = a$ or $x = -a$.

Steps to Solve an Absolute Value Equation

1. Isolate the absolute value to write the equation in the form $|x| = a$.
2. If $a < 0$, there is no solution.
3. Otherwise, rewrite the equation $|x| = a$ as $x = a$ or $x = -a$.
4. Solve the equations from step 3.
5. Check answers in the original equation.

The solution to the equation $|x| = |y|$ is $x = y$ or $x = -y$.

Examples

Example 1

$$|2x - 3| + 5 = 10$$
$$|2x - 3| = 5 \qquad \text{Isolate the absolute value.}$$

$2x - 3 = 5$	or	$2x - 3 = -5$
$2x = 8$	or	$2x = -2$
$x = 4$	or	$x = -1$

Example 2

$$|x + 2| + 5 = 1$$
$$|x + 2| = -4 \qquad \text{No solution}$$

Example 3

$$|2x - 1| = |x + 4|$$

$2x - 1 = x + 4$	or	$2x - 1 = -(x + 4)$
$x = 5$	or	$2x - 1 = -x - 4$
	or	$3x = -3$
	or	$x = -1$

Section 9.4   Absolute Value Inequalities

Key Concepts

Solutions to Absolute Value Inequalities

$$|x| > a \Rightarrow x < -a \quad \text{or} \quad x > a$$
$$|x| < a \Rightarrow -a < x < a$$

Examples

Example 1

$$|5x - 2| < 12$$
$$-12 < 5x - 2 < 12$$
$$-10 < 5x < 14$$
$$-2 < x < \frac{14}{5}$$

The solution is $\left(-2, \dfrac{14}{5}\right)$.

Test Point Method to Solve Inequalities

1. Find the boundary points of the inequality. (Boundary points are the real solutions to the related equation and points where the inequality is undefined.)
2. Plot the boundary points on the number line. This divides the number line into regions.
3. Select a test point from each region and substitute it into the original inequality.
   - If a test point makes the original inequality true, then that region is part of the solution set.
4. Test the boundary points in the original inequality.
   - If a boundary point makes the original inequality true, then that point is part of the solution set.

Example 2

$|x - 3| + 2 \geq 7$

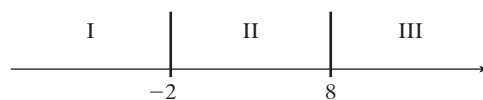$\quad |x - 3| \geq 5$ Isolate the absolute value.

$\quad |x - 3| = 5$ Related equation

$x - 3 = 5 \quad$ or $\quad x - 3 = -5$

$x = 8 \quad$ or $\quad x = -2$ Boundary points

```
        I        |        II        |        III
                 |                  |
  ───────────────┼──────────────────┼───────────────▶
                −2                   8
```
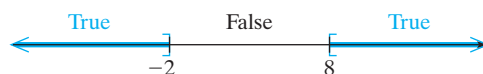
Region I:

Test $x = -3$: $|(-3) - 3| + 2 \overset{?}{\geq} 7$ True

Region II:

Test $x = 0$: $|(0) - 3| + 2 \overset{?}{\geq} 7$ False

Region III:

Test $x = 9$: $|(9) - 3| + 2 \overset{?}{\geq} 7$ True

```
        True          False          True
  ◀──────────────┤                ├──────────────▶
                −2                 8
```

The solution is $(-\infty, -2] \cup [8, \infty)$.

If *a* is *negative* (*a* < 0), then

1. $|x| < a$ has no solution.
2. $|x| > a$ is true for all real numbers.

Example 3

$|x + 5| > -2$

The solution is all real numbers because an absolute value will always be greater than a negative number.

$(-\infty, \infty)$

Section 9.5   Linear Inequalities in Two Variables

Key Concepts

A **linear inequality in two variables** is an inequality of the form $ax + by < c$, $ax + by > c$, $ax + by \leq c$, or $ax + by \geq c$.

Graphing a Linear Inequality in Two Variables

1. Solve for y, if possible.
2. Graph the related equation. Draw a dashed line if the inequality is strict, < or >. Otherwise, draw a solid line.
3. Shade above or below the line according to the following convention.
   - Shade *above* the line if the inequality is of the form $y > ax + b$ or $y \geq ax + b$.
   - Shade *below* the line if the inequality is of the form $y < ax + b$ or $y \leq ax + b$.

