Quadratic Equations and Functions

8

8.1 Square Root Property and Completing the Square
8.2 Quadratic Formula
8.3 Equations in Quadratic Form
8.4 Graphs of Quadratic Functions
8.5 Vertex of a Parabola and Applications

In Chapter 8 we revisit quadratic equations. In earlier chapters, we solved quadratic equations by factoring and applying the zero product rule. In this chapter, we present two additional techniques to solve quadratic equations. However, these techniques can be used to solve quadratic equations even if the equations are not factorable.

Complete the word scramble to familiarize yourself with the key terms for this chapter. As a hint, there is a clue for each word.

1. r a e q s u __ __ __ __ __ __ (________ root property)
2. e i t g c o m p l n __ __ __ __ __ __ __ __ __ __ (________ the square)
3. d q a t r i u a c __ __ __ __ __ __ __ __ __ (________ formula)
4. t a i i c i d s r m n n __ __ __ __ __ __ __ __ __ __ __ __ ($b^2 - 4ac$)
5. a o b r p l a a __ __ __ __ __ __ __ __ (shape of a quadratic function)
6. x t e e r v __ __ __ __ __ __ (maximum or minimum point on the graph of a parabola)
7. s x i a __ __ __ __ (________ of symmetry)

Section 8.1 Square Root Property and Completing the Square

Concepts

1. Solving Quadratic Equations by Using the Square Root Property
2. Solving Quadratic Equations by Completing the Square
3. Literal Equations

1. Solving Quadratic Equations by Using the Square Root Property

In Section 5.8 we learned how to solve a quadratic equation by factoring and applying the zero product rule. For example,

$$x^2 = 81$$

$$x^2 - 81 = 0 \qquad \text{Set one side equal to zero.}$$

$$(x - 9)(x + 9) = 0 \qquad \text{Factor.}$$

$$x - 9 = 0 \quad \text{or} \quad x + 9 = 0 \qquad \text{Set each factor equal to zero.}$$

$$x = 9 \quad \text{or} \quad x = -9$$

The solutions are $x = 9$ and $x = -9$.

It is important to note that the zero product rule can only be used if the equation is factorable. In this section and Section 8.2, we will learn how to solve quadratic equations, factorable and nonfactorable.

The first technique utilizes the **square root property**.

The Square Root Property

For any real number, k, if $x^2 = k$, then $x = \sqrt{k}$ or $x = -\sqrt{k}$.

Note: The solution may also be written as $x = \pm\sqrt{k}$, read x equals "plus or minus the square root of k."

Example 1 Solving Quadratic Equations by Using the Square Root Property

Use the square root property to solve the equations.

a. $x^2 = 81$ **b.** $3x^2 + 75 = 0$ **c.** $(w + 3)^2 = 20$

Solution:

a. $x^2 = 81$ The equation is in the form $x^2 = k$.

$x = \pm\sqrt{81}$ Apply the square root property.

$x = \pm 9$

The solutions are $x = 9$ and $x = -9$. Notice that this is the same solution obtained by factoring and applying the zero product rule.

Avoiding Mistakes:

A common mistake is to forget the $\pm$ symbol when solving the equation $x^2 = k$:

$$x = \pm\sqrt{k}$$

b. $3x^2 + 75 = 0$ Rewrite the equation to fit the form $x^2 = k$.

$3x^2 = -75$

$x^2 = -25$ The equation is now in the form $x^2 = k$.

$x = \pm\sqrt{-25}$ Apply the square root property.

$= \pm 5i$

The solutions are $x = 5i$ and $x = -5i$.

Check: $x = 5i$	Check: $x = -5i$
$3x^2 + 75 = 0$	$3x^2 + 75 = 0$
$3(5i)^2 + 75 \stackrel{?}{=} 0$	$3(-5i)^2 + 75 \stackrel{?}{=} 0$
$3(25i^2) + 75 \stackrel{?}{=} 0$	$3(25i^2) + 75 \stackrel{?}{=} 0$
$3(-25) + 75 \stackrel{?}{=} 0$	$3(-25) + 75 \stackrel{?}{=} 0$
$-75 + 75 = 0$ ✔	$-75 + 75 = 0$ ✔

c. $(w + 3)^2 = 20$ The equation is in the form $x^2 = k$, where $x = (w + 3)$.

$w + 3 = \pm\sqrt{20}$ Apply the square root property.

$w + 3 = \pm\sqrt{2^2 \cdot 5}$ Simplify the radical.

$w + 3 = \pm 2\sqrt{5}$

$w = -3 \pm 2\sqrt{5}$ Solve for w.

The solutions are $w = -3 + 2\sqrt{5}$ and $w = -3 - 2\sqrt{5}$.

Skill Practice Use the square root property to solve the equations.

1. $a^2 = 100$ **2.** $8x^2 + 72 = 0$ **3.** $(t - 5)^2 = 18$

2. Solving Quadratic Equations by Completing the Square

In Example 1(c) we used the square root property to solve an equation where the square of a binomial was equal to a constant.

$$\underbrace{(w + 3)^2}_{\text{square of a binomial}} = \underset{\text{constant}}{\underset{\uparrow}{20}}$$

The square of a binomial is the factored form of a perfect square trinomial. For example:

Perfect Square Trinomial		Factored Form
$x^2 + 10x + 25$	$\longrightarrow$	$(x + 5)^2$
$t^2 - 6t + 9$	$\longrightarrow$	$(t - 3)^2$
$p^2 - 14p + 49$	$\longrightarrow$	$(p - 7)^2$

For a perfect square trinomial with a leading coefficient of 1, the constant term is the square of one-half the linear term coefficient. For example:

$$x^2 + 10x + 25$$
$$[\tfrac{1}{2}(10)]^2$$

In general an expression of the form $x^2 + bx + n$ is a perfect square trinomial if $n = (\frac{1}{2}b)^2$. The process to create a perfect square trinomial is called **completing the square**.

Skill Practice Answers

1. $a = \pm 10$ **2.** $x = \pm 3i$
3. $t = 5 \pm 3\sqrt{2}$

Example 2 Completing the Square

Determine the value of n that makes the polynomial a perfect square trinomial. Then factor the expression as the square of a binomial.

a. $x^2 + 12x + n$ **b.** $x^2 - 26x + n$

c. $x^2 + 11x + n$ **d.** $x^2 - \frac{4}{7}x + n$

Solution:

The expressions are in the form $x^2 + bx + n$. The value of n equals the square of one-half the linear term coefficient $(\frac{1}{2}b)^2$.

a. $x^2 + 12x + n$

$x^2 + 12x + 36$ $\quad n = [\frac{1}{2}(12)]^2 = (6)^2 = 36.$

$(x + 6)^2$ $\quad$ Factored form

b. $x^2 - 26x + n$

$x^2 - 26x + 169$ $\quad n = [\frac{1}{2}(-26)]^2 = (-13)^2 = 169.$

$(x - 13)^2$ $\quad$ Factored form

c. $x^2 + 11x + n$

$x^2 + 11x + \frac{121}{4}$ $\quad n = [\frac{1}{2}(11)]^2 = (\frac{11}{2})^2 = \frac{121}{4}.$

$\left(x + \frac{11}{2}\right)^2$ $\quad$ Factored form

d. $x^2 - \frac{4}{7}x + n$

$x^2 - \frac{4}{7}x + \frac{4}{49}$ $\quad n = [\frac{1}{2}(-\frac{4}{7})]^2 = (-\frac{2}{7})^2 = \frac{4}{49}$

$\left(x - \frac{2}{7}\right)^2$ $\quad$ Factored form

Skill Practice Determine the value of n that makes the polynomial a perfect square trinomial. Then factor.

4. $x^2 + 20x + n$ **5.** $y^2 - 16y + n$

6. $a^2 - 5a + n$ **7.** $w^2 + \frac{7}{3}w + n$

The process of completing the square can be used to write a quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ in the form $(x - h)^2 = k$. Then the square root property can be used to solve the equation. The following steps outline the procedure.

Skill Practice Answers

4. $n = 100$; $(x + 10)^2$
5. $n = 64$; $(y - 8)^2$
6. $n = \frac{25}{4}$; $\left(a - \frac{5}{2}\right)^2$
7. $n = \frac{49}{36}$; $\left(w + \frac{7}{6}\right)^2$

Solving a Quadratic Equation in the Form $ax^2 + bx + c = 0 \quad (a \neq 0)$ by Completing the Square and Applying the Square Root Property

1. Divide both sides by a to make the leading coefficient 1.
2. Isolate the variable terms on one side of the equation.
3. Complete the square. (Add the square of one-half the linear term coefficient to both sides of the equation. Then factor the resulting perfect square trinomial.)
4. Apply the square root property and solve for x.

Example 3 Solving Quadratic Equations by Completing the Square and Applying the Square Root Property

Solve the quadratic equations by completing the square and applying the square root property.

a. $x^2 - 6x + 13 = 0$ **b.** $2x(2x - 10) = -30 + 6x$

Solution:

a. $x^2 - 6x + 13 = 0$

$x^2 - 6x + 13 = 0$ **Step 1:** Since the leading coefficient a is equal to 1, we do not have to divide by a. We can proceed to step 2.

$x^2 - 6x = -13$ **Step 2:** Isolate the variable terms on one side.

$x^2 - 6x + 9 = -13 + 9$ **Step 3:** To complete the square, add $[\frac{1}{2}(-6)]^2 = 9$ to both sides of the equation.

$(x - 3)^2 = -4$ Factor the perfect square trinomial.

$x - 3 = \pm\sqrt{-4}$ **Step 4:** Apply the square root property.

$x - 3 = \pm 2i$ Simplify the radical.

$x = 3 \pm 2i$ Solve for x.

The solutions are imaginary numbers and can be written as $x = 3 + 2i$ and $x = 3 - 2i$.

b. $2x(2x - 10) = -30 + 6x$

$4x^2 - 20x = -30 + 6x$ Clear parentheses.

$4x^2 - 26x + 30 = 0$ Write the equation in the form $ax^2 + bx + c = 0$.

$\frac{4x^2}{4} - \frac{26x}{4} + \frac{30}{4} = \frac{0}{4}$ **Step 1:** Divide both sides by the leading coefficient 4.

$x^2 - \frac{13}{2}x + \frac{15}{2} = 0$

$$x^2 - \frac{13}{2}x = -\frac{15}{2}$$

Step 2: Isolate the variable terms on one side.

$$x^2 - \frac{13}{2}x + \frac{169}{16} = -\frac{15}{2} + \frac{169}{16}$$

Step 3: Add $[\frac{1}{2}(-\frac{13}{2})]^2 = (-\frac{13}{4})^2 = \frac{169}{16}$ to both sides.

$$\left(x - \frac{13}{4}\right)^2 = -\frac{120}{16} + \frac{169}{16}$$

Factor the perfect square trinomial. Rewrite the right-hand side with a common denominator.

$$\left(x - \frac{13}{4}\right)^2 = \frac{49}{16}$$

$$x - \frac{13}{4} = \pm\sqrt{\frac{49}{16}}$$

Step 4: Apply the square root property.

$$x - \frac{13}{4} = \pm\frac{7}{4}$$

Simplify the radical.

$$x = \frac{13}{4} \pm \frac{7}{4}$$

$$x = \frac{13}{4} + \frac{7}{4} = \frac{20}{4} = 5$$

$$x = \frac{13}{4} - \frac{7}{4} = \frac{6}{4} = \frac{3}{2}$$

The solutions are rational numbers: $x = \frac{3}{2}$ and $x = 5$. The check is left to the reader.

Skill Practice Solve by completing the square and applying the square root property.

8. $z^2 - 4z - 2 = 0$ **9.** $2y(y - 1) = 3 - y$

TIP: In general, if the solutions to a quadratic equation are rational numbers, the equation can be solved by factoring and using the zero product rule. Consider the equation from Example 3(b).

$$2x(2x - 10) = -30 + 6x$$
$$4x^2 - 20x = -30 + 6x$$
$$4x^2 - 26x + 30 = 0$$
$$2(2x^2 - 13x + 15) = 0$$
$$2(x - 5)(2x - 3) = 0$$
$$x = 5 \quad \text{or} \quad x = \frac{3}{2}$$

Skill Practice Answers

8. $z = 2 \pm \sqrt{6}$

9. $y = \frac{3}{2}$ and $y = -1$

3. Literal Equations

Example 4 Solving a Literal Equation

Ignoring air resistance, the distance d (in meters) that an object falls in t sec is given by the equation

$$d = 4.9t^2 \qquad \text{where } t \geq 0$$

a. Solve the equation for t. Do not rationalize the denominator.

b. Using the equation from part (a), determine the amount of time required for an object to fall 500 m. Round to the nearest second.

Solution:

a. $d = 4.9t^2$

$\frac{d}{4.9} = t^2$ Isolate the quadratic term. The equation is in the form $t^2 = k$.

$t = \pm\sqrt{\frac{d}{4.9}}$ Apply the square root property.

$= \sqrt{\frac{d}{4.9}}$ Because $t \geq 0$, reject the negative solution.

b. $t = \sqrt{\frac{d}{4.9}}$

$= \sqrt{\frac{500}{4.9}}$ Substitute $d = 500$.

$t \approx 10.1$

The object will require approximately 10.1 sec to fall 500 m.

Skill Practice

10. The formula for the area of a circle is $A = \pi r^2$, where r is the radius.

a. Solve for r. (Do not rationalize the denominator.)

b. Use the equation from part (a) to find the radius when the area is 15.7 cm^2. (Use 3.14 for π and round to 2 decimal places.)

Skill Practice Answers

10a. $r = \sqrt{\frac{A}{\pi}}$

b. The radius is $\sqrt{5} \approx 2.24$ cm.

Section 8.1 Practice Exercises

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Study Skills Exercise

1. Define the key terms.

a. Square root property **b. Completing the square**

Concept 1: Solving Quadratic Equations by Using the Square Root Property

For Exercises 2–17, solve the equations by using the square root property.

2. $x^2 = 100$ **3.** $y^2 = 4$ **4.** $a^2 = 5$ **5.** $k^2 - 7 = 0$

6. $3v^2 + 33 = 0$ **7.** $-2m^2 = 50$ **8.** $(p - 5)^2 = 9$ **9.** $(q + 3)^2 = 4$

10. $(3x - 2)^2 - 5 = 0$ **11.** $(2y + 3)^2 - 7 = 0$ **12.** $(h - 4)^2 = -8$ **13.** $(t + 5)^2 = -18$

14. $6p^2 - 3 = 2$ **15.** $15 = 4 + 3w^2$ **16.** $\left(x - \frac{3}{2}\right)^2 + \frac{7}{4} = 0$ **17.** $\left(m + \frac{4}{5}\right)^2 - \frac{3}{25} = 0$

18. Given the equation $x^2 = k$, match the following statements.

a. If $k > 0$, then ______ **i.** there will be one real solution.

b. If $k < 0$, then ______ **ii.** there will be two real solutions.

c. If $k = 0$, then ______ **iii.** there will be two imaginary solutions.

19. State two methods that can be used to solve the equation $x^2 - 36 = 0$. Then solve the equation by using both methods.

20. State two methods that can be used to solve the equation $x^2 - 9 = 0$. Then solve the equation by using both methods.

Concept 2: Solving Quadratic Equations by Completing the Square

For Exercises 21–30, find the value of n so that the expression is a perfect square trinomial. Then factor the trinomial.

21. $x^2 - 6x + n$ **22.** $x^2 + 12x + n$ **23.** $t^2 + 8t + n$ **24.** $v^2 - 18v + n$

25. $c^2 - c + n$ **26.** $x^2 + 9x + n$ **27.** $y^2 + 5y + n$ **28.** $a^2 - 7a + n$

29. $b^2 + \frac{2}{5}b + n$ **30.** $m^2 - \frac{2}{7}m + n$

31. Summarize the steps used in solving a quadratic equation by completing the square and applying the square root property.

32. What types of quadratic equations can be solved by completing the square and applying the square root property?

For Exercises 33–52, solve the quadratic equation by completing the square and applying the square root property.

33. $t^2 + 8t + 15 = 0$ **34.** $m^2 + 6m + 8 = 0$ **35.** $x^2 + 6x = -16$ **36.** $x^2 - 4x = 3$

37. $p^2 + 4p + 6 = 0$ **38.** $q^2 + 2q + 2 = 0$ **39.** $y^2 - 3y - 10 = 0$ **40.** $-24 = -2y^2 + 2y$

41. $2a^2 + 4a + 5 = 0$ **42.** $3a^2 + 6a - 7 = 0$ **43.** $9x^2 - 36x + 40 = 0$ **44.** $9y^2 - 12y + 5 = 0$

45. $p^2 - \frac{2}{5}p = \frac{2}{25}$ **46.** $n^2 - \frac{2}{3}n = \frac{1}{9}$ **47.** $(2w + 5)(w - 1) = 2$ **48.** $(3p - 5)(p + 1) = -3$

49. $n(n - 4) = 7$ **50.** $m(m + 10) = 2$ **51.** $2x(x + 6) = 14$ **52.** $3x(x - 2) = 24$

Concept 3: Literal Equations

53. The distance (in feet) that an object falls in t sec is given by the equation $d = 16t^2$, where $t \geq 0$.

a. Solve the equation for t.

b. Using the equation from part (a), determine the amount of time required for an object to fall 1024 ft.

54. The volume of a can that is 4 in. tall is given by the equation $V = 4\pi r^2$, where r is the radius of the can, measured in inches.

a. Solve the equation for r. Do not rationalize the denominator.

b. Using the equation from part (a), determine the radius of a can with volume of 12.56 in.^{3} Use 3.14 for π.

For Exercises 55–60, solve for the indicated variable.

55. $A = \pi r^2$ for r $(r > 0)$

56. $E = mc^2$ for c $(c > 0)$

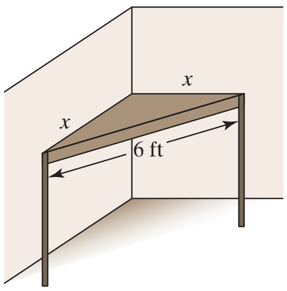
57. $a^2 + b^2 + c^2 = d^2$ for a $(a > 0)$

58. $a^2 + b^2 = c^2$ for b $(b > 0)$

59. $V = \frac{1}{3}\pi r^2 h$ for r $(r > 0)$

60. $V = \frac{1}{3}s^2 h$ for s

61. A corner shelf is to be made from a triangular piece of plywood, as shown in the diagram. Find the distance x that the shelf will extend along the walls. Assume that the walls are at right angles. Round the answer to a tenth of a foot.



62. A square has an area of 50 in.^{2} What are the lengths of the sides? (Round to 1 decimal place.)

63. The amount of money A in an account with an interest rate r compounded annually is given by

$$A = P(1 + r)^t$$

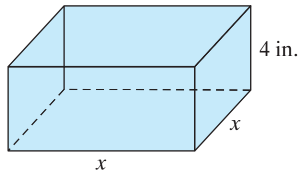
where P is the initial principal and t is the number of years the money is invested.

a. If a \$10,000 investment grows to \$11,664 after 2 years, find the interest rate.

b. If a \$6000 investment grows to \$7392.60 after 2 years, find the interest rate.

c. Jamal wants to invest \$5000. He wants the money to grow to at least \$6500 in 2 years to cover the cost of his son's first year at college. What interest rate does Jamal need for his investment to grow to \$6500 in 2 years? Round to the nearest hundredth of a percent.

64. The volume of a box with a square bottom and a height of 4 in. is given by $V(x) = 4x^2$, where x is the length (in inches) of the sides of the bottom of the box.



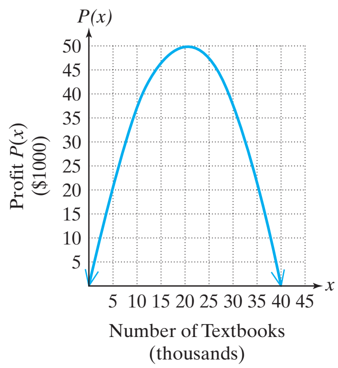
a. If the volume of the box is 289 in.^{3}, find the dimensions of the box.

b. Are there two possible answers to part (a)? Why or why not?

65. A textbook company has discovered that the profit for selling its books is given by

$$P(x) = -\frac{1}{8}x^2 + 5x$$

where x is the number of textbooks produced (in thousands) and $P(x)$ is the corresponding profit (in thousands of dollars). The graph of the function is shown at right.



a. Approximate the number of books required to make a profit of \$20,000. [*Hint:* Let $P(x) = 20$. Then complete the square to solve for x.] Round to one decimal place.

b. Why are there two answers to part (a)?

66. If we ignore air resistance, the distance (in feet) that an object travels in free fall can be approximated by $d(t) = 16t^2$, where t is the time in seconds after the object was dropped.

a. If the CN Tower in Toronto is 1815 ft high, how long will it take an object to fall from the top of the building? Round to one decimal place.

b. If the Renaissance Tower in Dallas is 886 ft high, how long will it take an object to fall from the top of the building? Round to one decimal place.

Quadratic Formula

Section 8.2

Concepts

1. Derivation of the Quadratic Formula
2. Solving Quadratic Equations by Using the Quadratic Formula
3. Using the Quadratic Formula in Applications
4. Discriminant
5. Mixed Review: Methods to Solve a Quadratic Equation

1. Derivation of the Quadratic Formula

If we solve a general quadratic equation $ax^2 + bx + c = 0\ (a \neq 0)$ by completing the square and using the square root property, the result is a formula that gives the solutions for x in terms of a, b, and c.

$$ax^2 + bx + c = 0$$
Begin with a quadratic equation in standard form.

$$\frac{ax^2}{a} + \frac{bx}{a} + \frac{c}{a} = \frac{0}{a}$$
Divide by the leading coefficient.

