Radicals and Complex Numbers

7

In this chapter we study radical expressions. This includes operations on square roots, cube roots, fourth roots, and so on. We also revisit the Pythagorean theorem and its applications.

As you work through this chapter, try to simplify the expressions or solve the equations. The following is a Sudoku puzzle. Use the clues given to fill in the boxes labeled a through n. Then fill in the remaining part of the grid so that every row, every column, and every 2×3 box contains the digits 1 through 6.

Clues

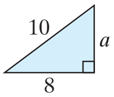
a. 2^2 b. $\sqrt[3]{27}$

c. $\sqrt[5]{32}$ d. $\sqrt[100]{1}$

e. Solution to $\sqrt{2x + 10} - 1 = 3$

f. Solution to $\sqrt{31 + x} - 1 = 5$

g. $\left| -\sqrt[4]{16} \right|$ h. Length of side a.



i. Solution to $\sqrt[3]{x + 123} = 5$

j. $125^{1/3}$ k. $(\sqrt{5} + 2)(\sqrt{5} - 2)$

l. $\left(\frac{1}{64}\right)^{-2/3} - \left(\frac{1}{169}\right)^{-1/2}$

m. $(3 + 2i) - (-4 - 3i) + (-2 - 5i)$

n. $\sqrt{10^2 - 8^2}$

		a			b
			c	d	
e	f			g	h
	i			j	k
	l	m			
n					

Section 7.1 Definition of an *n*th Root

Concepts

1. Definition of a Square Root
2. Definition of an *n*th Root
3. Roots of Variable Expressions
4. Pythagorean Theorem
5. Radical Functions

1. Definition of a Square Root

The inverse operation to squaring a number is to find its square roots. For example, finding a square root of 36 is equivalent to asking, "what number when squared equals 36?"

One obvious answer to this question is 6 because $(6)^2 = 36$, but -6 will also work, because $(-6)^2 = 36$.

Definition of a Square Root

b is a **square root** of a if $b^2 = a$.

TIP: All positive real numbers have two real-valued square roots: one positive and one negative. Zero has only one square root, which is 0 itself. Finally, for any negative real number, there are no real-valued square roots.

Example 1 Identifying Square Roots

Identify the square roots of the real numbers.

a. 25 **b.** 49 **c.** 0 **d.** -9

Solution:

a. 5 is a square root of 25 because $(5)^2 = 25$.
-5 is a square root of 25 because $(-5)^2 = 25$.

b. 7 is a square root of 49 because $(7)^2 = 49$.
-7 is a square root of 49 because $(-7)^2 = 49$.

c. 0 is a square root of 0 because $(0)^2 = 0$.

d. There are no real numbers that when squared will equal a negative number; therefore, there are no real-valued square roots of -9.

Skill Practice Identify the square roots of the real numbers.

1. 64 **2.** 16 **3.** 1 **4.** -100

Recall from Section 1.2 that the positive square root of a real number can be denoted with a **radical sign** $\sqrt{\ }$.

Notation for Positive and Negative Square Roots

Let a represent a positive real number. Then

1. $\sqrt{a}$ is the *positive* square root of a. The positive square root is also called the **principal square root**.
2. $-\sqrt{a}$ is the *negative* square root of a.
3. $\sqrt{0} = 0$

Skill Practice Answers

1. -8 and 8
2. -4 and 4
3. -1 and 1
4. No real-valued square roots

Example 2 Simplifying a Square Root

Simplify the square roots.

a. $\sqrt{36}$ **b.** $\sqrt{\frac{4}{9}}$ **c.** $\sqrt{0.04}$

Solution:

a. $\sqrt{36}$ denotes the positive square root of 36.

$\sqrt{36} = 6$

b. $\sqrt{\frac{4}{9}}$ denotes the positive square root of $\frac{4}{9}$.

$\sqrt{\frac{4}{9}} = \frac{2}{3}$

c. $\sqrt{0.04}$ denotes the positive square root of 0.04.

$\sqrt{0.04} = 0.2$

Skill Practice Simplify the square roots.

5. $\sqrt{81}$ **6.** $\sqrt{\frac{36}{49}}$ **7.** $\sqrt{0.09}$

The numbers 36, $\frac{4}{9}$, and 0.04 are **perfect squares** because their square roots are rational numbers.

Radicals that cannot be simplified to rational numbers are irrational numbers. Recall that an irrational number cannot be written as a terminating or repeating decimal. For example, the symbol $\sqrt{13}$ is used to represent the *exact* value of the square root of 13. The symbol $\sqrt{42}$ is used to represent the *exact* value of the square root of 42. These values can be approximated by a rational number by using a calculator.

$$\sqrt{13} \approx 3.605551275 \qquad \sqrt{42} \approx 6.480740698$$

TIP: Before using a calculator to evaluate a square root, try estimating the value first.

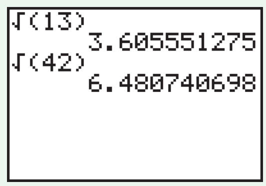
$\sqrt{13}$ must be a number between 3 and 4 because $\sqrt{9} < \sqrt{13} < \sqrt{16}$.

$\sqrt{42}$ must be a number between 6 and 7 because $\sqrt{36} < \sqrt{42} < \sqrt{49}$.

A negative number cannot have a real number as a square root because no real number when squared is negative. For example, $\sqrt{-25}$ is *not* a real number because there is no real number b for which $(b)^2 = -25$.

Calculator Connections

Use a calculator to approximate the values of $\sqrt{13}$ and $\sqrt{42}$.



Skill Practice Answers

5. 9 **6.** $\frac{6}{7}$ **7.** 0.3

Example 3 Evaluating Square Roots

Simplify the square roots, if possible.

a. $\sqrt{-144}$ **b.** $-\sqrt{144}$ **c.** $\sqrt{-0.01}$ **d.** $-\sqrt{\frac{1}{9}}$

Solution:

a. $\sqrt{-144}$ is *not* a real number.

b. $-\sqrt{144}$

$= -1 \cdot \sqrt{144}$

$= -1 \cdot 12$

$= -12$

TIP: For the expression $-\sqrt{144}$, the factor of -1 is *outside* the radical.

c. $\sqrt{-0.01}$ is *not* a real number.

d. $-\sqrt{\frac{1}{9}}$

$= -1 \cdot \sqrt{\frac{1}{9}}$

$= -1 \cdot \frac{1}{3}$

$= -\frac{1}{3}$

Skill Practice Simplify the square roots, if possible.

8. $-\sqrt{64}$ **9.** $\sqrt{-81}$ **10.** $\sqrt{-\frac{1}{4}}$ **11.** $-\sqrt{0.25}$

2. Definition of an *n*th Root

Finding a square root of a number is the inverse process of squaring a number. This concept can be extended to finding a third root (called a cube root), a fourth root, and in general an ***n*th root**.

Definition of an *n*th Root

b is an nth root of a if $b^n = a$.

For example, 2 is a fourth root of 16 because $2^4 = 16$.

The radical sign $\sqrt{}$ is used to denote the principal square root of a number. The symbol $\sqrt[n]{}$ is used to denote the principal nth root of a number. In the expression $\sqrt[n]{a}$, n is called the **index** of the radical, and a is called the **radicand**. For a square root, the index is 2, but it is usually not written ($\sqrt[2]{a}$ is denoted simply as $\sqrt{a}$). A radical with an index of 3 is called a **cube root**, denoted by $\sqrt[3]{a}$.

Skill Practice Answers

8. -8 **9.** Not a real number
10. Not a real number **11.** -0.5

Definition of $\sqrt[n]{a}$

1. If n is a positive *even* integer and $a > 0$, then $\sqrt[n]{a}$ is the principal (positive) nth root of a.
2. If $n > 1$ is an *odd* integer, then $\sqrt[n]{a}$ is the principal nth root of a.
3. If $n > 1$ is an integer, then $\sqrt[n]{0} = 0$.

For the purpose of simplifying radicals, it is helpful to know the following powers:

Perfect Cubes	Perfect Fourth Powers	Perfect Fifth Powers
$1^3 = 1$	$1^4 = 1$	$1^5 = 1$
$2^3 = 8$	$2^4 = 16$	$2^5 = 32$
$3^3 = 27$	$3^4 = 81$	$3^5 = 243$
$4^3 = 64$	$4^4 = 256$	$4^5 = 1024$
$5^3 = 125$	$5^4 = 625$	$5^5 = 3125$

Example 4 Identifying the Principal *n*th Root of a Real Number

Simplify the expressions, if possible.

a. $\sqrt{4}$ **b.** $\sqrt[3]{64}$ **c.** $\sqrt[5]{-32}$ **d.** $\sqrt[4]{81}$

e. $\sqrt[6]{1{,}000{,}000}$ **f.** $\sqrt{-100}$ **g.** $\sqrt[4]{-16}$

Solution:

a. $\sqrt{4} = 2$ because $(2)^2 = 4$

b. $\sqrt[3]{64} = 4$ because $(4)^3 = 64$

c. $\sqrt[5]{-32} = -2$ because $(-2)^5 = -32$

d. $\sqrt[4]{81} = 3$ because $(3)^4 = 81$

e. $\sqrt[6]{1{,}000{,}000} = 10$ because $(10)^6 = 1{,}000{,}000$

f. $\sqrt{-100}$ is not a real number. No real number when squared equals -100.

g. $\sqrt[4]{-16}$ is not a real number. No real number when raised to the fourth power equals -16.

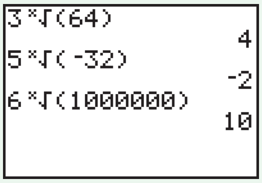
Skill Practice Simplify, if possible.

12. $\sqrt[4]{16}$ **13.** $\sqrt[3]{1000}$ **14.** $\sqrt[5]{-1}$ **15.** $\sqrt[5]{32}$

16. $\sqrt[5]{100{,}000}$ **17.** $\sqrt{-36}$ **18.** $\sqrt[3]{-27}$

Calculator Connections

A calculator can be used to approximate nth roots by using the $\sqrt[x]{\ }$ function. On most calculators, the index is entered first.



Examples 4(f) and 4(g) illustrate that an nth root of a negative quantity is not a real number if the index is even. This is because no real number raised to an even power is negative.

3. Roots of Variable Expressions

Finding an nth root of a variable expression is similar to finding an nth root of a numerical expression. For roots with an even index, however, particular care must be taken to obtain a nonnegative result.

Skill Practice Answers

12. 2 **13.** 10 **14.** -1
15. 2 **16.** 10
17. Not a real number
18. -3

Definition of $\sqrt[n]{a^n}$

1. If n is a positive *odd* integer, then $\sqrt[n]{a^n} = a$.
2. If n is a positive *even* integer, then $\sqrt[n]{a^n} = |a|$.

The absolute value bars are necessary for roots with an even index because the variable a may represent a positive quantity or a negative quantity. By using absolute value bars, $\sqrt[n]{a^n} = |a|$ is nonnegative and represents the principal nth root of a.

Example 5 Simplifying Expressions of the Form $\sqrt[n]{a^n}$

Simplify the expressions.

a. $\sqrt[4]{(-3)^4}$ **b.** $\sqrt[5]{(-3)^5}$ **c.** $\sqrt{(x+2)^2}$ **d.** $\sqrt[3]{(a+b)^3}$ **e.** $\sqrt{y^4}$

Solution:

a. $\sqrt[4]{(-3)^4} = |-3| = 3$ — Because this is an *even*-indexed root, absolute value bars are necessary to make the answer positive.

b. $\sqrt[5]{(-3)^5} = -3$ — This is an *odd*-indexed root, so absolute value bars are not used.

c. $\sqrt{(x+2)^2} = |x+2|$ — Because this is an *even*-indexed root, absolute value bars are necessary. The sign of the quantity $x + 2$ is unknown; however, $|x + 2| \geq 0$ regardless of the value of x.

d. $\sqrt[3]{(a+b)^3} = a + b$ — This is an *odd*-indexed root, so absolute value bars are not used.

e. $\sqrt{y^4} = \sqrt{(y^2)^2}$

$= |y^2|$ — Because this is an even-indexed root, use absolute value bars.

$= y^2$ — However, because y^2 is nonnegative, the absolute value bars are not necessary.

Skill Practice Simplify the expressions.

19. $\sqrt{(-4)^2}$ **20.** $\sqrt[3]{(-4)^3}$ **21.** $\sqrt{y^2}$

22. $\sqrt[4]{(a+1)^4}$ **23.** $\sqrt[4]{v^8}$

If n is an even integer, then $\sqrt[n]{a^n} = |a|$; however, if the variable a is assumed to be *nonnegative,* then the absolute value bars may be dropped. That is, $\sqrt[n]{a^n} = a$ provided $a \geq 0$. In many examples and exercises, we will make the assumption that the variables within a radical expression are positive real numbers. In such a case, the absolute value bars are not needed to evaluate $\sqrt[n]{a^n}$.

It is helpful to become familiar with the patterns associated with perfect squares and perfect cubes involving variable expressions.

Skill Practice Answers

19. 4 **20.** -4
21. $|y|$ **22.** $|a+1|$
23. v^2

The following powers of x are perfect squares:

Perfect Squares

$(x^1)^2 = x^2$

$(x^2)^2 = x^4$

$(x^3)^2 = x^6$

$(x^4)^2 = x^8$

TIP: In general, any expression raised to an even power (a multiple of 2) is a perfect square.

The following powers of x are perfect cubes:

Perfect Cubes

$(x^1)^3 = x^3$

$(x^2)^3 = x^6$

$(x^3)^3 = x^9$

$(x^4)^3 = x^{12}$

TIP: In general, any expression raised to a power that is a multiple of 3 is a perfect cube.

These patterns may be extended to higher powers.

Example 6 Simplifying *n*th Roots

Simplify the expressions. Assume that all variables are positive real numbers.

a. $\sqrt{y^8}$ **b.** $\sqrt[3]{27a^3}$ **c.** $\sqrt[5]{\dfrac{a^5}{b^5}}$ **d.** $-\sqrt[4]{\dfrac{81x^4y^8}{16}}$

Solution:

a. $\sqrt{y^8} = \sqrt{(y^4)^2} = y^4$

b. $\sqrt[3]{27a^3} = \sqrt[3]{(3a)^3} = 3a$

c. $\sqrt[5]{\dfrac{a^5}{b^5}} = \sqrt[5]{\left(\dfrac{a}{b}\right)^5} = \dfrac{a}{b}$

d. $-\sqrt[4]{\dfrac{81x^4y^8}{16}} = -\sqrt[4]{\left(\dfrac{3xy^2}{2}\right)^4} = -\dfrac{3xy^2}{2}$

Skill Practice Simplify the expressions. Assume all variables represent positive real numbers.

24. $\sqrt{t^6}$ **25.** $\sqrt[3]{y^{12}}$ **26.** $\sqrt[4]{x^{12}y^4}$ **27.** $\sqrt[5]{\dfrac{32}{b^{10}}}$

4. Pythagorean Theorem

In Section 5.8, we used the Pythagorean theorem in several applications. For the triangle shown in Figure 7-1, the **Pythagorean theorem** may be stated as $a^2 + b^2 = c^2$. In this formula, a and b are the legs of the right triangle, and c is the hypotenuse. Notice that the hypotenuse is the longest side of the right triangle and is opposite the 90° angle.

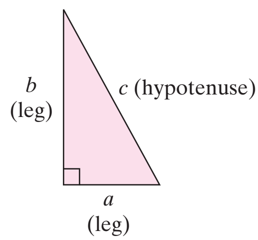


Figure 7-1

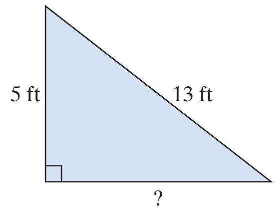
Skill Practice Answers

24. t^3 **25.** y^4

26. x^3y **27.** $\dfrac{2}{b^2}$

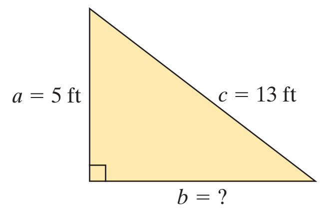
Example 7 Applying the Pythagorean Theorem

Use the Pythagorean theorem and the definition of the principal square root of a positive real number to find the length of the unknown side.



Solution:

Label the sides of the triangle.

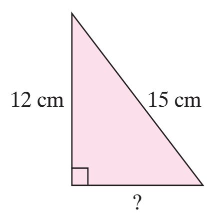


$a^2 + b^2 = c^2$ $(5)^2 + b^2 = (13)^2$	Apply the Pythagorean theorem.
$25 + b^2 = 169$	Simplify.
$b^2 = 169 - 25$	Isolate b^2.
$b^2 = 144$ $b = 12$	By definition, b must be one of the square roots of 144. Because b represents the length of a side of a triangle, choose the positive square root of 144.

The third side is 12 ft long.

Skill Practice

28. Use the Pythagorean theorem and the definition of the principal square root to find the length of the unknown side of the right triangle.



Example 8 Applying the Pythagorean Theorem

Two boats leave a dock at 12:00 noon. One travels due north at 6 mph, and the other travels due east at 8 mph (Figure 7-2). How far apart are the two boats after 2 hr?

Solution:

The boat traveling north travels a distance of (6 mph)(2 hr) = 12 mi. The boat traveling east travels a distance of (8 mph)(2 hr) = 16 mi. The

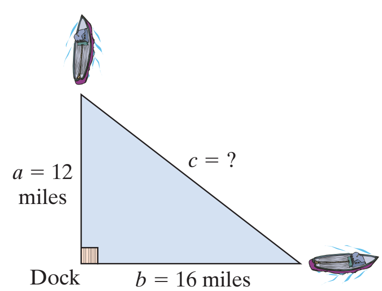


Figure 7-2

Skill Practice Answers

28. 9 cm

course of the boats forms a right triangle where the hypotenuse represents the distance between them.

$a^2 + b^2 = c^2$	
$(12)^2 + (16)^2 = c^2$	Apply the Pythagorean theorem.
$144 + 256 = c^2$	Simplify.
$400 = c^2$	
$20 = c$	By definition, c must be one of the square roots of 400. Choose the positive square root of 400 to represent distance between the two boats.

The boats are 20 mi apart.

Skill Practice

29. Two cars leave from the same place at the same time. One travels west at 40 mph, and the other travels north at 30 mph. How far apart are they after 2 hr?

5. Radical Functions

If n is an integer greater than 1, then a function written in the form $f(x) = \sqrt[n]{x}$ is called a **radical function**. Note that if n is an even integer, then the function will be a real number only if the radicand is nonnegative. Therefore, the domain is restricted to nonnegative real numbers, or equivalently, $[0, \infty)$. If n is an odd integer, then the domain is all real numbers.

Example 9 Determining the Domain of Radical Functions

For each function, write the domain in interval notation.

a. $g(t) = \sqrt[4]{t - 2}$ **b.** $h(a) = \sqrt[3]{a - 4}$ **c.** $k(x) = \sqrt{3 - 5x}$

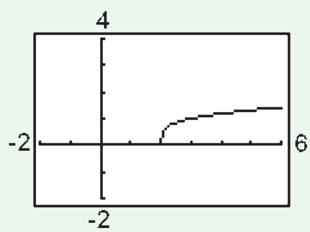
Solution:

a. $g(t) = \sqrt[4]{t - 2}$	The index is even. The radicand must be nonnegative.
$t - 2 \geq 0$	Set the radicand greater than or equal to zero.
$t \geq 2$	Solve for t.

The domain is $[2, \infty)$.

Calculator Connections

The domain of $g(t) = \sqrt[4]{t - 2}$ can be confirmed from its graph.



Skill Practice Answers

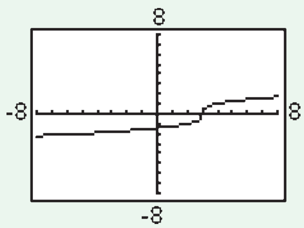
29. 100 mi

b. $h(a) = \sqrt[3]{a - 4}$ The index is odd; therefore, the domain is all real numbers.

The domain is $(-\infty, \infty)$.

Calculator Connections

The domain of $h(a) = \sqrt[3]{a - 4}$ can be confirmed from its graph.



c. $k(x) = \sqrt{3 - 5x}$ The index is even; therefore, the radicand must be nonnegative.

$3 - 5x \geq 0$ Set the radicand greater than or equal to zero.

$-5x \geq -3$ Solve for x.

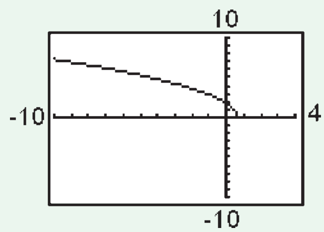
$\frac{-5x}{-5} \leq \frac{-3}{-5}$ Reverse the inequality sign, because we are dividing by a negative number.

$x \leq \frac{3}{5}$

The domain is $(-\infty, \frac{3}{5}]$.

Calculator Connections

The domain of the function defined by $k(x) = \sqrt{3 - 5x}$ can be confirmed from its graph.



Skill Practice For each function, write the domain in interval notation.

30. $f(x) = \sqrt{x + 5}$ **31.** $g(t) = \sqrt[3]{t - 9}$ **32.** $h(a) = \sqrt{1 - 2a}$

Skill Practice Answers

30. $[-5, \infty)$ **31.** $(-\infty, \infty)$
32. $(-\infty, \frac{1}{2}]$

Section 7.1 Practice Exercises

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Study Skills Exercise

1. Define the key terms.

a. **Square root** **b.** **Radical sign** **c.** **Principal square root** **d.** **Perfect square**

e. ***n*th root** **f.** **Index** **g.** **Radicand** **h.** **Cube root**

i. **Pythagorean theorem** **j.** **Radical function**

Concept 1: Definition of a Square Root

2. Simplify the expression $\sqrt[3]{8}$. Explain how you can check your answer.

3. **a.** Find the square roots of 64.
   b. Find $\sqrt{64}$.
   c. Explain the difference between the answers in part (a) and part (b).

4. **a.** Find the square roots of 121.
   b. Find $\sqrt{121}$.
   c. Explain the difference between the answers in part (a) and part (b).

5. **a.** What is the principal square root of 81?
   b. What is the negative square root of 81?

6. **a.** What is the principal square root of 100?
   b. What is the negative square root of 100?

7. Using the definition of a square root, explain why $\sqrt{-36}$ is not a real number.

For Exercises 8–19, evaluate the roots without using a calculator. Identify those that are not real numbers.

8. $\sqrt{25}$ **9.** $\sqrt{49}$ **10.** $-\sqrt{25}$ **11.** $-\sqrt{49}$

12. $\sqrt{-25}$ **13.** $\sqrt{-49}$ **14.** $\sqrt{\dfrac{100}{121}}$ **15.** $\sqrt{\dfrac{64}{9}}$

16. $\sqrt{0.64}$ **17.** $\sqrt{0.81}$ **18.** $-\sqrt{0.0144}$ **19.** $-\sqrt{0.16}$

Concept 2: Definition of an *n*th Root

20. Using the definition of an *n*th root, explain why $\sqrt[4]{-16}$ is not a real number.

For Exercises 21–22, evaluate the roots without using a calculator. Identify those that are not real numbers.

21. **a.** $\sqrt{64}$ **b.** $\sqrt[3]{64}$ **c.** $-\sqrt{64}$
d. $-\sqrt[3]{64}$ **e.** $\sqrt{-64}$ **f.** $\sqrt[3]{-64}$

22. **a.** $\sqrt{16}$ **b.** $\sqrt[4]{16}$ **c.** $-\sqrt{16}$
d. $-\sqrt[4]{16}$ **e.** $\sqrt{-16}$ **f.** $\sqrt[4]{-16}$

For Exercises 23–36, evaluate the roots without using a calculator. Identify those that are not real numbers.

23. $\sqrt[3]{-27}$ **24.** $\sqrt[3]{-125}$ **25.** $\sqrt[3]{\dfrac{1}{8}}$ **26.** $\sqrt[5]{\dfrac{1}{32}}$

27. $\sqrt[5]{32}$ **28.** $\sqrt[4]{1}$ **29.** $\sqrt[3]{-\dfrac{125}{64}}$ **30.** $\sqrt[3]{-\dfrac{8}{27}}$

31. $\sqrt[4]{-1}$ **32.** $\sqrt[6]{-1}$ **33.** $\sqrt[6]{1{,}000{,}000}$ **34.** $\sqrt[4]{10{,}000}$

35. $-\sqrt[3]{0.008}$ **36.** $-\sqrt[4]{0.0016}$

Concept 3: Roots of Variable Expressions

For Exercises 37–54, simplify the radical expressions.

37. $\sqrt{a^2}$ **38.** $\sqrt[4]{a^4}$ **39.** $\sqrt[3]{a^3}$ **40.** $\sqrt[5]{a^5}$

41. $\sqrt[6]{a^6}$ **42.** $\sqrt[7]{a^7}$ **43.** $\sqrt{x^4}$ **44.** $\sqrt[3]{y^{12}}$

45. $\sqrt{x^2y^4}$ **46.** $\sqrt[3]{(u+v)^3}$ **47.** $-\sqrt[3]{\dfrac{x^3}{y^3}},\quad y \neq 0$ **48.** $\sqrt[4]{\dfrac{a^4}{b^8}},\quad b \neq 0$

49. $\dfrac{2}{\sqrt[4]{x^4}}, \quad x \neq 0$ **50.** $\sqrt{(-5)^2}$ **51.** $\sqrt[3]{(-92)^3}$ **52.** $\sqrt[6]{(50)^6}$

53. $\sqrt[10]{(-2)^{10}}$ **54.** $\sqrt[5]{(-2)^5}$

For Exercises 55–70, simplify the expressions. Assume all variables are positive real numbers.