Examples

Example 1
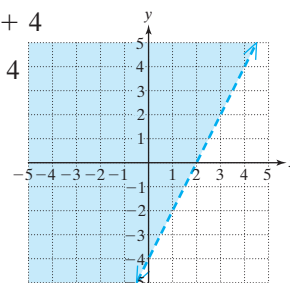
Graph the solution to the inequality $2x - y < 4$.
Graph the related equation, $y = 2x - 4$, with a dashed line.

Solve for y: $2x - y < 4$

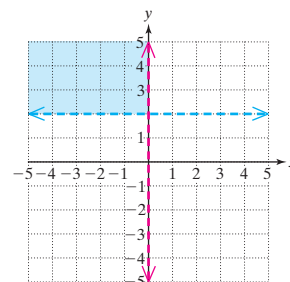$\qquad -y < -2x + 4$

$\qquad y > 2x - 4$

Shade above the line.

The union or intersection of two or more linear inequalities is the union or intersection of the solution sets.
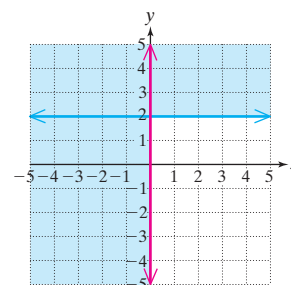
Example 2

Graph.

$$x < 0 \qquad \text{and} \qquad y > 2$$

Example 3

Graph.

$$x \le 0 \qquad \text{or} \qquad y \ge 2$$

Chapter 9   Review Exercises

Section 9.1

For Exercises 1–10, solve the compound inequalities. Write the solutions in interval notation.

1. $4m > -11 \qquad \text{and} \qquad 4m - 3 \le 13$

2. $4n - 7 < 1 \qquad \text{and} \qquad 7 + 3n \ge -8$

3. $-3y + 1 \ge 10 \qquad \text{and} \qquad -2y - 5 \le -15$

4. $\dfrac{1}{2} - \dfrac{h}{12} \le \dfrac{-7}{12} \qquad \text{and} \qquad \dfrac{1}{2} - \dfrac{h}{10} > -\dfrac{1}{5}$

5. $\dfrac{2}{3}t - 3 \le 1 \qquad \text{or} \qquad \dfrac{3}{4}t - 2 > 7$

6. $2(3x + 1) < -10 \qquad \text{or} \qquad 3(2x - 4) \ge 0$

7. $-7 < -7(2w + 3) \qquad \text{or} \qquad -2 < -4(3w - 1)$

8. $5(p + 3) + 4 > p - 1 \quad \text{or} \quad 4(p - 1) + 2 > p + 8$

9. $2 \ge -(b - 2) - 5b \ge -6$

10. $-4 \le \dfrac{1}{2}(x - 1) < -\dfrac{3}{2}$

11. The product of $\frac{1}{3}$ and the sum of a number and 3 is between -1 and 5. Find all such numbers.

12. Normal levels of total cholesterol vary according to age. For adults between 25 and 40 years old, the normal range is generally accepted to be between 140 and 225 mg/dL (milligrams per deciliter), inclusive.

     a. Write an inequality representing the normal range for total cholesterol for adults between 25 and 40 years old.

     b. Write a compound inequality representing abnormal ranges for total cholesterol for adults between 25 and 40 years old.

13. Normal levels of total cholesterol vary according to age. For adults younger than 25 years old, the normal range is generally accepted to be between 125 and 200 mg/dL, inclusive.

     a. Write an inequality representing the normal range for total cholesterol for adults younger than 25 years old.

     b. Write a compound inequality representing abnormal ranges for total cholesterol for adults younger than 25 years old.

14. In certain applications in statistics, a data value that is more than 3 standard deviations from the mean is said to be an *outlier* (a value unusually far from the average). If μ represents the mean of population and σ represents the population standard deviation, then the inequality $|x - \mu| > 3\sigma$ can be used to test whether a data value, x, is an outlier.

