

Rational Expressions and Rational Equations

6

6.1 Rational Expressions and Rational Functions

6.2 Multiplication and Division of Rational Expressions

6.3 Addition and Subtraction of Rational Expressions

6.4 Complex Fractions

Problem Recognition Exercises—Operations on Rational Expressions

6.5 Rational Equations

6.6 Applications of Rational Equations and Proportions

In this chapter, we define a rational expression as a ratio of two polynomials. First we focus on adding, subtracting, multiplying, and dividing rational expressions. Then the chapter concludes with solving rational equations and showing how they are used in applications.

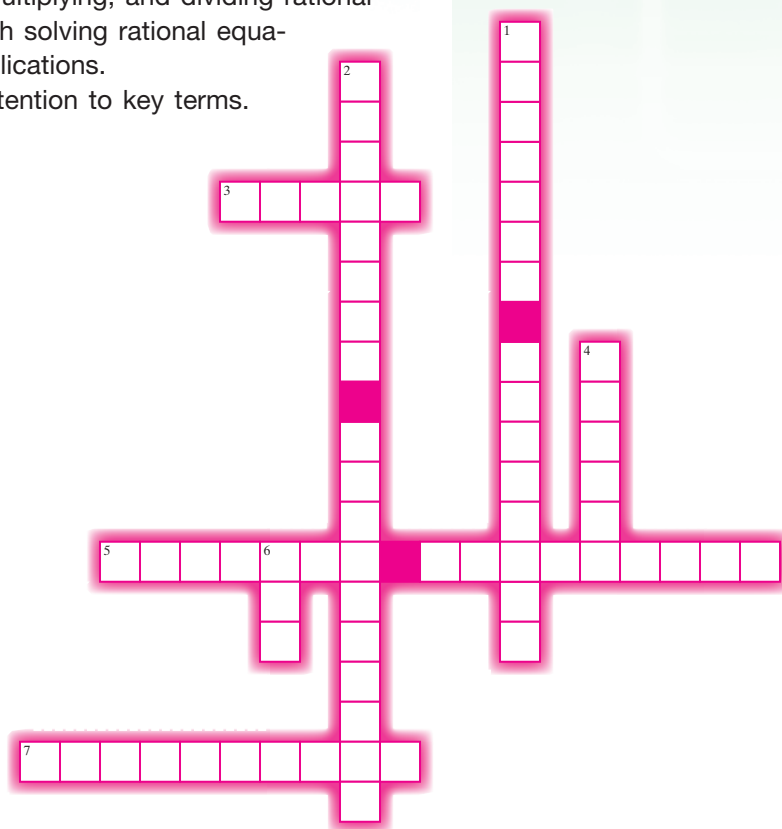
As you work through the chapter, pay attention to key terms. Then work through this crossword puzzle.

Across

- Expression of the form a/b
- Two triangles in which corresponding sides are proportional
- Equation of the form $a/b = c/d$

Down

- Fraction with fractions in numerator and denominator
- Ratio of two polynomials
- Set of real numbers for which an expression is defined
- Abbreviation for least common denominator



Section 6.1

Rational Expressions and Rational Functions

Concepts

1. Evaluating Rational Expressions
2. Domain of a Rational Expression
3. Simplifying Rational Expressions to Lowest Terms
4. Simplifying Ratios of -1
5. Rational Functions

1. Evaluating Rational Expressions

In Chapter 5, we introduced polynomials and polynomial functions. In this chapter, polynomials will be used to define rational expressions and rational functions.

Definition of a Rational Expression

An expression is a **rational expression** if it can be written in the form $\frac{p}{q}$, where p and q are polynomials and $q \neq 0$.

A rational expression is the ratio of two polynomials. The following expressions are examples of rational expressions.

$$\frac{5}{x}, \quad \frac{2y + 1}{y^2 - 4y + 3}, \quad \frac{c^3 + 5}{c - 4}, \quad \frac{a}{a + 10}$$

Example 1 Evaluating a Rational Expression

Evaluate the expression for the given values of x .

$$\frac{x - 2}{x + 3}$$

- a. 1 b. 2 c. 0 d. -1 e. -3

Solution:

$$\text{a. } \frac{(1) - 2}{(1) + 3} = \frac{-1}{4} \qquad \text{b. } \frac{(2) - 2}{(2) + 3} = \frac{0}{5} = 0 \qquad \text{c. } \frac{(0) - 2}{(0) + 3} = \frac{-2}{3}$$

$$\text{d. } \frac{(-1) - 2}{(-1) + 3} = \frac{-3}{2} \qquad \text{e. } \frac{(-3) - 2}{(-3) + 3} = \frac{-5}{0} \text{ (undefined)}$$