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$
Isolate the terms containing x.

$$x^2 + \frac{b}{a}x + \left(\frac{1}{2}\cdot\frac{b}{a}\right)^2 = \left(\frac{1}{2}\cdot\frac{b}{a}\right)^2 - \frac{c}{a}$$
Add the square of $\frac{1}{2}$ the linear term coefficient to both sides of the equation.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$
Factor the left side as a perfect square.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$
Combine fractions on the right side by getting a common denominator.

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$
Apply the square root property.

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$
Simplify the denominator.

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$
Subtract $\frac{b}{2a}$ from both sides.

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
Combine fractions.

The solution to the equation $ax^2 + bx + c = 0$ for x in terms of the coefficients a, b, and c is given by the **quadratic formula**.

The Quadratic Formula

For any quadratic equation of the form $ax^2 + bx + c = 0\ (a \neq 0)$ the solutions are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The quadratic formula gives us another technique to solve a quadratic equation. This method will work regardless of whether the equation is factorable or not factorable.

2. Solving Quadratic Equations by Using the Quadratic Formula

Example 1 Solving a Quadratic Equation by Using the Quadratic Formula

Solve the quadratic equation by using the quadratic formula.

$$2x^2 - 3x = 5$$

Solution:

$2x^2 - 3x = 5$

$2x^2 - 3x - 5 = 0$ Write the equation in the form $ax^2 + bx + c = 0$.

$a = 2, \quad b = -3, \quad c = -5$ Identify a, b, and c.

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Apply the quadratic formula.

$x = \dfrac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-5)}}{2(2)}$ Substitute $a = 2$, $b = -3$, and $c = -5$.

$= \dfrac{3 \pm \sqrt{9 + 40}}{4}$ Simplify.

$= \dfrac{3 \pm \sqrt{49}}{4}$

$= \dfrac{3 \pm 7}{4}$

$x = \dfrac{3 + 7}{4} = \dfrac{10}{4} = \dfrac{5}{2}$

$x = \dfrac{3 - 7}{4} = \dfrac{-4}{4} = -1$

There are two rational solutions, $x = \frac{5}{2}$ and $x = -1$. Both solutions check in the original equation.

TIP: Always remember to write the *entire* numerator over $2a$.

Skill Practice Solve the equation by using the quadratic formula.

1. $6x^2 - 5x = 4$

Example 2 Solving a Quadratic Equation by Using the Quadratic Formula

Solve the quadratic equation by using the quadratic formula.

$$-x(x - 6) = 11$$

Solution:

$-x(x - 6) = 11$

$-x^2 + 6x - 11 = 0$ Write the equation in the form $ax^2 + bx + c = 0$.

Skill Practice Answers

1. $x = -\frac{1}{2}; x = \frac{4}{3}$

$-1(-x^2 + 6x - 11) = -1(0)$

$x^2 - 6x + 11 = 0$

If the leading coefficient of the quadratic polynomial is negative, we suggest multiplying both sides of the equation by -1. Although this is not mandatory, it is generally easier to simplify the quadratic formula when the value of a is positive.

$a = 1, b = -6$, and $c = 11$ — Identify a, b, and c.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Apply the quadratic formula.

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(11)}}{2(1)}$$ Substitute $a = 1$, $b = -6$, and $c = 11$.

$$= \frac{6 \pm \sqrt{36 - 44}}{2}$$ Simplify.

$$= \frac{6 \pm \sqrt{-8}}{2}$$

$$= \frac{6 \pm 2i\sqrt{2}}{2}$$ Simplify the radical.

$$= \frac{2(3 \pm i\sqrt{2})}{2}$$ Factor the numerator.

$$= \frac{\cancel{2}(3 \pm i\sqrt{2})}{\cancel{2}}$$ Simplify the fraction to lowest terms.

$$= 3 \pm i\sqrt{2} \quad \begin{cases} x = 3 + i\sqrt{2} \\ x = 3 - i\sqrt{2} \end{cases}$$

There are two imaginary solutions, $x = 3 + i\sqrt{2}$ and $x = 3 - i\sqrt{2}$.

Skill Practice Solve the equation by using the quadratic formula.

2. $y(y + 3) = 2$

3. Using the Quadratic Formula in Applications

Example 3 Using the Quadratic Formula in an Application

A delivery truck travels south from Hartselle, Alabama, to Birmingham, Alabama, along Interstate 65. The truck then heads east to Atlanta, Georgia, along Interstate 20. The distance from Birmingham to Atlanta is 8 mi less than twice the distance from Hartselle to Birmingham. If the straight-line distance from Hartselle to Atlanta is 165 mi, find the distance from Hartselle to Birmingham and from Birmingham to Atlanta. (Round the answers to the nearest mile.)

Skill Practice Answers

2. $y = \dfrac{-3 \pm \sqrt{17}}{2}$

Solution:

The motorist travels due south and then due east. Therefore, the three cities form the vertices of a right triangle (Figure 8-1).

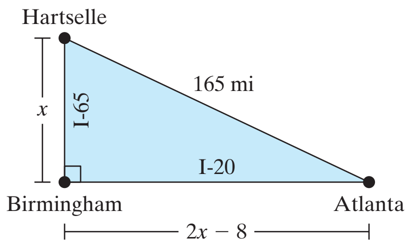


Figure 8-1

Let x represent the distance between Hartselle and Birmingham.

Then $2x - 8$ represents the distance between Birmingham and Atlanta.

Use the Pythagorean theorem to establish a relationship among the three sides of the triangle.

$$(x)^2 + (2x - 8)^2 = (165)^2$$

$$x^2 + 4x^2 - 32x + 64 = 27{,}225$$

$$5x^2 - 32x - 27{,}161 = 0$$ Write the equation in the form $ax^2 + bx + c = 0$.

$a = 5 \qquad b = -32 \qquad c = -27{,}161$ Identify a, b, and c.

$$x = \frac{-(-32) \pm \sqrt{(-32)^2 - 4(5)(-27{,}161)}}{2(5)}$$ Apply the quadratic formula.

$$x = \frac{32 \pm \sqrt{1024 + 543{,}220}}{10}$$ Simplify.

$$x = \frac{32 \pm \sqrt{544{,}244}}{10}$$

$$x = \frac{32 + \sqrt{544{,}244}}{10} \approx 76.97 \text{ mi} \qquad \text{or}$$

$$x = \frac{32 - \sqrt{544{,}244}}{10} \approx -70.57 \text{ mi}$$ We reject the negative solution because distance cannot be negative.

Recall that x represents the distance from Hartselle to Birmingham; therefore, to the nearest mile, the distance between Hartselle and Birmingham is 77 mi.

The distance between Birmingham and Atlanta is $2x - 8 = 2(77) - 8 = 146$ mi.

Skill Practice

3. Steve and Tammy leave a campground, hiking on two different trails. Steve heads south and Tammy heads east. By lunchtime they are 9 mi apart. Steve walked 3 mi more than twice as many miles as Tammy. Find the distance each person hiked. (Round to the nearest tenth of a mile.)

Skill Practice Answers

3. Tammy hiked 2.8 mi, and Steve hiked 8.6 mi.

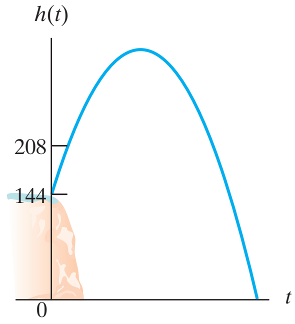


Figure 8-2

Example 4 Analyzing a Quadratic Function

A model rocket is launched straight up from the side of a 144-ft cliff (Figure 8-2). The initial velocity is 112 ft/sec. The height of the rocket $h(t)$ is given by

$$h(t) = -16t^2 + 112t + 144$$

where $h(t)$ is measured in feet and t is the time in seconds. Find the time(s) at which the rocket is 208 ft above the ground.

Solution:

$$h(t) = -16t^2 + 112t + 144$$

$$208 = -16t^2 + 112t + 144$$ Substitute 208 for $h(t)$.

$$16t^2 - 112t + 64 = 0$$ Write the equation in the form $at^2 + bt + c = 0$.

$$\frac{16t^2}{16} - \frac{112t}{16} + \frac{64}{16} = \frac{0}{16}$$ Divide by 16. This makes the coefficients smaller, and it is less cumbersome to apply the quadratic formula.

$$t^2 - 7t + 4 = 0$$ The equation is not factorable. Apply the quadratic formula.

$$t = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(4)}}{2(1)}$$ Let $a = 1$, $b = -7$, and $c = 4$.

$$= \frac{7 \pm \sqrt{33}}{2}$$

$$t = \frac{7 + \sqrt{33}}{2} \approx 6.37$$

$$t = \frac{7 - \sqrt{33}}{2} \approx 0.63$$

The rocket will reach a height of 208 ft after approximately 0.63 sec (on the way up) and after 6.37 sec (on the way down).

Skill Practice

4. An object is launched from the top of a 96-ft building with an initial velocity of 64 ft/sec. The height $h(t)$ of the rocket is given by

$$h(t) = -16t^2 + 64t + 96$$

Find the time it takes for the object to hit the ground. (*Hint:* $h(t) = 0$ when the object hits the ground.)

4. Discriminant

The radicand within the quadratic formula is the expression $b^2 - 4ac$. This is called the **discriminant**. The discriminant can be used to determine the number of solutions to a quadratic equation as well as whether the solutions are rational, irrational, or imaginary numbers.

Skill Practice Answers

4. $2 + \sqrt{10} \approx 5.16$ sec

Using the Discriminant to Determine the Number and Type of Solutions to a Quadratic Equation

Consider the equation $ax^2 + bx + c = 0$, where a, b, and c are rational numbers and $a \neq 0$. The expression $b^2 - 4ac$ is called the *discriminant.* Furthermore,

1. If $b^2 - 4ac > 0$, then there will be two real solutions.
   a. If $b^2 - 4ac$ is a perfect square, the solutions will be rational numbers.
   b. If $b^2 - 4ac$ is not a perfect square, the solutions will be irrational numbers.
2. If $b^2 - 4ac < 0$, then there will be two imaginary solutions.
3. If $b^2 - 4ac = 0$, then there will be one rational solution.

Example 5 Using the Discriminant

Use the discriminant to determine the type and number of solutions for each equation.

a. $2x^2 - 5x + 9 = 0$ **b.** $3x^2 = -x + 2$

c. $-2x(2x - 3) = -1$ **d.** $3.6x^2 = -1.2x - 0.1$

Solution:

For each equation, first write the equation in standard form $ax^2 + bx + c = 0$. Then determine the discriminant.

Equation	Discriminant	Solution Type and Number
a. $2x^2 - 5x + 9 = 0$	$b^2 - 4ac$ $= (-5)^2 - 4(2)(9)$ $= 25 - 72$ $= -47$	Because $-47 < 0$, there will be two imaginary solutions.
b. $3x^2 = -x + 2$ $3x^2 + x - 2 = 0$	$b^2 - 4ac$ $= (1)^2 - 4(3)(-2)$ $= 1 - (-24)$ $= 25$	Because $25 > 0$ and 25 is a perfect square, there will be two rational solutions.
c. $-2x(2x - 3) = -1$ $-4x^2 + 6x = -1$ $-4x^2 + 6x + 1 = 0$	$b^2 - 4ac$ $= (6)^2 - 4(-4)(1)$ $= 36 - (-16)$ $= 52$	Because $52 > 0$, but 52 is *not* a perfect square, there will be two irrational solutions.

d. $3.6x^2 = -1.2x - 0.1$

$3.6x^2 + 1.2x + 0.1 = 0$

$$\begin{aligned} &b^2 - 4ac \\ &= (1.2)^2 - 4(3.6)(0.1) \\ &= 1.44 - 1.44 \\ &= 0 \end{aligned}$$

Because the discriminant equals 0, there will be only one rational solution.

Skill Practice Determine the discriminant. Use the discriminant to determine the type and number of solutions for the equation.

5. $3y^2 + y + 3 = 0$

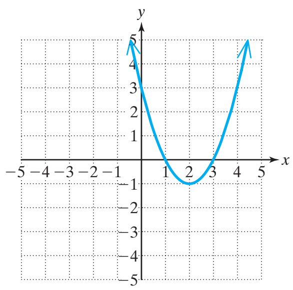
6. $4t^2 = 6t - 2$

7. $3t(t + 1) = 9$

8. $\frac{2}{3}x^2 - \frac{2}{3}x + \frac{1}{6} = 0$

With the discriminant we can determine the number of real-valued solutions to the equation $ax^2 + bx + c = 0$, and thus the number of x-intercepts to the function $f(x) = ax^2 + bx + c$. The following illustrations show the graphical interpretation of the three cases of the discriminant.

$f(x) = x^2 - 4x + 3$

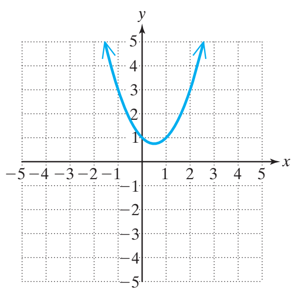


Use $x^2 - 4x + 3 = 0$ to find the value of the discriminant.

$$\begin{aligned} b^2 - 4ac &= (-4)^2 - 4(1)(3) \\ &= 4 \end{aligned}$$

Since the discriminant is positive, there are two real solutions to the quadratic equation. Therefore, there are two x-intercepts to the corresponding quadratic function, (1, 0) and (3, 0).

$f(x) = x^2 - x + 1$

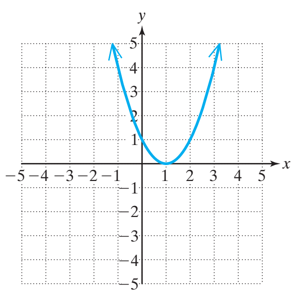


Use $x^2 - x + 1 = 0$ to find the value of the discriminant.

$$\begin{aligned} b^2 - 4ac &= (-1)^2 - 4(1)(1) \\ &= -3 \end{aligned}$$

Since the discriminant is negative, there are no real solutions to the quadratic equation. Therefore, there are no x-intercepts to the corresponding quadratic function.

$f(x) = x^2 - 2x + 1$



Use $x^2 - 2x + 1 = 0$ to find the value of the discriminant.

$$\begin{aligned} b^2 - 4ac &= (-2)^2 - 4(1)(1) \\ &= 0 \end{aligned}$$

Since the discriminant is zero, there is one real solution to the quadratic equation. Therefore, there is one x-intercept to the corresponding quadratic function, (1, 0).

Skill Practice Answers

5. -35; two imaginary solutions
6. 4; two rational solutions
7. 117; two irrational solutions
8. 0; one rational solution

Example 6 Finding the x- and y-Intercepts of a Quadratic Function

Given: $f(x) = \frac{1}{4}x^2 + \frac{1}{4}x + \frac{1}{2}$

a. Find the y-intercept. **b.** Find the x-intercept(s).

Solution:

a. The y-intercept is given by $f(0) = \frac{1}{4}(0)^2 + \frac{1}{4}(0) + \frac{1}{2}$

$$= \frac{1}{2}$$

The y-intercept is located at $(0, \frac{1}{2})$.

b. The x-intercepts are given by the real solutions to the equation $f(x) = 0$. In this case, we have

$$f(x) = \frac{1}{4}x^2 + \frac{1}{4}x + \frac{1}{2} = 0$$

$$4\left(\frac{1}{4}x^2 + \frac{1}{4}x + \frac{1}{2}\right) = 4(0)$$ Multiply by 4 to clear fractions.

$$x^2 + x + 2 = 0$$ The equation is in the form $ax^2 + bx + c = 0$, where $a = 1$, $b = 1$, and $c = 2$.

$$x = \frac{-(1) \pm \sqrt{(1)^2 - 4(1)(2)}}{2(1)}$$ Apply the quadratic formula.

$$= \frac{-1 \pm \sqrt{-7}}{2}$$ Simplify.

$$= -\frac{1}{2} \pm \frac{\sqrt{7}}{2}i$$

These solutions are *imaginary numbers*. Because there are no real solutions to the equation $f(x) = 0$, the function has no x-intercepts. The function is graphed in Figure 8-3.

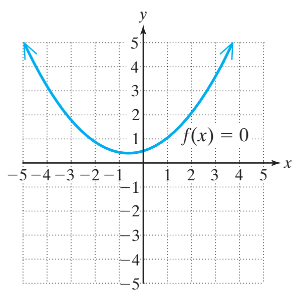


Figure 8-3

Skill Practice

9. Find the x- and y-intercepts of the function given by

$$f(x) = \frac{1}{2}x^2 - x - \frac{5}{2}$$

Skill Practice Answers

9. x-intercepts: $(1 + \sqrt{6}, 0) \approx (3.45, 0)$, $(1 - \sqrt{6}, 0) \approx (-1.45, 0)$; y-intercept: $\left(0, -\frac{5}{2}\right)$

5. Mixed Review: Methods to Solve a Quadratic Equation

Three methods have been presented to solve quadratic equations.

Methods to Solve a Quadratic Equation

- Factor and use the zero product rule (Section 5.8).
- Use the square root property. Complete the square, if necessary (Section 8.1).
- Use the quadratic formula (Section 8.2).

Using the zero product rule is often the simplest method, but it works only if you can factor the equation. The square root property and the quadratic formula can be used to solve any quadratic equation. Before solving a quadratic equation, take a minute to analyze it first. Each problem must be evaluated individually before choosing the most efficient method to find its solutions.

Example 7 Solving Quadratic Equations by Using Any Method

Solve the quadratic equations by using any method.

a. $(x+3)^2 + x^2 - 9x = 8$ **b.** $\dfrac{x^2}{2} + \dfrac{5}{2} = -x$ **c.** $(p-2)^2 - 11 = 0$

Solution:

a. $(x+3)^2 + x^2 - 9x = 8$

$x^2 + 6x + 9 + x^2 - 9x - 8 = 0$ Clear parentheses and write the equation in the form $ax^2 + bx + c = 0$.

$2x^2 - 3x + 1 = 0$ This equation is factorable.

$(2x-1)(x-1) = 0$ Factor.

$2x - 1 = 0$ or $x - 1 = 0$ Apply the zero product rule.

$x = \frac{1}{2}$ or $x = 1$ Solve for x.

This equation could have been solved by using any of the three methods, but factoring was the most efficient method.

b. $\dfrac{x^2}{2} + \dfrac{5}{2} = -x$ Clear fractions and write the equation in the form $ax^2 + bx + c = 0$.

$x^2 + 5 = -2x$

$x^2 + 2x + 5 = 0$ This equation does not factor. Use the quadratic formula.

$a = 1 \quad b = 2 \quad c = 5$ Identify a, b, and c.

$$x = \frac{-(2) \pm \sqrt{(2)^2 - 4(1)(5)}}{2(1)}$$ Apply the quadratic formula.

$$= \frac{-2 \pm \sqrt{-16}}{2}$$ Simplify.

$$= \frac{-2 \pm 4i}{2}$$ Simplify the radical.

$$= -1 \pm 2i$$ Reduce to lowest terms.

$$\frac{-2 \pm 4i}{2} = \frac{\cancel{2}(-1 \pm 2i)}{\cancel{2}}$$

This equation could also have been solved by completing the square and applying the square root property.

c. $(p - 2)^2 - 11 = 0$

$(p - 2)^2 = 11$ The equation is in the form $x^2 = k$, where $x = (p - 2)$.

$p - 2 = \pm\sqrt{11}$ Apply the square root property.

$p = 2 \pm \sqrt{11}$ Solve for p.

This problem could have been solved by the quadratic formula, but that would have involved clearing parentheses and collecting *like* terms first.

Skill Practice Solve the quadratic equations by using any method.

10. $2t(t - 1) + t^2 = 5$ **11.** $0.2x^2 + 1.1x = -0.3$ **12.** $(2y - 6)^2 = -12$

Skill Practice Answers

10. $t = -1; t = \frac{5}{3}$

11. $x = \frac{-11 \pm \sqrt{97}}{4}$

12. $y = 3 \pm i\sqrt{3}$

Section 8.2 Practice Exercises

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Study Skills Exercise

1. Define the key terms.

a. Quadratic formula **b. Discriminant**

Review Exercises

2. Show by substitution that $x = -3 + \sqrt{5}$ is a solution to $x^2 + 6x + 4 = 0$.

For Exercises 3–6, solve by completing the square and using the square root property.

3. $(x + 5)^2 = 49$ **4.** $16 = (2x - 3)^2$ **5.** $x^2 - 2x + 10 = 0$ **6.** $x^2 - 4x + 15 = 0$

For Exercises 7–10, simplify the expressions.

7. $\frac{16 - \sqrt{320}}{4}$ **8.** $\frac{18 + \sqrt{180}}{3}$ **9.** $\frac{14 - \sqrt{-147}}{7}$ **10.** $\frac{10 - \sqrt{-175}}{5}$

Concept 2: Solving Quadratic Equations by Using the Quadratic Formula

11. What form should a quadratic equation be in so that the quadratic formula can be applied?

12. Describe the circumstances in which the square root property can be used as a method for solving a quadratic equation.

13. Write the quadratic formula from memory.

For Exercises 14–39, solve the equation by using the quadratic formula.

14. $a^2 + 11a - 12 = 0$

15. $5b^2 - 14b - 3 = 0$

16. $9y^2 - 2y + 5 = 0$

17. $2t^2 + 3t - 7 = 0$

18. $12p^2 - 4p + 5 = 0$

19. $-5n^2 + 4n - 6 = 0$

20. $-z^2 = -2z - 35$

21. $12x^2 - 5x = 2$

22. $a^2 + 3a = 8$

23. $k^2 + 4 = 6k$

24. $25x^2 - 20x + 4 = 0$

25. $9y^2 = -12y - 4$

26. $w^2 - 6w + 14 = 0$

27. $2m^2 + 3m = -2$

28. $(x + 2)(x - 3) = 1$

29. $3y(y + 1) - 7y(y + 2) = 6$

30. $-4a^2 - 2a + 3 = 0$

31. $-2m^2 - 5m + 3 = 0$

32. $\frac{1}{2}y^2 + \frac{2}{3} = -\frac{2}{3}y$
(*Hint:* Clear the fractions first.)