The mean height, μ, of adult men is 69.0 in. (5′9″) and the standard deviation, σ, of the height of adult men is 3.0. Determine whether the heights of the following men are outliers.

a. Shaquille O'Neal, 7′1″ = 85 in.

b. Charlie Ward, 6′3″ = 75 in.

c. Elmer Fudd, 4′5″ = 53 in.

Mean
69 in.

53 in.   75 in.   85 in.
Elmer   Charlie   Shaq
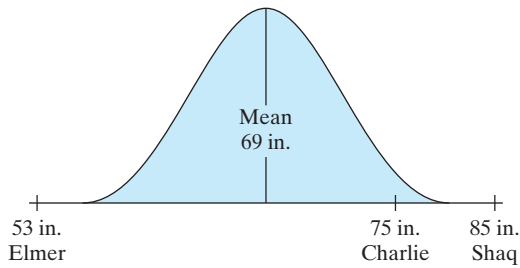
15. Explain the difference between the solution sets of the following compound inequalities.

a. $x \leq 5$ and $x \geq -2$

b. $x \leq 5$ or $x \geq -2$

Section 9.2

16. Solve the equation and inequalities. How do your answers to parts (a), (b), and (c) relate to the graph of $g(x) = x^2 - 4$?

a. $x^2 - 4 = 0$

b. $x^2 - 4 < 0$

c. $x^2 - 4 > 0$

$g(x) = x^2 - 4$
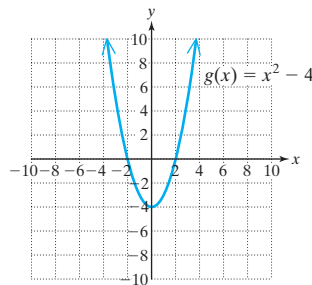
17. Solve the equation and inequalities. How do your answers to parts (a), (b), (c), and (d) relate to the graph of $k(x) = \dfrac{4x}{(x - 2)}$?

a. $\dfrac{4x}{x - 2} = 0$

b. For which values is $k(x)$ undefined?

c. $\dfrac{4x}{x - 2} \geq 0$

d. $\dfrac{4x}{x - 2} \leq 0$

$k(x) = \dfrac{4x}{x - 2}$
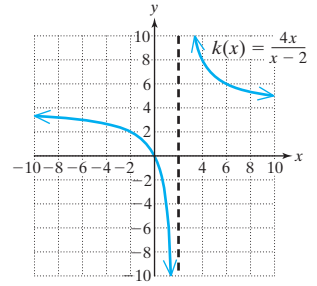
For Exercises 18–29, solve the inequalities. Write the answers in interval notation.

18. $w^2 - 4w - 12 < 0$

19. $t^2 + 6t + 9 \geq 0$

20. $\dfrac{12}{x + 2} \leq 6$

21. $\dfrac{8}{p - 1} \geq -4$

22. $3y(y - 5)(y + 2) > 0$

23. $-3c(c + 2)(2c - 5) < 0$

24. $-x^2 - 4x \leq 1$

25. $y^2 + 4y > 5$

26. $\dfrac{w + 1}{w - 3} > 1$

27. $\dfrac{2a}{a + 3} \leq 2$

28. $t^2 + 10t + 25 \leq 0$

29. $-x^2 - 4x < 4$

Section 9.3

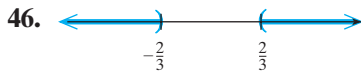For Exercises 30–41, solve the absolute value equations.

30. $|x| = 10$

31. $|x| = 17$

32. $|8.7 - 2x| = 6.1$

33. $|5.25 - 5x| = 7.45$

34. $16 = |x + 2| + 9$

35. $5 = |x - 2| + 4$

36. $|4x - 1| + 6 = 4$

37. $|3x - 1| + 7 = 3$

38. $\left|\dfrac{7x - 3}{5}\right| + 4 = 4$

39. $\left|\dfrac{4x + 5}{-2}\right| - 3 = -3$

40. $|3x - 5| = |2x + 1|$

41. $|8x + 9| = |8x - 1|$

42. Which absolute value expression represents the distance between 3 and -2 on the number line?

$$|3 - (-2)| \qquad |-2 - 3|$$

Section 9.4

43. Write the compound inequality $x < -5$ or $x > 5$ as an absolute value inequality.

44. Write the compound inequality $-4 < x < 4$ as an absolute value inequality.

For Exercises 45–46, write an absolute value inequality that represents the solution sets graphed here.