55. $\sqrt{x^2y^4}$ **56.** $\sqrt{16p^2}$ **57.** $\sqrt{\dfrac{a^6}{b^2}}$ **58.** $\sqrt{\dfrac{w^2}{z^4}}$

59. $-\sqrt{\dfrac{25}{q^2}}$ **60.** $-\sqrt{\dfrac{p^6}{81}}$ **61.** $\sqrt{9x^2y^4z^2}$ **62.** $\sqrt{4a^4b^2c^6}$

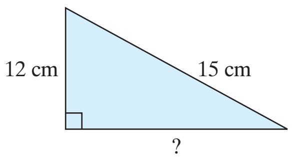
63. $\sqrt{\dfrac{h^2k^4}{16}}$ **64.** $\sqrt{\dfrac{4x^2}{y^8}}$ **65.** $-\sqrt[3]{\dfrac{t^3}{27}}$ **66.** $\sqrt[4]{\dfrac{16}{w^4}}$

67. $\sqrt[5]{32y^{10}}$ **68.** $\sqrt[3]{64x^6y^3}$ **69.** $\sqrt[6]{64p^{12}q^{18}}$ **70.** $\sqrt[4]{16r^{12}s^8}$

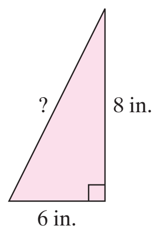
Concept 4: Pythagorean Theorem

For Exercises 71–74, find the length of the third side of each triangle by using the Pythagorean theorem.

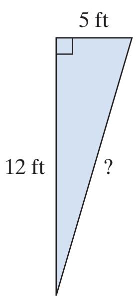
71.



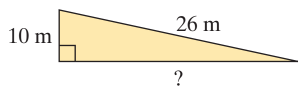
72.



73.



74.



For Exercises 75–78, use the Pythagorean theorem.

75. Roberto and Sherona began running from the same place at the same time. They ran along two different paths that formed right angles with each other. Roberto ran 4 mi and stopped, while Sherona ran 3 mi and stopped. How far apart were they when they stopped?

76. Leine and Laura began hiking from their campground. Laura headed south while Leine headed east. Laura walked 12 mi and Leine walked 5 mi. How far apart were they when they stopped walking?

77. Two mountain bikers take off from the same place at the same time. One travels north at 4 mph, and the other travels east at 3 mph. How far apart are they after 5 hr?

78. Professor Ortiz leaves campus on her bike, heading west at 6 mph. Professor Wilson leaves campus at the same time and walks south at 2.5 mph. How far apart are they after 4 hr?

Concept 5: Radical Functions

For Exercises 79–82, evaluate the function for the given values of x. Then write the domain of the function in interval notation.

79. $h(x) = \sqrt{x - 2}$
- **a.** $h(0)$
- **b.** $h(1)$
- **c.** $h(2)$
- **d.** $h(3)$
- **e.** $h(6)$

80. $k(x) = \sqrt{x + 1}$
- **a.** $k(-3)$
- **b.** $k(-2)$
- **c.** $k(-1)$
- **d.** $k(0)$
- **e.** $k(3)$

81. $g(x) = \sqrt[3]{x - 2}$
- **a.** $g(-6)$
- **b.** $g(1)$
- **c.** $g(2)$
- **d.** $g(3)$

82. $f(x) = \sqrt[3]{x + 1}$
- **a.** $f(-9)$
- **b.** $f(-2)$
- **c.** $f(0)$
- **d.** $f(7)$

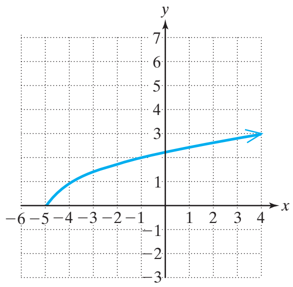
For each function defined in Exercises 83–86, write the domain in interval notation.

83. $q(x) = \sqrt{x + 5}$ **84.** $p(x) = \sqrt{1 - x}$ **85.** $R(x) = \sqrt[3]{x + 1}$ **86.** $T(x) = \sqrt{x - 10}$

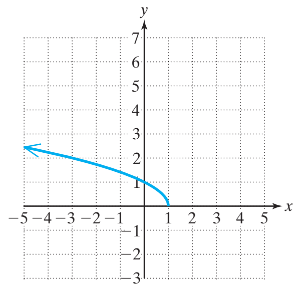
For Exercises 87–90, match the function with the graph. Use the domain information from Exercises 83–86.

87. $p(x) = \sqrt{1 - x}$ **88.** $q(x) = \sqrt{x + 5}$ **89.** $T(x) = \sqrt{x - 10}$ **90.** $R(x) = \sqrt[3]{x + 1}$

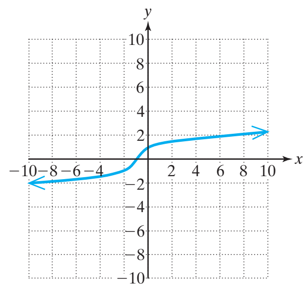
a.



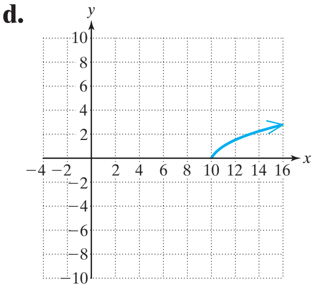
b.



c.



d.



Mixed Exercises

For Exercises 91–94, translate the English phrase to an algebraic expression.

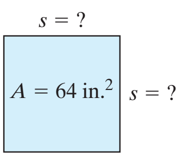
91. The sum of q and the square of p

92. The product of 11 and the cube root of x

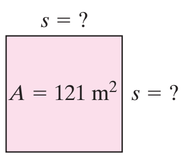
93. The quotient of 6 and the cube root of x

94. The difference of y and the principal square root of x

95. If a square has an area of 64 in.^{2}, then what are the lengths of the sides?



96. If a square has an area of 121 m^2, then what are the lengths of the sides?



Graphing Calculator Exercises

For Exercises 97–104, use a calculator to evaluate the expressions to four decimal places.

97. $\sqrt{69}$ **98.** $\sqrt{5798}$ **99.** $2 + \sqrt[3]{5}$ **100.** $3 - 2\sqrt[4]{10}$

101. $7\sqrt[4]{25}$ **102.** $-3\sqrt[3]{9}$ **103.** $\dfrac{3 - \sqrt{19}}{11}$ **104.** $\dfrac{5 + 2\sqrt{15}}{12}$

105. Graph $h(x) = \sqrt{x - 2}$ on the standard viewing window. Use the graph to confirm the domain found in Exercise 79.

106. Graph $k(x) = \sqrt{x + 1}$ on the standard viewing window. Use the graph to confirm the domain found in Exercise 80.

107. Graph $g(x) = \sqrt[3]{x - 2}$ on the standard viewing window. Use the graph to confirm the domain found in Exercise 81.

108. Graph $f(x) = \sqrt[3]{x + 1}$ on the standard viewing window. Use the graph to confirm the domain found in Exercise 82.

Section 7.2 Rational Exponents

Concepts

1. Definition of $a^{1/n}$ and $a^{m/n}$
2. Converting Between Rational Exponents and Radical Notation
3. Properties of Rational Exponents
4. Applications Involving Rational Exponents

1. Definition of $a^{1/n}$ and $a^{m/n}$

In Section 1.8, the properties for simplifying expressions with integer exponents were presented. In this section, the properties are expanded to include expressions with rational exponents. We begin by defining expressions of the form $a^{1/n}$.

Definition of $a^{1/n}$

Let a be a real number, and let n be an integer such that $n > 1$. If $\sqrt[n]{a}$ is a real number, then

$$a^{1/n} = \sqrt[n]{a}$$

Example 1 Evaluating Expressions of the Form $a^{1/n}$

Convert the expression to radical form and simplify, if possible.

a. $(-8)^{1/3}$ **b.** $81^{1/4}$ **c.** $-100^{1/2}$ **d.** $(-100)^{1/2}$ **e.** $16^{-1/2}$

Solution:

a. $(-8)^{1/3} = \sqrt[3]{-8} = -2$

b. $81^{1/4} = \sqrt[4]{81} = 3$

c. $-100^{1/2} = -1 \cdot 100^{1/2}$ The exponent applies only to the base of 100.

$= -1\sqrt{100}$

$= -10$

d. $(-100)^{1/2}$ is not a real number because $\sqrt{-100}$ is not a real number.

e. $16^{-1/2} = \dfrac{1}{16^{1/2}}$ Write the expression with a positive exponent. Recall that $b^{-n} = \dfrac{1}{b^n}$.

$= \dfrac{1}{\sqrt{16}}$

$= \dfrac{1}{4}$

Skill Practice Simplify.

1. $(-64)^{1/3}$ **2.** $16^{1/4}$ **3.** $-36^{1/2}$ **4.** $(-36)^{1/2}$ **5.** $64^{-1/3}$

If $\sqrt[n]{a}$ is a real number, then we can define an expression of the form $a^{m/n}$ in such a way that the multiplication property of exponents still holds true. For example,

$$16^{3/4} \nearrow (16^{1/4})^3 = (\sqrt[4]{16})^3 = (2)^3 = 8$$

$$16^{3/4} \searrow (16^3)^{1/4} = \sqrt[4]{16^3} = \sqrt[4]{4096} = 8$$

Definition of $a^{m/n}$

Let a be a real number, and let m and n be positive integers such that m and n share no common factors and $n > 1$. If $\sqrt[n]{a}$ is a real number, then

$$a^{m/n} = (a^{1/n})^m = (\sqrt[n]{a})^m \quad \text{and} \quad a^{m/n} = (a^m)^{1/n} = \sqrt[n]{a^m}$$

The rational exponent in the expression $a^{m/n}$ is essentially performing two operations. The numerator of the exponent raises the base to the mth power. The denominator takes the nth root.

Example 2 Evaluating Expressions of the Form $a^{m/n}$

Convert each expression to radical form and simplify.

a. $8^{2/3}$ **b.** $100^{5/2}$ **c.** $\left(\dfrac{1}{25}\right)^{3/2}$ **d.** $4^{-3/2}$ **e.** $(-81)^{3/4}$

Solution:

a. $8^{2/3} = (\sqrt[3]{8})^2$ Take the cube root of 8 and square the result.

$= (2)^2$ Simplify.

$= 4$

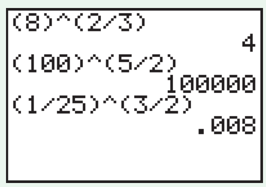
b. $100^{5/2} = (\sqrt{100})^5$ Take the square root of 100 and raise the result to the fifth power.

$= (10)^5$ Simplify.

$= 100{,}000$

Calculator Connections

A calculator can be used to confirm the results of Example 2(a)–2(c).



Skill Practice Answers

1. -4 **2.** 2 **3.** -6
4. Not a real number **5.** $\frac{1}{4}$

c. $\left(\frac{1}{25}\right)^{3/2} = \left(\sqrt{\frac{1}{25}}\right)^3$ Take the square root of $\frac{1}{25}$ and cube the result.

$= \left(\frac{1}{5}\right)^3$ Simplify.

$= \frac{1}{125}$

d. $4^{-3/2} = \frac{1}{4^{3/2}}$ Write the expression with positive exponents.

$= \frac{1}{(\sqrt{4})^3}$ Take the square root of 4 and cube the result.

$= \frac{1}{2^3}$ Simplify.

$= \frac{1}{8}$

e. $(-81)^{3/4}$ is not a real number because $\sqrt[4]{-81}$ is not a real number.

Skill Practice Convert each expression to radical form and simplify.

6. $9^{3/2}$ **7.** $8^{5/3}$ **8.** $32^{-4/5}$ **9.** $\left(\frac{1}{27}\right)^{4/3}$ **10.** $(-4)^{3/2}$

2. Converting Between Rational Exponents and Radical Notation

Example 3 Using Radical Notation and Rational Exponents

Convert each expression to radical notation. Assume all variables represent positive real numbers.

a. $a^{3/5}$ **b.** $(5x^2)^{1/3}$ **c.** $3y^{1/4}$ **d.** $z^{-3/4}$

Solution:

a. $a^{3/5} = \sqrt[5]{a^3}$

b. $(5x^2)^{1/3} = \sqrt[3]{5x^2}$

c. $3y^{1/4} = 3\sqrt[4]{y}$ Note that the coefficient 3 is not raised to the $\frac{1}{4}$ power.

d. $z^{-3/4} = \frac{1}{z^{3/4}} = \frac{1}{\sqrt[4]{z^3}}$

Skill Practice Convert each expression to radical notation. Assume all variables represent positive real numbers.

11. $t^{4/5}$ **12.** $(2y^3)^{1/4}$ **13.** $10p^{1/2}$ **14.** $q^{-2/3}$

Skill Practice Answers

6. 27 **7.** 32 **8.** $\frac{1}{16}$
9. $\frac{1}{81}$ **10.** Not a real number
11. $\sqrt[5]{t^4}$ **12.** $\sqrt[4]{2y^3}$
13. $10\sqrt{p}$ **14.** $\frac{1}{\sqrt[3]{q^2}}$

Example 4 **Using Radical Notation and Rational Exponents**

Convert each expression to an equivalent expression by using rational exponents. Assume that all variables represent positive real numbers.

a. $\sqrt[4]{b^3}$ **b.** $\sqrt{7a}$ **c.** $7\sqrt{a}$

Solution:

a. $\sqrt[4]{b^3} = b^{3/4}$

b. $\sqrt{7a} = (7a)^{1/2}$

c. $7\sqrt{a} = 7a^{1/2}$

Skill Practice Convert to an equivalent expression using rational exponents. Assume all variables represent positive real numbers.

15. $\sqrt[3]{x^2}$ **16.** $\sqrt{5y}$ **17.** $5\sqrt{y}$

3. Properties of Rational Exponents

In Section 1.8, several properties and definitions were introduced to simplify expressions with integer exponents. These properties also apply to rational exponents.

Properties of Exponents and Definitions

Let a and b be nonzero real numbers. Let m and n be rational numbers such that a^m, a^n, b^n, and b^m are real numbers.

Description	Property	Example
1. Multiplying like bases	$a^m a^n = a^{m+n}$	$x^{1/3}x^{4/3} = x^{5/3}$
2. Dividing like bases	$\dfrac{a^m}{a^n} = a^{m-n}$	$\dfrac{x^{3/5}}{x^{1/5}} = x^{2/5}$
3. The power rule	$(a^m)^n = a^{mn}$	$(2^{1/3})^{1/2} = 2^{1/6}$
4. Power of a product	$(ab)^m = a^m b^m$	$(xy)^{1/2} = x^{1/2}y^{1/2}$
5. Power of a quotient	$\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$	$\left(\dfrac{4}{25}\right)^{1/2} = \dfrac{4^{1/2}}{25^{1/2}} = \dfrac{2}{5}$

Description	Definition	Example
1. Negative exponents	$a^{-m} = \left(\dfrac{1}{a}\right)^m = \dfrac{1}{a^m}$	$(8)^{-1/3} = \left(\dfrac{1}{8}\right)^{1/3} = \dfrac{1}{2}$
2. Zero exponent	$a^0 = 1$	$5^0 = 1$

Example 5 **Simplifying Expressions with Rational Exponents**

Use the properties of exponents to simplify the expressions. Assume all variables represent positive real numbers.

a. $y^{2/5}y^{3/5}$ **b.** $\left(\dfrac{s^{1/2}t^{1/3}}{w^{3/4}}\right)^4$ **c.** $\left(\dfrac{81cd^{-2}}{3c^{-2}d^4}\right)^{1/3}$

Skill Practice Answers

15. $x^{2/3}$ **16.** $(5y)^{1/2}$
17. $5y^{1/2}$

Solution:

a. $y^{2/5}y^{3/5} = y^{(2/5)+(3/5)}$ Multiply like bases by adding exponents.

$= y^{5/5}$ Simplify.

$= y$

b. $\left(\dfrac{s^{1/2}t^{1/3}}{w^{3/4}}\right)^4 = \dfrac{s^{(1/2)\cdot 4}t^{(1/3)\cdot 4}}{w^{(3/4)\cdot 4}}$ Apply the power rule. Multiply exponents.

$= \dfrac{s^2t^{4/3}}{w^3}$ Simplify.

c. $\left(\dfrac{81cd^{-2}}{3c^{-2}d^4}\right)^{1/3} = (27c^{1-(-2)}d^{-2-4})^{1/3}$ Simplify inside parentheses. Subtract exponents.

$= (27c^3d^{-6})^{1/3}$

$= \left(\dfrac{27c^3}{d^6}\right)^{1/3}$ Rewrite using positive exponents.

$= \dfrac{27^{1/3}c^{3/3}}{d^{6/3}}$ Apply the power rule. Multiply exponents.

$= \dfrac{3c}{d^2}$ Simplify.

Skill Practice Use the properties of exponents to simplify the expressions. Assume all variables represent positive real numbers.

18. $x^{1/2} \cdot x^{3/4}$ **19.** $\left(\dfrac{a^{1/3}b^{1/2}}{c^{5/8}}\right)^6$ **20.** $\left(\dfrac{32y^2z^{-3}}{2y^{-2}z^5}\right)^{1/4}$

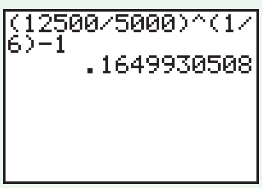
4. Applications Involving Rational Exponents

Calculator Connections

The expression

$$r = \left(\frac{12{,}500}{5000}\right)^{1/6} - 1$$

is easily evaluated on a graphing calculator.



Example 6 Applying Rational Exponents

Suppose P dollars in principal is invested in an account that earns interest annually. If after t years the investment grows to A dollars, then the annual rate of return r on the investment is given by

$$r = \left(\frac{A}{P}\right)^{1/t} - 1$$

Find the annual rate of return on \$5000 which grew to \$12,500 after 6 years.

Solution:

$r = \left(\dfrac{A}{P}\right)^{1/t} - 1$

$= \left(\dfrac{12{,}500}{5000}\right)^{1/6} - 1$ Substitute $A = \$12{,}500$, $P = \$5000$, and $t = 6$.

$= (2.5)^{1/6} - 1$

$\approx 1.165 - 1$

≈ 0.165 or 16.5% The annual rate of return is 16.5%.

Skill Practice Answers

18. $x^{5/4}$ **19.** $\dfrac{a^2b^3}{c^{15/4}}$ **20.** $\dfrac{2y}{z^2}$

Skill Practice

21. The radius r of a sphere of volume V is given by $r = \left(\frac{3V}{4\pi}\right)^{1/3}$. Find the radius of a sphere whose volume is 113.04 in.^{3} (Use 3.14 for π.)

Skill Practice Answers

21. Approximately 3 in.

Section 7.2 Practice Exercises

For the exercises in this set, assume that all variables represent positive real numbers unless otherwise stated.

Study Skills Exercises

1. Before you do your homework for this section, go back to Section 1.8 and review the properties of exponents. Do several problems from the Section 1.8 exercises. This will help you with the concepts in Section 7.2.

2. Define the key terms.

a. $a^{1/n}$ **b.** $a^{m/n}$

Review Exercises

3. Given: $\sqrt[3]{27}$

a. Identify the index.

b. Identify the radicand.

4. Given: $\sqrt{18}$

a. Identify the index.

b. Identify the radicand.

For Exercises 5–8, evaluate the radicals (if possible).

5. $\sqrt{25}$ **6.** $\sqrt[3]{8}$ **7.** $\sqrt[4]{81}$ **8.** $(\sqrt[4]{16})^3$

Concept 1: Definition of $a^{1/n}$ and $a^{m/n}$

For Exercises 9–18, convert the expressions to radical form and simplify.

9. $144^{1/2}$ **10.** $16^{1/4}$ **11.** $-144^{1/2}$ **12.** $-16^{1/4}$

13. $(-144)^{1/2}$ **14.** $(-16)^{1/4}$ **15.** $(-64)^{1/3}$ **16.** $(-32)^{1/5}$

17. $(25)^{-1/2}$ **18.** $(27)^{-1/3}$

19. Explain how to interpret the expression $a^{m/n}$ as a radical.

20. Explain why $(\sqrt[3]{8})^4$ is easier to evaluate than $\sqrt[3]{8^4}$.

For Exercises 21–24, simplify the expression, if possible.

21. a. $16^{3/4}$ **b.** $-16^{3/4}$ **c.** $(-16)^{3/4}$ **d.** $16^{-3/4}$ **e.** $-16^{-3/4}$ **f.** $(-16)^{-3/4}$

22. a. $81^{3/4}$ **b.** $-81^{3/4}$ **c.** $(-81)^{3/4}$ **d.** $81^{-3/4}$ **e.** $-81^{-3/4}$ **f.** $(-81)^{-3/4}$

23. **a.** $25^{3/2}$ **b.** $-25^{3/2}$ **c.** $(-25)^{3/2}$ **d.** $25^{-3/2}$ **e.** $-25^{-3/2}$ **f.** $(-25)^{-3/2}$

24. **a.** $4^{3/2}$ **b.** $-4^{3/2}$ **c.** $(-4)^{3/2}$ **d.** $4^{-3/2}$ **e.** $-4^{-3/2}$ **f.** $(-4)^{-3/2}$

For Exercises 25–50, simplify the expression.

25. $64^{-3/2}$ **26.** $81^{-3/2}$ **27.** $243^{3/5}$ **28.** $1^{5/3}$

29. $-27^{-4/3}$ **30.** $-16^{-5/4}$ **31.** $\left(\frac{100}{9}\right)^{-3/2}$ **32.** $\left(\frac{49}{100}\right)^{-1/2}$

33. $(-4)^{-3/2}$ **34.** $(-49)^{-3/2}$ **35.** $(-8)^{1/3}$ **36.** $(-9)^{1/2}$

37. $-8^{1/3}$ **38.** $-9^{1/2}$ **39.** $27^{-2/3}$ **40.** $125^{-1/3}$

41. $\frac{1}{36^{-1/2}}$ **42.** $\frac{1}{16^{-1/2}}$ **43.** $\frac{1}{1000^{-1/3}}$ **44.** $\frac{1}{81^{-3/4}}$

45. $\left(\frac{1}{8}\right)^{2/3} + \left(\frac{1}{4}\right)^{1/2}$ **46.** $\left(\frac{1}{8}\right)^{-2/3} + \left(\frac{1}{4}\right)^{-1/2}$ **47.** $\left(\frac{1}{16}\right)^{-3/4} - \left(\frac{1}{49}\right)^{-1/2}$ **48.** $\left(\frac{1}{16}\right)^{1/4} - \left(\frac{1}{49}\right)^{1/2}$

49. $\left(\frac{1}{4}\right)^{1/2} + \left(\frac{1}{64}\right)^{-1/3}$ **50.** $\left(\frac{1}{36}\right)^{1/2} + \left(\frac{1}{64}\right)^{-5/6}$

Concept 2: Converting Between Rational Exponents and Radical Notation

For Exercises 51–58, convert each expression to radical notation.

51. $q^{2/3}$ **52.** $t^{3/5}$ **53.** $6y^{3/4}$ **54.** $8b^{4/9}$

55. $(x^2y)^{1/3}$ **56.** $(c^2d)^{1/6}$ **57.** $(qr)^{-1/5}$ **58.** $(7x)^{-1/4}$

For Exercises 59–66, write each expression by using rational exponents rather than radical notation.

59. $\sqrt[3]{x}$ **60.** $\sqrt[4]{a}$ **61.** $10\sqrt{b}$ **62.** $-2\sqrt[3]{t}$

63. $\sqrt[3]{y^2}$ **64.** $\sqrt[6]{z^5}$ **65.** $\sqrt[4]{a^2b^3}$ **66.** $\sqrt{abc}$

Concept 3: Properties of Rational Exponents

For Exercises 67–90, simplify the expressions by using the properties of rational exponents. Write the final answer using positive exponents only.

67. $x^{1/4}x^{-5/4}$ **68.** $2^{2/3}2^{-5/3}$ **69.** $\frac{p^{5/3}}{p^{2/3}}$ **70.** $\frac{q^{5/4}}{q^{1/4}}$

71. $(y^{1/5})^{10}$ **72.** $(x^{1/2})^8$ **73.** $6^{-1/5}6^{3/5}$ **74.** $a^{-1/3}a^{2/3}$

75. $\frac{4t^{-1/3}}{t^{4/3}}$ **76.** $\frac{5s^{-1/3}}{s^{5/3}}$ **77.** $(a^{1/3}a^{1/4})^{12}$ **78.** $(x^{2/3}x^{1/2})^6$

79. $(5a^2c^{-1/2}d^{1/2})^2$ **80.** $(2x^{-1/3}y^2z^{5/3})^3$ **81.** $\left(\frac{x^{-2/3}}{y^{-3/4}}\right)^{12}$ **82.** $\left(\frac{m^{-1/4}}{n^{-1/2}}\right)^{-4}$

83. $\left(\frac{16w^{-2}z}{2wz^{-8}}\right)^{1/3}$

84. $\left(\frac{50p^{-1}q}{2pq^{-3}}\right)^{1/2}$

85. $(25x^2y^4z^6)^{1/2}$

86. $(8a^6b^3c^9)^{2/3}$

87. $(x^2y^{-1/3})^6(x^{1/2}yz^{2/3})^2$

88. $(a^{-1/3}b^{1/2})^4(a^{-1/2}b^{3/5})^{10}$

89. $\left(\frac{x^{3m}y^{2m}}{z^{5m}}\right)^{1/m}$

90. $\left(\frac{a^{4n}b^{3n}}{c^n}\right)^{1/n}$

Concept 4: Applications Involving Rational Exponents

91. If the area A of a square is known, then the length of its sides, s, can be computed by the formula $s = A^{1/2}$.

a. Compute the length of the sides of a square having an area of 100 in.^2

b. Compute the length of the sides of a square having an area of 72 in.^2 Round your answer to the nearest 0.1 in.

92. The radius r of a sphere of volume V is given by $r = \left(\frac{3V}{4\pi}\right)^{1/3}$. Find the radius of a sphere having a volume of 85 in.^3 Round your answer to the nearest 0.1 in.

93. If P dollars in principal grows to A dollars after t years with annual interest, then the interest rate is given by $r = \left(\frac{A}{P}\right)^{1/t} - 1$.

a. In one account, \$10,000 grows to \$16,802 after 5 years. Compute the interest rate. Round your answer to a tenth of a percent.

b. In another account \$10,000 grows to \$18,000 after 7 years. Compute the interest rate. Round your answer to a tenth of a percent.

c. Which account produced a higher average yearly return?

94. Is $(a + b)^{1/2}$ the same as $a^{1/2} + b^{1/2}$? If not, give a counterexample.

Expanding Your Skills

For Exercises 95–100, write the expression as a single radical.