33. $\frac{2}{3}p^2 - \frac{1}{6}p + \frac{1}{2} = 0$

34. $\frac{1}{5}h^2 + h + \frac{3}{5} = 0$

35. $\frac{1}{4}w^2 + \frac{7}{4}w + 1 = 0$

36. $0.01x^2 + 0.06x + 0.08 = 0$
(*Hint:* Clear the decimals first.)

37. $0.5y^2 - 0.7y + 0.2 = 0$

38. $0.3t^2 + 0.7t - 0.5 = 0$

39. $0.01x^2 + 0.04x - 0.07 = 0$

For Exercises 40–43, factor the expression. Then use the zero product rule and the quadratic formula to solve the equation. There should be three solutions to each equation.

40. a. Factor. $x^3 - 27$

b. Solve. $x^3 - 27 = 0$

41. a. Factor. $64x^3 + 1$

b. Solve. $64x^3 + 1 = 0$

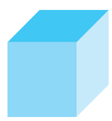
42. a. Factor. $3x^3 - 6x^2 + 6x$

b. Solve. $3x^3 - 6x^2 + 6x = 0$

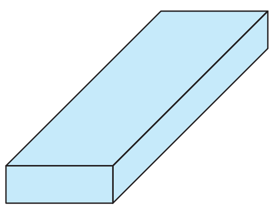
43. a. Factor. $5x^3 + 5x^2 + 10x$

b. Solve. $5x^3 + 5x^2 + 10x = 0$

44. The volume of a cube is 27 ft^3. Find the lengths of the sides.



45. The volume of a rectangular box is 64 ft^3. If the width is 3 times longer than the height, and the length is 9 times longer than the height, find the dimensions of the box.



Concept 3: Using the Quadratic Formula in Applications

46. The hypotenuse of a right triangle measures 4 in. One leg of the triangle is 2 in. longer than the other leg. Find the lengths of the legs of the triangle. Round to one decimal place.

47. The length of one leg of a right triangle is 1 cm more than twice the length of the other leg. The hypotenuse measures 6 cm. Find the lengths of the legs. Round to 1 decimal place.

48. The hypotenuse of a right triangle is 10.2 m long. One leg is 2.1 m shorter than the other leg. Find the lengths of the legs. Round to 1 decimal place.

49. The hypotenuse of a right triangle is 17 ft long. One leg is 3.4 ft longer than the other leg. Find the lengths of the legs.

50. The fatality rate (in fatalities per 100 million vehicle miles driven) can be approximated for drivers x years old according to the function, $F(x) = 0.0036x^2 - 0.35x + 9.2$. Source: U.S. Department of Transportation

a. Approximate the fatality rate for drivers 16 years old.

b. Approximate the fatality rate for drivers 40 years old.

c. Approximate the fatality rate for drivers 80 years old.

d. For what age(s) is the fatality rate approximately 2.5?

51. Mitch throws a baseball straight up in the air from a cliff that is 48 ft high. The initial velocity is 48 ft/sec. The height (in feet) of the object after t sec is given by $h(t) = -16t^2 + 48t + 48$. Find the time at which the height of the object is 64 ft.

52. An astronaut on the moon throws a rock into the air from the deck of a spacecraft that is 8 m high. The initial velocity of the rock is 2.4 m/sec. The height (in meters) of the rock after t sec is given by $h(t) = -0.8t^2 + 2.4t + 8$. Find the time at which the height of the rock is 6 m.

53. The braking distance (in feet) of a car going v mph is given by

$$d(v) = \frac{v^2}{20} + v \qquad v \geq 0$$

a. How fast would a car be traveling if its braking distance were 150 ft? Round to the nearest mile per hour.

b. How fast would a car be traveling if its braking distance were 100 ft? Round to the nearest mile per hour.

Concept 4: Discriminant

For Exercises 54–61:

a. Write the equation in the form $ax^2 + bx + c = 0, a > 0$.

b. Find the value of the discriminant.

c. Use the discriminant to determine the number and type of solutions.

54. $x^2 + 2x = -1$

55. $12y - 9 = 4y^2$

56. $19m^2 = 8m$

57. $2n - 5n^2 = 0$

58. $5p^2 - 21 = 0$

59. $3k^2 = 7$

60. $4n(n - 2) - 5n(n - 1) = 4$

61. $(2x + 1)(x - 3) = -9$

For Exercises 62–65, find the x- and y-intercepts of the quadratic function.

62. $f(x) = x^2 - 5x + 3$

63. $g(x) = -x^2 + x - 1$

64. $f(x) = 2x^2 + x + 5$

65. $h(x) = -3x^2 + 2x + 2$

Concept 5: Mixed Review: Methods to Solve a Quadratic Equation

For Exercises 66–83, solve the quadratic equations by using any method.

66. $a^2 + 3a + 4 = 0$

67. $4z^2 + 7z = 0$

68. $x^2 - 2 = 0$

69. $b^2 + 7 = 0$

70. $4y^2 + 8y - 5 = 0$

71. $k^2 - k + 8 = 0$

72. $\left(x + \frac{1}{2}\right)^2 + 4 = 0$

73. $(2y + 3)^2 = 9$

74. $2y(y - 3) = -1$

75. $w(w - 5) = 4$

76. $(2t + 5)(t - 1) = (t - 3)(t + 8)$

77. $(b - 1)(b + 4) = (3b + 2)(b + 1)$

78. $a^2 - 2a = 3$

79. $x^2 - x = 30$

80. $32z^2 - 20z - 3 = 0$

81. $8k^2 - 14k + 3 = 0$

82. $3p^2 - 27 = 0$

83. $5h^2 - 125 = 0$

Sometimes students shy away from completing the square and using the square root property to solve a quadratic equation. However, sometimes this process leads to a simple solution. For Exercises 84–85, solve the equations two ways.

a. Solve the equation by completing the square and applying the square root property.

b. Solve the equation by applying the quadratic formula.

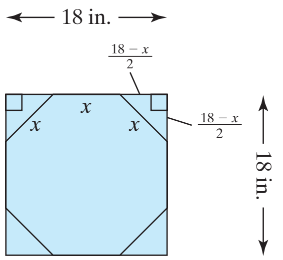
c. Which technique was easier for you?

84. $x^2 + 6x = 5$

85. $x^2 - 10x = -27$

Expanding Your Skills

86. An artist has been commissioned to make a stained glass window in the shape of a regular octagon. The octagon must fit inside an 18-in. square space. See the figure.



a. Let x represent the length of each side of the octagon. Verify that the legs of the small triangles formed by the corners of the square can be expressed as $\dfrac{18 - x}{2}$.

b. Use the Pythagorean theorem to set up an equation in terms of x that represents the relationship between the legs of the triangle and the hypotenuse.

c. Simplify the equation by clearing parentheses and clearing fractions.

d. Solve the resulting quadratic equation by using the quadratic formula. Use a calculator and round your answers to the nearest tenth of an inch.

e. There are two solutions for x. Which one is appropriate and why?

Graphing Calculator Exercises

87. Graph $Y_1 = 64x^3 + 1$. Compare the x-intercepts with the solutions to the equation $64x^3 + 1 = 0$ found in Exercise 41.

88. Graph $Y_1 = x^3 - 27$. Compare the x-intercepts with the solutions to the equation $x^3 - 27 = 0$ found in Exercise 40.

89. Graph $Y_1 = 5x^3 + 5x^2 + 10x$. Compare the x-intercepts with the solutions to the equation $5x^3 + 5x^2 + 10x = 0$ found in Exercise 43.

90. Graph $Y_1 = 3x^3 - 6x^2 + 6x$. Compare the x-intercepts with the solutions to the equation $3x^3 - 6x^2 + 6x = 0$ found in Exercise 42.

91. The recent population (in thousands) of Ecuador can be approximated by $P(t) = 1.12t^2 + 204.4t + 6697$, where $t = 0$ corresponds to the year 1974.

a. Approximate the number of people in Ecuador in the year 2000. (Round to the nearest thousand.)

b. If this trend continues, approximate the number of people in Ecuador in the year 2010. (Round to the nearest thousand.)

c. In what year after 1974 did the population of Ecuador reach 10 million? Round to the nearest year. (*Hint:* 10 million equals 10,000 thousands.)

d. Use a graphing calculator to graph the function P on the window $0 \le x \le 20$, $4000 \le y \le 12{,}000$. Use a *Trace* feature to approximate the year in which the population in Ecuador first exceeded 10 million (10,000 thousands).

Section 8.3 Equations in Quadratic Form

Concepts

1. Solving Equations by Using Substitution
2. Solving Equations Reducible to a Quadratic

1. Solving Equations by Using Substitution

We have learned to solve a variety of different types of equations, including linear, radical, rational, and polynomial equations. Sometimes, however, it is necessary to use a quadratic equation as a tool to solve other types of equations.

In Example 1, we will solve the equation $(2x^2 - 5)^2 - 16(2x^2 - 5) + 39 = 0$.

Notice that the terms in the equation are written in descending order by degree. Furthermore, the first two terms have the same base, $2x^2 - 5$, and the exponent on the first term is exactly double the exponent on the second term. The third term is a constant. An equation in this pattern is called **quadratic in form**.

exponent is double $\qquad$ 3^{rd} term is constant

$$(2x^2 - 5)^2 - 16(2x^2 - 5)^1 + 39 = 0.$$

To solve this equation we will use substitution as demonstrated in Example 1.

Example 1 Solving an Equation in Quadratic Form

Solve the equation.

$$(2x^2 - 5)^2 - 16(2x^2 - 5) + 39 = 0$$

Solution:

$(2x^2 - 5)^2 - 16(2x^2 - 5) + 39 = 0$ — If the substitution $u = (2x^2 - 5)$ is made, the equation becomes quadratic in the variable u.

Substitute $u = (2x^2 - 5)$.

$u^2 \quad - \quad 16u \quad + 39 = 0$ — The equation is in the form $au^2 + bu + c = 0$.

$(u - 13)(u - 3) = 0$ — The equation is factorable.

$u = 13 \quad \text{or} \quad u = 3$ — Apply the zero product rule.

Reverse substitute.

$2x^2 - 5 = 13 \quad \text{or} \quad 2x^2 - 5 = 3$

$2x^2 = 18 \quad \text{or} \quad 2x^2 = 8$

$x^2 = 9 \quad \text{or} \quad x^2 = 4$ — Write the equations in the form $x^2 = k$.

$x = \pm\sqrt{9} \quad \text{or} \quad x = \pm\sqrt{4}$ — Apply the square root property.

$= \pm 3 \quad \text{or} \quad = \pm 2$

The solutions are $x = 3, x = -3, x = 2, x = -2$. — Substituting these values in the original equation verifies that these are all valid solutions.

Skill Practice Solve the equation.

1. $(3t^2 - 10)^2 + 5(3t^2 - 10) - 14 = 0$

Avoiding Mistakes:

When using substitution, it is critical to reverse substitute to solve the equation in terms of the original variable.

For an equation written in descending order, notice that u was set equal to the variable factor on the middle term. This is generally the case.

Example 2 Solving an Equation in Quadratic Form

Solve the equation.

$$p^{2/3} - 2p^{1/3} = 8$$

Solution:

$p^{2/3} - 2p^{1/3} = 8$

$p^{2/3} - 2p^{1/3} - 8 = 0$ — Set the equation equal to zero.

Skill Practice Answers

1. $t = 1, t = -1, t = 2, t = -2$

$$(p^{1/3})^2 - 2(p^{1/3})^1 - 8 = 0$$

Make the substitution $u = p^{1/3}$.

Then the equation is in the form $au^2 + bu + c = 0$.

Substitute $u = p^{1/3}$.

$$u^2 - 2u - 8 = 0$$

$$(u - 4)(u + 2) = 0 \qquad \text{The equation is factorable.}$$

$$u = 4 \quad \text{or} \quad u = -2 \qquad \text{Apply the zero product rule.}$$

Reverse substitute.

$$p^{1/3} = 4 \quad \text{or} \quad p^{1/3} = -2$$

$$\sqrt[3]{p} = 4 \quad \text{or} \quad \sqrt[3]{p} = -2 \qquad \text{The equations are radical equations.}$$

$$(\sqrt[3]{p})^3 = (4)^3 \quad \text{or} \quad (\sqrt[3]{p})^3 = (-2)^3 \qquad \text{Cube both sides.}$$

$$p = 64 \quad \text{or} \quad p = -8$$

Check: $p = 64$

$$p^{2/3} - 2p^{1/3} = 8$$
$$(64)^{2/3} - 2(64)^{1/3} \stackrel{?}{=} 8$$
$$16 - 2(4) \stackrel{?}{=} 8$$
$$8 = 8 \checkmark$$

Check: $p = -8$

$$p^{2/3} - 2p^{1/3} = 8$$
$$(-8)^{2/3} - 2(-8)^{1/3} \stackrel{?}{=} 8$$
$$4 - 2(-2) \stackrel{?}{=} 8$$
$$4 + 4 = 8 \checkmark$$

The solutions are $p = 64$ and $p = -8$.

Skill Practice Solve the equation.

2. $y^{2/3} - y^{1/3} = 12$

Example 3 Solving a Quadratic Equation in Quadratic Form

Solve the equation.

$$x - \sqrt{x} - 12 = 0$$

Solution:

The equation can be solved by first isolating the radical and then squaring both sides (this is left as an exercise—see Exercise 24). However, this equation is also quadratic in form. By writing $\sqrt{x}$ as $x^{1/2}$, we see that the exponent on the first term is exactly double the exponent on the middle term.

$$x^1 - x^{1/2} - 12 = 0$$

$$u^2 - u - 12 = 0 \qquad \text{Let } u = x^{1/2}.$$

$$(u - 4)(u + 3) = 0 \qquad \text{Factor.}$$

$$u = 4 \quad \text{or} \quad u = -3 \qquad \text{Solve for } u.$$

$$x^{1/2} = 4 \quad \text{or} \quad x^{1/2} = -3 \qquad \text{Back substitute.}$$

$$\sqrt{x} = 4 \quad \text{or} \quad \cancel{\sqrt{x} = -3} \qquad \text{Solve each equation for } x. \text{ Notice that the principal square root of a number cannot equal } -3.$$

$$x = 16$$

The solution $x = 16$

Skill Practice Answers

2. $y = 64, y = -27$

Skill Practice

3. $z - \sqrt{z} - 2 = 0$

2. Solving Equations Reducible to a Quadratic

Some equations are reducible to a quadratic equation. In Example 4, we solve a polynomial equation by factoring. The resulting factors are quadratic.

Example 4 Solving a Polynomial Equation

Solve the equation.

$$4x^4 + 7x^2 - 2 = 0$$

Solution:

$4x^4 + 7x^2 - 2 = 0$	This is a polynomial equation.
$(4x^2 - 1)(x^2 + 2) = 0$	Factor.
$4x^2 - 1 = 0$ or $x^2 + 2 = 0$	Set each factor equal to zero. Notice that the two equations are quadratic. Each can be solved by the square root property.
$x^2 = \frac{1}{4}$ or $x^2 = -2$	
$x = \pm\sqrt{\frac{1}{4}}$ or $x = \pm\sqrt{-2}$	Apply the square root property.
$x = \pm\frac{1}{2}$ or $x = \pm i\sqrt{2}$	Simplify the radicals.

The equation has four solutions: $x = \frac{1}{2}$, $x = -\frac{1}{2}$, $x = i\sqrt{2}$, $x = -i\sqrt{2}$.

Skill Practice

4. $9x^4 + 35x^2 - 4 = 0$

Example 5 Solving a Rational Equation

Solve the equation.

$$\frac{3y}{y+2} - \frac{2}{y-1} = 1$$

Solution:

$\frac{3y}{y+2} - \frac{2}{y-1} = 1$ This is a rational equation. The LCD is $(y + 2)(y - 1)$.

$\left(\frac{3y}{y+2} - \frac{2}{y-1}\right) \cdot (y+2)(y-1) = 1 \cdot (y+2)(y-1)$ Multiply both sides by the LCD.

$$\frac{3y}{\cancel{y+2}} \cdot \cancel{(y+2)}(y-1) - \frac{2}{\cancel{y-1}} \cdot (y+2)\cancel{(y-1)} = 1 \cdot (y+2)(y-1)$$

$3y(y - 1) - 2(y + 2) = (y + 2)(y - 1)$ Clear fractions.

Skill Practice Answers

3. $z = 4$

4. $x = \pm\frac{1}{3}, x = \pm 2i$

$3y^2 - 3y - 2y - 4 = y^2 - y + 2y - 2$ Apply the distributive property.

$3y^2 - 5y - 4 = y^2 + y - 2$ The equation is quadratic.

$2y^2 - 6y - 2 = 0$ Write the equation in descending order.

$$\frac{2y^2}{2} - \frac{6y}{2} - \frac{2}{2} = \frac{0}{2}$$

$$y^2 - 3y - 1 = 0$$

Each coefficient in the equation is divisible by 2. Therefore, if we divide both sides by 2, the coefficients in the equation are smaller. This will make it easier to apply the quadratic formula.

$$y = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-1)}}{2(1)}$$ Apply the quadratic formula.

$$= \frac{3 \pm \sqrt{9 + 4}}{2}$$

$$= \frac{3 \pm \sqrt{13}}{2} \quad \nearrow y = \frac{3 + \sqrt{13}}{2} \qquad \searrow y = \frac{3 - \sqrt{13}}{2}$$

The values $y = \frac{3 + \sqrt{13}}{2}$ and $y = \frac{3 - \sqrt{13}}{2}$ are both defined in the original equation.

Skill Practice

5. $\frac{t}{2t - 1} - \frac{1}{t + 4} = 1$

Skill Practice Answers

5. $t = \frac{-5 \pm 3\sqrt{5}}{2}$

Section 8.3 Practice Exercises

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Study Skills Exercise

1. What is meant by an equation in **quadratic form**?

Review Exercises

For Exercises 2–7, solve the quadratic equations.

2. $16 = (2x - 3)^2$

3. $\left(x - \frac{3}{2}\right)^2 = \frac{7}{4}$

4. $n(n - 6) = -13$

5. $x(x + 8) = -16$

6. $6k^2 + 7k = 6$

7. $2x^2 - 8x - 44 = 0$

Concept 1: Solving Equations by Using Substitution

8. **a.** Solve the quadratic equation by factoring. $u^2 + 10u + 24 = 0$

b. Solve the equation by using substitution. $(y^2 + 5y)^2 + 10(y^2 + 5y) + 24 = 0$

9. **a.** Solve the quadratic equation by factoring. $u^2 - 2u - 35 = 0$

b. Solve the equation by using substitution. $(w^2 - 6w)^2 - 2(w^2 - 6w) - 35 = 0$

10. **a.** Solve the quadratic equation by factoring. $u^2 - 2u - 24 = 0$

b. Solve the equation by using substitution. $(x^2 - 5x)^2 - 2(x^2 - 5x) - 24 = 0$

11. **a.** Solve the quadratic equation by factoring. $u^2 - 4u + 3 = 0$

b. Solve the equation by using substitution. $(2p^2 + p)^2 - 4(2p^2 + p) + 3 = 0$

For Exercises 12–23, solve the equation by using substitution.

12. $(x^2 - 2x)^2 + 2(x^2 - 2x) = 3$

13. $(x^2 + x)^2 - 8(x^2 + x) = -12$

14. $m^{2/3} - m^{1/3} - 6 = 0$

15. $2n^{2/3} + 7n^{1/3} - 15 = 0$

16. $2t^{2/5} + 7t^{1/5} + 3 = 0$

17. $p^{2/5} + p^{1/5} - 2 = 0$

18. $y + 6\sqrt{y} = 16$

19. $p - 8\sqrt{p} = -15$

20. $2x + 3\sqrt{x} - 2 = 0$

21. $3t + 5\sqrt{t} - 2 = 0$

22. $16\left(\frac{x+6}{4}\right)^2 + 8\left(\frac{x+6}{4}\right) + 1 = 0$

23. $9\left(\frac{x+3}{2}\right)^2 - 6\left(\frac{x+3}{2}\right) + 1 = 0$

24. In Example 3, we solved the equation $x - \sqrt{x} - 12 = 0$ by using substitution. Now solve this equation by first isolating the radical and then squaring both sides. Don't forget to check the potential solutions in the original equation. Do you obtain the same solution as in Example 3?

Concept 2: Solving Equations Reducible to a Quadratic

For Exercises 25–34, solve the equations.

25. $t^4 + t^2 - 12 = 0$

26. $w^4 + 4w^2 - 45 = 0$

27. $x^2(9x^2 + 7) = 2$

28. $y^2(4y^2 + 17) = 15$

29. $\frac{y}{10} - 1 = -\frac{12}{5y}$

30. $\frac{x+5}{x} + \frac{x}{2} = \frac{x+19}{4x}$

31. $\frac{3x}{x+1} - \frac{2}{x-3} = 1$

32. $\frac{2t}{t-3} - \frac{1}{t+4} = 1$

33. $\frac{x}{2x-1} = \frac{1}{x-2}$

34. $\frac{z}{3z+2} = \frac{2}{z+1}$

Mixed Exercises

For Exercises 35–54, solve the equations.

35. $x^4 - 16 = 0$

36. $t^4 - 625 = 0$

37. $(4x + 5)^2 + 3(4x + 5) + 2 = 0$

38. $2(5x + 3)^2 - (5x + 3) - 28 = 0$

39. $4m^4 - 9m^2 + 2 = 0$

40. $x^4 - 7x^2 + 12 = 0$

41. $x^6 - 9x^3 + 8 = 0$

42. $x^6 - 26x^3 - 27 = 0$

43. $\sqrt{x^2 + 20} = 3\sqrt{x}$

44. $\sqrt{4t + 1} = t + 1$

45. $2\left(\frac{t-4}{3}\right)^2 - \left(\frac{t-4}{3}\right) - 3 = 0$

46. $\left(\frac{x+1}{5}\right)^2 - 3\left(\frac{x+1}{5}\right) - 10 = 0$

47. $x^{2/3} + x^{1/3} = 20$

48. $x^{2/5} - 3x^{1/5} = -2$

49. $m^4 + 2m^2 - 8 = 0$

50. $2c^4 + c^2 - 1 = 0$

51. $a^3 + 16a - a^2 - 16 = 0$ *Hint:* Factor by grouping first.