Skill Practice

1. Evaluate the expression for the given values of y .

$$\frac{3y}{y - 4}$$

- a. $y = -1$ b. $y = 10$ c. $y = 0$ d. $y = 4$ e. $y = -2$

2. Domain of a Rational Expression

The domain of an expression is the set of values that when substituted into the expression produces a real number. Therefore, the domain of a rational expression must exclude the values that make the denominator zero. For example:

Given $\frac{2}{x-3}$, the domain is the set of all real numbers, x , excluding $x = 3$. This can be written in set-builder notation as:

$$\{x \mid x \text{ is a real number and } x \neq 3\}$$

Skill Practice Answers

- 1a. $\frac{3}{5}$ b. 5 c. 0
 d. Undefined e. 1

Steps to Find the Domain of a Rational Expression

1. Set the denominator equal to zero and solve the resulting equation.
2. The domain is the set of all real numbers *excluding* the values found in step 1.

Example 2 Finding the Domain of a Rational Expression

Write the domain in set-builder notation.

a. $\frac{5}{4t-1}$ b. $\frac{x+4}{2x^2-11x+5}$ c. $\frac{x}{x^2+4}$

Solution:

a. $\frac{5}{4t-1}$ The expression is undefined when the denominator is zero.
 $4t-1=0$ Set the denominator equal to zero and solve for t .
 $4t=1$
 $t=\frac{1}{4}$ The value $t=\frac{1}{4}$ must be excluded from the domain.

Domain: $\{t \mid t \text{ is a real number and } t \neq \frac{1}{4}\}$

b. $\frac{x+4}{2x^2-11x+5}$
 $2x^2-11x+5=0$ Set the denominator equal to zero and solve for x .
 $(2x-1)(x-5)=0$ This is a factorable quadratic equation.
 $2x-1=0$ or $x-5=0$
 $x=\frac{1}{2}$ or $x=5$

Domain: $\{x \mid x \text{ is a real number and } x \neq \frac{1}{2}, x \neq 5\}$

TIP: The domain of a rational expression excludes values for which the denominator is zero. The values for which the numerator is zero *do not affect* the domain of the expression.

c. $\frac{x}{x^2+4}$

Because the quantity x^2 is nonnegative for any real number x , the denominator x^2+4 cannot equal zero; therefore, no real numbers are excluded from the domain.

Domain: $\{x \mid x \text{ is a real number}\}$

Skill Practice Write the domain in set-builder notation.

$$2. \frac{-6}{v-9} \quad 3. \frac{x+10}{2x^2-x-1} \quad 4. \frac{t}{t^2+1}$$

3. Simplifying Rational Expressions to Lowest Terms

As with fractions, it is often advantageous to simplify rational expressions to lowest terms. The method for simplifying rational expressions to lowest terms mirrors the process to simplify fractions. In each case, factor the numerator and denominator. Common factors in the numerator and denominator form a ratio of 1 and can be simplified.

Simplifying a fraction:
$$\frac{15}{35} \xrightarrow{\text{factor}} \frac{3 \cdot \overset{1}{\cancel{5}}}{7 \cdot \overset{1}{\cancel{5}}} = \frac{3}{7} (1) = \frac{3}{7}$$

Simplifying a rational expression:
$$\frac{x^2-x-12}{x^2-16} \xrightarrow{\text{factor}} \frac{(x+3)(\overset{1}{\cancel{x-4}})}{(x+4)(\overset{1}{\cancel{x-4}})} = \frac{(x+3)}{(x+4)} (1) = \frac{x+3}{x+4}$$

This process is stated formally as the fundamental principle of rational expressions.

Fundamental Principle of Rational Expressions

Let p , q , and r represent polynomials. Then

$$\frac{pr}{qr} = \frac{p}{q} \quad \text{for } q \neq 0 \text{ and } r \neq 0$$

The first example involves reducing a quotient of two monomials. This was also covered in Section 1.8.