45.
$$\overset{\text{(}\qquad\qquad\text{)}}{\underset{-6\qquad\qquad 6}{\rule{3cm}{0pt}}}$$

46.
$$\overset{\text{)}\qquad\qquad\text{(}}{\underset{-\frac{2}{3}\qquad\qquad\frac{2}{3}}{\rule{3cm}{0pt}}}$$

For Exercises 47–60, solve the absolute value inequalities. Graph the solution set and write the solution in interval notation.

47. $|x + 6| \geq 8$

48. $|x + 8| \leq 3$

49. $2|7x - 1| + 4 > 4$

50. $4|5x + 1| - 3 > -3$

51. $|3x + 4| - 6 \leq -4$

52. $|5x - 3| + 3 \leq 6$

53. $\left|\dfrac{x}{2} - 6\right| < 5$

54. $\left|\dfrac{x}{3} + 2\right| < 2$

55. $|4 - 2x| + 8 \geq 8$

56. $|9 + 3x| + 1 \geq 1$

57. $-2|5.2x - 7.8| < 13$

58. $-|2.5x + 15| < 7$

59. $|3x - 8| < -1$

60. $|x + 5| < -4$

61. State one possible situation in which an absolute value inequality will have no solution.

62. State one possible situation in which an absolute value inequality will have a solution of all real numbers.

63. The Neilsen ratings estimated that the percent, p, of the television viewing audience watching *American Idol* was 20% with a 3% margin of error. Solve the inequality $|p - 0.20| \leq 0.03$ and interpret the answer in the context of this problem.

64. The length, L, of a screw is supposed to be $3\frac{3}{8}$ in. Due to variation in the production equipment, there is a $\frac{1}{4}$-in. margin of error. Solve the inequality $|L - 3\frac{3}{8}| \leq \frac{1}{4}$ and interpret the answer in the context of this problem.
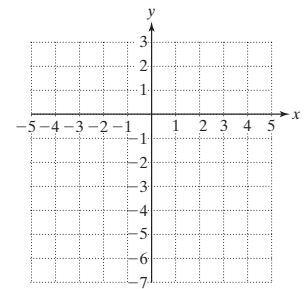
Section 9.5

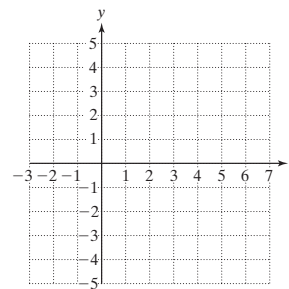For Exercises 65–72, solve the inequalities by graphing.
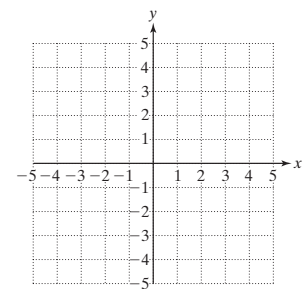
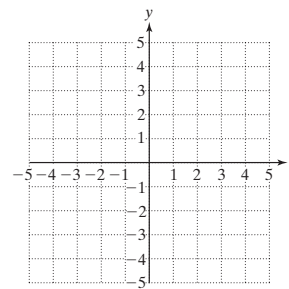**65.** $2x > -y + 5$
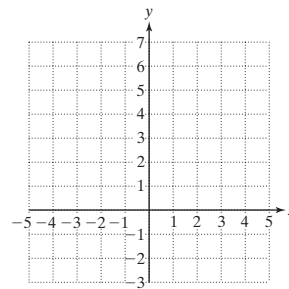
66. $2x \leq -8 - 3y$
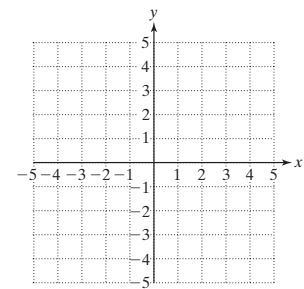
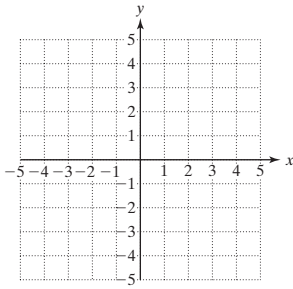**67.** $y \geq -\dfrac{2}{3}x + 3$
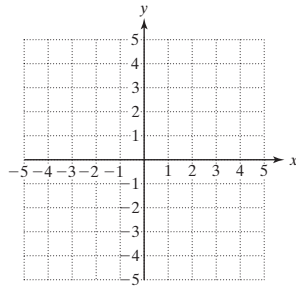
68. $y > \dfrac{3}{4}x - 2$