52. $b^3 + 9b - b^2 - 9 = 0$

53. $x^3 + 5x - 4x^2 - 20 = 0$

54. $y^3 + 8y - 3y^2 - 24 = 0$

Graphing Calculator Exercises

55. **a.** Solve the equation $x^4 + 4x^2 + 4 = 0$.

b. How many solutions are real and how many solutions are imaginary?

c. How many x-intercepts do you anticipate for the function defined by $y = x^4 + 4x^2 + 4$?

d. Graph $Y_1 = x^4 + 4x^2 + 4$ on a standard viewing window.

56. **a.** Solve the equation $x^4 - 2x^2 + 1 = 0$.

b. How many solutions are real and how many solutions are imaginary?

c. How many x-intercepts do you anticipate for the function defined by $y = x^4 - 2x^2 + 1$?

d. Graph $Y_1 = x^4 - 2x^2 + 1$ on a standard viewing window.

57. **a.** Solve the equation $x^4 - x^3 - 6x^2 = 0$.

b. How many solutions are real and how many solutions are imaginary?

c. How many x-intercepts do you anticipate for the function defined by $y = x^4 - x^3 - 6x^2$?

d. Graph $Y_1 = x^4 - x^3 - 6x^2$ on a standard viewing window.

58. **a.** Solve the equation $x^4 - 10x^2 + 9 = 0$.

b. How many solutions are real and how many solutions are imaginary?

c. How many x-intercepts do you anticipate for the function defined by $y = x^4 - 10x^2 + 9$?

d. Graph $Y_1 = x^4 - 10x^2 + 9$ on a standard viewing window.

Graphs of Quadratic Functions

Section 8.4

Concepts

1. Quadratic Functions of the Form $f(x) = x^2 + k$
2. Quadratic Functions of the Form $f(x) = (x - h)^2$
3. Quadratic Functions of the Form $f(x) = ax^2$
4. Quadratic Functions of the Form $f(x) = a(x - h)^2 + k$

In Section 5.8, we defined a quadratic function as a function of the form $f(x) = ax^2 + bx + c$ $(a \neq 0)$. We also learned that the graph of a quadratic function is a **parabola**. In this section we will learn how to graph parabolas.

A parabola opens upward if $a > 0$ (Figure 8-4) and opens downward if $a < 0$ (Figure 8-5). If a parabola opens upward, the **vertex** is the lowest point on the graph. If a parabola opens downward, the **vertex** is the highest point on the graph. The **axis of symmetry** is the vertical line that passes through the vertex.

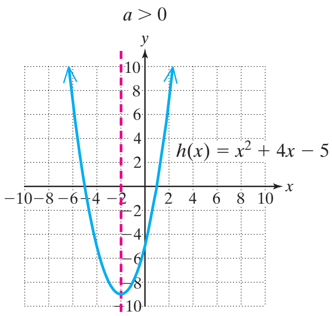


Figure 8-4

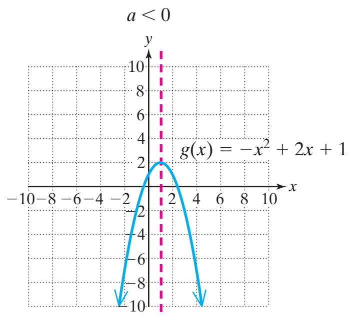


Figure 8-5

1. Quadratic Functions of the Form $f(x) = x^2 + k$

One technique for graphing a function is to plot a sufficient number of points on the function until the general shape and defining characteristics can be determined. Then sketch a curve through the points.

Example 1 Graphing Quadratic Functions of the Form $f(x) = x^2 + k$

Graph the functions f, g, and h on the same coordinate system.

$$f(x) = x^2 \qquad g(x) = x^2 + 1 \qquad h(x) = x^2 - 2$$

Solution:

Several function values for f, g, and h are shown in Table 8-1 for selected values of x. The corresponding graphs are pictured in Figure 8-6.

Table 8-1

x	$f(x) = x^2$	$g(x) = x^2 + 1$	$h(x) = x^2 - 2$
−3	9	10	7
−2	4	5	2
−1	1	2	−1
0	0	1	−2
1	1	2	−1
2	4	5	2
3	9	10	7

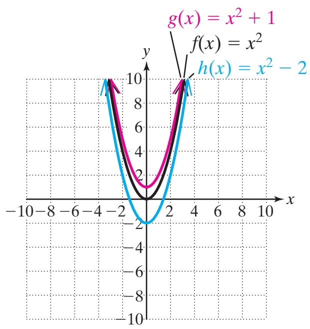
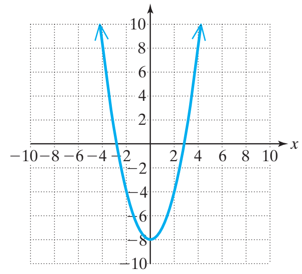


Figure 8-6

Notice that the graphs of $g(x) = x^2 + 1$ and $h(x) = x^2 - 2$ take on the same shape as $f(x) = x^2$. However, the y-values of g are 1 greater than the y-values of f. Hence the graph of $g(x) = x^2 + 1$ is the same as the graph of $f(x) = x^2$ shifted *up* 1 unit. Likewise the y-values of h are 2 less than those of f. The graph of $h(x) = x^2 - 2$ is the same as the graph of $f(x) = x^2$ shifted *down* 2 units.

Skill Practice

1. Refer to the graph of $f(x) = x^2 + k$ to determine the value of k.



The functions in Example 1 illustrate the following properties of quadratic functions of the form $f(x) = x^2 + k$.

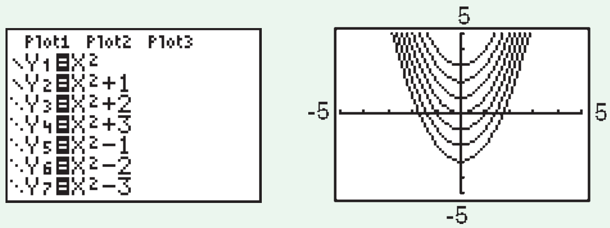
Graphs of $f(x) = x^2 + k$

If $k > 0$, then the graph of $f(x) = x^2 + k$ is the same as the graph of $y = x^2$ shifted *up* k units.

If $k < 0$, then the graph of $f(x) = x^2 + k$ is the same as the graph of $y = x^2$ shifted *down* $|k|$ units.

Calculator Connections

Try experimenting with a graphing calculator by graphing functions of the form $y = x^2 + k$ for several values of k.



Example 2 Graphing Quadratic Functions of the Form $f(x) = x^2 + k$

Sketch the functions defined by

a. $m(x) = x^2 - 4$ **b.** $n(x) = x^2 + \frac{7}{2}$

Solution:

a. $m(x) = x^2 - 4$

$m(x) = x^2 + (-4)$

Because $k = -4$, the graph is obtained by shifting the graph of $y = x^2$ down $|-4|$ units (Figure 8-7).

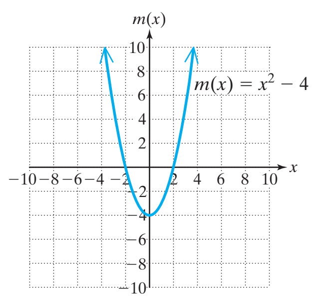


Figure 8-7

Skill Practice Answers

1. $k = -8$

b. $n(x) = x^2 + \dfrac{7}{2}$

Because $k = \frac{7}{2}$, the graph is obtained by shifting the graph of $y = x^2$ up $\frac{7}{2}$ units (Figure 8-8).

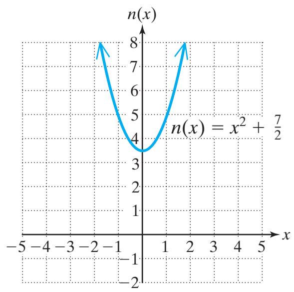


Figure 8-8

Skill Practice Graph the functions.

2. $g(x) = x^2 + 3$ **3.** $h(x) = x^2 - 5$

2. Quadratic Functions of the Form $f(x) = (x - h)^2$

The graph of $f(x) = x^2 + k$ represents a vertical shift (up or down) of the function $y = x^2$. Example 3 shows that functions of the form $f(x) = (x - h)^2$ represent a horizontal shift (left or right) of the function $y = x^2$.

Example 3 Graphing Quadratic Functions of the Form $f(x) = (x - h)^2$

Graph the functions f, g, and h on the same coordinate system.

$$f(x) = x^2 \qquad g(x) = (x + 1)^2 \qquad h(x) = (x - 2)^2$$

Solution:

Several function values for f, g, and h are shown in Table 8-2 for selected values of x. The corresponding graphs are pictured in Figure 8-9.

Table 8-2

x	$f(x) = x^2$	$g(x) = (x + 1)^2$	$h(x) = (x - 2)^2$
−4	16	9	36
−3	9	4	25
−2	4	1	16
−1	1	0	9
0	0	1	4
1	1	4	1
2	4	9	0
3	9	16	1
4	16	25	4
5	25	36	9

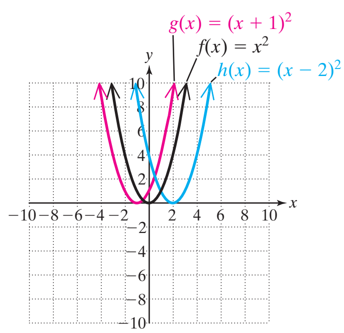
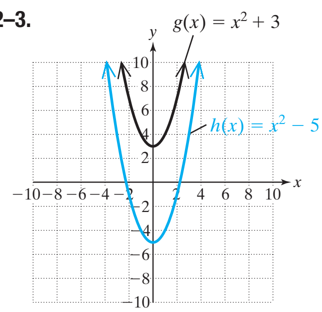


Figure 8-9

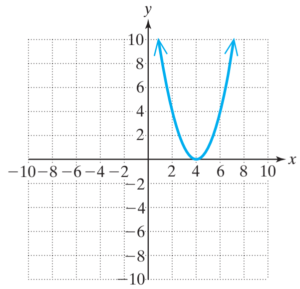
Skill Practice Answers

2–3.



Skill Practice

4. Refer to the graph of $f(x) = (x - h)^2$ to determine the value of h.



Example 3 illustrates the following properties of quadratic functions of the form $f(x) = (x - h)^2$.

Graphs of $f(x) = (x - h)^2$

If $h > 0$, then the graph of $f(x) = (x - h)^2$ is the same as the graph of $y = x^2$ shifted h units to the *right*.

If $h < 0$, then the graph of $f(x) = (x - h)^2$ is the same as the graph of $y = x^2$ shifted $|h|$ units to the *left*.

From Example 3 we have

$$h(x) = (x - 2)^2 \qquad \text{and} \qquad g(x) = (x + 1)^2$$
$$g(x) = [x - (-1)]^2$$

$y = x^2$ shifted 2 units to the right

$y = x^2$ shifted $|-1|$ unit to the left

Example 4 Graphing Functions of the Form $f(x) = (x - h)^2$

Sketch the functions p and q.

a. $p(x) = (x - 7)^2$ **b.** $q(x) = (x + 1.6)^2$

Solution:

a. $p(x) = (x - 7)^2$

Because $h = 7 > 0$, shift the graph of $y = x^2$ to the *right* 7 units (Figure 8-10).

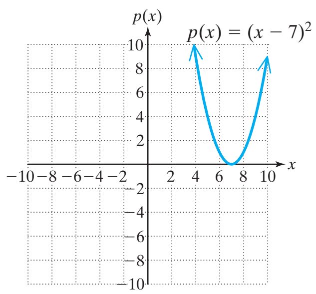


Figure 8-10

Skill Practice Answers

4. $h = 4$

b. $q(x) = (x + 1.6)^2$

$q(x) = [x - (-1.6)]^2$

Because $h = -1.6 < 0$, shift the graph of $y = x^2$ to the *left* 1.6 units (Figure 8-11).

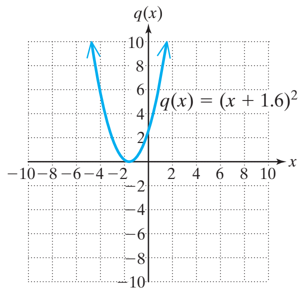


Figure 8-11

Skill Practice Graph the functions.

5. $g(x) = (x + 3)^2$ **6.** $h(x) = (x - 6)^2$

3. Quadratic Functions of the Form $f(x) = ax^2$

Examples 5 and 6 investigate functions of the form $f(x) = ax^2$ $(a \neq 0)$.

Example 5 Graphing Functions of the Form $f(x) = ax^2$

Graph the functions f, g, and h on the same coordinate system.

$$f(x) = x^2 \qquad g(x) = 2x^2 \qquad h(x) = \frac{1}{2}x^2$$

Solution:

Several function values for f, g, and h are shown in Table 8-3 for selected values of x. The corresponding graphs are pictured in Figure 8-12.

Table 8-3

x	$f(x) = x^2$	$g(x) = 2x^2$	$h(x) = \frac{1}{2}x^2$
-3	9	18	$\frac{9}{2}$
-2	4	8	2
-1	1	2	$\frac{1}{2}$
0	0	0	0
1	1	2	$\frac{1}{2}$
2	4	8	2
3	9	18	$\frac{9}{2}$

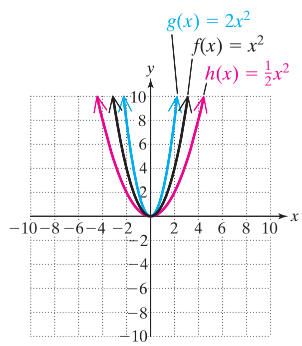


Figure 8-12

Skill Practice

7. Graph the functions f, g, and h on the same coordinate system.

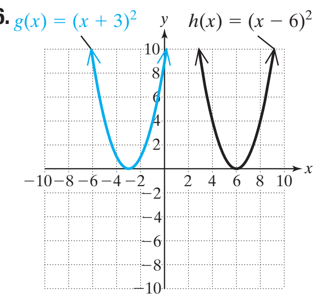
$$f(x) = x^2$$

$$g(x) = 3x^2$$

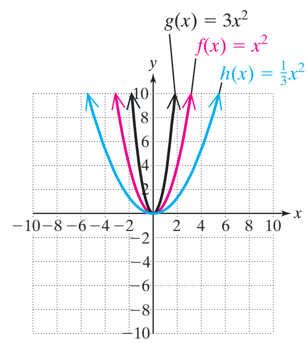
$$h(x) = \frac{1}{3}x^2$$

Skill Practice Answers

5–6.



7.



In Example 5, the function values defined by $g(x) = 2x^2$ are twice those of $f(x) = x^2$. The graph of $g(x) = 2x^2$ is the same as the graph of $f(x) = x^2$ *stretched vertically* by a factor of 2 (the graph appears narrower than $f(x) = x^2$).

In Example 5, the function values defined by $h(x) = \frac{1}{2}x^2$ are one-half those of $f(x) = x^2$. The graph of $h(x) = \frac{1}{2}x^2$ is the same as the graph of $f(x) = x^2$ *shrunk vertically* by a factor of $\frac{1}{2}$ (the graph appears wider than $f(x) = x^2$).

Example 6 Graphing Functions of the Form $f(x) = ax^2$

Graph the functions f, g, and h on the same coordinate system.

$$f(x) = -x^2 \qquad g(x) = -3x^2 \qquad h(x) = -\frac{1}{3}x^2$$

Solution:

Several function values for f, g, and h are shown in Table 8-4 for selected values of x. The corresponding graphs are pictured in Figure 8-13.

Table 8-4

x	$f(x) = -x^2$	$g(x) = -3x^2$	$h(x) = -\frac{1}{3}x^2$
-3	-9	-27	-3
-2	-4	-12	$-\frac{4}{3}$
-1	-1	-3	$-\frac{1}{3}$
0	0	0	0
1	-1	-3	$-\frac{1}{3}$
2	-4	-12	$-\frac{4}{3}$
3	-9	-27	-3

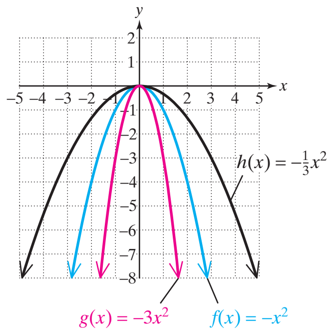


Figure 8-13

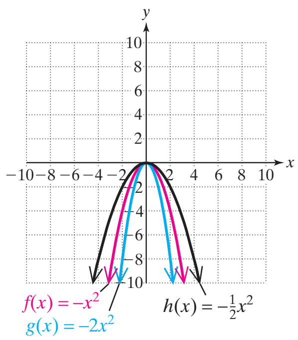
Skill Practice

8. Graph the functions f, g, and h on the same coordinate system.

$$f(x) = -x^2$$
$$g(x) = -2x^2$$
$$h(x) = -\frac{1}{2}x^2$$

Skill Practice Answers

8.



Example 6 illustrates that if the coefficient of the square term is negative, the parabola opens down. The graph of $g(x) = -3x^2$ is the same as the graph of $f(x) = -x^2$ with a *vertical stretch* by a factor of $|-3|$. The graph of $h(x) = -\frac{1}{3}x^2$ is the same as the graph of $f(x) = -x^2$ with a *vertical shrink* by a factor of $|-\frac{1}{3}|$.

Graphs of $f(x) = ax^2$

1. If $a > 0$, the parabola opens upward. Furthermore,
   - If $0 < a < 1$, then the graph of $f(x) = ax^2$ is the same as the graph of $y = x^2$ with a *vertical shrink* by a factor of a.
   - If $a > 1$, then the graph of $f(x) = ax^2$ is the same as the graph of $y = x^2$ with a *vertical stretch* by a factor of a.
2. If $a < 0$, the parabola opens downward. Furthermore,
   - If $0 < |a| < 1$, then the graph of $f(x) = ax^2$ is the same as the graph of $y = -x^2$ with a *vertical shrink* by a factor of $|a|$.
   - If $|a| > 1$, then the graph of $f(x) = ax^2$ is the same as the graph of $y = -x^2$ with a *vertical stretch* by a factor of $|a|$.

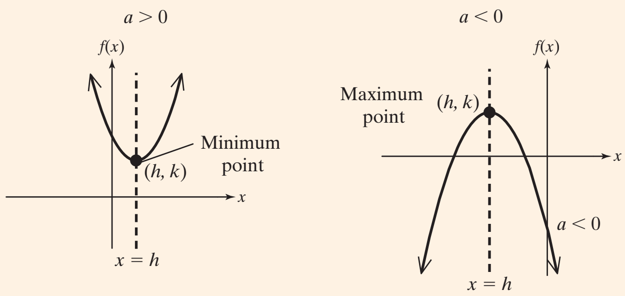
4. Quadratic Functions of the Form $f(x) = a(x - h)^2 + k$

We can summarize our findings from Examples 1–6 by graphing functions of the form $f(x) = a(x - h)^2 + k \quad (a \neq 0)$.

The graph of $y = x^2$ has its vertex at the origin $(0, 0)$. The graph of $f(x) = a(x - h)^2 + k$ is the same as the graph of $y = x^2$ shifted to the right or left h units and shifted up or down k units. Therefore, the vertex is shifted from $(0, 0)$ to (h, k). The axis of symmetry is the vertical line through the vertex, that is, the line $x = h$.

Graphs of $f(x) = a(x - h)^2 + k$

1. The vertex is located at (h, k).
2. The axis of symmetry is the line $x = h$.
3. If $a > 0$, the parabola opens upward, and k is the **minimum value** of the function.
4. If $a < 0$, the parabola opens downward, and k is the **maximum value** of the function.



Example 7 Graphing a Function of the Form $f(x) = a(x - h)^2 + k$

Given the function defined by

$$f(x) = 2(x - 3)^2 + 4$$

a. Identify the vertex.

b. Sketch the function.

c. Identify the axis of symmetry.

d. Identify the maximum or minimum value of the function.

Solution:

a. The function is in the form $f(x) = a(x - h)^2 + k$, where $a = 2$, $h = 3$, and $k = 4$. Therefore, the vertex is $(3, 4)$.

b. The graph of f is the same as the graph of $y = x^2$ shifted to the right 3 units, shifted up 4 units, and stretched vertically by a factor of 2 (Figure 8-14).

c. The axis of symmetry is the line $x = 3$.

d. Because $a > 0$, the function opens upward. Therefore, the minimum function value is 4. Notice that the minimum value is the minimum y-value on the graph.

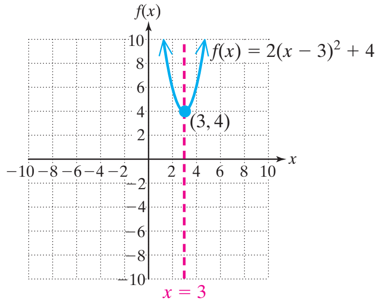
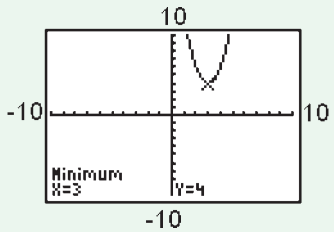


Figure 8-14

Calculator Connections

Some graphing calculators have *Minimum* and *Maximum* features that enable the user to approximate the minimum and maximum values of a function. Otherwise *Zoom* and *Trace* can be used.



Skill Practice

9. Given the function defined by $g(x) = 3(x + 1)^2 - 3$

a. Identify the vertex.

b. Sketch the graph.

c. Identify the axis of symmetry.

d. Identify the maximum or minimum value of the function.