Example 3 Simplifying a Rational Expression to Lowest Terms

Simplify the rational expression.
$$\frac{2x^2y^5}{8x^4y^3}$$

Solution:

$$\frac{2x^2y^5}{8x^4y^3} \quad \text{This expression has the restriction that } x \neq 0 \text{ and } y \neq 0.$$

$$= \frac{2x^2y^5}{2^3x^4y^3} \quad \text{Factor the denominator.}$$

$$= \frac{\overset{1}{\cancel{2}}x^2\overset{1}{\cancel{y^3}} \cdot y^2}{\overset{1}{\cancel{2}}x^2\overset{1}{\cancel{y^3}} \cdot 2^2x^2} \quad \text{Reduce the common factors whose ratio is 1.}$$

$$= \frac{y^2}{4x^2} \quad \text{provided that } x \neq 0 \text{ and } y \neq 0.$$

Skill Practice Answers

- $\{v \mid v \text{ is a real number and } v \neq 9\}$
- $\{x \mid x \text{ is a real number and } x \neq -\frac{1}{2}, x \neq 1\}$
- $\{t \mid t \text{ is a real number}\}$

Skill Practice Simplify the rational expression.

5. $\frac{9a^5b^3}{18a^8b}$

Example 4 Simplifying a Rational Expression to Lowest Terms

$$\frac{2x^3 + 12x^2 + 16x}{6x + 24}$$

- Factor the numerator and denominator.
- Determine the domain and write the domain in set-builder notation.
- Simplify the expression.

Solution:

a.
$$\begin{aligned} \frac{2x^3 + 12x^2 + 16x}{6x + 24} &= \frac{2x(x^2 + 6x + 8)}{6(x + 4)} && \text{Factor the numerator and denominator.} \\ &= \frac{2x(x + 4)(x + 2)}{6(x + 4)} \end{aligned}$$

- b. To determine the domain, set the denominator equal to zero and solve for x .

$$\begin{aligned} 6(x + 4) &= 0 && \text{Set } x + 4 = 0. \text{ Note that the factor of 6 will never} \\ x &= -4 && \text{equal 0.} \end{aligned}$$

The domain of the function is $\{x \mid x \text{ is a real number and } x \neq -4\}$.

c.
$$\begin{aligned} \frac{\cancel{2}x(x + 4)(x + 2)}{\cancel{2} \cdot 3(x + 4)} &= \frac{x(x + 2)}{3} && \text{Simplify the ratio of common factors to 1.} \\ & \text{provided } x \neq -4 \end{aligned}$$

Skill Practice Factor the numerator and the denominator to determine the domain. Then simplify the rational expression.

6. $\frac{a^2 + 3a - 28}{2a + 14}$

Avoiding Mistakes:

The domain of a rational expression is always determined *before* simplifying the expression.

It is important to note that the expressions

$$\frac{2x^3 + 12x^2 + 16x}{6x + 24} \quad \text{and} \quad \frac{x(x + 2)}{3}$$

Skill Practice Answers

5. $\frac{b^2}{2a^3}$, $a \neq 0$ and $b \neq 0$

6. $\{a \mid a \text{ is a real number and } a \neq -7\}; \frac{a - 4}{2}$

are equal for all values of x that make each expression a real number. Therefore,

$$\frac{2x^3 + 12x^2 + 16x}{6x + 24} = \frac{x(x + 2)}{3}$$

for all values of x except $x = -4$. (At $x = -4$, the original expression is undefined.) This is why the *domain of a rational expression is always determined before the expression is simplified*.

The purpose of simplifying a rational expression is to create an equivalent expression that is simpler to evaluate. Consider the expression from Example 4 in its original form and in its simplified form. If we substitute an arbitrary value of x into the expression (such as $x = 3$), we see that the simplified form is easier to evaluate.

	Original Expression	Simplified Form
	$\frac{2x^3 + 12x^2 + 16x}{6x + 24}$	$\frac{x(x + 2)}{3}$
Substitute $x = 3$ →	$\frac{2(3)^3 + 12(3)^2 + 16(3)}{6(3) + 24}$	$\frac{(3)[(3) + 2]}{3}$
	$= \frac{2(27) + 12(9) + 48}{18 + 24}$	$= \frac{3(5)}{3}$
	$= \frac{54 + 108 + 48}{42}$	$= 5$
	$= \frac{210}{42}$	
	$= 5$	

Example 5 Simplifying a Rational Expression to Lowest Terms

Simplify to lowest terms. $\frac{t^3 + 8}{t^2 + 6t + 8}$

Solution:

$$\frac{t^3 + 8}{t^2 + 6t + 8}$$

$$= \frac{(t + 2)(t^2 - 2t + 4)}{(t + 2)(t + 4)}$$

$$= \frac{\overset{1}{\cancel{t+2}}(t^2 - 2t + 4)}{\cancel{t+2}(t + 4)}$$

$$= \frac{t^2 - 2t + 4}{t + 4} \quad \text{provided } t \neq -2, t \neq -4$$

TIP: $t^3 + 8$ is a sum of cubes.