69. $x > -3$
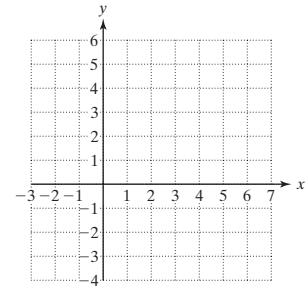
70. $x \leq 2$

71. $x \geq \frac{1}{2}y$
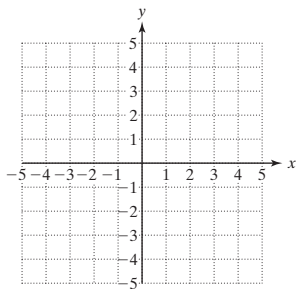
72. $x < \frac{2}{5}y$

76. $x \geq 0$, $\quad y \geq 0$, $\quad$ and $\quad y \leq -\frac{2}{3}x + 4$

For Exercises 73–76, graph the system of inequalities.

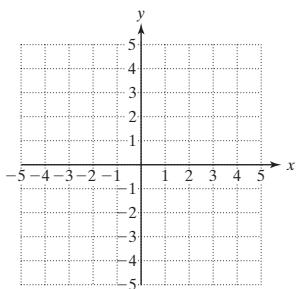**73.** $2x - y > -2$ $\quad$ and $\quad 2x - y \leq 2$
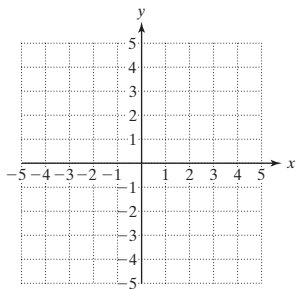
77. A pirate's treasure is buried on a small, uninhabited island in the eastern Caribbean. A shipwrecked sailor finds a treasure map at the base of a coconut palm tree. The treasure is buried within the intersection of three linear inequalities. The palm tree is located at the origin, and the positive y-axis is oriented due north. The scaling on the map is in 1-yd increments. Find the region where the sailor should dig for the treasure.

$$-2x + y \leq 4$$
$$y \leq -x + 6$$
$$y \geq 0$$

74. $3x + y \geq 6$ $\quad$ or $\quad 3x + y < -6$

75. $x \geq 0$, $\quad y \geq 0$, $\quad$ and $\quad y \geq -\frac{3}{2}x + 4$
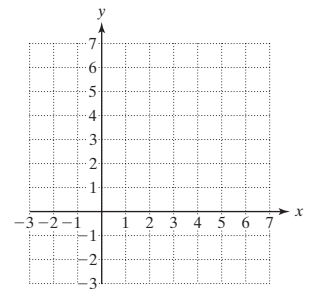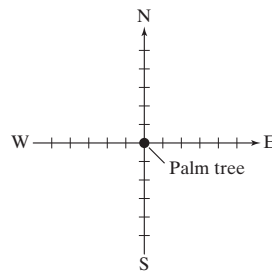
78. Suppose a farmer has 100 acres of land on which to grow oranges and grapefruit. Furthermore, because of demand from his customers, he wants to plant at least 4 times as many acres of orange trees as grapefruit trees.

Let x represent the number of acres of orange trees.