Example 8 Graphing a Function of the Form $f(x) = a(x - h)^2 + k$

Given the function defined by

$$g(x) = -(x + 2)^2 - \frac{7}{4}$$

a. Identify the vertex.

b. Sketch the function.

c. Identify the axis of symmetry.

d. Identify the maximum or minimum value of the function.

Solution:

a.
$$\begin{aligned} g(x) &= -(x + 2)^2 - \frac{7}{4} \\ &= -1[x - (-2)]^2 + \left(-\frac{7}{4}\right) \end{aligned}$$

The function is in the form $g(x) = a(x - h)^2 + k$, where $a = -1$, $h = -2$, and $k = -\frac{7}{4}$. Therefore, the vertex is $(-2, -\frac{7}{4})$.

b. The graph of g is the same as the graph of $y = x^2$ shifted to the left 2 units, shifted down $\frac{7}{4}$ units, and opening downward (Figure 8-15).

c. The axis of symmetry is the line $x = -2$.

d. The parabola opens downward, so the maximum function value is $-\frac{7}{4}$.

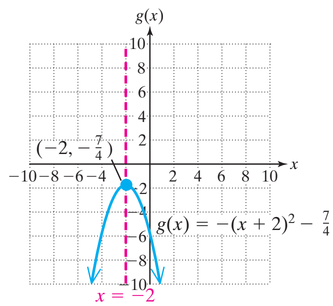
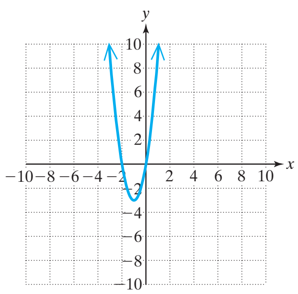


Figure 8-15

Skill Practice Answers

9a. Vertex: $(-1, -3)$

b.



c. Axis of symmetry: $x = -1$

d. Minimum value: -3

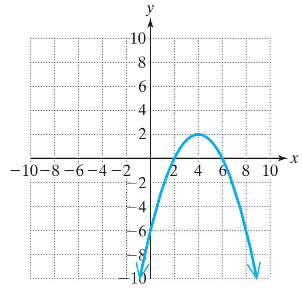
Skill Practice

10. Given the function defined by $h(x) = -\frac{1}{2}(x - 4)^2 + 2$

a. Identify the vertex.
b. Sketch the graph.
c. Identify the axis of symmetry.
d. Identify the maximum or minimum value of the function.

Skill Practice Answers

10a. Vertex: (4, 2)
b.



c. Axis of symmetry: $x = 4$
d. Maximum value: 2

Section 8.4 Practice Exercises

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Study Skills Exercise

1. Define the key terms.

a. Parabola **b. Vertex** **c. Axis of symmetry**

d. Maximum value **e. Minimum value**

Review Exercises

For Exercises 2–8, solve the equations.

2. $x^2 + x - 5 = 0$ **3.** $(y - 3)^2 = -4$ **4.** $\sqrt{2a + 7} = a + 1$ **5.** $5t(t - 2) = -3$

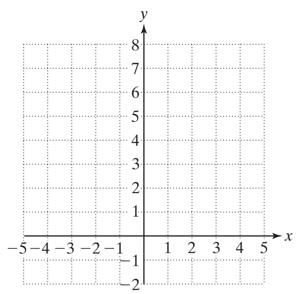
6. $2z^2 - 3z - 9 = 0$ **7.** $x^{2/3} + 5x^{1/3} + 6 = 0$ **8.** $m^2(m^2 + 6) = 27$

Concept 1: Quadratic Functions of the Form $f(x) = x^2 + k$

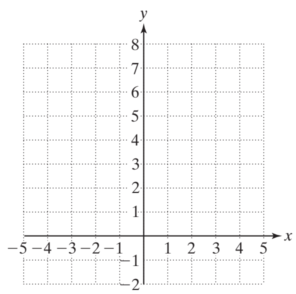
9. Describe how the value of k affects the graphs of functions of the form $f(x) = x^2 + k$.

For Exercises 10–19, graph the functions.

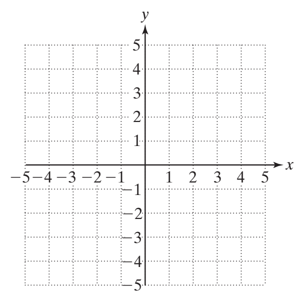
10. $g(x) = x^2 + 1$



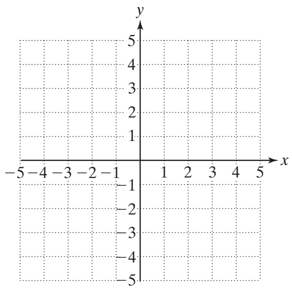
11. $f(x) = x^2 + 2$



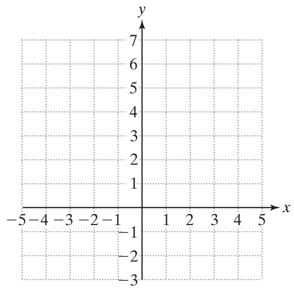
12. $p(x) = x^2 - 3$



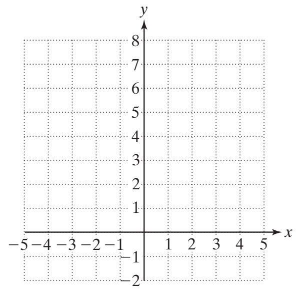
13. $q(x) = x^2 - 4$



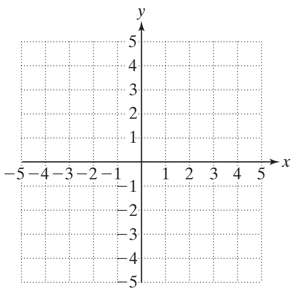
14. $T(x) = x^2 + \frac{3}{4}$



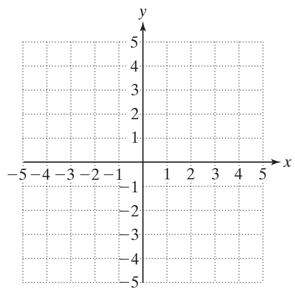
15. $S(x) = x^2 + \frac{3}{2}$



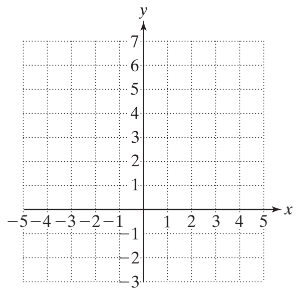
16. $M(x) = x^2 - \frac{5}{4}$



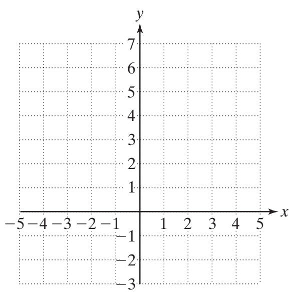
17. $n(x) = x^2 - \frac{1}{3}$



18. $P(x) = x^2 + \frac{1}{2}$



19. $Q(x) = x^2 + \frac{1}{4}$

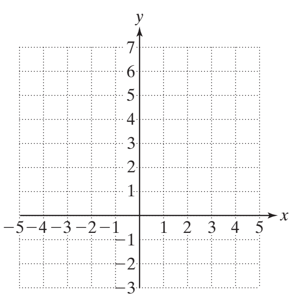


Concept 2: Quadratic Functions of the Form $f(x) = (x - h)^2$

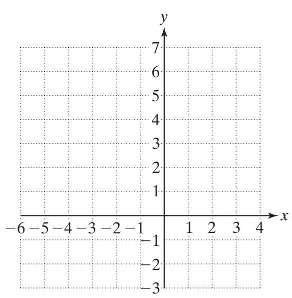
20. Describe how the value of h affects the graphs of functions of the form $f(x) = (x - h)^2$.

For Exercises 21–30, graph the functions.

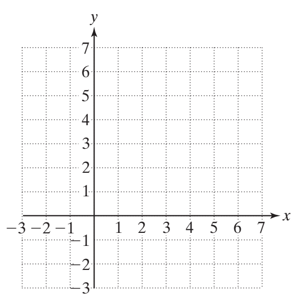
21. $r(x) = (x + 1)^2$



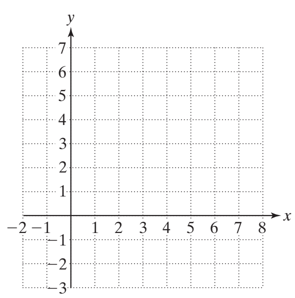
22. $h(x) = (x + 2)^2$



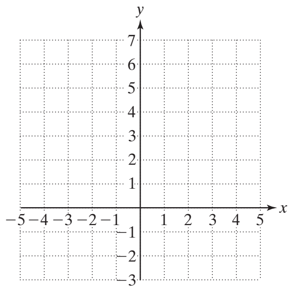
23. $k(x) = (x - 3)^2$



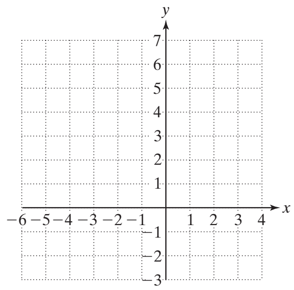
24. $L(x) = (x - 4)^2$



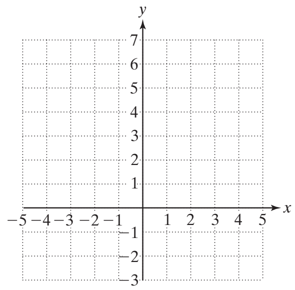
25. $A(x) = \left(x + \frac{3}{4}\right)^2$



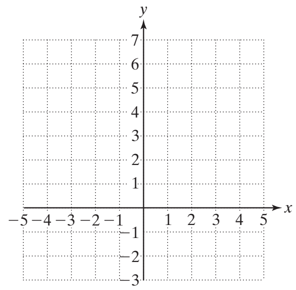
26. $r(x) = \left(x + \frac{3}{2}\right)^2$



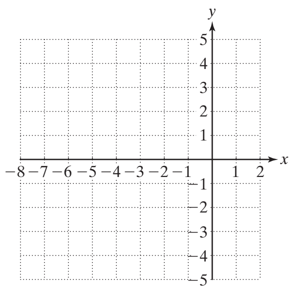
27. $W(x) = \left(x - \frac{5}{4}\right)^2$



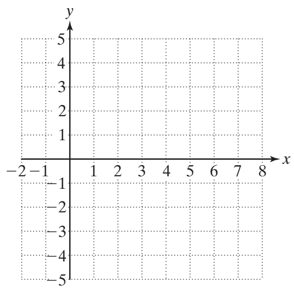
28. $V(x) = \left(x - \frac{1}{3}\right)^2$



29. $M(x) = (x + 4.5)^2$



30. $N(x) = (x - 3.5)^2$



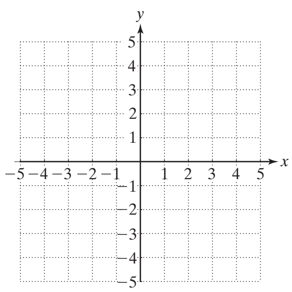
Concept 3: Quadratic Functions of the Form $f(x) = ax^2$

31. Describe how the value of a affects the graph of a function of the form $f(x) = ax^2$, where $a \neq 0$.

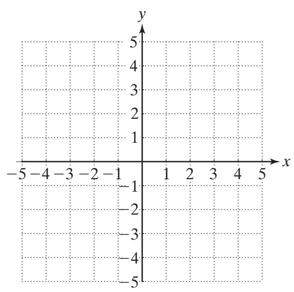
32. How do you determine whether the graph of a function defined by $h(x) = ax^2 + bx + c \quad (a \neq 0)$ opens up or down?

For Exercises 33–40, graph the functions.

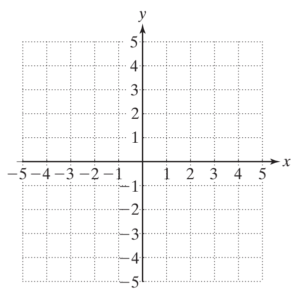
33. $f(x) = 2x^2$



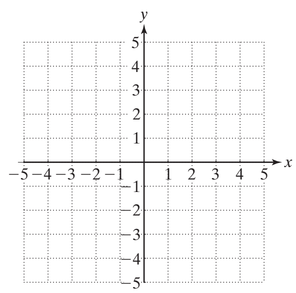
34. $g(x) = 3x^2$



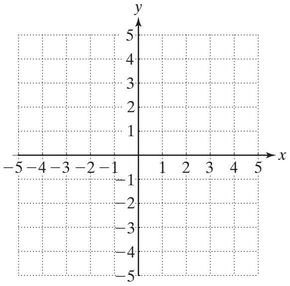
35. $h(x) = \frac{1}{2}x^2$



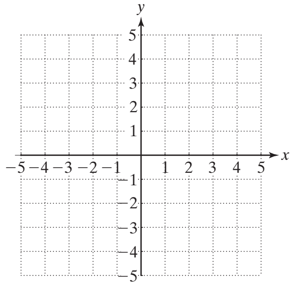
36. $f(x) = \frac{1}{3}x^2$



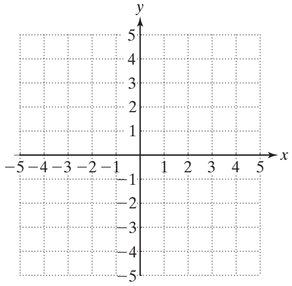
37. $c(x) = -x^2$



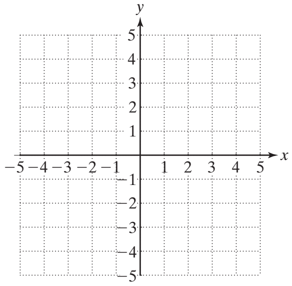
38. $g(x) = -2x^2$



39. $v(x) = -\frac{1}{3}x^2$



40. $f(x) = -\frac{1}{2}x^2$



Concept 4: Quadratic Functions of the Form $f(x) = a(x - h)^2 + k$

For Exercises 41–48, match the function with its graph.

41. $f(x) = -\frac{1}{4}x^2$

42. $g(x) = (x + 3)^2$

43. $k(x) = (x - 3)^2$

44. $h(x) = \frac{1}{4}x^2$

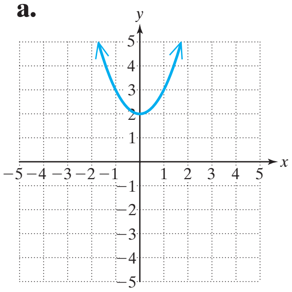
45. $t(x) = x^2 + 2$

46. $m(x) = x^2 - 4$

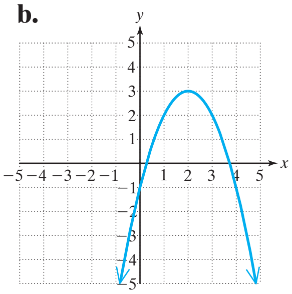
47. $p(x) = (x + 1)^2 - 3$

48. $n(x) = -(x - 2)^2 + 3$

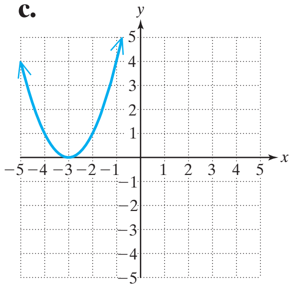
a.



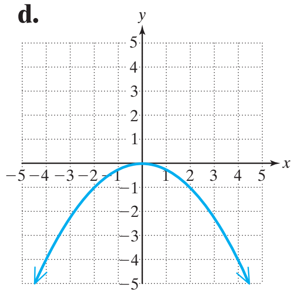
b.



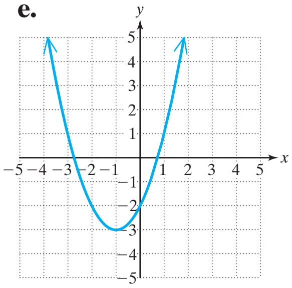
c.



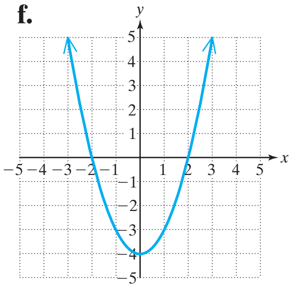
d.



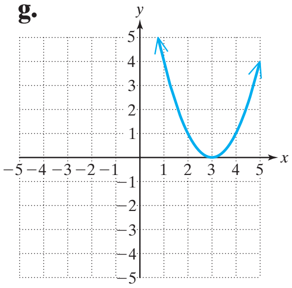
e.



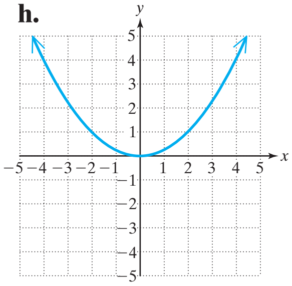
f.



g.

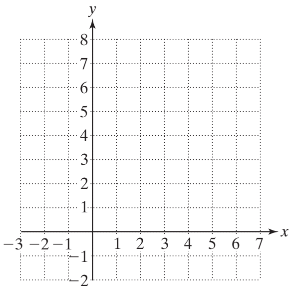


h.

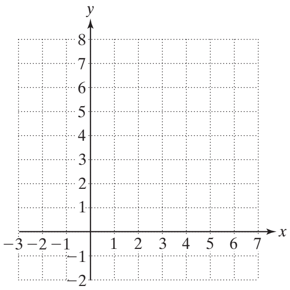


For Exercises 49–66, graph the parabola and the axis of symmetry. Label the coordinates of the vertex, and write the equation of the axis of symmetry.

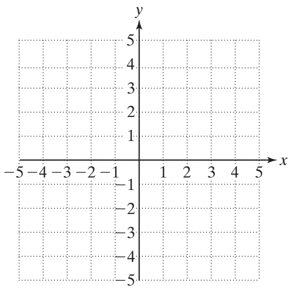
49. $y = (x - 3)^2 + 2$



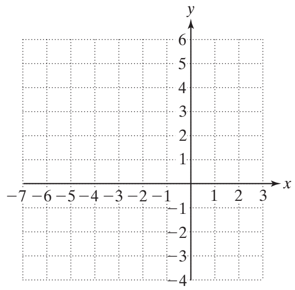
50. $y = (x - 2)^2 + 3$



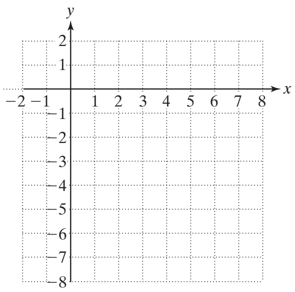
51. $y = (x + 1)^2 - 3$



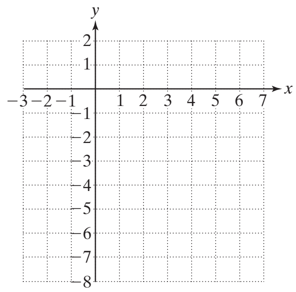
52. $y = (x + 3)^2 - 1$



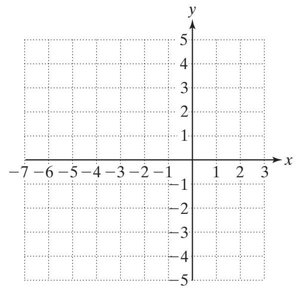
53. $y = -(x - 4)^2 - 2$



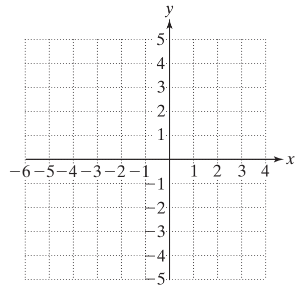
54. $y = -(x - 2)^2 - 4$



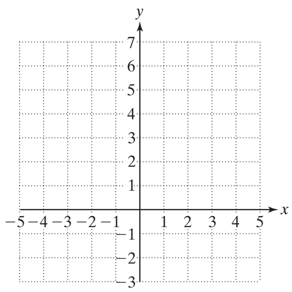
55. $y = -(x + 3)^2 + 3$



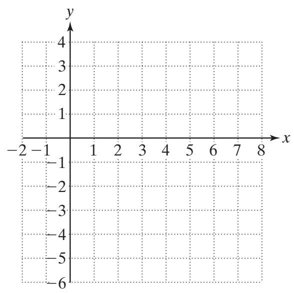
56. $y = -(x + 2)^2 + 2$



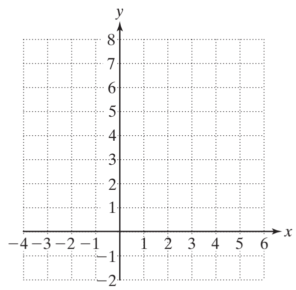
57. $y = (x + 1)^2 + 1$



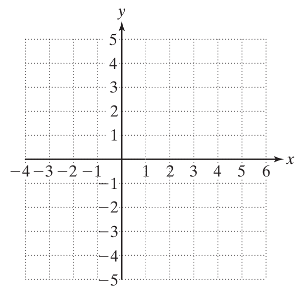
58. $y = (x - 4)^2 - 4$



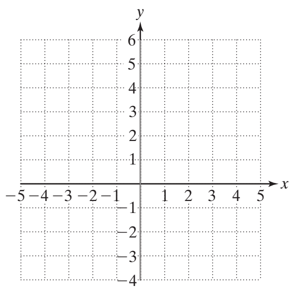
59. $y = 3(x - 1)^2$



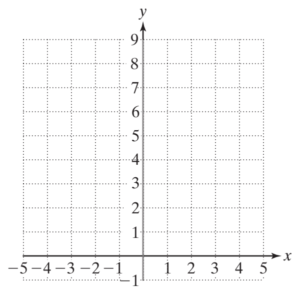
60. $y = -3(x - 1)^2$



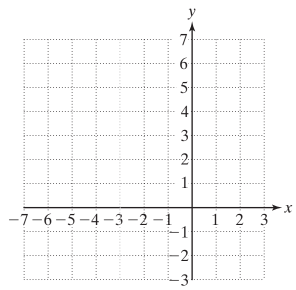
61. $y = -4x^2 + 3$



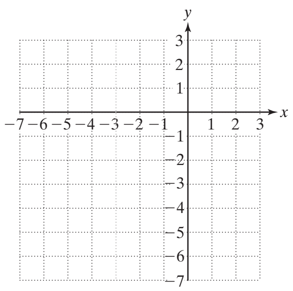
62. $y = 4x^2 + 3$



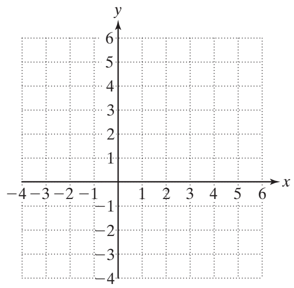
63. $y = 2(x + 3)^2 - 1$



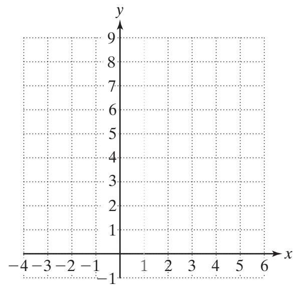
64. $y = -2(x + 3)^2 - 1$



65. $y = -\frac{1}{4}(x - 1)^2 + 2$



66. $y = \frac{1}{4}(x - 1)^2 + 2$



For Exercises 67–78, write the coordinates of the vertex and determine if the vertex is a maximum point or a minimum point. Then write the maximum or minimum value.