95. $\sqrt{\sqrt[3]{x}}$

96. $\sqrt[3]{\sqrt{x}}$

97. $\sqrt[4]{\sqrt{y}}$

98. $\sqrt{\sqrt[4]{y}}$

99. $\sqrt[5]{\sqrt[3]{w}}$

100. $\sqrt[3]{\sqrt[4]{w}}$

For Exercises 101–108, use a calculator to approximate the expressions and round to 4 decimal places, if necessary.

101. $9^{1/2}$

102. $125^{-1/3}$

103. $50^{-1/4}$

104. $(172)^{3/5}$

105. $\sqrt[3]{5^2}$

106. $\sqrt[4]{6^3}$

107. $\sqrt{10^3}$

108. $\sqrt[3]{16}$

Section 7.3 Simplifying Radical Expressions

Concepts

1. Multiplication and Division Properties of Radicals
2. Simplifying Radicals by Using the Multiplication Property of Radicals
3. Simplifying Radicals by Using the Division Property of Radicals

1. Multiplication and Division Properties of Radicals

You may have already noticed certain properties of radicals involving a product or quotient.

Multiplication and Division Properties of Radicals

Let a and b represent real numbers such that $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are both real. Then

1. $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$ *Multiplication property of radicals*
2. $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$ $b \neq 0$ *Division property of radicals*

Properties 1 and 2 follow from the properties of rational exponents.

$$\sqrt[n]{ab} = (ab)^{1/n} = a^{1/n}b^{1/n} = \sqrt[n]{a} \cdot \sqrt[n]{b}$$

$$\sqrt[n]{\frac{a}{b}} = \left(\frac{a}{b}\right)^{1/n} = \frac{a^{1/n}}{b^{1/n}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

The multiplication and division properties of radicals indicate that a product or quotient within a radicand can be written as a product or quotient of radicals, provided the roots are real numbers. For example:

$$\sqrt{144} = \sqrt{16} \cdot \sqrt{9}$$

$$\sqrt{\frac{25}{36}} = \frac{\sqrt{25}}{\sqrt{36}}$$

The reverse process is also true. A product or quotient of radicals can be written as a single radical provided the roots are real numbers and they have the same indices.

$$\sqrt{3} \cdot \sqrt{12} = \sqrt{36}$$

$$\frac{\sqrt[3]{8}}{\sqrt[3]{125}} = \sqrt[3]{\frac{8}{125}}$$

In algebra it is customary to simplify radical expressions as much as possible.

Simplified Form of a Radical

Consider any radical expression where the radicand is written as a product of prime factors. The expression is in *simplified form* if all the following conditions are met:

1. The radicand has no factor raised to a power greater than or equal to the index.
2. The radicand does not contain a fraction.
3. There are no radicals in the denominator of a fraction.

For example, the following radicals are not simplified.

1. The expression $\sqrt[3]{x^5}$ fails rule 1.
2. The expression $\sqrt{\dfrac{1}{4}}$ fails rule 2.
3. The expression $\dfrac{1}{\sqrt[3]{8}}$ fails rule 3.

2. Simplifying Radicals by Using the Multiplication Property of Radicals

The expression $\sqrt{x^2}$ is not simplified because it fails condition 1. Because x^2 is a perfect square, $\sqrt{x^2}$ is easily simplified:

$$\sqrt{x^2} = x \qquad \text{for } x \geq 0$$

However, how is an expression such as $\sqrt{x^9}$ simplified? This and many other radical expressions are simplified by using the multiplication property of radicals. The following examples illustrate how nth powers can be removed from the radicands of nth roots.

Example 1 **Using the Multiplication Property to Simplify a Radical Expression**

Use the multiplication property of radicals to simplify the expression $\sqrt{x^9}$. Assume $x \geq 0$.

Solution:

The expression $\sqrt{x^9}$ is equivalent to $\sqrt{x^8 \cdot x}$. Applying the multiplication property of radicals, we have

$\sqrt{x^9} = \sqrt{x^8 \cdot x}$

$= \sqrt{x^8} \cdot \sqrt{x}$ — Apply the multiplication property of radicals.

Note that x^8 is a perfect square because $x^8 = (x^4)^2$.

$= x^4\sqrt{x}$ — Simplify.

Skill Practice Simplify the expression. Assume that $a > 0$.

1. $\sqrt{a^{11}}$

In Example 1, the expression x^9 is not a perfect square. Therefore, to simplify $\sqrt{x^9}$, it was necessary to write the expression as the product of the largest perfect square and a remaining or "left-over" factor: $\sqrt{x^9} = \sqrt{x^8 \cdot x}$. This process also applies to simplifying nth roots, as shown in Example 2.

Example 2 **Using the Multiplication Property to Simplify a Radical Expression**

Use the multiplication property of radicals to simplify each expression. Assume all variables represent positive real numbers.

a. $\sqrt[4]{b^7}$ **b.** $\sqrt[3]{w^7z^9}$

Solution:

The goal is to rewrite each radicand as the product of the largest perfect square (perfect cube, perfect fourth power, and so on) and a left-over factor.

a. $\sqrt[4]{b^7} = \sqrt[4]{b^4 \cdot b^3}$ — b^4 is the largest perfect fourth power in the radicand.

$= \sqrt[4]{b^4} \cdot \sqrt[4]{b^3}$ — Apply the multiplication property of radicals.

$= b\sqrt[4]{b^3}$ — Simplify.

Skill Practice Answers

1. $a^5\sqrt{a}$

b. $\sqrt[3]{w^7z^9} = \sqrt[3]{(w^6z^9)\cdot(w)}$ $\quad w^6z^9$ is the largest perfect cube in the radicand.

$= \sqrt[3]{w^6z^9}\cdot\sqrt[3]{w}$ $\quad$ Apply the multiplication property of radicals.

$= w^2z^3\sqrt[3]{w}$ $\quad$ Simplify.

Skill Practice Simplify the expressions. Assume all variables represent positive real numbers.

2. $\sqrt[4]{v^{25}}$ **3.** $\sqrt[3]{p^8q^{12}}$

Each expression in Example 2 involves a radicand that is a product of variable factors. If a numerical factor is present, sometimes it is necessary to factor the coefficient before simplifying the radical.

Example 3 Using the Multiplication Property to Simplify Radicals

Use the multiplication property of radicals to simplify the expressions. Assume all variables represent positive real numbers.

a. $\sqrt{56}$ **b.** $6\sqrt{50}$ **c.** $\sqrt[3]{40x^3y^5z^7}$

Solution:

a. $\sqrt{56} = \sqrt{2^3\cdot 7}$ $\quad$ Factor the radicand.

$= \sqrt{(2^2)\cdot(2\cdot 7)}$ $\quad 2^2$ is the largest perfect square in the radicand.

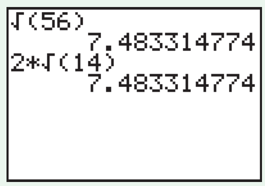
$= \sqrt{2^2}\cdot\sqrt{2\cdot 7}$ $\quad$ Apply the multiplication property of radicals.

$= 2\sqrt{14}$ $\quad$ Simplify.

2 | 56
2 | 28
2 | 14
7

Calculator Connections

A calculator can be used to support the solution to Example 3(a). The decimal approximation for $\sqrt{56}$ and $2\sqrt{14}$ agree for the first 10 digits. This in itself does not make $\sqrt{56} = 2\sqrt{14}$. It is the multiplication of property of radicals that guarantees that the expressions are equal.



b. $6\sqrt{50} = 6\sqrt{2\cdot 5^2}$ $\quad$ Factor the radicand.

$= 6\cdot\sqrt{5^2}\cdot\sqrt{2}$ $\quad$ Apply the multiplication property of radicals.

$= 6\cdot 5\cdot\sqrt{2}$ $\quad$ Simplify.

$= 30\sqrt{2}$ $\quad$ Simplify.

2 | 50
5 | 25
5

Avoiding Mistakes:

The multiplication property of radicals allows us to simplify a product of *factors* within a radical. For example:

$$\sqrt{x^2y^2} = \sqrt{x^2}\cdot\sqrt{y^2} = xy$$

However, this rule does not apply to *terms* that are added or subtracted within the radical. For example:

$$\sqrt{x^2 + y^2} \text{ and } \sqrt{x^2 - y^2}$$

cannot be simplified.

Skill Practice Answers

2. $v^6\sqrt[4]{v}$ **3.** $p^2q^4\sqrt[3]{p^2}$

c. $\sqrt[3]{40x^3y^5z^7}$

$= \sqrt[3]{2^3 5x^3y^5z^7}$ Factor the radicand.

2	40
2	20
2	10
	5

$= \sqrt[3]{(2^3x^3y^3z^6) \cdot (5y^2z)}$ $2^3x^3y^3z^6$ is the largest perfect cube.

$= \sqrt[3]{2^3x^3y^3z^6} \cdot \sqrt[3]{5y^2z}$ Apply the multiplication property of radicals.

$= 2xyz^2\sqrt[3]{5y^2z}$ Simplify.

Skill Practice Simplify the radicals. Assume all variables represent positive real numbers.

4. $\sqrt{24}$ **5.** $5\sqrt{18}$ **6.** $\sqrt[4]{32a^{10}b^{19}}$

3. Simplifying Radicals by Using the Division Property of Radicals

The division property of radicals indicates that a radical of a quotient can be written as the quotient of the radicals and vice versa, provided all roots are real numbers.

Example 4 Using the Division Property to Simplify Radicals

Simplify the expressions. Assume all variables represent positive real numbers.

a. $\sqrt{\dfrac{a^7}{a^3}}$ **b.** $\dfrac{\sqrt[3]{3}}{\sqrt[3]{81}}$ **c.** $\dfrac{7\sqrt{50}}{15}$ **d.** $\sqrt[4]{\dfrac{2c^5}{32cd^8}}$

Solution:

a. $\sqrt{\dfrac{a^7}{a^3}}$ The radicand contains a fraction. However, the fraction can be reduced to lowest terms.

$= \sqrt{a^4}$

$= a^2$ Simplify the radical.

b. $\dfrac{\sqrt[3]{3}}{\sqrt[3]{81}}$ The expression has a radical in the denominator.

$= \sqrt[3]{\dfrac{3}{81}}$ Because the radicands have a common factor, write the expression as a single radical (division property of radicals).

$= \sqrt[3]{\dfrac{1}{27}}$ Reduce to lowest terms.

$= \dfrac{1}{3}$ Simplify.

Skill Practice Answers

4. $2\sqrt{6}$ **5.** $15\sqrt{2}$

6. $2a^2b^4\sqrt[4]{2a^2b^3}$

c. $\dfrac{7\sqrt{50}}{15}$ Simplify $\sqrt{50}$.

$= \dfrac{7\sqrt{5^2 \cdot 2}}{15}$ 5^2 is the largest perfect square in the radicand.

$= \dfrac{7\sqrt{5^2} \cdot \sqrt{2}}{15}$ Multiplication property of radicals

$= \dfrac{7 \cdot 5\sqrt{2}}{15}$ Simplify the radicals.

$= \dfrac{7 \cdot \overset{1}{\cancel{5}}\sqrt{2}}{\underset{3}{\cancel{15}}}$ Reduce to lowest terms.

$= \dfrac{7\sqrt{2}}{3}$

d. $\sqrt[4]{\dfrac{2c^5}{32cd^8}}$ The radicand contains a fraction.

$= \sqrt[4]{\dfrac{c^4}{16d^8}}$ Simplify the factors in the radicand.

$= \dfrac{\sqrt[4]{c^4}}{\sqrt[4]{16d^8}}$ Apply the division property of radicals.

$= \dfrac{c}{2d^2}$ Simplify.

Skill Practice Simplify the expressions. Assume all variables represent positive real numbers.

7. $\sqrt{\dfrac{v^{21}}{v^5}}$ **8.** $\dfrac{\sqrt[5]{64}}{\sqrt[5]{2}}$ **9.** $\dfrac{2\sqrt{300}}{30}$ **10.** $\sqrt[3]{\dfrac{54x^{17}y}{2x^2y^{19}}}$

Skill Practice Answers

7. v^8 **8.** 2

9. $\dfrac{2\sqrt{3}}{3}$ **10.** $\dfrac{3x^5}{y^6}$

Section 7.3 Practice Exercises

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For the exercises in this set, assume that all variables represent positive real numbers unless otherwise stated.

Study Skills Exercise

1. The final exam is just around the corner. Your old tests and quizzes provide good material to study for the final exam. Use your old tests to make a list of the chapters on which you need to concentrate. Ask your professor for help if there are still concepts that you do not understand.

Review Exercises

For Exercises 2–4, simplify the expression. Write the answer with positive exponents only.

2. $(a^2b^{-4})^{1/2}\left(\frac{a}{b^{-3}}\right)$

3. $\left(\frac{p^4}{q^{-6}}\right)^{-1/2}(p^3q^{-2})$

4. $(x^{1/3}y^{5/6})^{-6}$

5. Write $x^{4/7}$ in radical notation.

6. Write $y^{2/5}$ in radical notation.

7. Write $\sqrt{y^9}$ by using rational exponents.

8. Write $\sqrt[3]{x^2}$ by using rational exponents.

Concept 2: Simplifying Radicals by Using the Multiplication Property of Radicals

For Exercises 9–30, simplify the radicals.

9. $\sqrt{x^{11}}$

10. $\sqrt{p^{15}}$

11. $\sqrt[3]{q^7}$

12. $\sqrt[3]{r^{17}}$

13. $\sqrt{a^5b^4}$

14. $\sqrt{c^9d^6}$

15. $-\sqrt[4]{x^8y^{13}}$

16. $-\sqrt[4]{p^{16}q^{17}}$

17. $\sqrt{28}$

18. $\sqrt{63}$

19. $\sqrt{80}$

20. $\sqrt{108}$

21. $\sqrt[3]{54}$

22. $\sqrt[3]{250}$

23. $\sqrt{25ab^3}$

24. $\sqrt{64m^5n^{20}}$

25. $\sqrt{18a^6b^3}$

26. $\sqrt{72m^5n^2}$

27. $\sqrt[3]{-16x^6yz^3}$

28. $\sqrt[3]{-192a^6bc^2}$

29. $\sqrt[4]{80w^4z^7}$

30. $\sqrt[4]{32p^8qr^5}$

Concept 3: Simplifying Radicals by Using the Division Property of Radicals

For Exercises 31–42, simplify the radicals.

31. $\sqrt{\frac{x^3}{x}}$

32. $\sqrt{\frac{y^5}{y}}$

33. $\frac{\sqrt{p^7}}{\sqrt{p^3}}$

34. $\frac{\sqrt{q^{11}}}{\sqrt{q^5}}$

35. $\sqrt{\frac{50}{2}}$

36. $\sqrt{\frac{98}{2}}$

37. $\frac{\sqrt[3]{3}}{\sqrt[3]{24}}$

38. $\frac{\sqrt[3]{3}}{\sqrt[3]{81}}$

39. $\frac{5\sqrt[3]{16}}{6}$

40. $\frac{7\sqrt{18}}{9}$

41. $\frac{5\sqrt[3]{72}}{12}$

42. $\frac{3\sqrt[3]{250}}{10}$

Mixed Exercises

For Exercises 43–58, simplify the radicals.

43. $5\sqrt{18}$

44. $2\sqrt{24}$

45. $-6\sqrt{75}$

46. $-8\sqrt{8}$

47. $\sqrt{25x^4y^3}$

48. $\sqrt{125p^3q^2}$

49. $\sqrt[3]{27x^2y^3z^4}$

50. $\sqrt[3]{108a^3bc^2}$

51. $\sqrt[3]{\frac{16a^2b}{2a^2b^4}}$

52. $\sqrt[4]{\frac{3s^2t^4}{10{,}000}}$

53. $\sqrt[5]{\frac{32x}{y^{10}}}$

54. $\sqrt[3]{\frac{-16j^3}{k^3}}$

55. $\frac{\sqrt{50x^3y}}{\sqrt{9y^4}}$

56. $\frac{\sqrt[3]{-27a^4}}{\sqrt[3]{8a}}$

57. $\sqrt{2^3a^{14}b^8c^{31}d^{22}}$

58. $\sqrt{7^5u^{12}v^{20}w^{65}x^{80}}$

For Exercises 59–62, write a mathematical expression for the English phrase and simplify.

59. The quotient of 1 and the cube root of w^6

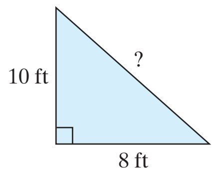
60. The principal square root of the quotient of h and 49

61. The principal square root of the quantity k raised to the third power

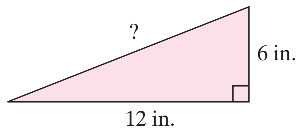
62. The cube root of $2x^4$

For Exercises 63–66, find the third side of the right triangle. Write your answer as a radical and simplify.

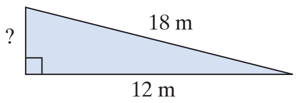
63.



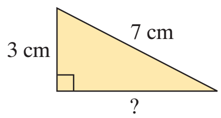
64.



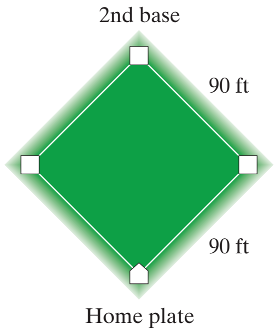
65.



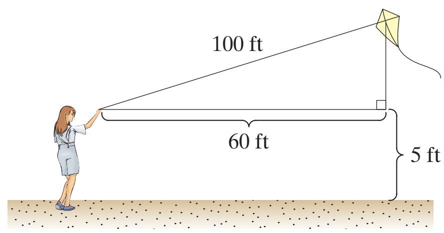
66.



67. On a baseball diamond, the bases are 90 ft apart. Find the exact distance from home plate to second base. Then round to the nearest tenth of a foot.

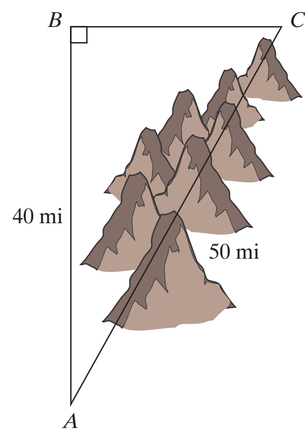


68. Linda is at the beach flying a kite. The kite is directly over a sand castle 60 ft away from Linda. If 100 ft of kite string is out (ignoring any sag in the string), how high is the kite? (Assume that Linda is 5 ft tall.) See figure.

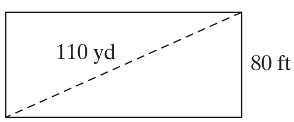


Expanding Your Skills

69. Tom has to travel from town A to town C across a small mountain range. He can travel one of two routes. He can travel on a four-lane highway from A to B and then from B to C at an average speed of 55 mph. Or he can travel on a two-lane road directly from town A to town C, but his average speed will be only 35 mph. If Tom is in a hurry, which route will take him to town C faster?



70. One side of a rectangular pasture is 80 ft in length. The diagonal distance is 110 yd. If fencing costs \$3.29 per foot, how much will it cost to fence the pasture?



Addition and Subtraction of Radicals

Section 7.4

Concepts

1. Definition of *Like* Radicals
2. Addition and Subtraction of Radicals

1. Definition of *Like* Radicals

Definition of *Like* Radicals

Two radical terms are said to be ***like* radicals** if they have the same index and the same radicand.

The following are pairs of *like* radicals:

same index

$-7a\sqrt{5}$ and $3a\sqrt{5}$ — Indices and radicands are the same.

same radicand

same index

$-\frac{1}{2}\sqrt[3]{15b}$ and $4\sqrt[3]{15b}$ — Indices and radicands are the same.

same radicand

These pairs are not *like* radicals:

different indices

$-2\sqrt{6}$ and $13\sqrt[4]{6}$ — Radicals have different indices.

$1.3cd\sqrt{3}$ and $-3.7cd\sqrt{10}$ — Radicals have different radicands.

different radicands

2. Addition and Subtraction of Radicals

To add or subtract *like* radicals, use the distributive property. For example:

$$2\sqrt{5} + 6\sqrt{5} = (2 + 6)\sqrt{5}$$
$$= 8\sqrt{5}$$
$$9\sqrt[3]{2y} - 4\sqrt[3]{2y} = (9 - 4)\sqrt[3]{2y}$$
$$= 5\sqrt[3]{2y}$$

Example 1 Adding and Subtracting Radicals

Add or subtract as indicated.

a. $6\sqrt{11} + 2\sqrt{11}$

b. $\sqrt{3} + \sqrt{3}$

c. $-2\sqrt[3]{ab} + 7\sqrt[3]{ab} - \sqrt[3]{ab}$

d. $\frac{1}{4}x\sqrt{3y} - \frac{3}{2}x\sqrt{3y}$

Solution:

a. $6\sqrt{11} + 2\sqrt{11}$

$= (6 + 2)\sqrt{11}$ Apply the distributive property.

$= 8\sqrt{11}$ Simplify.

b. $\sqrt{3} + \sqrt{3}$

$= 1\sqrt{3} + 1\sqrt{3}$ Note that $\sqrt{3} = 1\sqrt{3}$.

$= (1 + 1)\sqrt{3}$ Apply the distributive property.

$= 2\sqrt{3}$ Simplify.

Avoiding Mistakes:

The process of adding *like* radicals with the distributive property is similar to adding *like* terms. The end result is that the numerical coefficients are added and the radical factor is unchanged.

$$\sqrt{3} + \sqrt{3} = 1\sqrt{3} + 1\sqrt{3} = 2\sqrt{3}$$

Be careful: $\sqrt{3} + \sqrt{3} \neq \sqrt{6}$

In general: $\sqrt{x} + \sqrt{y} \neq \sqrt{x + y}$

c. $-2\sqrt[3]{ab} + 7\sqrt[3]{ab} - \sqrt[3]{ab}$

$= (-2 + 7 - 1)\sqrt[3]{ab}$ Apply the distributive property.

$= 4\sqrt[3]{ab}$ Simplify.

d. $\frac{1}{4}x\sqrt{3y} - \frac{3}{2}x\sqrt{3y}$

$= \left(\frac{1}{4} - \frac{3}{2}\right)x\sqrt{3y}$ Apply the distributive property.

$= \left(\frac{1}{4} - \frac{6}{4}\right)x\sqrt{3y}$ Get a common denominator.

$= -\frac{5}{4}x\sqrt{3y}$ Simplify.

Skill Practice Add or subtract as indicated.

1. $5\sqrt{6} - 8\sqrt{6}$ **2.** $\sqrt{10} + \sqrt{10}$ **3.** $5\sqrt[3]{xy} - 3\sqrt[3]{xy} + 7\sqrt[3]{xy}$

4. $\frac{5}{6}y\sqrt{2} + \frac{1}{4}y\sqrt{2}$

Example 2 shows that it is often necessary to simplify radicals before adding or subtracting.

Skill Practice Answers

1. $-3\sqrt{6}$ **2.** $2\sqrt{10}$

3. $9\sqrt[3]{xy}$ **4.** $\frac{13}{12}y\sqrt{2}$

Example 2 Adding and Subtracting Radicals

Simplify the radicals and add or subtract as indicated. Assume all variables represent positive real numbers.

a. $3\sqrt{8} + \sqrt{2}$

b. $8\sqrt{x^3y^2} - 3y\sqrt{x^3}$

c. $\sqrt{50x^2y^5} - 13y\sqrt{2x^2y^3} + xy\sqrt{98y^3}$

Solution:

a. $3\sqrt{8} + \sqrt{2}$ — The radicals are different. Try simplifying the radicals first.

$= 3 \cdot 2\sqrt{2} + \sqrt{2}$ — Simplify: $\sqrt{8} = \sqrt{2^3} = 2\sqrt{2}$

$= 6\sqrt{2} + \sqrt{2}$

$= (6 + 1)\sqrt{2}$ — Apply the distributive property.

$= 7\sqrt{2}$ — Simplify.

b. $8\sqrt{x^3y^2} - 3y\sqrt{x^3}$ — The radicals are different. Simplify the radicals first.

$= 8xy\sqrt{x} - 3xy\sqrt{x}$ — Simplify $\sqrt{x^3y^2} = xy\sqrt{x}$ and $\sqrt{x^3} = x\sqrt{x}$.

$= (8 - 3)xy\sqrt{x}$ — Apply the distributive property.

$= 5xy\sqrt{x}$ — Simplify.

c. $\sqrt{50x^2y^5} - 13y\sqrt{2x^2y^3} + xy\sqrt{98y^3}$ — Simplify each radical.

$= 5xy^2\sqrt{2y} - 13xy^2\sqrt{2y} + 7xy^2\sqrt{2y}$

$$\sqrt{50x^2y^5} = \sqrt{5^2 2x^2y^5} = 5xy^2\sqrt{2y}$$
$$-13y\sqrt{2x^2y^3} = -13xy^2\sqrt{2y}$$
$$xy\sqrt{98y^3} = xy\sqrt{7^2 2y^3} = 7xy^2\sqrt{2y}$$

$= (5 - 13 + 7)xy^2\sqrt{2y}$ — Apply the distributive property.

$= -xy^2\sqrt{2y}$

Skill Practice Simplify the radicals and add or subtract as indicated. Assume all variables represent positive real numbers.

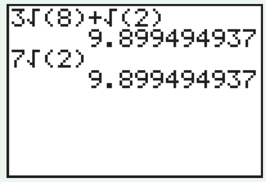
5. $\sqrt{75} + 2\sqrt{3}$

6. $4\sqrt{a^2b} - 6a\sqrt{b}$

7. $-3\sqrt{2y^3} + 5y\sqrt{18y} - 2\sqrt{50y^3}$

Calculator Connections

To check the solution to Example 2(a), use a calculator to evaluate the expressions $3\sqrt{8} + \sqrt{2}$ and $7\sqrt{2}$. The decimal approximations agree to 10 digits.



Skill Practice Answers

5. $7\sqrt{3}$ **6.** $-2a\sqrt{b}$

7. $2y\sqrt{2y}$

Section 7.4 Practice Exercises

For the exercises in this set, assume that all variables represent positive real numbers, unless otherwise stated.

Study Skills Exercises

1. Sometimes the problems on a test do not appear in the same order as the concepts appear in the text. In order to prepare better for a test, try to practice on problems taken from the book but placed in random order. Choose 30 problems from various chapters, randomize the order and use them to review for the test. Repeat the process several times for additional practice.