Let y represent the number of acres of grape-fruit trees.

a. Write two inequalities that express the fact that the farmer cannot use a negative number of acres to plant orange and grapefruit trees.

b. Write an inequality that expresses the fact that the total number of acres used for growing orange and grapefruit trees is at most 100.

c. Write an inequality that expresses the fact that the farmer wants to plant at least 4 times as many orange trees as grapefruit trees.

d. Sketch the inequalities in parts (a)–(c) to find the feasible region for the farmer's distribution of orange and grapefruit trees.
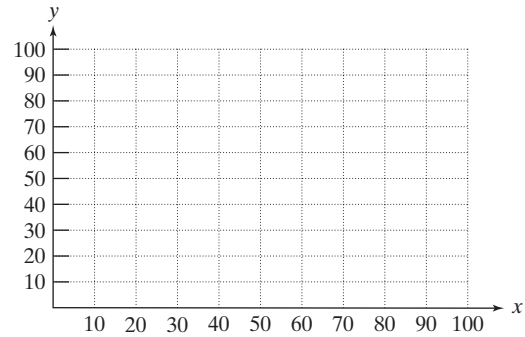
Chapter 9   Test

For Exercises 1–5, solve the compound inequalities. Write the answers in interval notation.

1. $-2 \le 3x - 1 \le 5$

2. $-\dfrac{3}{5}x - 1 \le 8$ or $-\dfrac{2}{3}x \ge 16$

3. $-2x - 3 > -3$ and $x + 3 \ge 0$

4. $5x + 1 \le 6$ or $2x + 4 > -6$

5. $2x - 3 > 1$ and $x + 4 < -1$

6. The normal range in humans of the enzyme adenosine deaminase (ADA), is between 9 and 33 IU (international units), inclusive. Let x represent the ADA level in international units.

a. Write an inequality representing the normal range for ADA.

b. Write a compound inequality representing abnormal ranges for ADA.

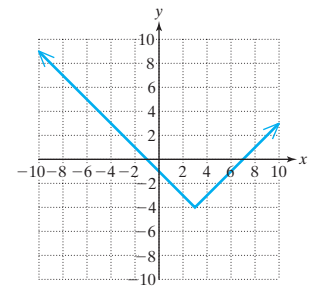For Exercises 7–12, solve the polynomial and rational inequalities.

7. $\dfrac{2x - 1}{x - 6} \le 0$ **8.** $50 - 2a^2 > 0$

9. $y^3 + 3y^2 - 4y - 12 < 0$ **10.** $\dfrac{3}{w + 3} > 2$

11. $5x^2 - 2x + 2 < 0$

12. $t^2 + 22t + 121 \le 0$

For Exercises 13–14, solve the absolute value equations.

13. $\left|\dfrac{1}{2}x + 3\right| - 4 = 4$

14. $|3x + 4| = |x - 12|$

15. Solve the following equation and inequalities. How do your answers to parts (a)–(c) relate to the graph of $f(x) = |x - 3| - 4$?
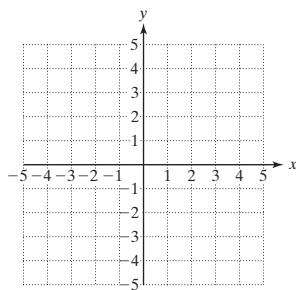
a. $|x - 3| - 4 = 0$

b. $|x - 3| - 4 < 0$

c. $|x - 3| - 4 > 0$

For Exercises 16–19, solve the absolute value inequalities. Write the answers in interval notation.

16. $|3 - 2x| + 6 < 2$ **17.** $|3x - 8| > 9$

18. $|0.4x + 0.3| - 0.2 < 7$
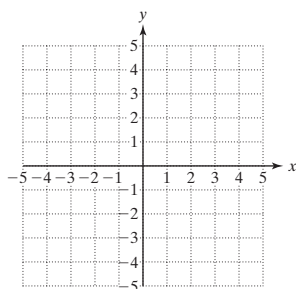
19. $|7 - 3x| + 1 > -3$

20. The mass of a small piece of metal is measured to be 15.41 g. If the measurement error is at most ± 0.01 g, write an absolute value inequality that represents the possible mass, x, of the piece of metal.