67. $f(x) = 4(x - 6)^2 - 9$

68. $g(x) = 3(x - 4)^2 - 7$

69. $p(x) = -\frac{2}{5}(x - 2)^2 + 5$

70. $h(x) = -\frac{3}{7}(x - 5)^2 + 10$

71. $k(x) = \frac{1}{2}(x + 8)^2$

72. $m(x) = \frac{2}{9}(x + 11)^2$

73. $n(x) = -6x^2 + \frac{21}{4}$

74. $q(x) = -4x^2 + \frac{1}{6}$

75. $A(x) = 2(x - 7)^2 - \frac{3}{2}$

76. $B(x) = 5(x - 3)^2 - \frac{1}{4}$

77. $F(x) = 7x^2$

78. $G(x) = -7x^2$

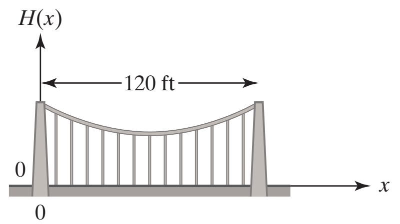
79. True or false: The function defined by $g(x) = -5x^2$ has a maximum value but no minimum value.

80. True or false: The function defined by $f(x) = 2(x - 5)^2$ has a maximum value but no minimum value.

81. True or false: If the vertex $(-2, 8)$ represents a minimum point, then the minimum value is -2.

82. True or false: If the vertex $(-2, 8)$ represents a maximum point, then the maximum value is 8.

83. A suspension bridge is 120 ft long. Its supporting cable hangs in a shape that resembles a parabola. The function defined by $H(x) = \frac{1}{90}(x - 60)^2 + 30$ (where $0 \le x \le 120$) approximates the height of the supporting cable a distance of x ft from the end of the bridge (see figure).

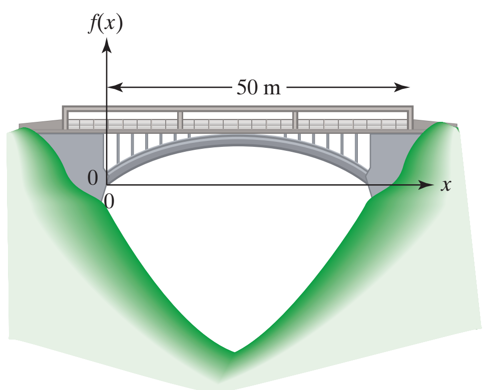


a. What is the location of the vertex of the parabolic cable?

b. What is the minimum height of the cable?

c. How high are the towers at either end of the supporting cable?

84. A 50-m bridge over a crevasse is supported by a parabolic arch. The function defined by $f(x) = -0.16(x - 25)^2 + 100$ (where $0 \le x \le 50$) approximates the height of the supporting arch x meters from the end of the bridge (see figure).



a. What is the location of the vertex of the arch?

b. What is the maximum height of the arch (relative to the origin)?

Graphing Calculator Exercises

For Exercises 85–88, verify the maximum and minimum points found in Exercises 67–70, by graphing each function on the calculator.

85. $Y_1 = 4(x - 6)^2 - 9$ (Exercise 67)

86. $Y_1 = 3(x - 4)^2 - 7$ (Exercise 68)

87. $Y_1 = -\frac{2}{5}(x - 2)^2 + 5$ (Exercise 69)

88. $Y_1 = -\frac{3}{7}(x - 5)^2 + 10$ (Exercise 70)

Vertex of a Parabola and Applications

Section 8.5

Concepts

1. Writing a Quadratic Function in the Form $f(x) = a(x - h)^2 + k$
2. Vertex Formula
3. Determining the Vertex and Intercepts of a Quadratic Function
4. Vertex of a Parabola: Applications

1. Writing a Quadratic Function in the Form $f(x) = a(x - h)^2 + k$

A quadratic function can be expressed as $f(x) = ax^2 + bx + c$ $(a \neq 0)$. However, by completing the square, we can write the function in the form $f(x) = a(x - h)^2 + k$. In this form, the vertex is easily recognized as (h, k).

The process to complete the square in this context is similar to the steps outlined in Section 8.1. However, all algebraic steps are performed on the right-hand side of the equation.

Example 1 Writing a Quadratic Function in the Form $f(x) = a(x - h)^2 + k$ $(a \neq 0)$

Given: $f(x) = x^2 + 8x + 13$

a. Write the function in the form $f(x) = a(x - h)^2 + k$.

b. Identify the vertex, axis of symmetry, and minimum function value.

Solution:

a. $f(x) = x^2 + 8x + 13$

$= 1(x^2 + 8x) + 13$ — Rather than dividing by the leading coefficient on both sides, we will factor out the leading coefficient from the variable terms on the right-hand side.

$= 1(x^2 + 8x \qquad) + 13$ — Next, complete the square on the expression within the parentheses: $[\frac{1}{2}(8)]^2 = 16$.

$= 1(x^2 + 8x + 16 - 16) + 13$ — Rather than add 16 to both sides of the function, we *add and subtract 16* within the parentheses on the right-hand side. This has the effect of adding 0 to the right-hand side.

Avoiding Mistakes:

Do not factor out the leading coefficient from the constant term.

$= 1(x^2 + 8x + 16) - 16 + 13$ Use the associative property of addition to regroup terms and isolate the perfect square trinomial within the parentheses.

$= (x + 4)^2 - 3$ Factor and simplify.

b. $f(x) = (x + 4)^2 - 3$

The vertex is $(-4, -3)$.

The axis of symmetry is $x = -4$.

Because $a > 0$, the parabola opens upward.

The minimum value is -3 (Figure 8-16).

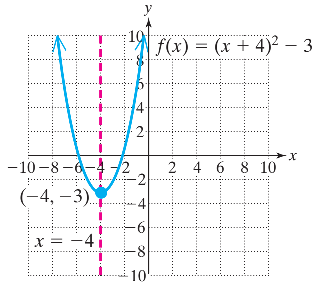


Figure 8-16

Skill Practice

1. Given: $f(x) = x^2 + 8x - 1$

a. Write the equation in the form $f(x) = a(x - h)^2 + k$.

b. Identify the vertex, axis of symmetry, and minimum value of the function.

Example 2 Analyzing a Quadratic Function

Given: $f(x) = -2x^2 + 12x - 16$

a. Write the function in the form $f(x) = a(x - h)^2 + k$.

b. Find the vertex, axis of symmetry, and maximum function value.

c. Find the x- and y-intercepts.

d. Graph the function.

Solution:

a. $f(x) = -2x^2 + 12x - 16$ To find the vertex, write the function in the form $f(x) = a(x - h)^2 + k$.

$= -2(x^2 - 6x \qquad) - 16$ Factor the leading coefficient from the variable terms.

$= -2(x^2 - 6x + 9 - 9) - 16$ Add and subtract the quantity $[\frac{1}{2}(-6)]^2 = 9$ within the parentheses.

$= -2(x^2 - 6x + 9) + (-2)(-9) - 16$ To remove the term -9 from the parentheses, we must first apply the distributive property. When -9 is removed from the parentheses, it carries with it a factor of -2.

$= -2(x - 3)^2 + 18 - 16$ Factor and simplify.

$= -2(x - 3)^2 + 2$

Skill Practice Answers

1a. $f(x) = (x + 4)^2 - 17$

b. Vertex: $(-4, -17)$; axis of symmetry: $x = -4$; minimum value: -17

b. $f(x) = -2(x - 3)^2 + 2$
The vertex is (3, 2). The axis of symmetry is $x = 3$. Because $a < 0$, the parabola opens downward and the maximum value is 2.

c. The y-intercept is given by $f(0) = -2(0)^2 + 12(0) - 16 = -16$.
The y-intercept is $(0, -16)$.

To find the x-intercept(s), find the real solutions to the equation $f(x) = 0$.

$$f(x) = -2x^2 + 12x - 16$$
$$0 = -2x^2 + 12x - 16 \qquad \text{Substitute } f(x) = 0.$$
$$0 = -2(x^2 - 6x + 8) \qquad \text{Factor.}$$
$$0 = -2(x - 4)(x - 2)$$
$$x = 4 \quad \text{or} \quad x = 2$$

The x-intercepts are (4, 0) and (2, 0).

d. Using the information from parts (a)–(c), sketch the graph (Figure 8-17).

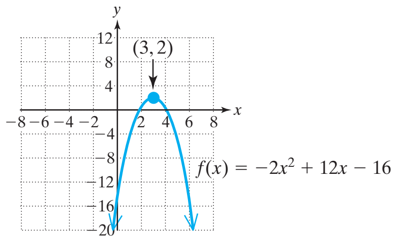


Figure 8-17

Skill Practice

2. Given: $g(x) = -x^2 + 6x - 5$
a. Write the equation in the form $g(x) = a(x - h)^2 + k$.
b. Identify the vertex, axis of symmetry, and maximum value of the function.
c. Determine the x- and y-intercepts.
d. Graph the function.

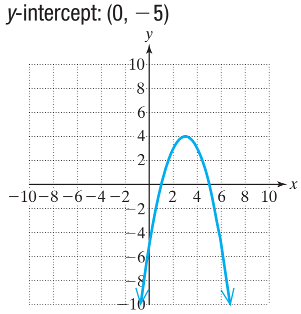
2. Vertex Formula

Completing the square and writing a quadratic function in the form $f(x) = a(x - h)^2 + k \quad (a \neq 0)$ is one method to find the vertex of a parabola. Another method is to use the vertex formula. The **vertex formula** can be derived by completing the square on the function defined by $f(x) = ax^2 + bx + c \quad (a \neq 0)$.

$$f(x) = ax^2 + bx + c \qquad (a \neq 0)$$
$$= a\left(x^2 + \frac{b}{a}x \qquad\qquad \right) + c \qquad \text{Factor } a \text{ from the variable terms.}$$
$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + c \qquad \text{Add and subtract } [\tfrac{1}{2}(b/a)]^2 = b^2/(4a^2) \text{ within the parentheses.}$$
$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + (a)\left(-\frac{b^2}{4a^2}\right) + c \qquad \text{Apply the distributive property and remove the term } -b^2/(4a^2) \text{ from the parentheses.}$$

Skill Practice Answers

2a. $g(x) = -(x - 3)^2 + 4$
b. Vertex: (3, 4); axis of symmetry: $x = 3$; maximum value: 4
c. x-intercepts: (5, 0) and (1, 0); y-intercept: $(0, -5)$
d.



$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$ Factor the trinomial and simplify.

$$= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$ Apply the commutative property of addition to reverse the last two terms.

$$= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac}{4a} - \frac{b^2}{4a}$$ Obtain a common denominator.

$$= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

$$= a\left[x - \left(-\frac{b}{2a}\right)\right]^2 + \frac{4ac - b^2}{4a}$$

$$\downarrow \qquad \downarrow$$

$$f(x) = a(x \quad - \quad h)^2 \quad + \quad k$$

The function is in the form $f(x) = a(x - h)^2 + k$, where

$$h = \frac{-b}{2a} \quad \text{and} \quad k = \frac{4ac - b^2}{4a}$$

Hence, the vertex is

$$\left(\frac{-b}{2a}, \frac{4ac - b^2}{4a}\right)$$

Although the y-coordinate of the vertex is given as $(4ac - b^2)/(4a)$, it is usually easier to determine the x-coordinate of the vertex first and then find y by evaluating the function at $x = -b/(2a)$.

The Vertex Formula

For $f(x) = ax^2 + bx + c \quad (a \neq 0)$, the vertex is given by

$$\left(\frac{-b}{2a}, \frac{4ac - b^2}{4a}\right) \quad \text{or} \quad \left(\frac{-b}{2a}, f\left(-\frac{b}{2a}\right)\right)$$

3. Determining the Vertex and Intercepts of a Quadratic Function

Example 3 Determining the Vertex and Intercepts of a Quadratic Function

Given: $h(x) = x^2 - 2x + 5$

a. Use the vertex formula to find the vertex.

b. Find the x- and y-intercepts.

c. Sketch the function.

Solution:

a. $h(x) = x^2 - 2x + 5$

$a = 1 \qquad b = -2 \qquad c = 5$ Identify a, b, and c.

The x-coordinate of the vertex is $\dfrac{-b}{2a} = \dfrac{-(-2)}{2(1)} = 1$.

The y-coordinate of the vertex is $h(1) = (1)^2 - 2(1) + 5 = 4$.

The vertex is $(1, 4)$.

b. The y-intercept is given by $h(0) = (0)^2 - 2(0) + 5 = 5$.

The y-intercept is $(0, 5)$.

To find the x-intercept(s), find the real solutions to the equation $h(x) = 0$.

$h(x) = x^2 - 2x + 5$

$0 = x^2 - 2x + 5$ This quadratic equation is not factorable. Apply the quadratic formula: $a = 1$, $b = -2$, $c = 5$.

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{4 - 20}}{2(1)}$$

$$= \frac{2 \pm \sqrt{-16}}{2}$$

$$= \frac{2 \pm 4i}{2}$$

$$= 1 \pm 2i$$

The solutions to the equation $h(x) = 0$ are not real numbers. Therefore, there are no x-intercepts.

c.

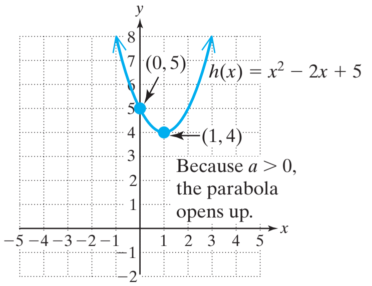


Figure 8-18

TIP: The location of the vertex and the direction that the parabola opens can be used to determine whether the function has any x-intercepts.

Given $h(x) = x^2 - 2x + 5$, the vertex $(1, 4)$ is *above* the x-axis. Furthermore, because $a > 0$, the parabola opens upward. Therefore, it is not possible for the function h to cross the x-axis (Figure 8-18).

Skill Practice

3. Given: $f(x) = x^2 + 4x + 6$

a. Use the vertex formula to find the vertex of the parabola.

b. Determine the x- and y-intercepts.

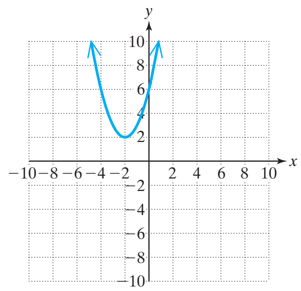
c. Sketch the graph.

Skill Practice Answers

3a. Vertex: $(-2, 2)$

b. x-intercepts: none; y-intercepts: $(0, 6)$

c.



4. Vertex of a Parabola: Applications

Example 4 Applying a Quadratic Function

The crew from Extravaganza Entertainment launches fireworks at an angle of 60° from the horizontal. The height of one particular type of display can be approximated by the following function:

$$h(t) = -16t^2 + 128\sqrt{3}t$$

where $h(t)$ is measured in feet and t is measured in seconds.

a. How long will it take the fireworks to reach their maximum height? Round to the nearest second.

b. Find the maximum height. Round to the nearest foot.

Solution:

$h(t) = -16t^2 + 128\sqrt{3}t$ — This parabola opens downward; therefore, the maximum height of the fireworks will occur at the vertex of the parabola.

$a = -16 \qquad b = 128\sqrt{3} \qquad c = 0$ — Identify a, b, and c, and apply the vertex formula.

The x-coordinate of the vertex is

$$\frac{-b}{2a} = \frac{-128\sqrt{3}}{2(-16)} = \frac{-128\sqrt{3}}{-32} \approx 6.9$$

The y-coordinate of the vertex is approximately

$$h(6.9) = -16(6.9)^2 + 128\sqrt{3}(6.9) \approx 768$$

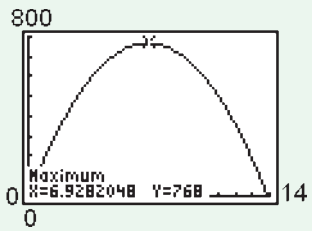
The vertex is (6.9, 768).

a. The fireworks will reach their maximum height in 6.9 sec.

b. The maximum height is 768 ft.

Calculator Connections

Some graphing calculators have *Minimum* and *Maximum* features that enable the user to approximate the minimum and maximum values of a function. Otherwise, *Zoom* and *Trace* can be used.



Skill Practice

4. An object is launched into the air with an initial velocity of 48 ft/sec from the top of a building 288 ft high. The height $h(t)$ of the object after t seconds is given by

$$h(t) = -16t^2 + 48t + 288$$

a. Find the time it takes for the object to reach its maximum height.

b. Find the maximum height.

Example 5 Applying a Quadratic Function

A group of students start a small company that produces CDs. The weekly profit (in dollars) is given by the function

$$P(x) = -2x^2 + 80x - 600$$

where x represents the number of CDs produced.

a. Find the x-intercepts of the profit function, and interpret the meaning of the x-intercepts in the context of this problem.

b. Find the y-intercept of the profit function, and interpret its meaning in the context of this problem.

Skill Practice Answers

4a. 1.5 sec **b.** 324 ft

c. Find the vertex of the profit function, and interpret its meaning in the context of this problem.

d. Sketch the profit function.

Solution:

a. $P(x) = -2x^2 + 80x - 600$ The x-intercepts are the real solutions of the equation $P(x) = 0$.

$$0 = -2x^2 + 80x - 600$$
$$0 = -2(x^2 - 40x + 300)$$
$$0 = -2(x - 10)(x - 30) \quad \text{Solve by factoring.}$$
$$x = 10 \quad \text{or} \quad x = 30$$

The x-intercepts are (10, 0) and (30, 0). The x-intercepts represent points where the profit is zero. These are called break-even points. The break-even points occur when 10 CDs are produced and also when 30 CDs are produced.

b. The y-intercept is $P(0) = -2(0)^2 + 80(0) - 600 = -600$. The y-intercept is $(0, -600)$. The y-intercept indicates that if no CDs are produced, the company has a \$600 loss.

c. The x-coordinate of the vertex is

$$\frac{-b}{2a} = \frac{-80}{2(-2)} = 20$$

The y-coordinate is $P(20) = -2(20)^2 + 80(20) - 600 = 200$.

The vertex is (20, 200).
A maximum weekly profit of \$200 is obtained when 20 CDs are produced.

d. Using the information from parts (a)–(c), sketch the profit function (Figure 8-19).

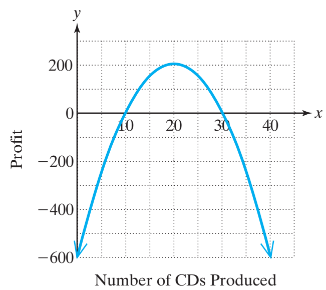


Figure 8-19

Skill Practice

5. The weekly profit function for a tutoring service is given by $P(x) = -4x^2 + 120x - 500$, where x is the number of hours that the tutors are scheduled to work and $P(x)$ represents profit in dollars.

a. Find the x-intercepts of the profit function, and interpret their meaning in the context of this problem.

b. Find the y-intercept of the profit function, and interpret its meaning in the context of this problem.

c. Find the vertex of the profit function, and interpret its meaning in the context of this problem.

d. Sketch the profit function.

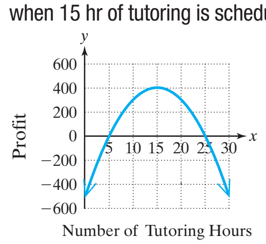
Skill Practice Answers

5a. x-intercepts: (5, 0) and (25, 0). The x-intercepts represent the break-even points, where the profit is 0.

b. y-intercept: $(0, -500)$. If the service does no tutoring, it will have a \$500 loss.

c. Vertex: (15, 400). A maximum weekly profit of \$400 is obtained when 15 hr of tutoring is scheduled.

d.



Example 6 Applying a Quadratic Function

The average number of visits to office-based physicians is a function of the age of the patient.

$$N(x) = 0.0014x^2 - 0.0658x + 2.65$$

where x is a patient's age in years and $N(x)$ is the average number of doctor visits per year (Figure 8-20). Find the age for which the number of visits to office-based physicians is a minimum.

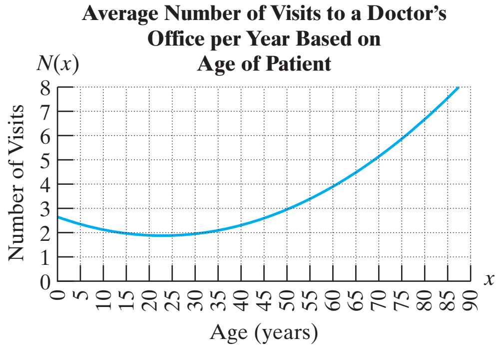


Figure 8-20

(Source: U.S. National Center for Health Statistics.)