2. Define the key term ***like* radicals**.

Review Exercises

For Exercises 3–6, simplify the radicals.

3. $\sqrt[3]{-16s^4t^9}$ **4.** $-\sqrt[4]{x^7y^4}$ **5.** $\sqrt{36a^2b^3}$ **6.** $\dfrac{\sqrt[3]{7b^8}}{\sqrt[3]{56b^2}}$

7. Write the expression $(4x^2)^{3/2}$ as a radical and simplify.

8. Convert to rational exponents and simplify. $\sqrt[5]{3^5x^{15}y^{10}}$

For Exercises 9–10, simplify the expressions. Write the answer with positive exponents only.

9. $y^{2/3}y^{1/4}$ **10.** $(x^{1/2}y^{-3/4})^{-4}$

Concept 1: Definition of *Like* Radicals

For Exercises 11–12, determine if the radical terms are *like*.

11. a. $\sqrt{2}$ and $\sqrt[3]{2}$

b. $\sqrt{2}$ and $3\sqrt{2}$

c. $\sqrt{2}$ and $\sqrt{5}$

12. a. $7\sqrt[3]{x}$ and $\sqrt[3]{x}$

b. $\sqrt[3]{x}$ and $\sqrt[4]{x}$

c. $2\sqrt[4]{x}$ and $x\sqrt[4]{2}$

13. Explain the similarities between the following pairs of expressions.

a. $7\sqrt{5} + 4\sqrt{5}$ and $7x + 4x$

b. $-2\sqrt{6} - 9\sqrt{3}$ and $-2x - 9y$

14. Explain the similarities between the following pairs of expressions.

a. $-4\sqrt{3} + 5\sqrt{3}$ and $-4z + 5z$

b. $13\sqrt{7} - 18$ and $13a - 18$

Concept 2: Addition and Subtraction of Radicals

For Exercises 15–32, add or subtract the radical expressions, if possible.

15. $3\sqrt{5} + 6\sqrt{5}$
16. $5\sqrt{a} + 3\sqrt{a}$
17. $3\sqrt[3]{t} - 2\sqrt[3]{t}$
18. $6\sqrt[3]{7} - 2\sqrt[3]{7}$
19. $6\sqrt{10} - \sqrt{10}$
20. $13\sqrt{11} - \sqrt{11}$
21. $\sqrt[4]{3} + 7\sqrt[4]{3} - \sqrt[4]{14}$
22. $2\sqrt{11} + 3\sqrt{13} + 5\sqrt{11}$
23. $8\sqrt{x} + 2\sqrt{y} - 6\sqrt{x}$
24. $10\sqrt{10} - 8\sqrt{10} + \sqrt{2}$
25. $\sqrt[3]{ab} + a\sqrt[3]{b}$
26. $x\sqrt[4]{y} - y\sqrt[4]{x}$
27. $\sqrt{2t} + \sqrt[3]{2t}$
28. $\sqrt[4]{5c} + \sqrt[3]{5c}$
29. $\frac{5}{6}z\sqrt[3]{6} + \frac{7}{9}z\sqrt[3]{6}$
30. $\frac{3}{4}a\sqrt[4]{b} + \frac{1}{6}a\sqrt[4]{b}$
31. $0.81x\sqrt{y} - 0.11x\sqrt{y}$
32. $7.5\sqrt{pq} - 6.3\sqrt{pq}$

33. Explain the process for adding the two radicals. Then find the sum. $3\sqrt{2} + 7\sqrt{50}$

34. Explain the process for adding the two radicals. Then find the sum. $\sqrt{8} + \sqrt{32}$

For Exercises 35–60, add or subtract the radical expressions as indicated.

35. $\sqrt{36} + \sqrt{81}$
36. $3\sqrt{80} - 5\sqrt{45}$
37. $2\sqrt{12} + \sqrt{48}$
38. $5\sqrt{32} + 2\sqrt{50}$
39. $4\sqrt{7} + \sqrt{63} - 2\sqrt{28}$
40. $8\sqrt{3} - 2\sqrt{27} + \sqrt{75}$
41. $5\sqrt{18} + \sqrt{32} - 4\sqrt{50}$
42. $7\sqrt{72} - \sqrt{8} + 4\sqrt{50}$
43. $\sqrt[3]{81} - \sqrt[3]{24}$
44. $17\sqrt[3]{81} - 2\sqrt[3]{24}$
45. $3\sqrt{2a} - \sqrt{8a} - \sqrt{72a}$
46. $\sqrt{12t} - \sqrt{27t} + 5\sqrt{3t}$
47. $2s^2\sqrt[3]{s^2t^6} + 3t^2\sqrt[3]{8s^8}$
48. $4\sqrt[3]{x^4} - 2x\sqrt[3]{x}$
49. $7\sqrt[3]{x^4} - x\sqrt[3]{x}$
50. $6\sqrt[3]{y^{10}} - 3y^2\sqrt[3]{y^4}$
51. $5p\sqrt{20p^2} + p^2\sqrt{80}$
52. $2q\sqrt{48q^2} - \sqrt{27q^4}$
53. $\sqrt[3]{a^2b} - \sqrt[3]{8a^2b}$
54. $w\sqrt{80} - 3\sqrt{125w^2}$
55. $11\sqrt[3]{54cd^3} - 2\sqrt[3]{2cd^3} + d\sqrt[3]{16c}$
56. $x\sqrt[3]{64x^5y^2} - x^2\sqrt[3]{x^2y^2} + 5\sqrt[3]{x^8y^2}$
57. $\frac{3}{2}ab\sqrt{24a^3} + \frac{4}{3}\sqrt{54a^5b^2} - a^2b\sqrt{150a}$
58. $mn\sqrt{72n} + \frac{2}{3}n\sqrt{8m^2n} - \frac{5}{6}\sqrt{50m^2n^3}$
59. $x\sqrt[3]{16} - 2\sqrt[3]{27x} + \sqrt[3]{54x^3}$
60. $5\sqrt[4]{y^5} - 2y\sqrt[4]{y} + \sqrt[4]{16y^7}$

Mixed Exercises

For Exercises 61–66, answer true or false. If an answer is false, explain why or give a counterexample.

61. $\sqrt{x} + \sqrt{y} = \sqrt{x + y}$
62. $\sqrt{x} + \sqrt{x} = 2\sqrt{x}$
63. $5\sqrt[3]{x} + 2\sqrt[3]{x} = 7\sqrt[3]{x}$
64. $6\sqrt{x} + 5\sqrt[3]{x} = 11\sqrt{x}$
65. $\sqrt{y} + \sqrt{y} = \sqrt{2y}$
66. $\sqrt{c^2 + d^2} = c + d$

For Exercises 67–70, translate the English phrase to an algebraic expression. Simplify each expression, if possible.

67. The sum of the principal square root of 48 and the principal square root of 12

68. The sum of the cube root of 16 and the cube root of 2

69. The difference of 5 times the cube root of x^6 and the square of x

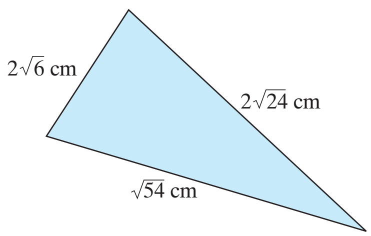
70. The sum of the cube of y and the principal fourth root of y^{12}

For Exercises 71–74, write an English phrase that translates the mathematical expression. (Answers may vary.)

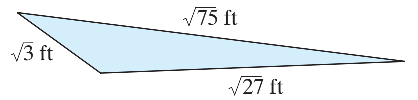
71. $\sqrt{18} - 5^2$ **72.** $4^3 - \sqrt[3]{4}$ **73.** $\sqrt[4]{x} + y^3$ **74.** $a^4 + \sqrt{a}$

For Exercises 75–76, find the exact value of the perimeter, and then approximate the value to 1 decimal place.

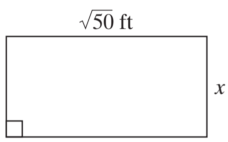
75.



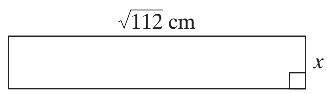
76.



77. The figure has perimeter $14\sqrt{2}$ ft. Find the value of x.



78. The figure has perimeter $12\sqrt{7}$. Find the value of x.

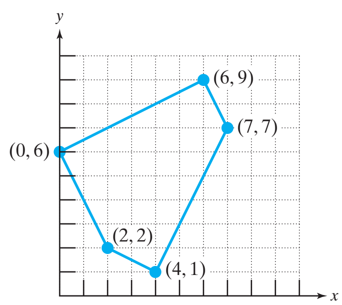


Expanding Your Skills

79. a. An irregularly shaped garden is shown in the figure. All distances are expressed in yards. Find the perimeter. *Hint:* Use the Pythagorean theorem to find the length of each side. Write the final answer in radical form.

b. Approximate your answer to 2 decimal places.

c. If edging costs \$1.49 per foot and sales tax is 6%, find the total cost of edging the garden.



Multiplication of Radicals

Section 7.5

Concepts

1. Multiplication Property of Radicals

In this section we will learn how to multiply radicals by using the multiplication property of radicals first introduced in Section 7.3.

The Multiplication Property of Radicals

Let a and b represent real numbers such that $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are both real. Then

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

To multiply two radical expressions, we use the multiplication property of radicals along with the commutative and associative properties of multiplication.

Example 1 Multiplying Radical Expressions

Multiply the expressions and simplify the result. Assume all variables represent positive real numbers.

a. $(3\sqrt{2})(5\sqrt{6})$ **b.** $(2x\sqrt{y})(-7\sqrt{xy})$ **c.** $(15c\sqrt[3]{cd})\left(\frac{1}{3}\sqrt[3]{cd^2}\right)$

Solution:

a. $(3\sqrt{2})(5\sqrt{6})$

$= (3 \cdot 5)(\sqrt{2} \cdot \sqrt{6})$	Commutative and associative properties of multiplication
$= 15\sqrt{12}$	Multiplication property of radicals
$= 15\sqrt{2^2 3}$	
$= 15 \cdot 2\sqrt{3}$	Simplify the radical.
$= 30\sqrt{3}$	

b. $(2x\sqrt{y})(-7\sqrt{xy})$

$= (2x)(-7)(\sqrt{y} \cdot \sqrt{xy})$	Commutative and associative properties of multiplication
$= -14x\sqrt{xy^2}$	Multiplication property of radicals
$= -14xy\sqrt{x}$	Simplify the radical.

c. $(15c\sqrt[3]{cd})\left(\frac{1}{3}\sqrt[3]{cd^2}\right)$

$= \left(15c \cdot \frac{1}{3}\right)(\sqrt[3]{cd} \cdot \sqrt[3]{cd^2})$	Commutative and associative properties of multiplication
$= 5c\sqrt[3]{c^2d^3}$	Multiplication property of radicals
$= 5cd\sqrt[3]{c^2}$	Simplify the radical.

Skill Practice Multiply the expressions and simplify the results. Assume all variables represent positive real numbers.

1. $\sqrt{5} \cdot \sqrt{3}$ **2.** $(4\sqrt{6})(-2\sqrt{6})$

3. $(3ab\sqrt{b})(-2\sqrt{ab})$ **4.** $(2\sqrt[3]{4ab})(5\sqrt[3]{2a^2b})$

When multiplying radical expressions with more than one term, we use the distributive property.

Example 2 Multiplying Radical Expressions

Multiply the radical expressions. Assume all variables represent positive real numbers.

a. $3\sqrt{11}(2 + \sqrt{11})$ **b.** $(\sqrt{5} + 3\sqrt{2})(2\sqrt{5} - \sqrt{2})$

c. $(2\sqrt{14} + \sqrt{7})(6 - \sqrt{14} + 8\sqrt{7})$ **d.** $(-10a\sqrt{b} + 7b)(a\sqrt{b} + 2b)$

Solution:

a. $3\sqrt{11}(2 + \sqrt{11})$

$= 3\sqrt{11} \cdot (2) + 3\sqrt{11} \cdot \sqrt{11}$ Apply the distributive property.

$= 6\sqrt{11} + 3\sqrt{11^2}$ Multiplication property of radicals

$= 6\sqrt{11} + 3 \cdot 11$ Simplify the radical.

$= 6\sqrt{11} + 33$

b. $(\sqrt{5} + 3\sqrt{2})(2\sqrt{5} - \sqrt{2})$

$= 2\sqrt{5^2} - \sqrt{10} + 6\sqrt{10} - 3\sqrt{2^2}$ Apply the distributive property.

$= 2 \cdot 5 + 5\sqrt{10} - 3 \cdot 2$ Simplify radicals and combine *like* radicals.

$= 10 + 5\sqrt{10} - 6$

$= 4 + 5\sqrt{10}$ Combine *like* terms.

c. $(2\sqrt{14} + \sqrt{7})(6 - \sqrt{14} + 8\sqrt{7})$

$= 12\sqrt{14} - 2\sqrt{14^2} + 16\sqrt{7^2 \cdot 2} + 6\sqrt{7} - \sqrt{7^2 \cdot 2} + 8\sqrt{7^2}$ Apply the distributive property.

$= 12\sqrt{14} - 2 \cdot 14 + 16 \cdot 7\sqrt{2} + 6\sqrt{7} - 7\sqrt{2} + 8 \cdot 7$ Simplify the radicals.

$= 12\sqrt{14} - 28 + 112\sqrt{2} + 6\sqrt{7} - 7\sqrt{2} + 56$ Simplify.

$= 12\sqrt{14} + 105\sqrt{2} + 6\sqrt{7} + 28$ Combine *like* terms.

Skill Practice Answers

1. $\sqrt{15}$ **2.** -48

3. $-6ab^2\sqrt{a}$ **4.** $20a\sqrt[3]{b^2}$

d. $(-10a\sqrt{b} + 7b)(a\sqrt{b} + 2b)$

$= -10a^2\sqrt{b^2} - 20ab\sqrt{b} + 7ab\sqrt{b} + 14b^2$ Apply the distributive property.

$= -10a^2b - 13ab\sqrt{b} + 14b^2$ Simplify and combine *like* terms.

Skill Practice Multiply the radical expressions. Assume all variables represent positive real numbers.

5. $5\sqrt{5}(2\sqrt{5} - 2)$

6. $(2\sqrt{3} - 3\sqrt{10})(\sqrt{3} + 2\sqrt{10})$

7. $(\sqrt{6} + 3\sqrt{2})(-4\sqrt{6} + 2\sqrt{2})$

8. $(x\sqrt{y} + y)(3x\sqrt{y} - 2y)$

2. Expressions of the Form $(\sqrt[n]{a})^n$

The multiplication property of radicals can be used to simplify an expression of the form $(\sqrt{a})^2$, where $a \geq 0$.

$$(\sqrt{a})^2 = \sqrt{a} \cdot \sqrt{a} = \sqrt{a^2} = a \qquad \text{where } a \geq 0$$

This logic can be applied to nth roots. If $\sqrt[n]{a}$ is a real number, then $(\sqrt[n]{a})^n = a$.

Example 3 Simplifying Radical Expressions

Simplify the expressions. Assume all variables represent positive real numbers.

a. $(\sqrt{11})^2$ **b.** $(\sqrt[5]{z})^5$ **c.** $(\sqrt[3]{pq})^3$

Solution:

a. $(\sqrt{11})^2 = 11$ **b.** $(\sqrt[5]{z})^5 = z$ **c.** $(\sqrt[3]{pq})^3 = pq$

Skill Practice Simplify.

9. $(\sqrt{14})^2$ **10.** $(\sqrt[7]{q})^7$ **11.** $(\sqrt[5]{3z})^5$

3. Special Case Products

From Example 2, you may have noticed a similarity between multiplying radical expressions and multiplying polynomials.

In Section 5.2 we learned that the square of a binomial results in a perfect square trinomial:

$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a - b)^2 = a^2 - 2ab + b^2$$

The same patterns occur when squaring a radical expression with two terms.

Example 4 Squaring a Two-Term Radical Expression

Square the radical expressions. Assume all variables represent positive real numbers.

a. $(\sqrt{d} + 3)^2$ **b.** $(5\sqrt{y} - \sqrt{2})^2$

Skill Practice Answers

5. $50 - 10\sqrt{5}$
6. $-54 + \sqrt{30}$
7. $-12 - 20\sqrt{3}$
8. $3x^2y + xy\sqrt{y} - 2y^2$
9. 14 **10.** q **11.** $3z$

Solution:

a. $(\sqrt{d} + 3)^2$ — This expression is in the form $(a + b)^2$, where $a = \sqrt{d}$ and $b = 3$.

$$a^2 + 2ab + b^2$$

$$= (\sqrt{d})^2 + 2(\sqrt{d})(3) + (3)^2$$ Apply the formula $(a + b)^2 = a^2 + 2ab + b^2$.

$$= d + 6\sqrt{d} + 9$$ Simplify.

TIP: The product $(\sqrt{d} + 3)^2$ can also be found by using the distributive property:

$$(\sqrt{d} + 3)^2 = (\sqrt{d} + 3)(\sqrt{d} + 3) = \sqrt{d} \cdot \sqrt{d} + \sqrt{d} \cdot 3 + 3 \cdot \sqrt{d} + 3 \cdot 3$$
$$= \sqrt{d^2} + 3\sqrt{d} + 3\sqrt{d} + 9$$
$$= d + 6\sqrt{d} + 9$$

b. $(5\sqrt{y} - \sqrt{2})^2$ — This expression is in the form $(a - b)^2$, where $a = 5\sqrt{y}$ and $b = \sqrt{2}$.

$$a^2 - 2ab + b^2$$

$$= (5\sqrt{y})^2 - 2(5\sqrt{y})(\sqrt{2}) + (\sqrt{2})^2$$ Apply the formula $(a - b)^2 = a^2 - 2ab + b^2$.

$$= 25y - 10\sqrt{2y} + 2$$ Simplify.

Skill Practice Square the radical expressions. Assume all variables represent positive real numbers.

12. $(\sqrt{a} - 5)^2$ **13.** $(4\sqrt{x} + 3)^2$

Recall from Section 5.2 that the product of two conjugate binomials results in a difference of squares.

$$(a + b)(a - b) = a^2 - b^2$$

The same pattern occurs when multiplying two conjugate radical expressions.

Example 5 Multiplying Conjugate Radical Expressions

Multiply the radical expressions. Assume all variables represent positive real numbers.

a. $(\sqrt{3} + 2)(\sqrt{3} - 2)$ **b.** $\left(\frac{1}{3}\sqrt{s} - \frac{3}{4}\sqrt{t}\right)\left(\frac{1}{3}\sqrt{s} + \frac{3}{4}\sqrt{t}\right)$

Skill Practice Answers

12. $a - 10\sqrt{a} + 25$
13. $16x + 24\sqrt{x} + 9$

Solution:

a. $(\sqrt{3} + 2)(\sqrt{3} - 2)$ — The expression is in the form $(a + b)(a - b)$, where $a = \sqrt{3}$ and $b = 2$.

$a^2 - b^2$

$= (\sqrt{3})^2 - (2)^2$ — Apply the formula $(a + b)(a - b) = a^2 - b^2$.

$= 3 - 4$ — Simplify.

$= -1$

TIP: The product $(\sqrt{3} + 2)(\sqrt{3} - 2)$ can also be found by using the distributive property.

$$(\sqrt{3} + 2)(\sqrt{3} - 2) = \sqrt{3} \cdot \sqrt{3} + \sqrt{3} \cdot (-2) + 2 \cdot \sqrt{3} + 2 \cdot (-2)$$
$$= 3 - 2\sqrt{3} + 2\sqrt{3} - 4$$
$$= 3 - 4$$
$$= -1$$

b. $\left(\frac{1}{3}\sqrt{s} - \frac{3}{4}\sqrt{t}\right)\left(\frac{1}{3}\sqrt{s} + \frac{3}{4}\sqrt{t}\right)$ — This expression is in the form $(a - b)(a + b)$, where $a = \frac{1}{3}\sqrt{s}$ and $b = \frac{3}{4}\sqrt{t}$.

$a^2 - b^2$

$= \left(\frac{1}{3}\sqrt{s}\right)^2 - \left(\frac{3}{4}\sqrt{t}\right)^2$ — Apply the formula $(a + b)(a - b) = a^2 - b^2$.

$= \frac{1}{9}s - \frac{9}{16}t$ — Simplify.

Skill Practice Multiply the conjugates. Assume all variables represent positive real numbers.

14. $(\sqrt{5} + 3)(\sqrt{5} - 3)$ **15.** $\left(\frac{1}{2}\sqrt{a} + \frac{2}{5}\sqrt{b}\right)\left(\frac{1}{2}\sqrt{a} - \frac{2}{5}\sqrt{b}\right)$

4. Multiplying Radicals with Different Indices

The product of two radicals can be simplified provided the radicals have the same index. If the radicals have different indices, then we can use the properties of rational exponents to obtain a common index.

Example 6 Multiplying Radicals with Different Indices

Multiply the expressions. Write the answers in radical form.

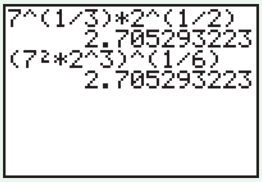
a. $\sqrt[3]{5} \cdot \sqrt[4]{5}$ **b.** $\sqrt[3]{7} \cdot \sqrt{2}$

Skill Practice Answers

14. -4 **15.** $\frac{1}{4}a - \frac{4}{25}b$

Calculator Connections

To approximate expressions such as $\sqrt[3]{7} \cdot \sqrt{2}$ on a calculator, you can enter the expressions using rational exponents. The original problem and simplified form show agreement to 10 digits.



Solution:

a. $\sqrt[3]{5} \cdot \sqrt[4]{5}$

$= 5^{1/3}5^{1/4}$	Rewrite each expression with rational exponents.
$= 5^{(1/3)+(1/4)}$	Because the bases are equal, we can add the exponents.
$= 5^{(4/12)+(3/12)}$	Write the fractions with a common denominator.
$= 5^{7/12}$	Simplify the exponent.
$= \sqrt[12]{5^7}$	Rewrite the expression as a radical.

b. $\sqrt[3]{7} \cdot \sqrt{2}$

$= 7^{1/3}2^{1/2}$	Rewrite each expression with rational exponents.
$= 7^{2/6}2^{3/6}$	Write the rational exponents with a common denominator.
$= (7^2 2^3)^{1/6}$	Apply the power rule of exponents.
$= \sqrt[6]{7^2 2^3}$	Rewrite the expression as a single radical.

Skill Practice Multiply the expressions. Write the answers in radical form. Assume all variables represent positive real numbers.

16. $\sqrt{x} \cdot \sqrt[3]{x}$ **17.** $\sqrt[4]{a^3} \cdot \sqrt[3]{b}$

Skill Practice Answers

16. $\sqrt[6]{x^5}$ **17.** $\sqrt[12]{a^9b^4}$

Section 7.5 Practice Exercises

For the exercises in this set, assume that all variables represent positive real numbers, unless otherwise stated.

Study Skills Exercise

1. In addition to studying the material for a final exam, here are some other activities that students can use to prepare for a test. Circle the importance of each statement.

	not important	somewhat important	very important
a. Get a good night sleep the night before the test.	1	2	3
b. Eat a good meal before the test.	1	2	3
c. Wear comfortable clothing on the day of the test.	1	2	3
d. Arrive early to class on the day of the test.	1	2	3

Review Exercises

For Exercises 2–3, simplify the radicals.

2. $\sqrt[3]{-16x^5y^6z^7}$

3. $-\sqrt{20a^2b^3c}$

For Exercises 4–6, simplify the expressions. Write the answer with positive exponents only.

4. $x^{1/3}y^{1/4}x^{-1/6}y^{1/3}$

5. $p^{1/8}q^{1/2}p^{-1/4}q^{3/2}$

6. $\dfrac{b^{1/4}}{b^{3/2}}$

For Exercises 7–8, add or subtract as indicated.

7. $-2\sqrt[3]{7} + 4\sqrt[3]{7}$

8. $4\sqrt{8x^3} - x\sqrt{50x}$

Concept 1: Multiplication Property of Radicals

For Exercises 9–44, multiply the radical expressions.

9. $\sqrt[3]{7} \cdot \sqrt[3]{3}$

10. $\sqrt[4]{6} \cdot \sqrt[4]{2}$

11. $\sqrt{2} \cdot \sqrt{10}$

12. $\sqrt[3]{4} \cdot \sqrt[3]{12}$

13. $\sqrt[4]{16} \cdot \sqrt[4]{64}$

14. $\sqrt{5x^3} \cdot \sqrt{10x^4}$

15. $(4\sqrt[3]{4})(2\sqrt[3]{5})$

16. $(2\sqrt{5})(3\sqrt{7})$

17. $(8a\sqrt{b})(-3\sqrt{ab})$

18. $(p\sqrt[4]{q^3})(\sqrt[4]{pq})$

19. $\sqrt{30} \cdot \sqrt{12}$

20. $\sqrt{20} \cdot \sqrt{54}$

21. $\sqrt{6x}\sqrt{12x}$

22. $(\sqrt{3ab^2})(\sqrt{21a^2b})$

23. $\sqrt{5a^3b^2}\sqrt{20a^3b^3}$

24. $\sqrt[3]{m^2n^2} \cdot \sqrt[3]{48m^4n^2}$

25. $(4\sqrt{3xy^3})(-2\sqrt{6x^3y^2})$

26. $(2\sqrt[4]{3x})(4\sqrt[4]{27x^6})$

27. $(\sqrt[3]{4a^2b})(\sqrt[3]{2ab^3})(\sqrt[3]{54a^2b})$

28. $(\sqrt[3]{9x^3y})(\sqrt[3]{6xy})(\sqrt[3]{8x^2y^5})$

29. $\sqrt{3}(4\sqrt{3} - 6)$

30. $3\sqrt{5}(2\sqrt{5} + 4)$

31. $\sqrt{2}(\sqrt{6} - \sqrt{3})$

32. $\sqrt{5}(\sqrt{3} + \sqrt{7})$

33. $-3\sqrt{x}(\sqrt{x} + 7)$

34. $-2\sqrt{y}(8 - \sqrt{y})$

35. $(\sqrt{3} + 2\sqrt{10})(4\sqrt{3} - \sqrt{10})$

36. $(8\sqrt{7} - \sqrt{5})(\sqrt{7} + 3\sqrt{5})$

37. $(\sqrt{x} + 4)(\sqrt{x} - 9)$

38. $(\sqrt{w} - 2)(\sqrt{w} - 9)$

39. $(\sqrt[3]{y} + 2)(\sqrt[3]{y} - 3)$

40. $(4 + \sqrt[5]{p})(5 + \sqrt[5]{p})$

41. $(\sqrt{a} - 3\sqrt{b})(9\sqrt{a} - \sqrt{b})$

42. $(11\sqrt{m} + 4\sqrt{n})(\sqrt{m} + \sqrt{n})$

43. $(\sqrt{p} + 2\sqrt{q})(8 + 3\sqrt{p} - \sqrt{q})$

44. $(5\sqrt{s} - \sqrt{t})(\sqrt{s} + 5 + 6\sqrt{t})$

Concept 2: Expressions of the Form $(\sqrt[n]{a})^n$

For Exercises 45–52, simplify the expressions. Assume all variables represent positive real numbers.