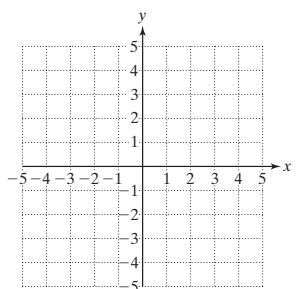**21.** Graph the solution to the inequality $2x - 5y \geq 10$.
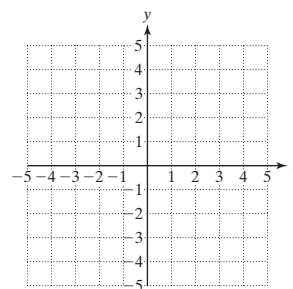
24. After menopause, women are at higher risk for hip fractures as a result of low calcium. As early as their teen years, women need at least 1200 mg of calcium per day (the USDA recommended daily allowance). One 8-oz glass of skim milk contains 300 mg of calcium, and one Antacid (regular strength) contains 400 mg of calcium. Let x represent the number of 8-oz glasses of milk that a woman drinks per day. Let y represent the number of Antacid tablets (regular strength) that a woman takes per day.

a. Write two inequalities that express the fact that the number of glasses of milk and the number of Antacid taken each day cannot be negative.

b. Write a linear inequality in terms of x and y for which the daily calcium intake is at least 1200 mg.

c. Graph the inequalities.

For Exercises 22–23, graph the solution to the compound inequality.

22. $x + y < 3$ and $3x - 2y > -6$

23. $5x \leq 5$ or $x + y \leq 0$

Chapters 1–9   Cumulative Review Exercises

1. Perform the indicated operations.
$$(2x - 3)(x - 4) - (x - 5)^2$$

2. Solve the equation. $-9m + 3 = 2m(m - 4)$

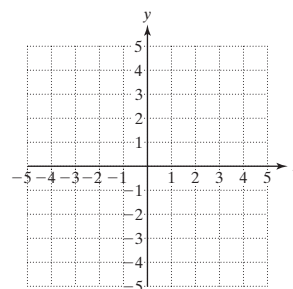For Exercises 3–4, solve the equation and inequalities. Write the solution to the inequalities in interval notation.

3. a. $2|3 - p| - 4 = 2$

  b. $2|3 - p| - 4 < 2$

  c. $2|3 - p| - 4 > 2$

4. a. $\left|\dfrac{y - 2}{4}\right| - 6 = -3$

  b. $\left|\dfrac{y - 2}{4}\right| - 6 < -3$

  c. $\left|\dfrac{y - 2}{4}\right| - 6 > -3$

5. Graph the inequality. $4x - y > 12$

6. The time in minutes required for a rat to run through a maze depends on the number of trials, n, that the rat has practiced.

$$t(n) = \frac{3n + 15}{n + 1} \qquad n \geq 1$$

a. Find $t(1)$, $t(50)$, and $t(500)$, and interpret the results in the context of this problem. Round to 2 decimal places, if necessary.

b. Does there appear to be a limiting time in which the rat can complete the maze?

c. How many trials are required so that the rat is able to finish the maze in under 5 min?

7. a. Solve the inequality. $2x^2 + x - 10 \geq 0$

b. How does the answer in part (a) relate to the graph of the function $f(x) = 2x^2 + x - 10$?
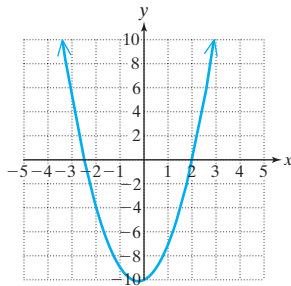
8. Shade the region defined by the compound inequality.

$$3x + y \leq -2 \qquad \text{or} \qquad y \geq 1$$
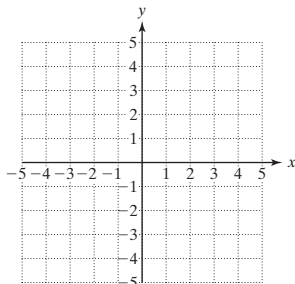
9. Simplify the expression.

$$2 - 3(x - 5) + 2[4 - (2x + 6)]$$

10. McDonald's corporation is the world's largest food service retailer. At the end of 1996, McDonald's operated 2.1×10^4 restaurants in over 100 countries. Worldwide sales in 1996 were nearly $\$3.18 \times 10^{10}$. Find the average sales per restaurant in 1996. Write the answer in scientific notation.