Solution:

$$N(x) = 0.0014x^2 - 0.0658x + 2.65$$

$$\frac{-b}{2a} = \frac{-(-0.0658)}{2(0.0014)} = 23.5 \qquad \text{Find the } x\text{-coordinate of the vertex.}$$

The average number of visits to office-based physicians is lowest for people approximately 23.5 years old.

Skill Practice

6. In a recent year, an Internet company started up, became successful, and then quickly went bankrupt. The price of a share of the company's stock is given by the function

$$S(x) = -2.25x^2 + 40.5x + 42.75$$

where x is the number of months that the stock was on the market. Find the number of months at which the stock price was a maximum.

Skill Practice Answers

6. The maximum price for the stock occurred after 9 months.

Section 8.5 Practice Exercises

Boost *your* GRADE at mathzone.com! MathZone

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- Self-Tests
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- e-Professors
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Study Skills Exercise

1. Write the **vertex formula**.

Review Exercises

2. How does the graph of $f(x) = -2x^2$ compare with the graph of $y = x^2$?

3. How does the graph of $p(x) = \frac{1}{4}x^2$ compare with the graph of $y = x^2$?

4. How does the graph of $Q(x) = x^2 - \frac{8}{3}$ compare with the graph of $y = x^2$?

5. How does the graph of $r(x) = x^2 + 7$ compare with the graph of $y = x^2$?

6. How does the graph of $s(x) = (x - 4)^2$ compare with the graph of $y = x^2$?

7. How does the graph of $t(x) = (x + 10)^2$ compare with the graph of $y = x^2$?

8. Find the coordinates of the vertex of the graph of $g(x) = 2(x + 3)^2 - 4$.

For Exercises 9–16, find the value of n to complete the square.

9. $x^2 - 8x + n$
10. $x^2 + 4x + n$
11. $y^2 + 7y + n$
12. $a^2 - a + n$
13. $b^2 + \frac{2}{9}b + n$
14. $m^2 - \frac{2}{7}m + n$
15. $t^2 - \frac{1}{3}t + n$
16. $p^2 + \frac{1}{4}p + n$

Concept 1: Writing a Quadratic Function in the Form $f(x) = a(x - h)^2 + k$

For Exercises 17–30, write the function in the form $f(x) = a(x - h)^2 + k$ by completing the square. Then identify the vertex.

17. $g(x) = x^2 - 8x + 5$
18. $h(x) = x^2 + 4x + 5$
19. $n(x) = 2x^2 + 12x + 13$
20. $f(x) = 4x^2 + 16x + 19$
21. $p(x) = -3x^2 + 6x - 5$
22. $q(x) = -2x^2 + 12x - 11$
23. $k(x) = x^2 + 7x - 10$
24. $m(x) = x^2 - x - 8$
25. $f(x) = x^2 + 8x + 1$
26. $g(x) = x^2 + 5x - 2$
27. $F(x) = 5x^2 + 10x + 1$
28. $G(x) = 4x^2 + 4x - 7$
29. $P(x) = -2x^2 + x$
30. $Q(x) = 3x^2 - 12x$

Concept 2: Vertex Formula

For Exercises 31–44, find the vertex by using the vertex formula.

31. $Q(x) = x^2 - 4x + 7$
32. $T(x) = x^2 - 8x + 17$
33. $r(x) = -3x^2 - 6x - 5$
34. $s(x) = -2x^2 - 12x - 19$
35. $N(x) = x^2 + 8x + 1$
36. $M(x) = x^2 + 6x - 5$
37. $m(x) = \frac{1}{2}x^2 + x + \frac{5}{2}$
38. $n(x) = \frac{1}{2}x^2 + 2x + 3$
39. $k(x) = -x^2 + 2x + 2$
40. $h(x) = -x^2 + 4x - 3$
41. $f(x) = 2x^2 + 4x + 6$
42. $g(x) = 3x^2 + 12x + 9$
43. $A(x) = -\frac{1}{3}x^2 + x$
44. $B(x) = -\frac{2}{3}x^2 - 2x$

Concept 3: Determining the Vertex and Intercepts of a Quadratic Function

For Exercises 45–50

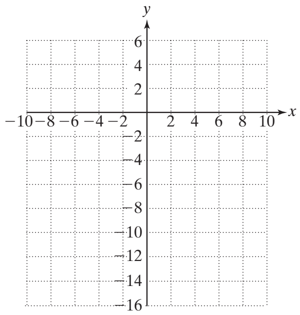
a. Find the vertex.

b. Find the y-intercept.

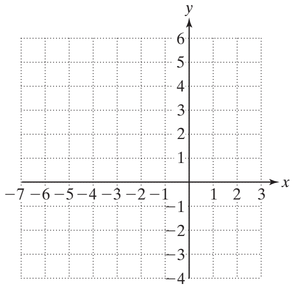
c. Find the x-intercept(s), if they exist.

d. Use this information to graph the function.

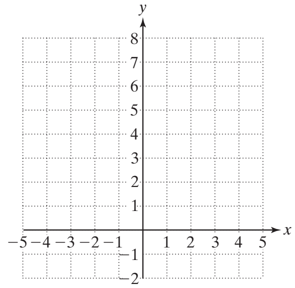
45. $y = x^2 + 9x + 8$



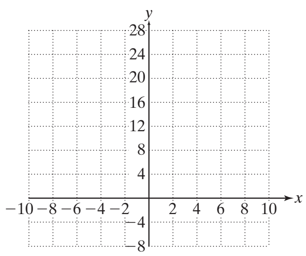
46. $y = x^2 + 7x + 10$



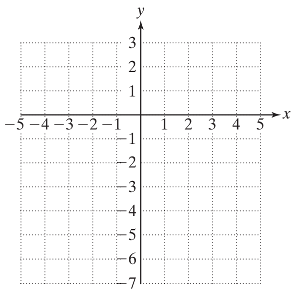
47. $y = 2x^2 - 2x + 4$



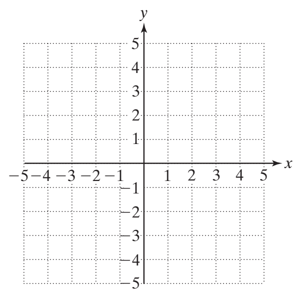
48. $y = 2x^2 - 12x + 19$



49. $y = -x^2 + 3x - \frac{9}{4}$



50. $y = -x^2 - \frac{3}{2}x - \frac{9}{16}$



Concept 4: Vertex of a Parabola: Applications

51. Mia sells used MP3 players. The average cost to package MP3 players is given by the equation $C(x) = 2x^2 - 40x + 2200$, where x is the number of MP3 players she packages in a week. How many players must she package to minimize her average cost?

52. Ben sells used iPods. The average cost to package iPods is given by the equation $C(x) = 3x^2 - 120x + 1300$, where x is the number of iPods packaged per month. Determine the number of iPods that Ben needs to package to minimize the average cost.

53. The pressure x in an automobile tire can affect its wear. Both overinflated and underinflated tires can lead to poor performance and poor mileage. For one particular tire, the function P represents the number of miles that a tire lasts (in thousands) for a given pressure x.

$$P(x) = -0.857x^2 + 56.1x - 880$$

where x is the tire pressure in pounds per square inch (psi).

a. Find the tire pressure that will yield the maximum mileage. Round to the nearest pound per square inch.

b. What is the maximum number of miles that a tire can last? Round to the nearest thousand.

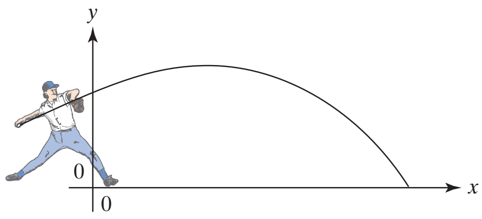
54. A baseball player throws a ball, and the height of the ball (in feet) can be approximated by

$$y(x) = -0.011x^2 + 0.577x + 5$$

where x is the horizontal position of the ball measured in feet from the origin.

a. For what value of x will the ball reach its highest point? Round to the nearest foot.

b. What is the maximum height of the ball? Round to the nearest tenth of a foot.



55. For a fund-raising activity, a charitable organization produces cookbooks to sell in the community. The profit (in dollars) depends on the number of cookbooks produced, x, according to

$$p(x) = -\frac{1}{50}x^2 + 12x - 550, \text{ where } x \geq 0.$$

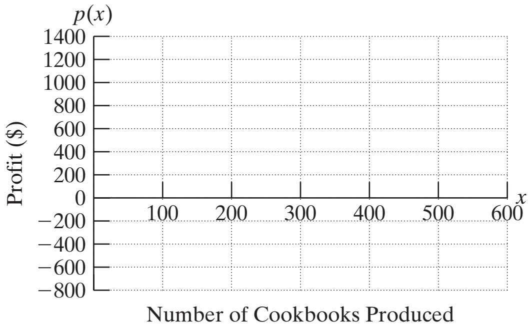
a. How much profit is made when 100 cookbooks are produced?

b. Find the y-intercept of the profit function, and interpret its meaning in the context of this problem.

c. How many cookbooks must be produced for the organization to break even? (*Hint:* Find the x-intercepts.)

d. Find the vertex.

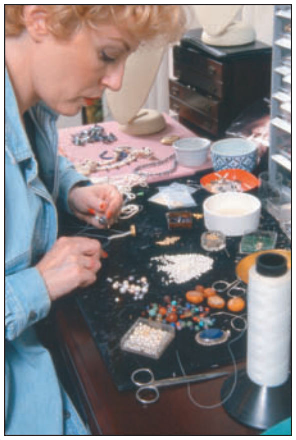
e. Sketch the function.



f. How many cookbooks must be produced to maximize profit? What is the maximum profit?

56. A jewelry maker sells bracelets at art shows. The profit (in dollars) depends on the number of bracelets produced, x, according to

$$p(x) = -\frac{1}{10}x^2 + 42x - 1260 \text{ where } x \geq 0.$$



a. How much profit does the jeweler make when 10 bracelets are produced?

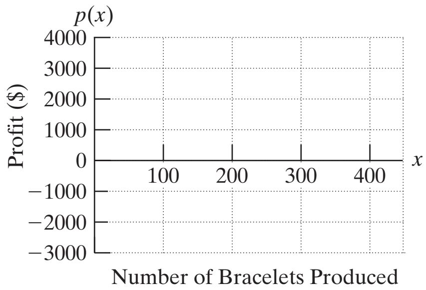
b. Find the y-intercept of the profit function, and interpret its meaning in the context of this problem.

c. How many bracelets must be produced for the jeweler to break even? Round to the nearest whole unit.

d. Find the vertex.

e. Sketch the function.

f. How many bracelets must be produced to maximize profit? What is the maximum profit?



57. Gas mileage depends in part on the speed of the car. The gas mileage of a subcompact car is given by the function $m(x) = -0.04x^2 + 3.6x - 49$, where $20 \leq x \leq 70$ represents the speed in miles per hour and $m(x)$ is given in miles per gallon. At what speed will the car get the maximum gas mileage?

58. Gas mileage depends in part on the speed of the car. The gas mileage of a luxury car is given by the function $L(x) = -0.015x^2 + 1.44x - 21$, where $25 \leq x \leq 70$ represents the speed in miles per hour and $L(x)$ is given in miles per gallon. At what speed will the car get the maximum gas mileage?

59. Tetanus bacillus bacterium is cultured to produce tetanus toxin used in an inactive form for the tetanus vaccine. The amount of toxin produced per batch increases with time and then becomes unstable. The amount of toxin (in grams) as a function of time t (in hours) can be approximated by the following function.

$$b(t) = -\frac{1}{1152}t^2 + \frac{1}{12}t$$

How many hours will it take to produce the maximum yield?

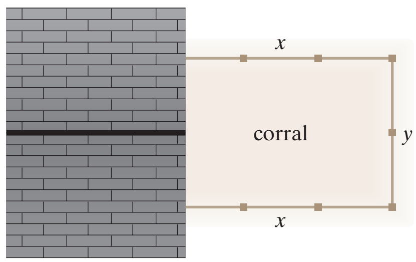
60. The bacterium *Pseudomonas aeruginosa* is cultured with an initial population of 10^4 active organisms. The population of active bacteria increases up to a point, and then due to a limited food supply and an increase of waste products, the population of living organisms decreases. Over the first 48 hr, the population can be approximated by the following function.

$$P(t) = -1718.75t^2 + 82{,}500t + 10{,}000 \qquad \text{where } 0 \leq t \leq 48$$

Find the time required for the population to reach its maximum value.

Expanding Your Skills

61. A farmer wants to fence a rectangular corral adjacent to the side of a barn; however, she has only 200 ft of fencing and wants to enclose the largest possible area. See the figure.



a. If x represents the length of the corral and y represents the width, explain why the dimensions of the corral are subject to the constraint $2x + y = 200$.

b. The area of the corral is given by $A = xy$. Use the constraint equation from part (a) to express A as a function of x, where $0 < x < 100$.

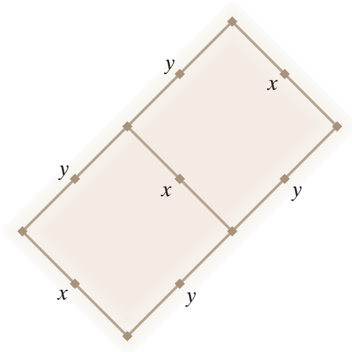
c. Use the function from part (b) to find the dimensions of the corral that will yield the maximum area. [*Hint:* Find the vertex of the function from part (b).]

62. A veterinarian wants to construct two equal-sized pens of maximum area out of 240 ft of fencing. See the figure.

a. If x represents the length of each pen and y represents the width of each pen, explain why the dimensions of the pens are subject to the constraint $3x + 4y = 240$.

b. The area of each individual pen is given by $A = xy$. Use the constraint equation from part (a) to express A as a function of x, where $0 < x < 80$.

c. Use the function from part (b) to find the dimensions of an individual pen that will yield the maximum area. [*Hint:* Find the vertex of the function from part (b).]



Graphing Calculator Exercises

For Exercises 63–68, graph the functions in Exercises 45–50 on a graphing calculator. Use the *Max* or *Min* feature or *Zoom* and *Trace* to approximate the vertex.

63. $Y_1 = x^2 + 9x + 8$ (Exercise 45)

64. $Y_1 = x^2 + 7x + 10$ (Exercise 46)

65. $Y_1 = 2x^2 - 2x + 4$ (Exercise 47)

66. $Y_1 = 2x^2 - 12x + 19$ (Exercise 48)

67. $Y_1 = -x^2 + 3x - \frac{9}{4}$ (Exercise 49)

68. $Y_1 = -x^2 - \frac{3}{2}x - \frac{9}{16}$ (Exercise 50)

Chapter 8 SUMMARY

Section 8.1 Square Root Property and Completing the Square

Key Concepts

The **square root property** states that

If $x^2 = k$ then $x = \pm\sqrt{k}$

Follow these steps to solve a quadratic equation in the form $ax^2 + bx + c = 0\,(a \neq 0)$ by completing the square and applying the square root property:

1. Divide both sides by a to make the leading coefficient 1.
2. Isolate the variable terms on one side of the equation.
3. Complete the square: Add the square of one-half the linear term coefficient to both sides of the equation. Then factor the resulting perfect square trinomial.
4. Apply the square root property and solve for x.

Examples

Example 1

$$(x - 5)^2 = -13$$

$$x - 5 = \pm\sqrt{-13} \qquad \text{(square root property)}$$

$$x = 5 \pm i\sqrt{13}$$

Example 2

$$2x^2 - 6x - 5 = 0$$

$$\frac{2x^2}{2} - \frac{6x}{2} - \frac{5}{2} = \frac{0}{2}$$

$$x^2 - 3x = \frac{5}{2}$$

$$\textit{Note: } \left[\frac{1}{2} \cdot (-3)\right]^2 = \frac{9}{4}$$

$$x^2 - 3x + \frac{9}{4} = \frac{5}{2} + \frac{9}{4}$$

$$x^2 - 3x + \frac{9}{4} = \frac{10}{4} + \frac{9}{4}$$

$$\left(x - \frac{3}{2}\right)^2 = \frac{19}{4}$$

$$x - \frac{3}{2} = \pm\sqrt{\frac{19}{4}}$$

$$x = \frac{3}{2} \pm \frac{\sqrt{19}}{2} \quad \text{or} \quad x = \frac{3 \pm \sqrt{19}}{2}$$

Section 8.2 Quadratic Formula

Key Concepts

The solutions to a quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ are given by the **quadratic formula**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The **discriminant** of a quadratic equation $ax^2 + bx + c = 0$ is $b^2 - 4ac$. If a, b, and c are rational numbers, then

1. If $b^2 - 4ac > 0$, then there will be two real solutions. Moreover,
   a. If $b^2 - 4ac$ is a perfect square, the solutions will be rational numbers.
   b. If $b^2 - 4ac$ is not a perfect square, the solutions will be irrational numbers.
2. If $b^2 - 4ac < 0$, then there will be two imaginary solutions.
3. If $b^2 - 4ac = 0$, then there will be one rational solution.

Three methods to solve a quadratic equation are

1. Factoring and applying the zero product rule.
2. Completing the square and applying the square root property.
3. Using the quadratic formula.

Example

Example 1

$$3x^2 - 2x + 4 = 0$$

$a = 3 \qquad b = -2 \qquad c = 4$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(4)}}{2(3)}$$

$$= \frac{2 \pm \sqrt{4 - 48}}{6}$$

$$= \frac{2 \pm \sqrt{-44}}{6}$$

The discriminant is -44. Therefore, there will be two imaginary solutions.

$$= \frac{2 \pm 2i\sqrt{11}}{6}$$

$$= \frac{1 \pm i\sqrt{11}}{3}$$

Section 8.3 Equations in Quadratic Form

Key Concepts

Substitution can be used to solve equations that are in quadratic form.

Examples

Example 1

$x^{2/3} - x^{1/3} - 12 = 0$

Let $u = x^{1/3}$. Therefore, $u^2 = (x^{1/3})^2 = x^{2/3}$

$u^2 - u - 12 = 0$

$(u - 4)(u + 3) = 0$

$u = 4 \quad$ or $\quad u = -3$

$x^{1/3} = 4 \quad$ or $\quad x^{1/3} = -3$

$x = 64 \quad$ or $\quad x = -27 \quad$ Cube both sides.

Section 8.4 Graphs of Quadratic Functions

Key Concepts

A quadratic function of the form $f(x) = x^2 + k$ shifts the graph of $y = x^2$ up k units if $k > 0$ and down $|k|$ units if $k < 0$.

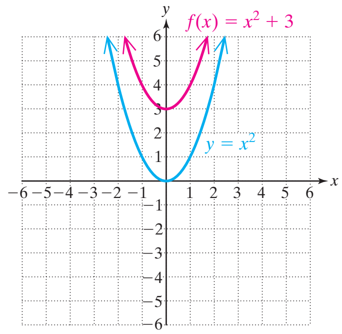
A quadratic function of the form $f(x) = (x - h)^2$ shifts the graph of $y = x^2$ right h units if $h > 0$ and left $|h|$ units if $h < 0$.

The graph of a quadratic function of the form $f(x) = ax^2$ is a parabola that opens up when $a > 0$ and opens down when $a < 0$. If $|a| > 1$, the graph of $y = x^2$ is stretched vertically by a factor of $|a|$. If $0 < |a| < 1$, the graph of $y = x^2$ is shrunk vertically by a factor of $|a|$.

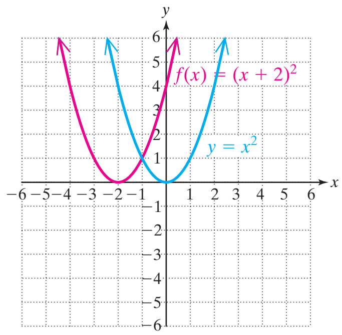
A quadratic function of the form $f(x) = a(x - h)^2 + k$ has vertex (h, k). If $a > 0$, the vertex represents the minimum point. If $a < 0$, the vertex represents the maximum point.

Examples

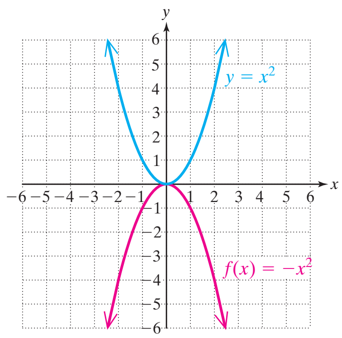
Example 1



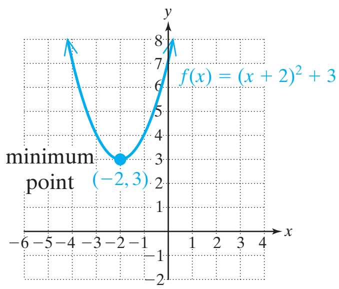
Example 2



Example 3



Example 4



Section 8.5 Vertex of a Parabola and Applications

Key Concepts

Completing the square is a technique used to write a quadratic function $f(x) = ax^2 + bx + c \quad (a \neq 0)$ in the form $f(x) = a(x - h)^2 + k$ for the purpose of identifying the vertex (h, k).

Examples

Example 1

$$
\begin{aligned}
f(x) &= 3x^2 + 6x + 11 \\
&= 3(x^2 + 2x \qquad\quad) + 11 \\
&= 3(x^2 + 2x + 1 - 1) + 11 \\
&= 3(x^2 + 2x + 1) - 3 + 11 \\
&= 3(x + 1)^2 + 8 \\
&= 3[x - (-1)]^2 + 8
\end{aligned}
$$

The vertex is $(-1, 8)$. Because $a = 3 > 0$, the parabola opens upward and the vertex $(-1, 8)$ is a minimum point.