45. $(\sqrt{15})^2$

46. $(\sqrt{58})^2$

47. $(\sqrt{3y})^2$

48. $(\sqrt{19yz})^2$

49. $(\sqrt[3]{6})^3$

50. $(\sqrt[5]{24})^5$

51. $\sqrt{709} \cdot \sqrt{709}$

52. $\sqrt{401} \cdot \sqrt{401}$

Concept 3: Special Case Products

53. **a.** Write the formula for the product of two conjugates. $(x + y)(x - y) =$

b. Multiply $(x + 5)(x - 5)$.

54. **a.** Write the formula for squaring a binomial. $(x + y)^2 =$

b. Multiply $(x + 5)^2$.

For Exercises 55–66, multiply the special products.

55. $(\sqrt{3} + x)(\sqrt{3} - x)$

56. $(y + \sqrt{6})(y - \sqrt{6})$

57. $(\sqrt{6} + \sqrt{2})(\sqrt{6} - \sqrt{2})$

58. $(\sqrt{15} + \sqrt{5})(\sqrt{15} - \sqrt{5})$

59. $(8\sqrt{x} + 2\sqrt{y})(8\sqrt{x} - 2\sqrt{y})$

60. $(4\sqrt{s} + 11\sqrt{t})(4\sqrt{s} - 11\sqrt{t})$

61. $(\sqrt{13} + 4)^2$

62. $(6 - \sqrt{11})^2$

63. $(\sqrt{p} - \sqrt{7})^2$

64. $(\sqrt{q} + \sqrt{2})^2$

65. $(\sqrt{2a} - 3\sqrt{b})^2$

66. $(\sqrt{3w} + 4\sqrt{z})^2$

Mixed Exercises

For Exercises 67–74, identify each statement as true or false. If an answer is false, explain why.

67. $\sqrt{3} \cdot \sqrt{2} = \sqrt{6}$

68. $\sqrt{5} \cdot \sqrt[3]{2} = \sqrt{10}$

69. $(x - \sqrt{5})^2 = x - 5$

70. $3(2\sqrt{5x}) = 6\sqrt{5x}$

71. $5(3\sqrt{4x}) = 15\sqrt{20x}$

72. $\dfrac{\sqrt{5x}}{5} = \sqrt{x}$

73. $\dfrac{3\sqrt{x}}{3} = \sqrt{x}$

74. $(\sqrt{t} - 1)(\sqrt{t} + 1) = t - 1$

For Exercises 75–82, square the expressions.

75. $\sqrt{39}$

76. $\sqrt{21}$

77. $-\sqrt{6x}$

78. $-\sqrt{8a}$

79. $\sqrt{3x + 1}$

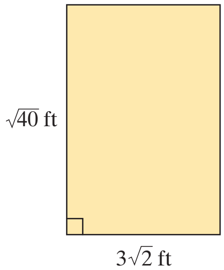
80. $\sqrt{x - 1}$

81. $\sqrt{x + 3} - 4$

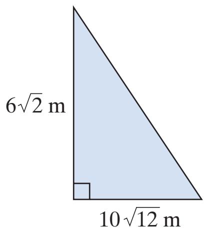
82. $\sqrt{x + 1} + 3$

For Exercises 83–86, find the exact area.

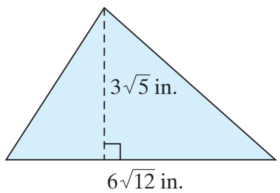
83.



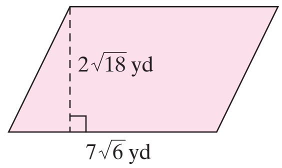
84.



85.



86.



Concept 4: Multiplying Radicals with Different Indices

For Exercises 87–96, multiply or divide the radicals with different indices. Write the answers in radical form and simplify.

87. $\sqrt{x} \cdot \sqrt[4]{x}$

88. $\sqrt[3]{y} \cdot \sqrt{y}$

89. $\sqrt[5]{2z} \cdot \sqrt[3]{2z}$

90. $\sqrt[3]{5w} \cdot \sqrt[4]{5w}$

91. $\sqrt[3]{p^2} \cdot \sqrt{p^3}$

92. $\sqrt[4]{q^3} \cdot \sqrt[3]{q^2}$

93. $\dfrac{\sqrt{u^3}}{\sqrt[3]{u}}$

94. $\dfrac{\sqrt{v^5}}{\sqrt[4]{v}}$

95. $\dfrac{\sqrt{(a + b)}}{\sqrt[3]{(a + b)}}$

96. $\dfrac{\sqrt[3]{(q - 1)}}{\sqrt[4]{(q - 1)}}$

For Exercises 97–100, multiply the radicals with different indices.

97. $\sqrt[3]{x} \cdot \sqrt[6]{y}$

98. $\sqrt{a} \cdot \sqrt[6]{b}$

99. $\sqrt[4]{8} \cdot \sqrt{3}$

100. $\sqrt{11} \cdot \sqrt[6]{2}$

Expanding Your Skills

101. Multiply $(\sqrt[3]{a} + \sqrt[3]{b})(\sqrt[3]{a^2} - \sqrt[3]{ab} + \sqrt[3]{b^2})$.

102. Multiply $(\sqrt[3]{a} - \sqrt[3]{b})(\sqrt[3]{a^2} + \sqrt[3]{ab} + \sqrt[3]{b^2})$.

Rationalization

Section 7.6

Concepts

1. Simplified Form of a Radical

In this section we will learn additional techniques to simplify a radical.

Simplified Form of a Radical

Consider any radical expression in which the radicand is written as a product of prime factors. The expression is in simplified form if all the following conditions are met:

1. The radicand has no factor raised to a power greater than or equal to the index.
2. The radicand does not contain a fraction.
3. No radicals are in the denominator of a fraction.

The third condition restricts radicals from the denominator of an expression. The process to remove a radical from the denominator is called **rationalizing the denominator**. In many cases, rationalizing the denominator creates an expression that is computationally simpler. In Examples 3 and 4 we will show that

$$\frac{6}{\sqrt{3}} = 2\sqrt{3} \qquad \text{and} \qquad \frac{-2}{2 + \sqrt{6}} = 2 - \sqrt{6}$$

We will demonstrate the process to rationalize the denominator as two separate cases:

- Rationalizing the denominator (one term)
- Rationalizing the denominator (two terms involving square roots)

2. Rationalizing the Denominator—One Term

To begin the first case, recall that the nth root of a perfect nth power simplifies completely.

$$\sqrt{x^2} = x \qquad x \geq 0$$
$$\sqrt[3]{x^3} = x$$
$$\sqrt[4]{x^4} = x \qquad x \geq 0$$
$$\sqrt[5]{x^5} = x$$
$$\vdots$$

Therefore, to rationalize a radical expression, use the multiplication property of radicals to create an nth root of an nth power.

Example 1 Rationalizing Radical Expressions

Fill in the missing radicand to rationalize the radical expressions. Assume all variables represent positive real numbers.

a. $\sqrt{a} \cdot \sqrt{?} = \sqrt{a^2} = a$ **b.** $\sqrt[3]{y} \cdot \sqrt[3]{?} = \sqrt[3]{y^3} = y$

c. $\sqrt[4]{2z^3} \cdot \sqrt[4]{?} = \sqrt[4]{2^4 z^4} = 2z$

Solution:

a. $\sqrt{a} \cdot \sqrt{?} = \sqrt{a^2} = a$ What multiplied by $\sqrt{a}$ will equal $\sqrt{a^2}$?

$\sqrt{a} \cdot \sqrt{a} = \sqrt{a^2} = a$

b. $\sqrt[3]{y} \cdot \sqrt[3]{?} = \sqrt[3]{y^3} = y$ What multiplied by $\sqrt[3]{y}$ will equal $\sqrt[3]{y^3}$?

$\sqrt[3]{y} \cdot \sqrt[3]{y^2} = \sqrt[3]{y^3} = y$

c. $\sqrt[4]{2z^3} \cdot \sqrt[4]{?} = \sqrt[4]{2^4z^4} = 2z$ What multiplied by $\sqrt[4]{2z^3}$ will equal $\sqrt[4]{2^4z^4}$?

$\sqrt[4]{2z^3} \cdot \sqrt[4]{2^3z} = \sqrt[4]{2^4z^4} = 2z$

Skill Practice Fill in the missing radicand to rationalize the radical expression.

1. $\sqrt{7} \cdot \sqrt{?}$ **2.** $\sqrt[5]{t^2} \cdot \sqrt[5]{?}$ **3.** $\sqrt[3]{5x^2} \cdot \sqrt[3]{?}$

To rationalize the denominator of a radical expression, multiply the numerator and denominator by an appropriate expression to create an *n*th root of an *n*th power in the denominator.

Example 2 Rationalizing the Denominator—One Term

Simplify the expression

$$\frac{5}{\sqrt[3]{a}} \qquad a \neq 0$$

Solution:

To remove the radical from the denominator, a cube root of a perfect cube is needed in the denominator. Multiply numerator and denominator by $\sqrt[3]{a^2}$ because $\sqrt[3]{a} \cdot \sqrt[3]{a^2} = \sqrt[3]{a^3} = a$.

$$\frac{5}{\sqrt[3]{a}} = \frac{5}{\sqrt[3]{a}} \cdot \frac{\sqrt[3]{a^2}}{\sqrt[3]{a^2}}$$

$$= \frac{5\sqrt[3]{a^2}}{\sqrt[3]{a^3}}$$ Multiply the radicals.

$$= \frac{5\sqrt[3]{a^2}}{a}$$ Simplify.

Skill Practice Simplify the expression. Assume $y > 0$.

4. $\frac{2}{\sqrt[4]{y}}$

Note that for $a \neq 0$, the expression $\frac{\sqrt[3]{a^2}}{\sqrt[3]{a^2}} = 1$. In Example 2, multiplying the expression $\frac{5}{\sqrt[3]{a}}$ by this ratio does not change its value.

Skill Practice Answers

1. 7 **2.** t^3 **3.** 5^2x

4. $\frac{2\sqrt[4]{y^3}}{y}$

Example 3 Rationalizing the Denominator—One Term

Simplify the expressions. Assume all variables represent positive real numbers.

a. $\dfrac{6}{\sqrt{3}}$ **b.** $\sqrt{\dfrac{y^5}{7}}$ **c.** $\dfrac{15}{\sqrt[3]{25s}}$ **d.** $\dfrac{\sqrt{125p^3}}{\sqrt{5p}}$

Solution:

a. To rationalize the denominator, a square root of a perfect square is needed. Multiply numerator and denominator by $\sqrt{3}$ because $\sqrt{3}\cdot\sqrt{3} = \sqrt{3^2} = 3$.

$$\frac{6}{\sqrt{3}} = \frac{6}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}$$ Rationalize the denominator.

$$= \frac{6\sqrt{3}}{\sqrt{3^2}}$$ Multiply the radicals.

$$= \frac{6\sqrt{3}}{3}$$ Simplify.

$$= 2\sqrt{3}$$ Reduce to lowest terms.

b. $\sqrt{\dfrac{y^5}{7}}$ The radical contains an irreducible fraction.

$$= \frac{\sqrt{y^5}}{\sqrt{7}}$$ Apply the division property of radicals.

$$= \frac{y^2\sqrt{y}}{\sqrt{7}}$$ Remove factors from the radical in the numerator.

$$= \frac{y^2\sqrt{y}}{\sqrt{7}}\cdot\frac{\sqrt{7}}{\sqrt{7}}$$ Rationalize the denominator. *Note:* $\sqrt{7}\cdot\sqrt{7} = \sqrt{7^2} = 7$.

$$= \frac{y^2\sqrt{7y}}{\sqrt{7^2}}$$

$$= \frac{y^2\sqrt{7y}}{7}$$ Simplify.

Avoiding Mistakes:

A factor within a radicand cannot be simplified with a factor outside the radicand. For example, $\dfrac{\sqrt{7y}}{7}$ cannot be simplified.

c. $\dfrac{15}{\sqrt[3]{25s}}$

$$= \frac{15}{\sqrt[3]{5^2 s}}\cdot\frac{\sqrt[3]{5s^2}}{\sqrt[3]{5s^2}}$$ Because $25 = 5^2$, one additional factor of 5 is needed to form a perfect cube. Two additional factors of s are needed to make a perfect cube. Multiply numerator and denominator by $\sqrt[3]{5s^2}$.

$$= \frac{15\sqrt[3]{5s^2}}{\sqrt[3]{5^3 s^3}}$$

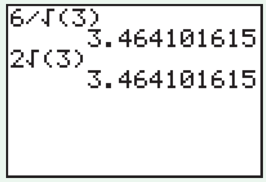
$$= \frac{15\sqrt[3]{5s^2}}{5s}$$ Simplify the perfect cube.

$$= \frac{\overset{3}{\cancel{15}}\sqrt[3]{5s^2}}{\underset{1}{\cancel{5}}s}$$ Reduce to lowest terms.

$$= \frac{3\sqrt[3]{5s^2}}{s}$$

Calculator Connections

A calculator can be used to support the solution to a simplified radical. The calculator approximations of the expressions $6/\sqrt{3}$ and $2\sqrt{3}$ agree to 10 digits.



TIP: In the expression $\frac{15\sqrt[3]{5s^2}}{5s}$, the factor of 15 and the factor of 5 may be reduced because both are outside the radical.

$$\frac{15\sqrt[3]{5s^2}}{5s} = \frac{15}{5} \cdot \frac{\sqrt[3]{5s^2}}{s} = \frac{3\sqrt[3]{5s^2}}{s}$$

d. $\frac{\sqrt{125p^3}}{\sqrt{5p}}$ Notice that the radicands in the numerator and denominator share common factors.

$= \sqrt{\frac{125p^3}{5p}}$ Rewrite the expression by using the division property of radicals.

$= \sqrt{25p^2}$ Simplify the fraction within the radicand.

$= 5p$ Simplify the radical.

Skill Practice Simplify the expressions.

5. $\frac{12}{\sqrt{2}}$ **6.** $\sqrt{\frac{8}{3}}$ **7.** $\frac{18}{\sqrt[3]{3y^2}}$ **8.** $\frac{\sqrt[3]{16x^4}}{\sqrt[3]{2x}}$

3. Rationalizing the Denominator—Two Terms

Example 4 demonstrates how to rationalize a two-term denominator involving square roots.

First recall from the multiplication of polynomials that the product of two conjugates results in a difference of squares.

$$(a + b)(a - b) = a^2 - b^2$$

If either a or b has a square root factor, the expression will simplify without a radical. That is, the expression is *rationalized.* For example:

$$\begin{aligned}(2 + \sqrt{6})(2 - \sqrt{6}) &= (2)^2 - (\sqrt{6})^2 \\ &= 4 - 6 \\ &= -2\end{aligned}$$

Example 4 **Rationalizing the Denominator—Two Terms**

Simplify the expression by rationalizing the denominator.

$$\frac{-2}{2 + \sqrt{6}}$$

Solution:

$\frac{-2}{2 + \sqrt{6}}$

$= \frac{(-2)}{(2 + \sqrt{6})} \cdot \frac{(2 - \sqrt{6})}{(2 - \sqrt{6})}$ Multiply the numerator and denominator by the conjugate of the denominator.

conjugates

Skill Practice Answers

5. $6\sqrt{2}$ **6.** $\frac{2\sqrt{6}}{3}$

7. $\frac{6\sqrt[3]{9y}}{y}$ **8.** $2x$

$$= \frac{-2(2 - \sqrt{6})}{(2)^2 - (\sqrt{6})^2}$$ In the denominator, apply the formula $(a + b)(a - b) = a^2 - b^2$.

$$= \frac{-2(2 - \sqrt{6})}{4 - 6}$$ Simplify.

$$= \frac{-2(2 - \sqrt{6})}{-2}$$

$$= \frac{\cancel{-2}(2 - \sqrt{6})}{\cancel{-2}}$$

$$= 2 - \sqrt{6}$$

Skill Practice Simplify by rationalizing the denominator.

9. $\dfrac{4}{3 + \sqrt{5}}$

TIP: Recall that only factors within a fraction may be simplified (not terms).

$$\frac{\cancel{a}b}{\cancel{a}} = b$$

but

$\dfrac{a + b}{a}$ cannot be simplified.

Example 5 Rationalizing the Denominator—Two Terms

Rationalize the denominator of the expression. Assume all variables represent positive real numbers and $c \neq d$.

$$\frac{\sqrt{c} + \sqrt{d}}{\sqrt{c} - \sqrt{d}}$$

Solution:

$$\frac{\sqrt{c} + \sqrt{d}}{\sqrt{c} - \sqrt{d}}$$

$$= \frac{(\sqrt{c} + \sqrt{d})}{(\sqrt{c} - \sqrt{d})} \cdot \frac{(\sqrt{c} + \sqrt{d})}{(\sqrt{c} + \sqrt{d})}$$ Multiply numerator and denominator by the conjugate of the denominator.

conjugates

$$= \frac{(\sqrt{c} + \sqrt{d})^2}{(\sqrt{c})^2 - (\sqrt{d})^2}$$ In the denominator apply the formula $(a + b)(a - b) = a^2 - b^2$.

$$= \frac{(\sqrt{c} + \sqrt{d})^2}{c - d}$$ Simplify.

$$= \frac{(\sqrt{c})^2 + 2\sqrt{c}\sqrt{d} + (\sqrt{d})^2}{c - d}$$ In the numerator apply the formula $(a + b)^2 = a^2 + 2ab + b^2$.

$$= \frac{c + 2\sqrt{cd} + d}{c - d}$$

Skill Practice Simplify by rationalizing the denominator. Assume that $y \geq 0$.

10. $\dfrac{\sqrt{3} - \sqrt{y}}{\sqrt{3} + \sqrt{y}}$

Skill Practice Answers

9. $3 - \sqrt{5}$

10. $\dfrac{3 - 2\sqrt{3y} + y}{3 - y}$

Section 7.6 Practice Exercises

For the exercises in this set, assume that all variables represent positive real numbers unless otherwise stated.

Study Skills Exercise

1. Define the key term **rationalizing the denominator**.

Review Exercises

2. Multiply and simplify. $\sqrt{x} \cdot \sqrt{x}$

For Exercises 3–10, perform the indicated operations.

3. $2y\sqrt{45} + 3\sqrt{20y^2}$ **4.** $3x\sqrt{72x} - 9\sqrt{50x^3}$ **5.** $(-6\sqrt{y} + 3)(3\sqrt{y} + 1)$ **6.** $(\sqrt{w} + 12)(2\sqrt{w} - 4)$

7. $(8 - \sqrt{t})^2$ **8.** $(\sqrt{p} + 4)^2$ **9.** $(\sqrt{2} + \sqrt{7})(\sqrt{2} - \sqrt{7})$ **10.** $(\sqrt{3} + 5)(\sqrt{3} - 5)$

Concept 2: Rationalizing the Denominator—One Term

The radical expressions in Exercises 11–18 have radicals in the denominator. Fill in the missing radicands to rationalize the radical expressions in the denominators.

11. $\dfrac{x}{\sqrt{5}} = \dfrac{x}{\sqrt{5}} \cdot \dfrac{\sqrt{?}}{\sqrt{?}}$ **12.** $\dfrac{2}{\sqrt{x}} = \dfrac{2}{\sqrt{x}} \cdot \dfrac{\sqrt{?}}{\sqrt{?}}$

13. $\dfrac{7}{\sqrt[3]{x}} = \dfrac{7}{\sqrt[3]{x}} \cdot \dfrac{\sqrt[3]{?}}{\sqrt[3]{?}}$ **14.** $\dfrac{5}{\sqrt[4]{y}} = \dfrac{5}{\sqrt[4]{y}} \cdot \dfrac{\sqrt[4]{?}}{\sqrt[4]{?}}$

15. $\dfrac{8}{\sqrt{3z}} = \dfrac{8}{\sqrt{3z}} \cdot \dfrac{\sqrt{??}}{\sqrt{??}}$ **16.** $\dfrac{10}{\sqrt{7w}} = \dfrac{10}{\sqrt{7w}} \cdot \dfrac{\sqrt{??}}{\sqrt{??}}$

17. $\dfrac{1}{\sqrt[4]{2a^2}} = \dfrac{1}{\sqrt[4]{2a^2}} \cdot \dfrac{\sqrt[4]{??}}{\sqrt[4]{??}}$ **18.** $\dfrac{1}{\sqrt[3]{6b^2}} = \dfrac{1}{\sqrt[3]{6b^2}} \cdot \dfrac{\sqrt[3]{??}}{\sqrt[3]{??}}$

For Exercises 19–50, rationalize the denominator.

19. $\dfrac{1}{\sqrt{3}}$ **20.** $\dfrac{1}{\sqrt{7}}$ **21.** $\sqrt{\dfrac{1}{x}}$ **22.** $\sqrt{\dfrac{1}{z}}$

23. $\dfrac{6}{\sqrt{2y}}$ **24.** $\dfrac{9}{\sqrt{3t}}$ **25.** $\dfrac{-2a}{\sqrt{a}}$ **26.** $\dfrac{-7b}{\sqrt{b}}$

27. $\dfrac{6}{\sqrt{8}}$ **28.** $\dfrac{2}{\sqrt{48}}$ **29.** $\dfrac{3}{\sqrt[3]{2}}$ **30.** $\dfrac{2}{\sqrt[3]{7}}$

31. $\dfrac{-6}{\sqrt[4]{x}}$ **32.** $\dfrac{-2}{\sqrt[5]{y}}$ **33.** $\dfrac{7}{\sqrt[3]{4}}$ **34.** $\dfrac{1}{\sqrt[3]{9}}$

35. $\sqrt[3]{\dfrac{4}{w^2}}$ **36.** $\sqrt[3]{\dfrac{5}{z^2}}$ **37.** $\sqrt[4]{\dfrac{16}{3}}$ **38.** $\sqrt[4]{\dfrac{81}{8}}$

39. $\dfrac{2}{\sqrt[3]{4x^2}}$ **40.** $\dfrac{6}{\sqrt[3]{3y^2}}$ **41.** $\sqrt[3]{\dfrac{16x^3}{y}}$ **42.** $\sqrt{\dfrac{5}{9x}}$

43. $\dfrac{\sqrt{x^4y^5}}{\sqrt{10x}}$ **44.** $\sqrt[4]{\dfrac{10x^2}{15xy^3}}$ **45.** $\dfrac{8}{7\sqrt{24}}$ **46.** $\dfrac{5}{3\sqrt{50}}$

47. $\dfrac{1}{\sqrt{x^7}}$ **48.** $\dfrac{1}{\sqrt{y^5}}$ **49.** $\dfrac{2}{\sqrt{8x^5}}$ **50.** $\dfrac{6}{\sqrt{27t^7}}$

Concept 3: Rationalizing the Denominator—Two Terms

51. What is the conjugate of $\sqrt{2} - \sqrt{6}$?

52. What is the conjugate of $\sqrt{11} + \sqrt{5}$?

53. What is the conjugate of $\sqrt{x} + 23$?

54. What is the conjugate of $17 - \sqrt{y}$?

For Exercises 55–74, rationalize the denominators.

55. $\dfrac{4}{\sqrt{2}+3}$ **56.** $\dfrac{6}{4-\sqrt{3}}$ **57.** $\dfrac{8}{\sqrt{6}-2}$ **58.** $\dfrac{-12}{\sqrt{5}-3}$

59. $\dfrac{\sqrt{7}}{\sqrt{3}+2}$ **60.** $\dfrac{\sqrt{8}}{\sqrt{3}+1}$ **61.** $\dfrac{-1}{\sqrt{p}+\sqrt{q}}$ **62.** $\dfrac{6}{\sqrt{a}-\sqrt{b}}$

63. $\dfrac{x-5}{\sqrt{x}+\sqrt{5}}$ **64.** $\dfrac{y-2}{\sqrt{y}-\sqrt{2}}$ **65.** $\dfrac{-7}{2\sqrt{a}-5\sqrt{b}}$ **66.** $\dfrac{-4}{3\sqrt{w}-2\sqrt{z}}$

67. $\dfrac{3\sqrt{5}-\sqrt{3}}{\sqrt{3}+\sqrt{5}}$ **68.** $\dfrac{2\sqrt{5}+\sqrt{6}}{\sqrt{6}-\sqrt{5}}$ **69.** $\dfrac{3\sqrt{10}}{2+\sqrt{10}}$ **70.** $\dfrac{4\sqrt{7}}{3+\sqrt{7}}$

71. $\dfrac{2\sqrt{3}+\sqrt{7}}{3\sqrt{3}-\sqrt{7}}$ **72.** $\dfrac{5\sqrt{2}-\sqrt{5}}{5\sqrt{2}+\sqrt{5}}$ **73.** $\dfrac{\sqrt{5}+4}{2-\sqrt{5}}$ **74.** $\dfrac{3+\sqrt{2}}{\sqrt{2}-5}$

Mixed Exercises

For Exercises 75–78, translate the English phrase to an algebraic expression. Then simplify the expression.

75. 16 divided by the cube root of 4

76. 21 divided by the principal fourth root of 27

77. 4 divided by the difference of x and the principal square root of 2

78. 8 divided by the sum of y and the principal square root of 3

79. The time T (in seconds) for a pendulum to make one complete swing back and forth is approximated by

$$T(x) = 2\pi\sqrt{\frac{x}{32}}$$

where x is the length of the pendulum in feet.