Recall: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

$$t^3 + 8 = (t + 2)(t^2 - 2t + 4)$$

The restrictions on the domain are $t \neq -2$ and $t \neq -4$.

Reduce common factors whose ratio is 1.

Skill Practice Simplify to lowest terms.

7. $\frac{8y^2 - 14y + 3}{2y^2 - y - 3}$

Skill Practice Answers

7. $\frac{4y - 1}{y + 1}$ provided $y \neq \frac{3}{2}, y \neq -1$

Avoiding Mistakes: The fundamental principle of rational expressions indicates that common *factors* in the numerator and denominator may be simplified.

$$\frac{pr}{qr} = \frac{p}{q} \cdot \frac{r}{r} = \frac{p}{q} \cdot (1) = \frac{p}{q}$$

Because this property is based on the identity property of multiplication, reducing applies only to factors (remember that factors are multiplied). Therefore, terms that are added or subtracted cannot be reduced. For example:

$$\frac{3x}{3y} = \frac{\overset{1}{\cancel{3x}}}{\cancel{3y}} = (1) \cdot \frac{x}{y} = \frac{x}{y}$$

↑
Reduce common factor.

However, $\frac{x+3}{y+3}$ cannot be simplified.

↑
cannot reduce common terms

4. Simplifying Ratios of -1

When two factors are identical in the numerator and denominator, they form a ratio of 1 and can be simplified. Sometimes we encounter two factors that are *opposites* and form a ratio of -1 . For example:

Simplified Form

Details/Notes

$$\frac{-5}{5} = -1$$

The ratio of a number and its opposite is -1 .

$$\frac{100}{-100} = -1$$

The ratio of a number and its opposite is -1 .

$$\frac{x+7}{-x-7} = -1$$

$$\frac{x+7}{-x-7} = \frac{x+7}{-1(x+7)} = \frac{\overset{1}{\cancel{x+7}}}{-1(\cancel{x+7})} = \frac{1}{-1} = -1$$

Factor out -1 .

$$\frac{2-x}{x-2} = -1$$

$$\frac{2-x}{x-2} = \frac{-1(-2+x)}{x-2} = \frac{-1(\overset{1}{\cancel{x-2}})}{\cancel{x-2}} = \frac{-1}{1} = -1$$

Recognizing factors that are opposites is useful when simplifying rational expressions. For example, $a - b$ and $b - a$ are opposites because the opposite of $a - b$ can be written $-(a - b) = -a + b = b - a$. Therefore, in general, $\frac{a-b}{b-a} = -1$, provided $a \neq b$.

Example 6 Simplifying Rational Expressions

Simplify the rational expressions to lowest terms.

a. $\frac{x-5}{25-x^2}$

b. $\frac{2x-6}{15-5x}$

Solution:

$$\begin{aligned} \text{a. } & \frac{x-5}{25-x^2} \\ &= \frac{x-5}{(5-x)(5+x)} && \text{Factor. The restrictions on } x \text{ are } x \neq 5 \text{ and } x \neq -5. \\ &= \frac{\overset{-1}{x-5}}{\cancel{(5-x)}(5+x)} && \text{Notice that } x-5 \text{ and } 5-x \text{ are opposites and form a ratio of } -1. \\ &= (-1)\left(\frac{1}{5+x}\right) && \text{In general, } \frac{a-b}{b-a} = -1. \\ &= -\frac{1}{5+x} \text{ or } -\frac{1}{x+5} && \text{provided } x \neq 5, x \neq -5 \end{aligned}$$

TIP: The factor of -1 may be applied in front of the rational expression, or it may be applied to the numerator or to the denominator. Therefore, the final answer may be written in several forms.

$$-\frac{1}{x+5} \quad \text{or} \quad \frac{-1}{x+5} \quad \text{or} \quad \frac{1}{-(x+5)}$$

$$\begin{aligned} \text{b. } & \frac{2x-6}{15-5x} \\ &= \frac{2(x-3)}{5(3-x)} && \text{Factor. The restriction on } x \text{ is } x \neq 3. \\ &= \frac{\overset{-1}{2(x-3)}}{5(\cancel{3-x})} && \text{Recall: } \frac{a-b}{b-a} = -1. \\ &= -\frac{2}{5} && \text{provided } x \neq 3 \end{aligned}$$

Skill Practice Simplify the expressions.

$$\text{8. } \frac{20-5x}{x^2-x-12} \qquad \text{9. } \frac{b^2-a^2}{a^2+ab-2b^2}$$

5. Rational Functions

Thus far in the text, we have introduced several types of functions including constant, linear, and quadratic functions. Now we will introduce another category of functions called rational functions.