The **vertex formula** finds the vertex of a quadratic function $f(x) = ax^2 + bx + c \quad (a \neq 0)$.

The vertex is

$$\left(\frac{-b}{2a}, \frac{4ac - b^2}{4a}\right) \quad \text{or} \quad \left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$$

Example 2

$f(x) = -5x^2 + 4x - 1$

$a = -5 \qquad b = 4 \qquad c = -1$

$$x = \frac{-4}{2(-5)} = \frac{2}{5}$$

$$f\left(\frac{2}{5}\right) = -5\left(\frac{2}{5}\right)^2 + 4\left(\frac{2}{5}\right) - 1 = -\frac{1}{5}$$

The vertex is $(\frac{2}{5}, -\frac{1}{5})$. Because $a = -5 < 0$, the parabola opens downward and the vertex $(\frac{2}{5}, -\frac{1}{5})$ is a maximum point.

Chapter 8 Review Exercises

Section 8.1

For Exercises 1–8, solve the equations by using the square root property.

1. $x^2 = 5$

2. $2y^2 = -8$

3. $a^2 = 81$

4. $3b^2 = -19$

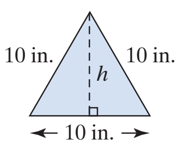
5. $(x - 2)^2 = 72$

6. $(2x - 5)^2 = -9$

7. $(3y - 1)^2 = 3$

8. $3(m - 4)^2 = 15$

9. The length of each side of an equilateral triangle is 10 in. Find the height of the triangle. Round the answer to the nearest tenth of an inch.



10. Use the square root property to find the length of the sides of a square whose area is 81 in.^{2}

11. Use the square root property to find the length of the sides of a square whose area is 150 in.^{2} Round the answer to the nearest tenth of an inch.

For Exercises 12–15, find the value of n so that the expression is a perfect square trinomial. Then factor the trinomial.

12. $x^2 + 16x + n$

13. $x^2 - 9x + n$

14. $y^2 + \frac{1}{2}y + n$

15. $z^2 - \frac{2}{5}z + n$

For Exercises 16–21, solve the equation by completing the square and applying the square root property.

16. $w^2 + 4w + 13 = 0$

17. $4y^2 - 12y + 13 = 0$

18. $3x^2 + 2x = 1$

19. $b^2 + \frac{7}{2}b = 2$

20. $2x^2 = 12x + 6$

21. $-t^2 + 8t - 25 = 0$

Section 8.2

22. Explain how the discriminant can determine the type and number of solutions to a quadratic equation with rational coefficients.

For Exercises 23–28, determine the type (rational, irrational, or imaginary) and number of solutions for the equations by using the discriminant.

23. $x^2 - 5x = -6$

24. $2y^2 = -3y$

25. $z^2 + 23 = 17z$

26. $a^2 + a + 1 = 0$

27. $10b + 1 = -25b^2$

28. $3x^2 + 15 = 0$

For Exercises 29–36, solve the equations by using the quadratic formula.

29. $y^2 - 4y + 1 = 0$

30. $m^2 - 5m + 25 = 0$

31. $6a(a - 1) = 10 + a$

32. $3x(x - 3) = x - 8$

33. $b^2 - \frac{4}{25} = \frac{3}{5}b$

34. $k^2 + 0.4k = 0.05$

35. $-32 + 4x - x^2 = 0$

36. $8y - y^2 = 10$

For Exercises 37–40, solve using any method.

37. $3x^2 - 4x = 6$

38. $\frac{b}{8} - \frac{2}{b} = \frac{3}{4}$

39. $y^2 + 14y = -46$

40. $(a + 1)^2 = 11$

41. The landing distance that a certain plane will travel on a runway is determined by the initial landing speed at the instant the plane touches down. The function D relates landing distance in feet to initial landing speed s:

$$D(s) = \frac{1}{10}s^2 - 3s + 22 \text{ for } s \geq 50$$

where s is in feet per second.

a. Find the landing distance for a plane traveling 150 ft/sec at touchdown.

b. If the landing speed is too fast, the pilot may run out of runway. If the speed is too slow, the plane may stall. Find the maximum initial landing speed of a plane for a runway that is 1000 ft long. Round to 1 decimal place.

42. The recent population (in thousands) of Kenya can be approximated by $P(t) = 4.62t^2 + 564.6t + 13{,}128$, where t is the number of years since 1974.

a. If this trend continues, approximate the number of people in Kenya in the year 2025.

b. In what year after 1974 will the population of Kenya reach 50 million? (*Hint:* 50 million equals 50,000 thousand.)

Section 8.3

For Exercises 43–52, solve the equations by using substitution, if necessary.

43. $x - 4\sqrt{x} - 21 = 0$

44. $n - 6\sqrt{n} + 8 = 0$

45. $y^4 - 11y^2 + 18 = 0$

46. $2m^4 - m^2 - 3 = 0$

47. $t^{2/5} + t^{1/5} - 6 = 0$

48. $p^{2/5} - 3p^{1/5} + 2 = 0$

49. $\frac{2t}{t + 1} + \frac{-3}{t - 2} = 1$

50. $\frac{1}{m - 2} - \frac{m}{m + 3} = 2$

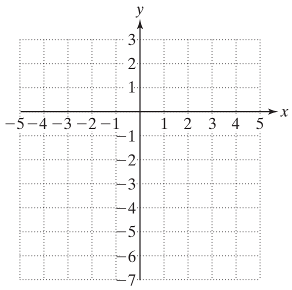
51. $(x^2 + 5)^2 + 2(x^2 + 5) - 8 = 0$

52. $(x^2 - 3)^2 - 5(x^2 - 3) + 4 = 0$

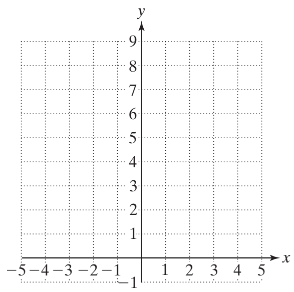
Section 8.4

For Exercises 53–60, graph the functions.

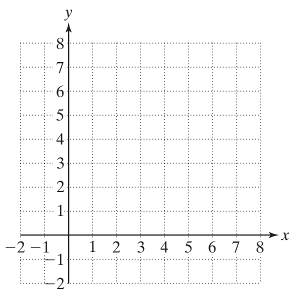
53. $g(x) = x^2 - 5$



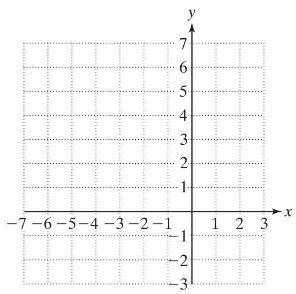
54. $f(x) = x^2 + 3$



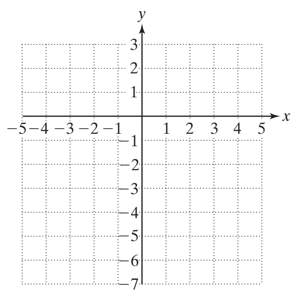
55. $h(x) = (x - 5)^2$



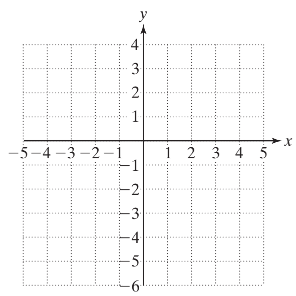
56. $k(x) = (x + 3)^2$



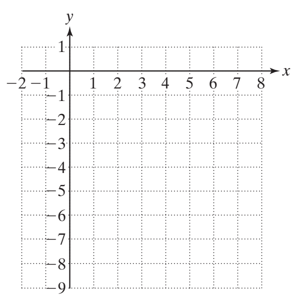
57. $m(x) = -2x^2$



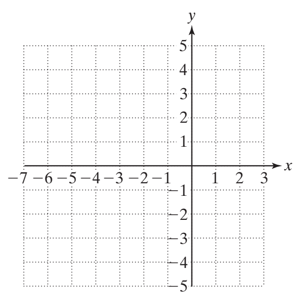
58. $n(x) = -4x^2$



59. $p(x) = -2(x - 5)^2 - 5$



60. $q(x) = -4(x + 3)^2 + 3$



For Exercises 61–62, write the coordinates of the vertex of the parabola and determine if the vertex is a maximum point or a minimum point. Then write the maximum or the minimum value.

61. $t(x) = \frac{1}{3}(x - 4)^2 + \frac{5}{3}$

62. $s(x) = -\frac{5}{7}(x - 1)^2 - \frac{1}{7}$

For Exercises 63–64, write the equation of the axis of symmetry of the parabola.

63. $a(x) = -\frac{3}{2}\left(x + \frac{2}{11}\right)^2 - \frac{4}{13}$

64. $w(x) = -\frac{4}{3}\left(x - \frac{3}{16}\right)^2 + \frac{2}{9}$

Section 8.5

For Exercises 65–68, write the function in the form $f(x) = a(x - h)^2 + k$ by completing the square. Then write the coordinates of the vertex.

65. $z(x) = x^2 - 6x + 7$

66. $b(x) = x^2 - 4x - 44$

67. $p(x) = -5x^2 - 10x - 13$

68. $q(x) = -3x^2 - 24x - 54$

For Exercises 69–72, find the coordinates of the vertex of each function by using the vertex formula.

69. $f(x) = -2x^2 + 4x - 17$

70. $g(x) = -4x^2 - 8x + 3$

71. $m(x) = 3x^2 - 3x + 11$

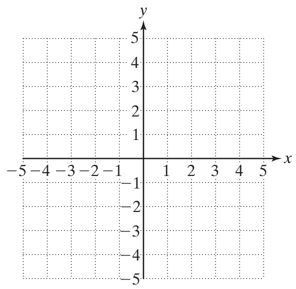
72. $n(x) = 3x^2 + 2x - 7$

73. For the quadratic equation $y = -(x + 2)^2 + 4$

a. Write the coordinates of the vertex.

b. Find the x- and y-intercepts.

c. Use this information to sketch a graph of the parabola.



74. The height of a projectile fired vertically into the air from the ground is given by the equation $h(t) = -16t^2 + 96t$, where t represents the number of seconds that the projectile has been in the air. How long will it take the projectile to reach its maximum height?

75. The weekly profit for a small catering service is given by $P(x) = -4x^2 + 1200x$, where x is the number of meals prepared. Find the number of meals that should be prepared to obtain the maximum profit.

Chapter 8 Test

For Exercises 1–3, solve the equation by using the square root property.

1. $(x + 3)^2 = 25$

2. $(p - 2)^2 = 12$

3. $(m + 1)^2 = -1$

4. Find the value of n so that the expression is a perfect square trinomial. Then factor the trinomial $d^2 + 7d + n$.

For Exercises 5–6, solve the equation by completing the square and applying the square root property.

5. $2x^2 + 12x - 36 = 0$

6. $2x^2 = 3x - 7$

For Exercises 7–8

a. Write the equation in standard form $ax^2 + bx + c = 0$.

b. Identify a, b, and c.

c. Find the discriminant.

d. Determine the number and type (rational, irrational, or imaginary) of solutions.

7. $x^2 - 3x = -12$

8. $y(y - 2) = -1$

For Exercises 9–10, solve the equation by using the quadratic formula.

9. $3x^2 - 4x + 1 = 0$

10. $x(x + 6) = -11 - x$

11. The base of a triangle is 3 ft less than twice the height. The area of the triangle is 14 ft^2. Find the base and the height. Round the answers to the nearest tenth of a foot.

12. A circular garden has an area of approximately 450 ft^2. Find the radius. Round the answer to the nearest tenth of a foot.

For Exercises 13–15, solve the equation by using substitution, if necessary.

13. $x - \sqrt{x} - 6 = 0$

14. $y^{2/3} + 2y^{1/3} = 8$

15. $(3y - 8)^2 - 13(3y - 8) + 30 = 0$

16. $p^4 - 15p^2 = -54$

17. $3 = \dfrac{y}{2} - \dfrac{1}{y + 1}$

For Exercises 18–21, find the x- and y-intercepts of the function. Then match the function with its graph.

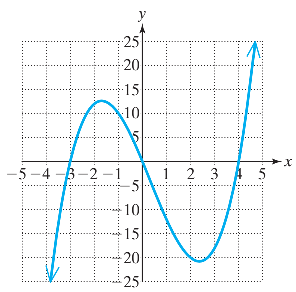
18. $f(x) = x^2 - 6x + 8$

19. $k(x) = x^3 + 4x^2 - 9x - 36$

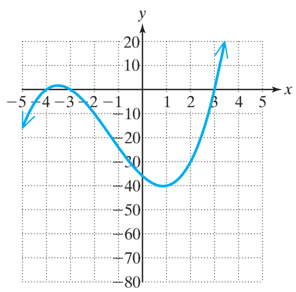
20. $p(x) = -2x^2 - 8x - 6$

21. $q(x) = x^3 - x^2 - 12x$

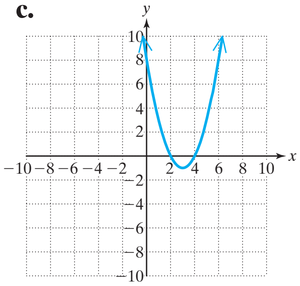
a.



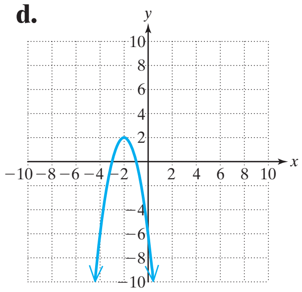
b.



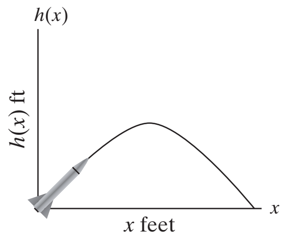
c.



d.



22. A child launches a toy rocket from the ground. The height of the rocket can be determined by its horizontal distance from the launch pad x by



$$h(x) = -\frac{x^2}{256} + x$$

where x and $h(x)$ are in feet.

How many feet from the launch pad will the rocket hit the ground?

23. The recent population (in millions) of India can be approximated by $P(t) = 0.135t^2 + 12.6t + 600$, where $t = 0$ corresponds to the year 1974.

a. If this trend continues, approximate the number of people in India in the year 2014.

b. Approximate the year in which the population of India reached 1 billion (1000 million). (Round to the nearest year.)

24. Explain the relationship between the graphs of $y = x^2$ and $y = x^2 - 2$.

25. Explain the relationship between the graphs of $y = x^2$ and $y = (x + 3)^2$.

26. Explain the relationship between the graphs of $y = 4x^2$ and $y = -4x^2$.

27. Given the function defined by

$$f(x) = -(x - 4)^2 + 2$$

a. Identify the vertex of the parabola.

b. Does this parabola open upward or downward?

c. Does the vertex represent the maximum or minimum point of the function?

d. What is the maximum or minimum value of the function f?

e. What is the axis of symmetry for this parabola?

28. For the function defined by $g(x) = 2x^2 - 20x + 51$, find the vertex by using two methods.

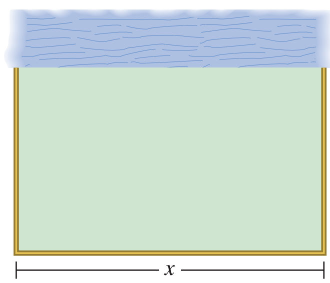
a. Complete the square to write $g(x)$ in the form $g(x) = a(x - h)^2 + k$. Identify the vertex.

b. Use the vertex formula to find the vertex.

29. A farmer has 400 ft of fencing with which to enclose a rectangular field. The field is situated such that one of its sides is adjacent to a river and requires no fencing. The area of the field (in square feet) can be modeled by

$$A(x) = -\frac{x^2}{2} + 200x$$

where x is the length of the side parallel to the river (measured in feet).



Use the function to determine the maximum area that can be enclosed.

Chapters 1–8 Cumulative Review Exercises

1. Given: $A = \{2, 4, 6, 8, 10\}$ and $B = \{2, 8, 12, 16\}$

a. Find $A \cup B$. **b.** Find $A \cap B$.

2. Perform the indicated operations and simplify.

$$(2x^2 - 5) - (x + 3)(5x - 2)$$

3. Simplify completely. $4^0 - \left(\frac{1}{2}\right)^{-3} - 81^{1/2}$

4. Perform the indicated operations. Write the answer in scientific notation:

$$(3.0 \times 10^{12})(6.0 \times 10^{-3})$$

5. a. Factor completely. $x^3 + 2x^2 - 9x - 18$

b. Divide by using long division. Identify the quotient and remainder.

$$(x^3 + 2x^2 - 9x - 18) \div (x - 3)$$

6. Multiply. $(\sqrt{x} - \sqrt{2})(\sqrt{x} + \sqrt{2})$

7. Simplify. $\dfrac{4}{\sqrt{2x}}$

8. Jacques invests a total of \$10,000 in two mutual funds. After 1 year, one fund produced 12% growth, and the other lost 3%. Find the amount invested in each fund if the total investment grew by \$900.

9. Solve the system of equations.

$$\frac{1}{9}x - \frac{1}{3}y = -\frac{13}{9}$$

$$x - \frac{1}{2}y = \frac{9}{2}$$

10. An object is fired straight up into the air from an initial height of 384 ft with an initial velocity of 160 ft/sec. The height in feet is given by

$$h(t) = -16t^2 + 160t + 384$$

where t is the time in seconds after launch.

a. Find the height of the object after 3 sec.

b. Find the height of the object after 7 sec.

c. Find the time required for the object to hit the ground.

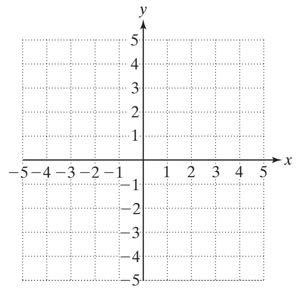
11. Solve the equation. $(x - 3)^2 + 16 = 0$

12. Solve the equation. $2x^2 + 5x - 1 = 0$

13. What number would have to be added to the quantity $x^2 + 10x$ to make it a perfect square trinomial?

14. Factor completely. $2x^3 + 250$

15. Graph the line.
$3x - 5y = 10$

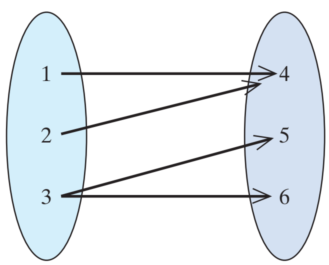


16. a. Find the x-intercepts of the function defined by $g(x) = 2x^2 - 9x + 10$.

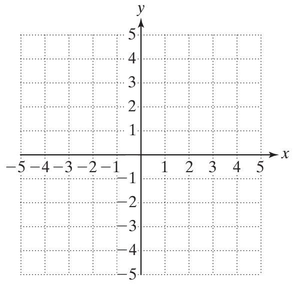
b. What is the y-intercept of $y = g(x)$?

17. Michael Jordan was the NBA leading scorer for 10 of 12 seasons between 1987 and 1998. In his 1998 season, he scored a total of 2357 points consisting of 1-point free throws, 2-point field goals, and 3-point field goals. He scored 286 more 2-point shots than he did free throws. The number of 3-point shots was 821 less than the number of 2-point shots. Determine the number of free throws, 2-point shots, and 3-point shots scored by Michael Jordan during his 1998 season.

18. Explain why this relation is *not* a function.



19. Graph the function defined by $f(x) = \dfrac{1}{x}$.



20. The quantity y varies directly as x and inversely as z. If $y = 15$ when $x = 50$ and $z = 10$, find y when $x = 65$ and $z = 5$.

21. The total number of flights (including passenger flights and cargo flights) at a large airport can be approximated by $F(x) = 300{,}000 + 0.008x$, where x is the number of passengers.

- **a.** Is this function linear, quadratic, constant, or other?
- **b.** Find the y-intercept and interpret its meaning in the context of this problem.
- **c.** What is the slope of the function and what does the slope mean in the context of this problem?

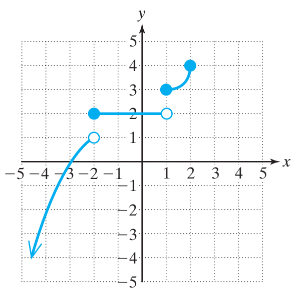
22. Given the function defined by $g(x) = \sqrt{2 - x}$, find the function values (if they exist) over the set of real numbers.

a. $g(-7)$ **b.** $g(0)$ **c.** $g(3)$

23. Let $m(x) = \sqrt{x + 4}$ and $n(x) = x^2 + 2$. Find

a. The domain of m **b.** The domain of n

24. Consider the function $y = f(x)$ graphed here. Find



a. The domain **b.** The range

c. $f(-2)$ **d.** $f(1)$ **e.** $f(0)$

f. For what value(s) of x is $f(x) = 0$?

25. Solve. $\sqrt{8x + 5} = \sqrt{2x} + 2$

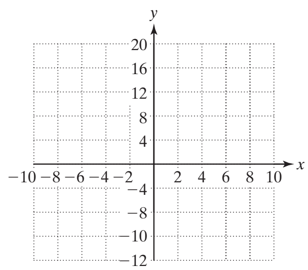
26. Solve for f. $\dfrac{1}{p} + \dfrac{1}{q} = \dfrac{1}{f}$

27. Solve. $\dfrac{15}{t^2 - 2t - 8} = \dfrac{1}{t - 4} + \dfrac{2}{t + 2}$

28. Simplify. $\dfrac{y - \dfrac{4}{y - 3}}{y - 4}$

29. Given: the function defined by $f(x) = 2(x - 3)^2 + 1$

- **a.** Write the coordinates of the vertex.
- **b.** Does the graph of the function open upward or downward?
- **c.** Write the coordinates of the y-intercept.
- **d.** Find the x-intercepts, if possible.
- **e.** Sketch the function.



30. Find the vertex of the parabola.

$$f(x) = x^2 - 16x + 2$$