Determine the exact time required for one swing for a pendulum that is 1 ft long. Then approximate the time to the nearest hundredth of a second.

80. An object is dropped off a building x meters tall. The time T (in seconds) required for the object to hit the ground is given by

$$T(x) = \sqrt{\frac{10x}{49}}$$

Find the exact time required for the object to hit the ground if it is dropped off the First National Plaza Building in Chicago, a height of 230 m. Then round the time to the nearest hundredth of a second.

Expanding Your Skills

For Exercises 81–86, simplify each term of the expression. Then add or subtract as indicated.

81. $\frac{\sqrt{6}}{2} + \frac{1}{\sqrt{6}}$

82. $\frac{1}{\sqrt{7}} + \sqrt{7}$

83. $\sqrt{15} - \sqrt{\frac{3}{5}} + \sqrt{\frac{5}{3}}$

84. $\sqrt{\frac{6}{2}} - \sqrt{12} + \sqrt{\frac{2}{6}}$

85. $\sqrt[3]{25} + \frac{3}{\sqrt[3]{5}}$

86. $\frac{1}{\sqrt[3]{4}} + \sqrt[3]{54}$

For Exercises 87–90, rationalize the numerator by multiplying both numerator and denominator by the conjugate of the numerator.

87. $\frac{\sqrt{3} + 6}{2}$

88. $\frac{\sqrt{7} - 2}{5}$

89. $\frac{\sqrt{a} - \sqrt{b}}{\sqrt{a} + \sqrt{b}}$

90. $\frac{\sqrt{p} + \sqrt{q}}{\sqrt{p} - \sqrt{q}}$

Section 7.7 Radical Equations

Concepts

1. Solutions to Radical Equations
2. Solving Radical Equations Involving One Radical
3. Solving Radical Equations Involving More than One Radical
4. Applications of Radical Equations and Functions

1. Solutions to Radical Equations

An equation with one or more radicals containing a variable is called a **radical equation**. For example, $\sqrt[3]{x} = 5$ is a radical equation. Recall that $(\sqrt[n]{a})^n = a$, provided $\sqrt[n]{a}$ is a real number. The basis of solving a radical equation is to eliminate the radical by raising both sides of the equation to a power equal to the index of the radical.

To solve the equation $\sqrt[3]{x} = 5$, cube both sides of the equation.

$$\sqrt[3]{x} = 5$$
$$(\sqrt[3]{x})^3 = (5)^3$$
$$x = 125$$

By raising each side of a radical equation to a power equal to the index of the radical, a new equation is produced. Note, however, that some of or all the solutions to the new equation may *not* be solutions to the original radical equation. For this reason, it is necessary to *check all potential solutions* in the original equation. For example, consider the equation $x = 4$. By squaring both sides we produce a quadratic equation.

$$x = 4$$

Square both sides.

$$(x)^2 = (4)^2$$ ← Squaring both sides produces a quadratic equation.

$$x^2 = 16$$ ← Solving this equation, we find two solutions. However, the solution $x = -4$ does not check in the original equation.

$$x^2 - 16 = 0$$

$$(x - 4)(x + 4) = 0$$

$x = 4$ or $x \not= -4$ ($x = -4$ does not check and is called an **extraneous solution**)

Steps to Solve a Radical Equation

1. Isolate the radical. If an equation has more than one radical, choose one of the radicals to isolate.
2. Raise each side of the equation to a power equal to the index of the radical.
3. Solve the resulting equation. If the equation still has a radical, repeat steps 1 and 2.

*4. Check the potential solutions in the original equation.

*Extraneous solutions can arise only when both sides of the equation are raised to an *even power.* Therefore, an equation with odd-index roots will not have an extraneous solution. However, it is still recommended that you check *all* potential solutions regardless of the type of root.

2. Solving Radical Equations Involving One Radical

Example 1 Solving Equations Containing One Radical

Solve the equations.

a. $\sqrt{p} + 5 = 9$ **b.** $\sqrt{3x - 2} + 4 = 5$ **c.** $(w - 1)^{1/3} - 2 = 2$

d. $7 = \sqrt[4]{x + 3} + 9$ **e.** $y + \sqrt{y - 2} = 8$

Solution:

a. $\sqrt{p} + 5 = 9$

$\sqrt{p} = 4$ Isolate the radical.

$(\sqrt{p})^2 = 4^2$ Because the index is 2, square both sides.

$p = 16$

Check: $p = 16$ Check $p = 16$ as a potential solution.

$\sqrt{p} + 5 = 9$

$\sqrt{16} + 5 \overset{?}{=} 9$

$4 + 5 = 9$ ✔ True, $p = 16$ is a solution to the original equation.

b. $\sqrt{3x - 2} + 4 = 5$

$\sqrt{3x - 2} = 1$	Isolate the radical.
$(\sqrt{3x - 2})^2 = (1)^2$	Because the index is 2, square both sides.
$3x - 2 = 1$	Simplify.
$3x = 3$	Solve the resulting equation.
$x = 1$	
Check: $x = 1$	Check $x = 1$ as a potential solution.
$\sqrt{3x - 2} + 4 = 5$	
$\sqrt{3(1) - 2} + 4 \stackrel{?}{=} 5$	
$\sqrt{1} + 4 \stackrel{?}{=} 5$	
$5 = 5$ ✔	True, $x = 1$ is a solution to the original equation.

c. $(w - 1)^{1/3} - 2 = 2$ Note that $(w - 1)^{1/3} = \sqrt[3]{w - 1}$.

$\sqrt[3]{w - 1} - 2 = 2$	
$\sqrt[3]{w - 1} = 4$	Isolate the radical.
$(\sqrt[3]{w - 1})^3 = (4)^3$	Because the index is 3, cube both sides.
$w - 1 = 64$	Simplify.
$w = 65$	
Check: $w = 65$	
$(w - 1)^{1/3} - 2 = 2$	Check $w = 65$ as a potential solution.
$\sqrt[3]{65 - 1} - 2 \stackrel{?}{=} 2$	
$\sqrt[3]{64} - 2 \stackrel{?}{=} 2$	
$4 - 2 \stackrel{?}{=} 2$	
$2 = 2$ ✔	True, $w = 65$ is a solution to the original equation.

TIP: After isolating the radical in Example 1(d), the equation shows a fourth root equated to a negative number:

$$-2 = \sqrt[4]{x + 3}$$

By definition, a principal fourth root of any real number must be nonnegative. Therefore, there can be no real solution to this equation.

d. $7 = \sqrt[4]{x + 3} + 9$

$-2 = \sqrt[4]{x + 3}$	Isolate the radical.
$(-2)^4 = (\sqrt[4]{x + 3})^4$	Because the index is 4, raise both sides to the fourth power.
$16 = x + 3$	
$x = 13$	Solve for x.
Check: $x = 13$	
$7 = \sqrt[4]{x + 3} + 9$	
$7 \stackrel{?}{=} \sqrt[4]{(13) + 3} + 9$	
$7 \stackrel{?}{=} \sqrt[4]{16} + 9$	
$7 \neq 2 + 9$	$x = 13$ is *not* a solution to the original equation.

The equation $7 = \sqrt[4]{x + 3} + 9$ has no solution.

e. $y + \sqrt{y - 2} = 8$

$\sqrt{y-2} = 8 - y$	Isolate the radical.
$(\sqrt{y-2})^2 = (8-y)^2$	Because the index is 2, square both sides.
	Note that $(8 - y)^2 = (8 - y)(8 - y) = 64 - 16y + y^2$.
$y - 2 = 64 - 16y + y^2$	Simplify.
$0 = y^2 - 17y + 66$	The equation is quadratic. Set one side equal to zero. Write the other side in descending order.
$0 = (y - 11)(y - 6)$	Factor.
$y - 11 = 0$ or $y - 6 = 0$	Set each factor equal to zero.
$y = 11$ or $y = 6$	Solve.

Check: $y = 11$

$y + \sqrt{y-2} = 8$

$11 + \sqrt{11 - 2} \stackrel{?}{=} 8$

$11 + \sqrt{9} \stackrel{?}{=} 8$

$11 + 3 \stackrel{?}{=} 8$

$14 \neq 8$

$y = 11$ is not a solution to the original equation.

Check: $y = 6$

$y + \sqrt{y-2} = 8$

$6 + \sqrt{6 - 2} \stackrel{?}{=} 8$

$6 + \sqrt{4} \stackrel{?}{=} 8$

$6 + 2 \stackrel{?}{=} 8$

$8 = 8$ ✔

True, $y = 6$ is a solution to the original equation.

Skill Practice Solve the equations.

1. $\sqrt{x} - 3 = 2$ **2.** $\sqrt{5y + 1} - 2 = 4$ **3.** $(t + 2)^{1/3} + 5 = 3$

4. $\sqrt[4]{b - 1} + 6 = 3$ **5.** $\sqrt{x + 1} + 5 = x$

Avoiding Mistakes:

Be sure to square both sides of the equation, not the individual terms.

3. Solving Radical Equations Involving More than One Radical

Example 2 Solving Equations with Two Radicals

Solve the radical equation.

$$\sqrt[3]{2x - 4} = \sqrt[3]{1 - 8x}$$

Solution:

$\sqrt[3]{2x-4} = \sqrt[3]{1-8x}$	
$(\sqrt[3]{2x-4})^3 = (\sqrt[3]{1-8x})^3$	Because the index is 3, cube both sides.
$2x - 4 = 1 - 8x$	Simplify.
$10x - 4 = 1$	Solve the resulting equation.
$10x = 5$	
$x = \frac{1}{2}$	Solve for x.

Skill Practice Answers

1. $x = 25$ **2.** $y = 7$

3. $t = -10$

4. No solution ($b = 82$ does not check)

5. $x = 8$ ($x = 3$ does not check.)

Check: $x = \frac{1}{2}$

$$\sqrt[3]{2x - 4} = \sqrt[3]{1 - 8x}$$
$$\sqrt[3]{2\left(\frac{1}{2}\right) - 4} \stackrel{?}{=} \sqrt[3]{1 - 8\left(\frac{1}{2}\right)}$$
$$\sqrt[3]{1 - 4} \stackrel{?}{=} \sqrt[3]{1 - 4}$$
$$\sqrt[3]{-3} = \sqrt[3]{-3} \checkmark$$

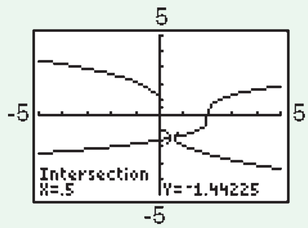
Therefore, $x = \frac{1}{2}$ is a solution to the original equation.

Calculator Connections

The expressions on the right- and left-hand sides of the equation $\sqrt[3]{2x - 4} = \sqrt[3]{1 - 8x}$ are each functions of x. Consider the graphs of the functions:

$$Y_1 = \sqrt[3]{2x - 4} \quad \text{and} \quad Y_2 = \sqrt[3]{1 - 8x}$$

The x-coordinate of the point of intersection of the two functions is the solution to the equation $\sqrt[3]{2x - 4} = \sqrt[3]{1 - 8x}$. The point of intersection can be approximated by using *Zoom* and *Trace* or by using an *Intersect* function.



Skill Practice Solve the equation.

6. $\sqrt[5]{2y - 1} = \sqrt[5]{10y + 3}$

Example 3 Solving Equations with Two Radicals

Solve the equation. $\sqrt{3m + 1} - \sqrt{m + 4} = 1$

Solution:

$$\sqrt{3m + 1} - \sqrt{m + 4} = 1$$

$\sqrt{3m + 1} = \sqrt{m + 4} + 1$ Isolate one of the radicals.

$(\sqrt{3m + 1})^2 = (\sqrt{m + 4} + 1)^2$ Square both sides.

$3m + 1 = m + 4 + 2\sqrt{m + 4} + 1$

Note: $(\sqrt{m + 4} + 1)^2$
$= (\sqrt{m + 4})^2 + 2(1)\sqrt{m + 4} + (1)^2$
$= m + 4 + 2\sqrt{m + 4} + 1$

$3m + 1 = m + 5 + 2\sqrt{m + 4}$ Combine *like* terms.

$2m - 4 = 2\sqrt{m + 4}$ Isolate the radical again.

$m - 2 = \sqrt{m + 4}$ Divide both sides by 2.

$(m - 2)^2 = (\sqrt{m + 4})^2$ Square both sides again.

Skill Practice Answers

6. $y = -\frac{1}{2}$

$m^2 - 4m + 4 = m + 4$	The resulting equation is quadratic.
$m^2 - 5m = 0$	Set the quadratic equation equal to zero.
$m(m - 5) = 0$	Factor.

$m = 0$ or $m = 5$

Check: $m = 0$

$\sqrt{3(0) + 1} - \sqrt{(0) + 4} \stackrel{?}{=} 1$

$\sqrt{1} - \sqrt{4} \stackrel{?}{=} 1$

$1 - 2 \neq 1$ Does not check

Check: $m = 5$

$\sqrt{3(5) + 1} - \sqrt{(5) + 4} = 1$

$\sqrt{16} - \sqrt{9} \stackrel{?}{=} 1$

$4 - 3 = 1$ ✔

The solution is $m = 5$ (the value $m = 0$ does not check).

Skill Practice Solve the equation.

7. $\sqrt{3c + 1} - \sqrt{c - 1} = 2$

4. Applications of Radical Equations and Functions

Example 4 **Applying a Radical Equation in Geometry**

For a pyramid with a square base, the length of a side of the base b is given by

$$b = \sqrt{\frac{3V}{h}}$$

where V is the volume and h is the height.

The Pyramid of the Pharoah Khufu (known as the Great Pyramid) at Giza has a square base (Figure 7-3). If the distance around the bottom of the pyramid is 921.6 m and the height is 146.6 m, what is the volume of the pyramid?

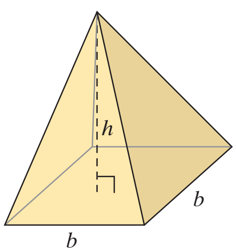


Figure 7-3

Skill Practice Answers

7. $c = 1$; $c = 5$

Solution:

The length of a side b (in meters) is given by $\frac{921.6}{4} = 230.4$ m.

$$b = \sqrt{\frac{3V}{h}}$$

$$230.4 = \sqrt{\frac{3V}{146.6}} \qquad \text{Substitute } b = 230.4 \text{ and } h = 146.6.$$

$$(230.4)^2 = \left(\sqrt{\frac{3V}{146.6}}\right)^2 \qquad \text{Because the index is 2, square both sides.}$$

$$53{,}084.16 = \frac{3V}{146.6} \qquad \text{Simplify.}$$

$$(53{,}084.16)(146.6) = \frac{3V}{146.6}(146.6) \qquad \text{Multiply both sides by 146.6.}$$

$$(53{,}084.16)(146.6) = 3V$$

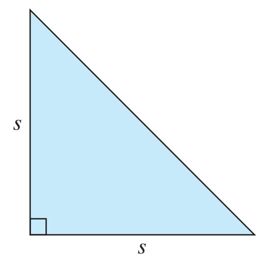
$$\frac{(53{,}084.16)(146.6)}{3} = \frac{3V}{3} \qquad \text{Divide both sides by 3.}$$

$$2{,}594{,}046 \approx V$$

The volume of the Great Pyramid at Giza is approximately 2,594,046 m^3.

Skill Practice

8. The length of the legs, s, of an isosceles right triangle is $s = \sqrt{2A}$, where A is the area. If the lengths of the legs are 9 in., find the area.



Example 5 Applying a Radical Function

On a certain surface, the speed $s(x)$ (in miles per hour) of a car before the brakes were applied can be approximated from the length of its skid marks x (in feet) by

$$s(x) = 3.8\sqrt{x} \qquad x \geq 0$$

See Figure 7-4.

Skill Practice Answers

8. 40.5 $in.^2$

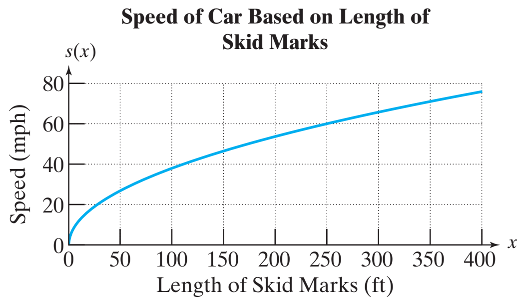


Figure 7-4

a. Find the speed of a car before the brakes were applied if its skid marks are 361 ft long.

b. How long would you expect the skid marks to be if the car had been traveling the speed limit of 50 mph? (Round to the nearest foot.)

Solution:

a. $s(x) = 3.8\sqrt{x}$

$s(361) = 3.8\sqrt{361}$ Substitute $x = 361$.

$= 3.8(19)$

$= 72.2$

If the skid marks are 361 ft, the car was traveling approximately 72.2 mph before the brakes were applied.

b. $s(x) = 3.8\sqrt{x}$

$50 = 3.8\sqrt{x}$ Substitute $s(x) = 50$ and solve for x.

$\frac{50}{3.8} = \sqrt{x}$ Isolate the radical.

$\left(\frac{50}{3.8}\right)^2 = x$

$x \approx 173$

If the car had been going the speed limit (50 mph), then the length of the skid marks would have been approximately 173 ft.

Skill Practice

9. When an object is dropped from a height of 64 ft, the time $t(h)$ in seconds it takes to reach a height h in feet is given by

$$t(h) = \frac{1}{4}\sqrt{64 - h}$$

a. Find the time to reach a height of 28 ft from the ground.

b. What is the height after 1 sec?

Skill Practice Answers

9a. 1.5 sec **b.** 48 ft

Section 7.7 Practice Exercises

Study Skills Exercise

1. Define the key terms.

a. Radical equation **b. Extraneous solution**

Review Exercises

2. Identify the equation as linear or quadratic. Then solve the equation.

a. $2x + 3 = 23$ **b.** $2x^2 - 9x = 5$

For Exercises 3–6, simplify the radical expressions, if possible. Assume all variables represent positive real numbers.

3. $\sqrt{\dfrac{9w^3}{16}}$ **4.** $\sqrt{\dfrac{a^2}{3}}$ **5.** $\sqrt[3]{54c^4}$ **6.** $\sqrt{\dfrac{49}{5t^3}}$

For Exercises 7–10, simplify each expression. Assume all radicands represent positive real numbers.

7. $(\sqrt{4x} - 6)^2$ **8.** $(\sqrt{5y} + 2)^2$ **9.** $(\sqrt[3]{9p} + 7)^3$ **10.** $(\sqrt[3]{4t} + 13)^3$

Concept 2: Solving Radical Equations Involving One Radical

For Exercises 11–26, solve the equations.

11. $\sqrt{4x} = 6$ **12.** $\sqrt{2x} = 8$ **13.** $\sqrt{5y + 1} = 4$ **14.** $\sqrt{9z - 5} = 11$

15. $(2z - 3)^{1/2} = 9$ **16.** $(8 + 3a)^{1/2} = 5$ **17.** $\sqrt[3]{x - 2} - 3 = 0$ **18.** $\sqrt[3]{2x - 5} - 2 = 0$

19. $(15 - w)^{1/3} = -5$ **20.** $(k + 18)^{1/3} = -2$ **21.** $3 + \sqrt{x - 16} = 0$ **22.** $12 + \sqrt{2x + 1} = 0$

23. $2\sqrt{6a + 7} - 2a = 0$ **24.** $2\sqrt{3 - w} - w = 0$ **25.** $\sqrt[4]{2x - 5} = -1$ **26.** $\sqrt[4]{x + 16} = -4$

For Exercises 27–30, assume all variables represent positive real numbers.

27. Solve for V: $r = \sqrt[3]{\dfrac{3V}{4\pi}}$ **28.** Solve for V: $r = \sqrt{\dfrac{V}{h\pi}}$

29. Solve for h^2: $r = \pi\sqrt{r^2 + h^2}$ **30.** Solve for d: $s = 1.3\sqrt{d}$

For Exercises 31–36, square the expression as indicated.

31. $(a + 5)^2$ **32.** $(b + 7)^2$ **33.** $(\sqrt{5a} - 3)^2$

34. $(2 + \sqrt{b})^2$ **35.** $(\sqrt{r - 3} + 5)^2$ **36.** $(2 - \sqrt{2t - 4})^2$

For Exercises 37–42, solve the radical equations, if possible.

37. $\sqrt{a^2 + 2a + 1} = a + 5$ **38.** $\sqrt{b^2 - 5b - 8} = b + 7$ **39.** $\sqrt{25w^2 - 2w - 3} = 5w - 4$

40. $\sqrt{4p^2 - 2p + 1} = 2p - 3$ **41.** $\sqrt{5y + 1} + 2 = y + 3$ **42.** $\sqrt{2x - 2} + 3 = x + 2$

Concept 3: Solving Radical Equations Involving More than One Radical

For Exercises 43–64, solve the radical equations, if possible.

43. $\sqrt[4]{h+4} = \sqrt[4]{2h-5}$

44. $\sqrt[4]{3b+6} = \sqrt[4]{7b-6}$

45. $\sqrt[3]{5a+3} - \sqrt[3]{a-13} = 0$

46. $\sqrt[3]{k-8} - \sqrt[3]{4k+1} = 0$

47. $\sqrt{5a-9} = \sqrt{5a} - 3$

48. $\sqrt{8+b} = 2 + \sqrt{b}$

49. $\sqrt{2h+5} - \sqrt{2h} = 1$

50. $\sqrt{3k-5} - \sqrt{3k} = -1$

51. $\sqrt{t-9} - \sqrt{t} = 3$

52. $\sqrt{y-16} - \sqrt{y} = 4$

53. $6 = \sqrt{x^2+3} - x$

54. $2 = \sqrt{y^2+5} - y$

55. $\sqrt{3t-7} = 2 - \sqrt{3t+1}$

56. $\sqrt{p-6} = \sqrt{p+2} - 4$

57. $\sqrt{z+1} + \sqrt{2z+3} = 1$

58. $\sqrt{2y+6} = \sqrt{7-2y} + 1$

59. $\sqrt{6m+7} - \sqrt{3m+3} = 1$

60. $\sqrt{5w+1} - \sqrt{3w} = 1$

61. $2 + 2\sqrt{2t+3} + 2\sqrt{3t-5} = 0$

62. $6 + 3\sqrt{3x+1} + 3\sqrt{x-1} = 0$

63. $4\sqrt{y} + 6 = 13$

64. $\sqrt{5x-8} = 2\sqrt{x-1}$

Concept 4: Applications of Radical Equations and Functions

65. If an object is dropped from an initial height h, its velocity at impact with the ground is given by

$$v = \sqrt{2gh}$$

where g is the acceleration due to gravity and h is the initial height.

a. Find the initial height (in feet) of an object if its velocity at impact is 44 ft/sec. (Assume that the acceleration due to gravity is $g = 32$ ft/sec^2.)

b. Find the initial height (in meters) of an object if its velocity at impact is 26 m/sec. (Assume that the acceleration due to gravity is $g = 9.8$ m/sec^2.) Round to the nearest tenth of a meter.

66. The time T (in seconds) required for a pendulum to make one complete swing back and forth is approximated by

$$T = 2\pi\sqrt{\frac{L}{g}}$$

where g is the acceleration due to gravity and L is the length of the pendulum (in feet).

a. Find the length of a pendulum that requires 1.36 sec to make one complete swing back and forth. (Assume that the acceleration due to gravity is $g = 32$ ft/sec^2.) Round to the nearest tenth of a foot.

b. Find the time required for a pendulum to complete one swing back and forth if the length of the pendulum is 4 ft. (Assume that the acceleration due to gravity is $g = 32$ ft/sec^2.) Round to the nearest tenth of a second.

67. The time $t(d)$ in seconds it takes an object to drop d meters is given by

$$t(d) = \sqrt{\frac{d}{4.9}}$$

a. Approximate the height of the Texas Commerce Tower in Houston if it takes an object 7.89 sec to drop from the top. Round to the nearest meter.

b. Approximate the height of the Shanghai World Financial Center if it takes an object 9.69 sec to drop from the top. Round to the nearest meter.

68. The airline cost for x thousand passengers to travel round trip from New York to Atlanta is given by

$$C(x) = \sqrt{0.3x + 1}$$

where $C(x)$ is measured in millions of dollars and $x \geq 0$.

a. Find the airline's cost for 10,000 passengers ($x = 10$) to travel from New York to Atlanta.

b. If the airline charges $320 per passenger, find the profit made by the airline for flying 10,000 passengers from New York to Atlanta.

c. Approximate the number of passengers who traveled from New York to Atlanta if the total cost for the airline was $4 million.

Expanding Your Skills

69. The number of hours needed to cook a turkey that weighs x lb can be approximated by

$$t(x) = 0.90\sqrt[5]{x^3}$$

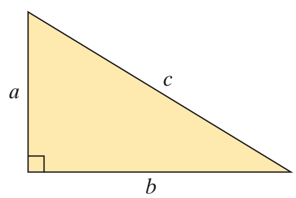
where $t(x)$ is the time in hours and x is the weight of the turkey in pounds.

a. Find the weight of a turkey that cooked for 4 hr. Round to the nearest pound.

b. Find $t(18)$ and interpret the result. Round to the nearest tenth of an hour.

For Exercises 70–73, use the Pythagorean theorem to find a, b, or c.

$$a^2 + b^2 = c^2$$



70. Find b when $a = 2$ and $c = y$.

71. Find b when $a = h$ and $c = 5$.

72. Find a when $b = x$ and $c = 8$.

73. Find a when $b = 14$ and $c = k$.

Graphing Calculator Exercises

74. Refer to Exercise 12. Graph Y_1 and Y_2 on a viewing window defined by $-10 \le x \le 40$ and $-5 \le y \le 10$.

$$Y_1 = \sqrt{2x} \qquad \text{and} \qquad Y_2 = 8$$

Use an *Intersect* feature to approximate the x-coordinate of the point of intersection of the two graphs to support your solution to Exercise 12.

75. Refer to Exercise 11. Graph Y_1 and Y_2 on a viewing window defined by $-10 \le x \le 20$ and $-5 \le y \le 10$.

$$Y_1 = \sqrt{4x} \qquad \text{and} \qquad Y_2 = 6$$

Use an *Intersect* feature to approximate the x-coordinate of the point of intersection of the two graphs to support your solution to Exercise 11.