11. a. Divide the polynomials.

$$\frac{2x^4 - x^3 + 5x - 7}{x^2 + 2x - 1}$$

Identify the quotient and remainder.

b. Check your answer by multiplying.

12. The area of a trapezoid is given by $A = \frac{1}{2}h(b_1 + b_2)$.

a. Solve for b_1.

b. Find b_1 when $h = 4$ cm, $b_2 = 6$ cm, and $A = 32$ cm^2.

13. The speed of a car varies inversely as the time to travel a fixed distance. A car traveling the speed limit of 60 mph travels between two points in 10 sec. How fast is a car moving if it takes only 8 sec to cover the same distance?

14. Two angles are supplementary. One angle measures $9°$ more than twice the other angle. Find the measures of the angles.

15. Chemique invests $3000 less in an account earning 5% simple interest than she does in an account bearing 6.5% simple interest. At the end of one year, she earns a total $770 in interest. Find the amount invested in each account.

16. Determine algebraically whether the lines are parallel, perpendicular, or neither.
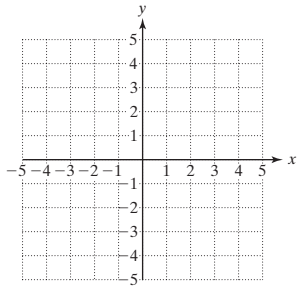
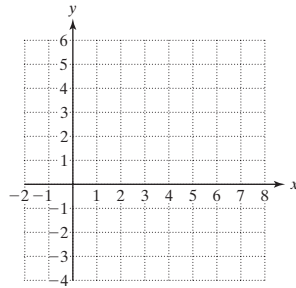$$4x - 2y = 5$$
$$-3x + 6y = 10$$

17. Solve the inequality.

$$\frac{-2}{x - 5} \geq 3$$

For Exercises 18–19, find the x- and y-intercepts and slope (if they exist) of the lines. Then graph the lines.

18. $3x + 5 = 8$

19. $\frac{1}{2}x + y = 4$

20. Find an equation of the line with slope $-\frac{2}{3}$ passing through the point $(4, -7)$. Write the final answer in slope-intercept form.

21. Solve the system of equations.

$$3x + y = z + 2$$
$$y = 1 - 2x$$
$$3z = -2y$$

22. Identify the order of the matrices.

a. $\begin{bmatrix} 2 & 4 & 5 \\ -1 & 0 & 1 \\ 9 & 2 & 3 \\ 3 & 0 & 1 \end{bmatrix}$ **b.** $\begin{bmatrix} 5 & 6 & 3 \\ 6 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix}$

23. Against a headwind, a plane can travel 6240 mi in 13 hr. On the return trip flying with the same wind, the plane can fly 6240 mi in 12 hr. Find the wind speed and the speed of the plane in still air.

24. The profit that a company makes manufacturing computer desks is given by

$$P(x) = -\tfrac{1}{5}(x - 20)(x - 650) \qquad x \geq 0$$

where x is the number of desks produced and $P(x)$ is the profit in dollars.

a. Is this function constant, linear, or quadratic?

b. Find $P(0)$ and interpret the result in the context of this problem.

c. Find the values of x where $P(x) = 0$. Interpret the results in the context of this problem.

25. Given $h(x) = \sqrt{50 - x}$, find the domain of h.

26. Simplify completely. $\dfrac{x^{-1} - y^{-1}}{y^{-2} - x^{-2}}$

27. Divide. $\dfrac{a^3 + 64}{16 - a^2} \div \dfrac{a^3 - 4a^2 + 16a}{a^2 - 3a - 4}$

28. Perform the indicated operations.

$$\frac{1}{x^2 - 7x + 10} + \frac{1}{x^2 + 8x - 20}$$

29. Solve the system of equations using Cramer's rule.

$$x + 5y = 6$$
$$-2x + y = -12$$

30. Factor the expression.

$$3x^3 - 6x^2 - 12x + 24$$