76. Refer to Exercise 44. Graph Y_1 and Y_2 on a viewing window defined by $-5 \le x \le 20$ and $-1 \le y \le 4$.

$$Y_1 = \sqrt[4]{3x + 6} \qquad \text{and} \qquad Y_2 = \sqrt[4]{7x - 6}$$

Use an *Intersect* feature to approximate the x-coordinate of the point of intersection of the two graphs to support your solution to Exercise 44.

77. Refer to Exercise 43. Graph Y_1 and Y_2 on a viewing window defined by $-5 \le x \le 20$ and $-1 \le y \le 4$.

$$Y_1 = \sqrt[4]{x + 4} \qquad \text{and} \qquad Y_2 = \sqrt[4]{2x - 5}$$

Use an *Intersect* feature to approximate the x-coordinate of the point of intersection of the two graphs to support your solution to Exercise 43.

Complex Numbers

Section 7.8

Concepts

1. Definition of i
2. Powers of i
3. Definition of a Complex Number
4. Addition, Subtraction, and Multiplication of Complex Numbers
5. Division and Simplification of Complex Numbers

1. Definition of i

In Section 7.1, we learned that there are no real-valued square roots of a negative number. For example, $\sqrt{-9}$ is not a real number because no real number when squared equals -9. However, the square roots of a negative number are defined over another set of numbers called the **imaginary numbers**. The foundation of the set of imaginary numbers is the definition of the imaginary number i as $i = \sqrt{-1}$.

Definition of i

$$\boldsymbol{i} = \sqrt{-1}$$

Note: From the definition of i, it follows that $i^2 = -1$.

Using the imaginary number i, we can define the square root of any negative real number.

Definition of $\sqrt{-b}$ for $b > 0$

Let b be a real number such that $b > 0$. Then $\sqrt{-b} = i\sqrt{b}$.

Example 1 Simplifying Expressions in Terms of i

Simplify the expressions in terms of i.

a. $\sqrt{-64}$ **b.** $\sqrt{-50}$ **c.** $-\sqrt{-4}$ **d.** $\sqrt{-29}$

Solution:

a. $\sqrt{-64} = i\sqrt{64}$

$= 8i$

b. $\sqrt{-50} = i\sqrt{50}$

$= i\sqrt{5^2 \cdot 2}$

$= 5i\sqrt{2}$

c. $-\sqrt{-4} = -1 \cdot \sqrt{-4}$

$= -1 \cdot 2i$

$= -2i$

d. $\sqrt{-29} = i\sqrt{29}$

Avoiding Mistakes:

In an expression such as $i\sqrt{29}$, the i is usually written in front of the square root. The expression $\sqrt{29}\, i$ is also correct, but may be misinterpreted as $\sqrt{29i}$ (with i incorrectly placed under the radical).

Skill Practice Simplify the expressions in terms of i.

1. $\sqrt{-81}$ **2.** $\sqrt{-20}$ **3.** $\sqrt{-7}$ **4.** $-\sqrt{-36}$

The multiplication and division properties of radicals were presented in Sections 7.3 and 7.5 as follows:

If a and b represent real numbers such that $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are both real, then

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \qquad \text{and} \qquad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \qquad b \neq 0$$

The conditions that $\sqrt[n]{a}$ and $\sqrt[n]{b}$ must both be real numbers prevent us from applying the multiplication and division properties of radicals for square roots with a negative radicand. Therefore, to multiply or divide radicals with a negative radicand, write the radical in terms of the imaginary number i first. This is demonstrated in Example 2.

Example 2 Simplifying a Product or Quotient in Terms of i

Simplify the expressions.

a. $\dfrac{\sqrt{-100}}{\sqrt{-25}}$ **b.** $\sqrt{-25} \cdot \sqrt{-9}$ **c.** $\sqrt{-5} \cdot \sqrt{-5}$

Solution:

a. $\dfrac{\sqrt{-100}}{\sqrt{-25}}$

$= \dfrac{10i}{5i}$ Simplify each radical in terms of i *before* dividing.

$= 2$ Simplify.

Skill Practice Answers

1. $9i$ **2.** $2i\sqrt{5}$ **3.** $i\sqrt{7}$ **4.** $-6i$

b. $\sqrt{-25} \cdot \sqrt{-9}$

$= 5i \cdot 3i$ — Simplify each radical in terms of i first *before* multiplying.

$= 15i^2$ — Multiply.

$= 15(-1)$ — Recall that $i^2 = -1$.

$= -15$ — Simplify.

c. $\sqrt{-5} \cdot \sqrt{-5}$

$= i\sqrt{5} \cdot i\sqrt{5}$

$= i^2 \cdot (\sqrt{5})^2$

$= -1 \cdot 5$

$= -5$

Skill Practice Simplify the expressions.

5. $\dfrac{\sqrt{-36}}{\sqrt{-9}}$ **6.** $\sqrt{-16} \cdot \sqrt{-49}$ **7.** $\sqrt{-2} \cdot \sqrt{-2}$

Avoiding Mistakes:

In Example 2, we wrote the radical expressions in terms of i first, before multiplying or dividing. If we had mistakenly applied the multiplication or division property first, we would have obtained an incorrect answer.

Correct: $\sqrt{-25} \cdot \sqrt{-9}$

$= (5i)(3i) = 15i^2$

$= 15(-1) = -15$ ← correct

Be careful: $\sqrt{-25} \cdot \sqrt{-9}$

$\neq \sqrt{225} = 15$ ← (incorrect answer)

$\sqrt{-25}$ and $\sqrt{-9}$ are not real numbers. Therefore, the multiplication property of radicals cannot be applied.

2. Powers of *i*

From the definition of $i = \sqrt{-1}$, it follows that

$i = i$

$i^2 = -1$

$i^3 = -i$ because $i^3 = i^2 \cdot i = (-1)i = -i$

$i^4 = 1$ because $i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$

$i^5 = i$ because $i^5 = i^4 \cdot i = (1)i = i$

$i^6 = -1$ because $i^6 = i^4 \cdot i^2 = (1)(-1) = -1$

This pattern of values $i, -1, -i, 1, i, -1, -i, 1, \ldots$ continues for all subsequent powers of i. Here is a list of several powers of i.

Skill Practice Answers

5. 2 **6.** -28 **7.** -2

Powers of *i*

$i^1 = i$	$i^5 = i$	$i^9 = i$
$i^2 = -1$	$i^6 = -1$	$i^{10} = -1$
$i^3 = -i$	$i^7 = -i$	$i^{11} = -i$
$i^4 = 1$	$i^8 = 1$	$i^{12} = 1$

To simplify higher powers of i, we can decompose the expression into multiples of i^4 ($i^4 = 1$) and write the remaining factors as i, i^2, or i^3.

Example 3 Simplifying Powers of *i*

Simplify the powers of i.

a. i^{13} **b.** i^{18} **c.** i^{107} **d.** i^{32}

Solution:

a. $i^{13} = (i^{12}) \cdot (i)$

$= (i^4)^3 \cdot (i)$

$= (1)^3(i)$ Recall that $i^4 = 1$.

$= i$ Simplify.

b. $i^{18} = (i^{16}) \cdot (i^2)$

$= (i^4)^4 \cdot (i^2)$

$= (1)^4 \cdot (-1)$ $i^4 = 1$ and $i^2 = -1$

$= -1$ Simplify.

c. $i^{107} = (i^{104}) \cdot (i^3)$

$= (i^4)^{26}(i^3)$

$= (1)^{26}(-i)$ $i^4 = 1$ and $i^3 = -i$

$= -i$ Simplify.

d. $i^{32} = (i^4)^8$

$= (1)^8$ $i^4 = 1$

$= 1$ Simplify.

Skill Practice Simplify the powers of i.

8. i^8 **9.** i^{22} **10.** i^{45} **11.** i^{31}

3. Definition of a Complex Number

We have already learned the definitions of the integers, rational numbers, irrational numbers, and real numbers. In this section, we define the complex numbers.

Skill Practice Answers

8. 1 **9.** -1 **10.** i
11. $-i$

Definition of a Complex Number

A **complex number** is a number of the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$.

Notes:

- If $b = 0$, then the complex number $a + bi$ is a real number.
- If $b \neq 0$, then we say that $a + bi$ is an imaginary number.
- The complex number $a + bi$ is said to be written in standard form. The quantities a and b are called the real and imaginary parts (respectively) of the complex number.
- The complex numbers $a - bi$ and $a + bi$ are called **conjugates**.

From the definition of a complex number, it follows that all real numbers are complex numbers and all imaginary numbers are complex numbers. Figure 7-5 illustrates the relationship among the sets of numbers we have learned so far.

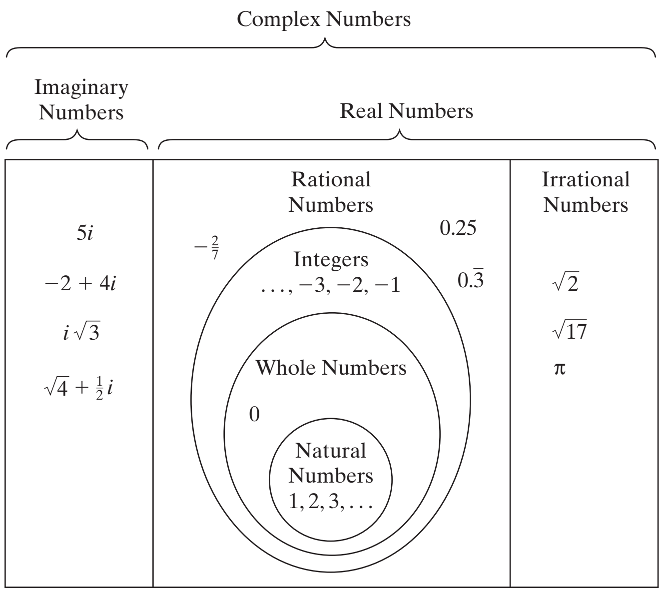


Figure 7-5

Example 4 Identifying the Real and Imaginary Parts of a Complex Number

Identify the real and imaginary parts of the complex numbers.

a. $-8 + 2i$ **b.** $\dfrac{3}{2}$ **c.** $-1.75i$

Solution:

a. $-8 + 2i$ — -8 is the real part, and 2 is the imaginary part.

b. $\dfrac{3}{2} = \dfrac{3}{2} + 0i$ — Rewrite $\frac{3}{2}$ in the form $a + bi$. $\frac{3}{2}$ is the real part, and 0 is the imaginary part.

c. $-1.75i$

$= 0 + -1.75i$ — Rewrite $-1.75i$ in the form $a + bi$. 0 is the real part, and -1.75 is the imaginary part.

Skill Practice Identify the real and imaginary parts of the complex numbers.

12. $22 - 14i$ **13.** -50 **14.** $15i$

TIP: Example 4(b) illustrates that a real number is also a complex number.

$$\frac{3}{2} = \frac{3}{2} + 0i$$

Example 4(c) illustrates that an imaginary number is also a complex number.

$$-1.75i = 0 + -1.75i$$

4. Addition, Subtraction, and Multiplication of Complex Numbers

The operations of addition, subtraction, and multiplication of real numbers also apply to imaginary numbers. To add or subtract complex numbers, combine the real parts and combine the imaginary parts. The commutative, associative, and distributive properties that apply to real numbers also apply to complex numbers.

Example 5 Adding, Subtracting, and Multiplying Complex Numbers

a. Add: $(1 - 5i) + (-3 + 7i)$

b. Subtract: $(-\frac{1}{4} + \frac{3}{5}i) - (\frac{1}{2} - \frac{1}{10}i)$

c. Multiply: $(10 - 5i)(2 + 3i)$

d. Multiply: $(1.2 + 0.5i)(1.2 - 0.5i)$

Solution:

a. $(1 - 5i) + (-3 + 7i) = (1 + -3) + (-5 + 7)i$ (real parts: 1 and -3; imaginary parts: $-5i$ and $7i$) Add real parts. Add imaginary parts.

$= -2 + 2i$ Simplify.

b. $\left(-\frac{1}{4} + \frac{3}{5}i\right) - \left(\frac{1}{2} - \frac{1}{10}i\right) = -\frac{1}{4} + \frac{3}{5}i - \frac{1}{2} + \frac{1}{10}i$ Apply the distributive property.

$= \left(-\frac{1}{4} - \frac{1}{2}\right) + \left(\frac{3}{5} + \frac{1}{10}\right)i$ Add real parts. Add imaginary parts.

$= \left(-\frac{1}{4} - \frac{2}{4}\right) + \left(\frac{6}{10} + \frac{1}{10}\right)i$ Get common denominators.

$= -\frac{3}{4} + \frac{7}{10}i$ Simplify.

Skill Practice Answers

12. real: 22; imaginary: -14
13. real: -50; imaginary: 0
14. real: 0; imaginary: 15

c. $(10 - 5i)(2 + 3i)$

$= (10)(2) + (10)(3i) + (-5i)(2) + (-5i)(3i)$ Apply the distributive property.

$= 20 + 30i - 10i - 15i^2$

$= 20 + 20i - (15)(-1)$ Recall $i^2 = -1$.

$= 20 + 20i + 15$

$= 35 + 20i$ Write in the form $a + bi$.

d. $(1.2 + 0.5i)(1.2 - 0.5i)$

The expressions $(1.2 + 0.5i)$ and $(1.2 - 0.5i)$ are conjugates. The product is a difference of squares.

$$(a + b)(a - b) = a^2 - b^2$$

$(1.2 + 0.5i)(1.2 - 0.5i) = (1.2)^2 - (0.5i)^2$ Apply the formula, where $a = 1.2$ and $b = 0.5i$.

$= 1.44 - 0.25i^2$

$= 1.44 - 0.25(-1)$ Recall $i^2 = -1$.

$= 1.44 + 0.25$

$= 1.69$

TIP: The complex numbers $(1.2 + 0.5i)$ and $(1.2 - 0.5i)$ can also be multiplied by using the distributive property:

$$(1.2 + 0.5i)(1.2 - 0.5i) = 1.44 - 0.6i + 0.6i - 0.25i^2$$
$$= 1.44 - 0.25(-1)$$
$$= 1.69$$

Skill Practice Perform the indicated operations.

15. $\left(\frac{1}{2} - \frac{1}{4}i\right) + \left(\frac{3}{5} + \frac{2}{3}i\right)$

16. $(-6 + 11i) - (-9 - 12i)$

17. $(4 - 6i)(2 - 3i)$

18. $(1.5 + 0.8i)(1.5 - 0.8i)$

5. Division and Simplification of Complex Numbers

The product of a complex number and its conjugate is a real number. For example:

$$(5 + 3i)(5 - 3i) = 25 - 9i^2$$
$$= 25 - 9(-1)$$
$$= 25 + 9$$
$$= 34$$

To divide by a complex number, multiply the numerator and denominator by the conjugate of the denominator. This produces a real number in the denominator so that the resulting expression can be written in the form $a + bi$.

Skill Practice Answers

15. $\frac{11}{10} + \frac{5}{12}i$ **16.** $3 + 23i$

17. $-10 - 24i$ **18.** 2.89

Example 6 Dividing by a Complex Number

Divide the complex numbers and write the answer in the form $a + bi$.

$$\frac{4 - 3i}{5 + 2i}$$

Solution:

$$\frac{4 - 3i}{5 + 2i}$$

Multiply the numerator and denominator by the conjugate of the denominator:

$$\frac{(4 - 3i)}{(5 + 2i)} \cdot \frac{(5 - 2i)}{(5 - 2i)} = \frac{(4)(5) + (4)(-2i) + (-3i)(5) + (-3i)(-2i)}{(5)^2 - (2i)^2}$$

$$= \frac{20 - 8i - 15i + 6i^2}{25 - 4i^2}$$ Simplify numerator and denominator.

$$= \frac{20 - 23i + 6(-1)}{25 - 4(-1)}$$ Recall $i^2 = -1$.

$$= \frac{20 - 23i - 6}{25 + 4}$$

$$= \frac{14 - 23i}{29}$$ Simplify.

$$= \frac{14}{29} - \frac{23i}{29}$$ Write in the form $a + bi$.

Skill Practice Divide the complex numbers. Write the answer in the form $a + bi$.

19. $\dfrac{2 + i}{3 - 2i}$

Example 7 Simplifying Complex Numbers

Simplify the complex numbers.

a. $\dfrac{6 + \sqrt{-18}}{9}$ **b.** $\dfrac{4 - \sqrt{-36}}{2}$

Solution:

a. $$\frac{6 + \sqrt{-18}}{9} = \frac{6 + i\sqrt{18}}{9}$$ Write the radical in terms of i.

$$= \frac{6 + 3i\sqrt{2}}{9}$$ Simplify $\sqrt{18} = 3\sqrt{2}$.

$$= \frac{3(2 + i\sqrt{2})}{9}$$ Factor the numerator.

$$= \frac{\overset{1}{\cancel{3}}(2 + i\sqrt{2})}{\underset{3}{\cancel{9}}}$$ Simplify.

$$= \frac{2 + i\sqrt{2}}{3} \text{ or } \frac{2}{3} + \frac{\sqrt{2}}{3}i$$

Skill Practice Answers

19. $\dfrac{4}{13} + \dfrac{7}{13}i$

b. $\dfrac{4 - \sqrt{-36}}{2} = \dfrac{4 - i\sqrt{36}}{2}$ Write the radical in terms of i.

$= \dfrac{4 - 6i}{2}$ Simplify $\sqrt{36} = 6$.

$= \dfrac{2(2 - 3i)}{2}$ Factor the numerator.

$= \dfrac{\overset{1}{\cancel{2}}(2 - 3i)}{\underset{1}{\cancel{2}}}$ Simplify.

$= 2 - 3i$

Skill Practice Simplify the complex numbers.

20. $\dfrac{8 - \sqrt{-24}}{6}$ **21.** $\dfrac{-12 + \sqrt{-64}}{4}$

Skill Practice Answers

20. $\dfrac{4 - i\sqrt{6}}{3}$ or $\dfrac{4}{3} - \dfrac{\sqrt{6}}{3}i$

21. $-3 + 2i$

Section 7.8 Practice Exercises

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Study Skills Exercises

1. Compare the process for dividing two imaginary numbers such as $\dfrac{1 + 2i}{3 - i}$ with the process of rationalizing the radical expression $\dfrac{1 + \sqrt{2}}{\sqrt{3} - 1}$. What do the processes have in common?

2. Define the key terms.

 a. Imaginary numbers **b.** ***i*** **c. Complex number** **d. Conjugate**

Review Exercises

For Exercises 3–6, perform the indicated operations.

3. $-2\sqrt{5} - 3\sqrt{50} + \sqrt{125}$
4. $\sqrt[3]{2x}(\sqrt[3]{2x} - \sqrt[3]{4x^2})$
5. $(3 - \sqrt{x})(3 + \sqrt{x})$
6. $(\sqrt{5} + \sqrt{2})^2$

For Exercises 7–10, solve the equations.

7. $\sqrt[3]{3p + 7} - \sqrt[3]{2p - 1} = 0$
8. $\sqrt[3]{t - 5} - \sqrt[3]{2t + 1} = 0$
9. $\sqrt{36c + 15} = 6\sqrt{c} + 1$
10. $\sqrt{4a + 29} = 2\sqrt{a} + 5$

Concept 1: Definition of *i*

11. Simplify the expressions $\sqrt{-1}$ and $-\sqrt{1}$.
12. Simplify i^2.

For Exercises 13–34, simplify the expressions.

13. $\sqrt{-144}$ **14.** $\sqrt{-81}$ **15.** $\sqrt{-3}$ **16.** $\sqrt{-17}$

17. $\sqrt{-20}$ **18.** $\sqrt{-75}$ **19.** $2\sqrt{-25} \cdot 3\sqrt{-4}$ **20.** $(-4\sqrt{-9})(-3\sqrt{-1})$

21. $3\sqrt{-18} + 5\sqrt{-32}$ **22.** $5\sqrt{-45} + 3\sqrt{-80}$ **23.** $7\sqrt{-63} - 4\sqrt{-28}$ **24.** $7\sqrt{-3} - 4\sqrt{-27}$

25. $\sqrt{-7} \cdot \sqrt{-7}$ **26.** $\sqrt{-11} \cdot \sqrt{-11}$ **27.** $\sqrt{-9} \cdot \sqrt{-16}$ **28.** $\sqrt{-25} \cdot \sqrt{-36}$

29. $\sqrt{-15} \cdot \sqrt{-6}$ **30.** $\sqrt{-12} \cdot \sqrt{-50}$ **31.** $\dfrac{\sqrt{-50}}{\sqrt{25}}$ **32.** $\dfrac{\sqrt{-27}}{\sqrt{9}}$

33. $\dfrac{\sqrt{-90}}{\sqrt{10}}$ **34.** $\dfrac{\sqrt{-125}}{\sqrt{45}}$

Concept 2: Powers of *i*

For Exercises 35–46, simplify the powers of i.

35. i^7 **36.** i^{38} **37.** i^{64} **38.** i^{75}

39. i^{41} **40.** i^{25} **41.** i^{52} **42.** i^0

43. i^{23} **44.** i^{103} **45.** i^6 **46.** i^{82}

Concept 3: Definition of a Complex Number

47. What is the conjugate of a complex number $a + bi$?

48. True or false?

a. Every real number is a complex number.

b. Every complex number is a real number.

For Exercises 49–56, identify the real and imaginary parts of the complex number.

49. $-5 + 12i$ **50.** $22 - 16i$ **51.** $-6i$ **52.** $10i$

53. 35 **54.** -1 **55.** $\dfrac{3}{5} + i$ **56.** $-\dfrac{1}{2} - \dfrac{1}{4}i$

Concept 4: Addition, Subtraction, and Multiplication of Complex Numbers

For Exercises 57–80, perform the indicated operations. Write the answer in the form $a + bi$.

57. $(2 - i) + (5 + 7i)$ **58.** $(5 - 2i) + (3 + 4i)$ **59.** $\left(\dfrac{1}{2} + \dfrac{2}{3}i\right) - \left(\dfrac{1}{5} - \dfrac{5}{6}i\right)$

60. $\left(\dfrac{11}{10} - \dfrac{7}{5}i\right) - \left(-\dfrac{2}{5} + \dfrac{3}{5}i\right)$ **61.** $(1 + 3i) + (4 - 3i)$ **62.** $(-2 + i) + (1 - i)$

63. $(2 + 3i) - (1 - 4i) + (-2 + 3i)$ **64.** $(2 + 5i) - (7 - 2i) + (-3 + 4i)$

65. $(8i)(3i)$ **66.** $(2i)(4i)$ **67.** $6i(1 - 3i)$ **68.** $-i(3 + 4i)$

69. $(2 - 10i)(3 + 2i)$

70. $(4 + 7i)(1 - i)$

71. $(-5 + 2i)(5 + 2i)$

72. $(4 - 11i)(-4 - 11i)$

73. $(4 + 5i)^2$

74. $(3 - 2i)^2$

75. $(2 + i)(3 - 2i)(4 + 3i)$

76. $(3 - i)(3 + i)(4 - i)$

77. $(-4 - 6i)^2$

78. $(-3 - 5i)^2$

79. $\left(-\frac{1}{2} - \frac{3}{4}i\right)\left(-\frac{1}{2} + \frac{3}{4}i\right)$

80. $\left(-\frac{2}{3} + \frac{1}{6}i\right)\left(-\frac{2}{3} - \frac{1}{6}i\right)$

Concept 5: Division and Simplification of Complex Numbers

For Exercises 81–94, divide the complex numbers. Write the answer in the form $a + bi$.

81. $\frac{2}{1 + 3i}$

82. $\frac{-2}{3 + i}$

83. $\frac{-i}{4 - 3i}$

84. $\frac{3 - 3i}{1 - i}$

85. $\frac{5 + 2i}{5 - 2i}$

86. $\frac{7 + 3i}{4 - 2i}$

87. $\frac{3 + 7i}{-2 - 4i}$

88. $\frac{-2 + 9i}{-1 - 4i}$

89. $\frac{13i}{-5 - i}$

90. $\frac{15i}{-2 - i}$

91. $\frac{2 + 3i}{6i}$ (*Hint:* The denominator can be written as $0 + 6i$. Therefore, the conjugate is $0 - 6i$, or simply $-6i$.)

92. $\frac{4 - i}{2i}$

93. $\frac{-10 + i}{i}$

94. $\frac{-6 - i}{-i}$

For Exercises 95–100, simplify the complex numbers.

95. $\frac{2 + \sqrt{-16}}{8}$

96. $\frac{6 - \sqrt{-4}}{4}$

97. $\frac{-6 + \sqrt{-72}}{6}$

98. $\frac{-20 + \sqrt{-500}}{10}$

99. $\frac{-8 - \sqrt{-48}}{4}$

100. $\frac{-18 - \sqrt{-72}}{3}$

Chapter 7 SUMMARY

Section 7.1 Definition of an *n*th root

Key Concepts

b is an **nth root** of a if $b^n = a$.

The expression $\sqrt{a}$ represents the principal square root of a.

The expression $\sqrt[n]{a}$ represents the principal nth root of a.

$\sqrt[n]{a^n} = |a|$ if n is even.

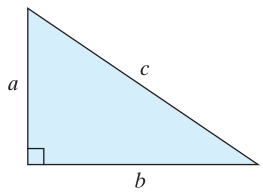
$\sqrt[n]{a^n} = a$ if n is odd.

$\sqrt[n]{a}$ is not a real number if $a < 0$ and n is even.

$f(x) = \sqrt[n]{x}$ defines a **radical function**.

The Pythagorean Theorem

$a^2 + b^2 = c^2$



Examples

Example 1

2 is a square root of 4.

-2 is a square root of 4.

-3 is a cube root of -27.

Example 2

$\sqrt{36} = 6$ $\sqrt[3]{-64} = -4$

Example 3

$\sqrt[4]{(x + 3)^4} = |x + 3|$ $\sqrt[5]{(x + 3)^5} = x + 3$

Example 4

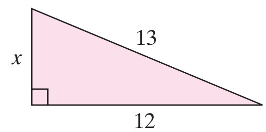
$\sqrt[4]{-16}$ is not a real number.

Example 5

For $g(x) = \sqrt{x}$ the domain is $[0, \infty)$.

For $h(x) = \sqrt[3]{x}$ the domain is $(-\infty, \infty)$.

Example 6



$$x^2 + 12^2 = 13^2$$
$$x^2 + 144 = 169$$
$$x^2 = 25$$
$$x = \sqrt{25}$$
$$x = 5$$

Section 7.2 Rational Exponents

Key Concepts

Let a be a real number and n be an integer such that $n > 1$. If $\sqrt[n]{a}$ exists, then

$$a^{1/n} = \sqrt[n]{a}$$

$$a^{m/n} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

All properties of integer exponents hold for rational exponents, provided the roots are real-valued.

Examples

Example 1

$121^{1/2} = \sqrt{121} = 11$

Example 2

$27^{2/3} = \left(\sqrt[3]{27}\right)^2 = (3)^2 = 9$

Example 3

$$\begin{aligned} &p^{1/3} \cdot p^{1/4} \\ &\quad = p^{1/3 + 1/4} \\ &\quad = p^{4/12 + 3/12} \\ &\quad = p^{7/12} \\ &\quad = \sqrt[12]{p^7} \end{aligned}$$

Section 7.3 Simplifying Radical Expressions

Key Concepts

Let a and b represent real numbers such that $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are both real. Then

$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$ Multiplication property

$\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$ Division property

A radical expression whose radicand is written as a product of prime factors is in simplified form if all the following conditions are met:

1. The radicand has no factor raised to a power greater than or equal to the index.
2. The radicand does not contain a fraction.
3. No radicals are in the denominator of a fraction.

Examples

Example 1

$\sqrt{3} \cdot \sqrt{5} = \sqrt{15}$

Example 2

$$\sqrt{\frac{x}{9}} = \frac{\sqrt{x}}{\sqrt{9}} = \frac{\sqrt{x}}{3}$$

Example 3

$$\begin{aligned} &\sqrt[3]{16x^5y^7} \\ &\quad = \sqrt[3]{2^4x^5y^7} \\ &\quad = \sqrt[3]{2^3x^3y^6 \cdot 2x^2y} \\ &\quad = \sqrt[3]{2^3x^3y^6} \cdot \sqrt[3]{2x^2y} \\ &\quad = 2xy^2\sqrt[3]{2x^2y} \end{aligned}$$

Section 7.4 Addition and Subtraction of Radicals

Key Concepts

***Like* radicals** have radical factors with the same index and the same radicand.

Use the distributive property to add and subtract *like* radicals.

Examples

Example 1

$3x\sqrt{7} - 5x\sqrt{7} + x\sqrt{7}$

$= (3 - 5 + 1) \cdot x\sqrt{7}$

$= -x\sqrt{7}$

Example 2

$x\sqrt[4]{16x} - 3\sqrt[4]{x^5}$

$= 2x\sqrt[4]{x} - 3x\sqrt[4]{x}$

$= (2 - 3)x\sqrt[4]{x}$

$= -x\sqrt[4]{x}$

Section 7.5 Multiplication of Radicals

Key Concepts

The Multiplication Property of Radicals

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, then

$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$

To multiply or divide radicals with different indices, convert to rational exponents and use the properties of exponents.

Examples

Example 1

$3\sqrt{2}(\sqrt{2} + 5\sqrt{7} - \sqrt{6})$

$= 3\sqrt{4} + 15\sqrt{14} - 3\sqrt{12}$

$= 3 \cdot 2 + 15\sqrt{14} - 3 \cdot 2\sqrt{3}$

$= 6 + 15\sqrt{14} - 6\sqrt{3}$

Example 2

$\sqrt{p} \cdot \sqrt[5]{p^2}$

$= p^{1/2} \cdot p^{2/5}$

$= p^{5/10} \cdot p^{4/10}$

$= p^{9/10}$

$= \sqrt[10]{p^9}$

Section 7.6 Rationalization

Key Concepts

The process of removing a radical from the denominator of an expression is called **rationalizing the denominator**.

Rationalizing a denominator with one term

Rationalizing a denominator with two terms involving square roots

Examples

Example 1

Rationalize:

$$\frac{4}{\sqrt[4]{2y^3}}$$

$$= \frac{4}{\sqrt[4]{2y^3}} \cdot \frac{\sqrt[4]{2^3y}}{\sqrt[4]{2^3y}}$$

$$= \frac{4\sqrt[4]{8y}}{\sqrt[4]{2^4y^4}}$$

$$= \frac{4\sqrt[4]{8y}}{2y}$$

$$= \frac{2\sqrt[4]{8y}}{y}$$

Example 2

Rationalize the denominator:

$$\frac{\sqrt{2}}{\sqrt{x} - \sqrt{3}}$$

$$= \frac{\sqrt{2}}{(\sqrt{x} - \sqrt{3})} \cdot \frac{(\sqrt{x} + \sqrt{3})}{(\sqrt{x} + \sqrt{3})}$$

$$= \frac{\sqrt{2x} + \sqrt{6}}{x - 3}$$

Section 7.7 Radical Equations

Key Concepts

Steps to Solve a Radical Equation

1. Isolate the radical. If an equation has more than one radical, choose one of the radicals to isolate.
2. Raise each side of the equation to a power equal to the index of the radical.
3. Solve the resulting equation. If the equation still has a radical, repeat steps 1 and 2.
4. Check the potential solutions in the original equation.

Examples

Example 1

Solve:

$$\sqrt{b-5} - \sqrt{b+3} = 2$$

$$\sqrt{b-5} = \sqrt{b+3} + 2$$

$$(\sqrt{b-5})^2 = (\sqrt{b+3} + 2)^2$$

$$b - 5 = b + 3 + 4\sqrt{b+3} + 4$$

$$b - 5 = b + 7 + 4\sqrt{b+3}$$

$$-12 = 4\sqrt{b+3}$$

$$-3 = \sqrt{b+3}$$

$$(-3)^2 = (\sqrt{b+3})^2$$

$$9 = b + 3$$

$$6 = b$$

Check:

$$\sqrt{6-5} - \sqrt{6+3} \stackrel{?}{=} 2$$

$$\sqrt{1} - \sqrt{9} \stackrel{?}{=} 2$$

$$1 - 3 \neq 2 \quad \text{Does not check.}$$

No solution.

Section 7.8 Complex Numbers

Key Concepts

$\boldsymbol{i} = \sqrt{-1}$ and $i^2 = -1$

For a real number $b > 0$, $\sqrt{-b} = i\sqrt{b}$

A **complex number** is in the form $a + bi$, where a and b are real numbers. The a is called the real part, and the b is called the imaginary part.

To add or subtract complex numbers, combine the real parts and combine the imaginary parts.

Examples

Example 1

$$\sqrt{-4} \cdot \sqrt{-9}$$

$$= (2i)(3i)$$

$$= 6i^2$$

$$= -6$$

Example 2

$$(3 - 5i) - (2 + i) + (3 - 2i)$$

$$= 3 - 5i - 2 - i + 3 - 2i$$

$$= 4 - 8i$$

Multiply complex numbers by using the distributive property.

Example 3

$(1 + 6i)(2 + 4i)$

$= 2 + 4i + 12i + 24i^2$

$= 2 + 16i + 24(-1)$

$= -22 + 16i$

Divide complex numbers by multiplying the numerator and denominator by the **conjugate** of the denominator.

Example 4

$$\frac{3}{2 - 5i}$$

$$= \frac{3}{2 - 5i} \cdot \frac{(2 + 5i)}{(2 + 5i)} = \frac{6 + 15i}{4 - 25i^2}$$

$$= \frac{6 + 15i}{29} \quad \text{or} \quad \frac{6}{29} + \frac{15}{29}i$$

Chapter 7 Review Exercises

For the exercises in this set, assume that all variables represent positive real numbers unless otherwise stated.

Section 7.1

1. True or false?
   a. The principal nth root of an even-indexed root is always positive.
   b. The principal nth root of an odd-indexed root is always positive.
2. Explain why $\sqrt{(-3)^2} \neq -3$.
3. Are the following statements true or false?
   a. $\sqrt{a^2 + b^2} = a + b$
   b. $\sqrt{(a + b)^2} = a + b$

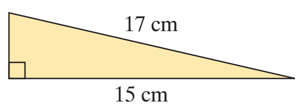
For Exercises 4–6, simplify the radicals.

4. $\sqrt{\frac{50}{32}}$ 5. $\sqrt[4]{625}$ 6. $\sqrt{(-6)^2}$

7. Evaluate the function values for $f(x) = \sqrt{x - 1}$.
   a. $f(10)$ b. $f(1)$ c. $f(8)$
   d. Write the domain of f in interval notation.
8. Evaluate the function values for $g(t) = \sqrt{5 + t}$.
   a. $g(-5)$ b. $g(-4)$ c. $g(4)$
   d. Write the domain of g in interval notation.
9. Translate the English expression to an algebraic expression: Four more than the quotient of the cube root of $2x$ and the principal fourth root of $2x$.

For Exercises 10–11, simplify the expression. Assume that x and y represent *any* real number.

10. a. $\sqrt{x^2}$ b. $\sqrt[3]{x^3}$
    c. $\sqrt[4]{x^4}$ d. $\sqrt[5]{(x + 1)^5}$
11. a. $\sqrt{4y^2}$ b. $\sqrt[3]{27y^3}$
    c. $\sqrt[100]{y^{100}}$ d. $\sqrt[101]{y^{101}}$
12. Use the Pythagorean theorem to find the length of the third side of the triangle.



Section 7.2

13. Are the properties of exponents the same for rational exponents and integer exponents? Give an example. (Answers may vary.)

14. In the expression $x^{m/n}$ what does n represent?

15. Explain the process of eliminating a negative exponent from an algebraic expression.

For Exercises 16–20, simplify the expressions. Write the answer with positive exponents only.

16. $(-125)^{1/3}$

17. $16^{-1/4}$

18. $\left(\frac{1}{16}\right)^{-3/4} - \left(\frac{1}{8}\right)^{-2/3}$

19. $(b^{1/2} \cdot b^{1/3})^{12}$

20. $\left(\frac{x^{-1/4}y^{-1/3}z^{3/4}}{2^{1/3}x^{-1/3}y^{2/3}}\right)^{-12}$

For Exercises 21–22, rewrite the expressions by using rational exponents.

21. $\sqrt[4]{x^3}$

22. $\sqrt[3]{2y^2}$

For Exercises 23–25, use a calculator to approximate the expressions to 4 decimal places.

23. $10^{1/3}$

24. $17.8^{2/3}$

25. $147^{4/5}$

26. An initial investment of P dollars is made in an account in which the return is compounded quarterly. The amount in the account can be determined by

$$A = P\left(1 + \frac{r}{4}\right)^{t/3}$$

where r is the annual rate of return and t is the time in months.

When she is 20 years old, Jenna invests \$5000 in a mutual fund that grows by an average of 11% per year compounded quarterly. How much money does she have

a. After 6 months?

b. After 1 year?

c. At age 40?

d. At age 50?

e. At age 65?

Section 7.3

27. List the criteria for a radical expression to be simplified.

For Exercises 28–31, simplify the radicals.

28. $\sqrt{108}$

29. $\sqrt[4]{x^5yz^4}$

30. $\sqrt{5x} \cdot \sqrt{20x}$

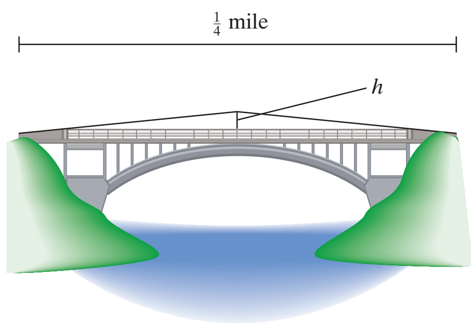
31. $\sqrt[3]{\frac{-16x^7y^6}{z^9}}$

32. Write an English phrase that describes the following mathematical expressions: (Answers may vary.)

a. $\sqrt{\frac{2}{x}}$

b. $(x + 1)^3$

33. An engineering firm made a mistake when building a $\frac{1}{4}$-mi bridge in the Florida Keys. The bridge was made without adequate expansion joints to prevent buckling during the heat of summer. During mid-June, the bridge expanded 1.5 ft, causing a vertical bulge in the middle. Calculate the height of the bulge h in feet. (*Note:* 1 mi = 5280 ft.) Round to the nearest foot.



Section 7.4

34. Complete the following statement: Radicals may be added or subtracted if . . .

For Exercises 35–38, determine whether the radicals may be combined, and explain your answer.

35. $\sqrt[3]{2x} - 2\sqrt{2x}$

36. $2 + \sqrt{x}$

37. $\sqrt[4]{3xy} + 2\sqrt[4]{3xy}$

38. $-4\sqrt{32} + 7\sqrt{50}$

For Exercises 39–42, add or subtract as indicated.

39. $4\sqrt{7} - 2\sqrt{7} + 3\sqrt{7}$

40. $2\sqrt[3]{64} + 3\sqrt[3]{54} - 16$

41. $\sqrt{50} + 7\sqrt{2} - \sqrt{8}$

42. $x\sqrt[3]{16x^2} - 4\sqrt[3]{2x^5} + 5x\sqrt[3]{54x^2}$

For Exercises 43–44, answer true or false. If an answer is false, explain why. Assume all variables represent positive real numbers.

43. $5 + 3\sqrt{x} = 8\sqrt{x}$

44. $\sqrt{y} + \sqrt{y} = \sqrt{2y}$

Section 7.5

For Exercises 45–56, multiply the radicals and simplify the answer.

45. $\sqrt{3} \cdot \sqrt{12}$

46. $\sqrt[4]{4} \cdot \sqrt[4]{8}$

47. $-2\sqrt{3}(\sqrt{3} - 3\sqrt{3})$

48. $-3\sqrt{5}(2\sqrt{3} - \sqrt{5})$

49. $(2\sqrt{x} - 3)(2\sqrt{x} + 3)$

50. $(\sqrt{y} + 4)(\sqrt{y} - 4)$

51. $(\sqrt{7y} - \sqrt{3x})^2$

52. $(2\sqrt{3w} + 5)^2$

53. $(-\sqrt{z} - \sqrt{6})(2\sqrt{z} + 7\sqrt{6})$

54. $(3\sqrt{a} - \sqrt{5})(\sqrt{a} + 2\sqrt{5})$

55. $\sqrt[3]{u} \cdot \sqrt{u^5}$

56. $\sqrt{2} \cdot \sqrt[4]{w^3}$

Section 7.6

For Exercises 57–64, rationalize the denominator.

57. $\sqrt{\dfrac{7}{2y}}$

58. $\sqrt{\dfrac{5}{3w}}$

59. $\dfrac{4}{\sqrt[3]{9p^2}}$

60. $\dfrac{-2}{\sqrt[3]{2x}}$

61. $\dfrac{-5}{\sqrt{15} - \sqrt{10}}$

62. $\dfrac{-6}{\sqrt{7} - \sqrt{5}}$

63. $\dfrac{t - 3}{\sqrt{t} - \sqrt{3}}$

64. $\dfrac{w - 7}{\sqrt{w} - \sqrt{7}}$

65. Translate the mathematical expression to an English phrase. (Answers may vary.)

$$\frac{\sqrt{2}}{x^2}$$

Section 7.7

Solve the radical equations in Exercises 66–73, if possible.

66. $\sqrt{2y} = 7$

67. $\sqrt{a - 6} - 5 = 0$

68. $\sqrt[3]{2w - 3} + 5 = 2$

69. $\sqrt[4]{p + 12} - \sqrt[4]{5p - 16} = 0$

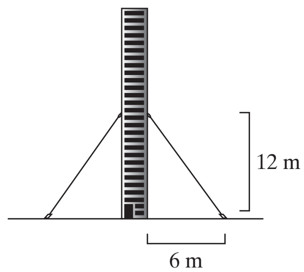
70. $\sqrt{t} + \sqrt{t - 5} = 5$

71. $\sqrt{8x + 1} = -\sqrt{x - 13}$

72. $\sqrt{2m^2 + 4} - \sqrt{9m} = 0$

73. $\sqrt{x + 2} = 1 - \sqrt{2x + 5}$

74. A tower is supported by stabilizing wires. Find the exact length of each wire, and then round to the nearest tenth of a meter.



75. The velocity, $v(d)$, of an ocean wave depends on the water depth d as the wave approaches land.

$$v(d) = \sqrt{32d}$$

where $v(d)$ is in feet per second and d is in feet.

a. Find $v(20)$ and interpret its value. Round to 1 decimal place.

b. Find the depth of the water at a point where a wave is traveling at 16 ft/sec.

Section 7.8

76. Define a complex number.

77. Define an imaginary number.

78. Consider the following expressions.

$$\frac{3}{4 + 6i} \quad \text{and} \quad \frac{3}{\sqrt{4} + \sqrt{6}}$$

Compare the process of dividing by a complex number to the process of rationalizing the denominator.

For Exercises 79–82, rewrite the expressions in terms of i.

79. $\sqrt{-16}$

80. $-\sqrt{-5}$

81. $\sqrt{-75} \cdot \sqrt{-3}$

82. $\dfrac{-\sqrt{-24}}{\sqrt{6}}$

For Exercises 83–86, simplify the powers of i.

83. i^{38}

84. i^{101}

85. i^{19}

86. $i^{1000} + i^{1002}$

For Exercises 87–90, perform the indicated operations. Write the final answer in the form $a + bi$.

87. $(-3 + i) - (2 - 4i)$

88. $(1 + 6i)(3 - i)$

89. $(4 - 3i)(4 + 3i)$

90. $(5 - i)^2$

For Exercises 91–92, write the expressions in the form $a + bi$, and determine the real and imaginary parts.

91. $\dfrac{17 - 4i}{-4}$

92. $\dfrac{-16 - 8i}{8}$

For Exercises 93–94, divide and simplify. Write the final answer in the form $a + bi$.

93. $\dfrac{2 - i}{3 + 2i}$

94. $\dfrac{10 + 5i}{2 - i}$

For Exercises 95–96, simplify the expression.

95. $\dfrac{-8 + \sqrt{-40}}{12}$

96. $\dfrac{6 - \sqrt{-144}}{3}$

Chapter 7 Test

1. a. What is the principal square root of 36?

b. What is the negative square root of 36?

2. Which of the following are real numbers?

a. $-\sqrt{100}$ **b.** $\sqrt{-100}$

c. $-\sqrt[3]{1000}$ **d.** $\sqrt[3]{-1000}$

3. Simplify.

a. $\sqrt[3]{y^3}$ **b.** $\sqrt[4]{y^4}$

For Exercises 4–11, simplify the radicals. Assume that all variables represent positive numbers.

4. $\sqrt[4]{81}$

5. $\sqrt{\dfrac{16}{9}}$

6. $\sqrt[3]{32}$

7. $\sqrt{a^4b^3c^5}$

8. $\sqrt{3x} \cdot \sqrt{6x^3}$

9. $\sqrt{\dfrac{32w^6}{3w}}$

10. $\sqrt[6]{7} \cdot \sqrt{y}$

11. $\dfrac{\sqrt[3]{10}}{\sqrt[4]{10}}$

12. a. Evaluate the function values $f(-8)$, $f(-6)$, $f(-4)$, and $f(-2)$ for $f(x) = \sqrt{-2x - 4}$.

b. Write the domain of f in interval notation.

13. Use a calculator to evaluate $\dfrac{-3 - \sqrt{5}}{17}$ to 4 decimal places.

For Exercises 14–15, simplify the expressions. Assume that all variables represent positive numbers.

14. $-27^{1/3}$

15. $\dfrac{t^{-1} \cdot t^{1/2}}{t^{1/4}}$

16. Add or subtract as indicated

$$3\sqrt{5} + 4\sqrt{5} - 2\sqrt{20}$$

17. Multiply the radicals.

a. $3\sqrt{x}(\sqrt{2} - \sqrt{5})$

b. $(2\sqrt{5} - 3\sqrt{x})(4\sqrt{5} + \sqrt{x})$

18. Rationalize the denominator. Assume $x > 0$.

a. $\dfrac{-2}{\sqrt[3]{x}}$ **b.** $\dfrac{\sqrt{x} + 2}{3 - \sqrt{x}}$

19. Rewrite the expressions in terms of i.

a. $\sqrt{-8}$ **b.** $2\sqrt{-16}$ **c.** $\dfrac{2+\sqrt{-8}}{4}$

For Exercises 20–26, perform the indicated operation and simplify completely. Write the final answer in the form $a + bi$.

20. $(3 - 5i) - (2 + 6i)$

21. $(4 + i)(8 + 2i)$

22. $\sqrt{-16} \cdot \sqrt{-49}$

23. $(4 - 7i)^2$

24. $(2 - 10i)(2 + 10i)$

25. $\dfrac{3 - 2i}{3 - 4i}$

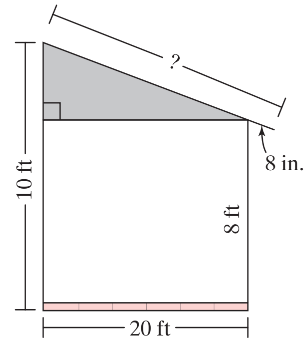
26. $(10 + 3i)[(-5i + 8) - (5 - 3i)]$

27. If the volume V of a sphere is known, the radius of the sphere can be computed by

$$r(V) = \sqrt[3]{\frac{3V}{4\pi}}.$$

Find $r(10)$ to 2 decimal places. Interpret the meaning in the context of the problem.

28. A patio 20 ft wide has a slanted roof, as shown in the picture. Find the length of the roof if there is an 8-in. overhang. Round the answer to the nearest foot.



For Exercises 29–31, solve the radical equation.

29. $\sqrt[3]{2x + 5} = -3$

30. $\sqrt{5x + 8} = \sqrt{5x - 1} + 1$

31. $\sqrt{t + 7} - \sqrt{2t - 3} = 2$

Chapters 1–7 Cumulative Review Exercises

1. Simplify the expression.
$6^2 - 2[5 - 8(3 - 1) + 4 \div 2]$

2. Simplify the expression.
$3x - 3(-2x + 5) - 4y + 2(3x + 5) - y$

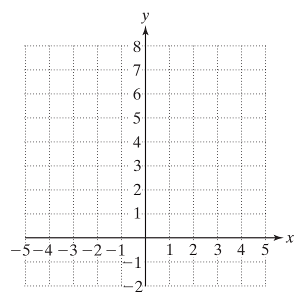
3. Solve the equation: $9(2y + 8) = 20 - (y + 5)$

4. Solve the inequality. Write the answer in interval notation.

$$2a - 4 < -14$$

5. Write an equation of the line that is parallel to the line $2x + y = 9$ and passes through the point $(3, -1)$. Write the answer in slope-intercept form.

6. On the same coordinate system, graph the line $2x + y = 9$ and the line that you derived in Exercise 5. Verify that these two lines are indeed parallel.



7. Solve the system of equations by using the addition method.

$$2x - 3y = 0$$
$$-4x + 3y = -1$$

8. Determine if $(2, -2, \frac{1}{2})$ is a solution to the system.

$$\begin{aligned} 2x + y - 4z &= 0 \\ x - y + 2z &= 5 \\ 3x + 2y + 2z &= 4 \end{aligned}$$

9. Write a system of linear equations from the augmented matrix. Use x, y, and z for the variables.

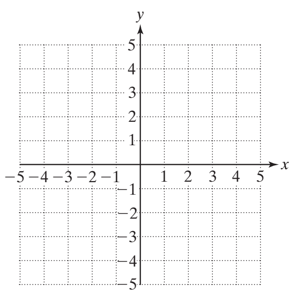
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 8 \end{array}\right]$$

10. Given the function defined by $f(x) = 4x - 2$.

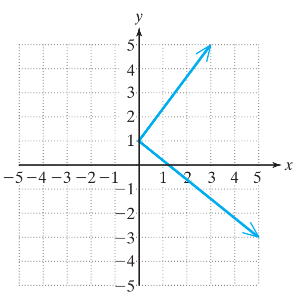
a. Find $f(-2), f(0), f(4)$, and $f(\frac{1}{2})$.

b. Write the ordered pairs that correspond to the function values in part (a).

c. Graph $y = f(x)$.



11. Determine if the graph defines y as a function of x.



12. Simplify the expression. Write the final answer with positive exponents only.

$$\left(\frac{a^3b^{-1}c^3}{ab^{-5}c^2}\right)^2$$

13. Simplify the expression. Write the final answer with positive exponents only.

$$\left(\frac{a^{3/2}b^{-1/4}c^{1/3}}{ab^{-5/4}c^0}\right)^{12}$$

14. Multiply or divide as indicated, and write the answer in scientific notation.

a. $(3.5 \times 10^7)(4 \times 10^{-12})$

b. $\dfrac{6.28 \times 10^5}{2.0 \times 10^{-4}}$

15. Multiply the polynomials $(2x + 5)(x - 3)$. What is the degree of the product?

16. Perform the indicated operations and simplify. $\sqrt{3}(\sqrt{5} + \sqrt{6} + \sqrt{3})$

17. Divide $(x^2 - x - 12) \div (x + 3)$.

18. Simplify and subtract: $\sqrt[4]{\dfrac{1}{16}} - \sqrt[3]{\dfrac{8}{27}}$

19. Simplify: $\sqrt[3]{\dfrac{54c^4}{cd^3}}$

20. Add: $4\sqrt{45b^3} + 5b\sqrt{80b}$

21. Divide: $\dfrac{13i}{3 + 2i}$ Write the answer in the form $a + bi$.

22. Solve the equation.

$$\frac{5}{y - 2} - \frac{3}{y - 4} = \frac{6}{y^2 - 6y + 8}$$

23. Add: $\dfrac{3}{x^2 + 5x} + \dfrac{-2}{x^2 - 25}$

24. Divide: $\dfrac{a + 10}{2a^2 - 11a - 6} \div \dfrac{a^2 + 12a + 20}{6 - a}$

25. Perform the indicated operations. $(-5x^2 - 4x + 8) - (3x - 5)^2$

26. Simplify. $\dfrac{-4}{\sqrt{3} - \sqrt{5}}$

27. Divide. $\dfrac{4}{3 - 5i}$

28. Solve. $12x^2 + 4x - 21 = 0$

29. Factor. $x^2 + 6x + 9 - y^2$

30. Factor. $x^6 + 8$