Skill Practice Answers

$$\begin{aligned} \text{8. } & \frac{-5}{x+3} \\ \text{9. } & \frac{-(a+b)}{a+2b} \text{ or } -\frac{a+b}{a+2b} \end{aligned}$$

Definition of a Rational Function

A function f is a **rational function** if it can be written in the form $f(x) = \frac{p(x)}{q(x)}$, where p and q are polynomial functions and $q(x) \neq 0$.

For example, the functions f , g , h , and k are rational functions.

$$f(x) = \frac{1}{x}, \quad g(x) = \frac{2}{x-3}, \quad h(x) = \frac{a+6}{a^2-5}, \quad k(x) = \frac{x+4}{2x^2-11x+5}$$

In Section 4.3, we introduced the rational function defined by $f(x) = \frac{1}{x}$. Recall that $f(x) = \frac{1}{x}$ has a restriction on its domain that $x \neq 0$ and the graph of $f(x) = \frac{1}{x}$ has a vertical asymptote at $x = 0$ (Figure 6-1).

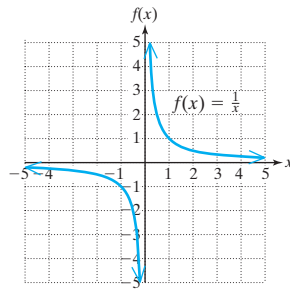


Figure 6-1

To evaluate, simplify, and find the domain of rational functions, we follow procedures similar to those shown for rational expressions.

Example 7 Evaluating a Rational Function

Given $g(x) = \frac{2}{x-3}$

- Find the function values (if they exist) $g(0)$, $g(1)$, $g(2)$, $g(2.5)$, $g(2.9)$, $g(3)$, $g(3.1)$, $g(3.5)$, $g(4)$, and $g(5)$.
- Write the domain of the function in interval notation.

Solution:

$$\begin{aligned} \text{a. } g(0) &= \frac{2}{(0)-3} = -\frac{2}{3} & g(3) &= \frac{2}{(3)-3} = \frac{2}{0} \quad \text{undefined} \\ g(1) &= \frac{2}{(1)-3} = -1 & g(3.1) &= \frac{2}{(3.1)-3} = \frac{2}{0.1} = 20 \\ g(2) &= \frac{2}{(2)-3} = -2 & g(3.5) &= \frac{2}{(3.5)-3} = \frac{2}{0.5} = 4 \\ g(2.5) &= \frac{2}{(2.5)-3} = \frac{2}{-0.5} = -4 & g(4) &= \frac{2}{(4)-3} = 2 \\ g(2.9) &= \frac{2}{(2.9)-3} = \frac{2}{-0.1} = -20 & g(5) &= \frac{2}{(5)-3} = 1 \end{aligned}$$

$$\text{b. } g(x) = \frac{2}{x-3}$$

$$x - 3 = 0$$

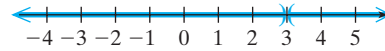
$$x = 3$$

$$\text{Domain: } (-\infty, 3) \cup (3, \infty)$$

The value of the function is undefined when the denominator equals zero.

Set the denominator equal to zero and solve for x .

The value $x = 3$ must be excluded from the domain.



Skill Practice

10. Given $h(x) = \frac{x+2}{x+6}$, find the function values.

- a. $h(0)$ b. $h(2)$ c. $h(-2)$ d. $h(-6)$
 e. Write the domain of h in interval notation.

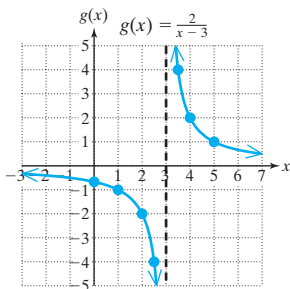


Figure 6-2

The graph of $g(x) = \frac{2}{x-3}$ is shown in Figure 6-2. Notice that the function has a vertical asymptote at $x = 3$ where the function is undefined (Figure 6-2).

Calculator Connections

A *Table* feature can be used to check the function values in Example 7, $g(x) = \frac{2}{x-3}$

X	Y1
0	-.6667
1	-1
2	-2
3	ERROR
4	1
5	.6667

X=0

The graph of a rational function may be misleading on a graphing calculator. For example, $g(x) = 2/(x-3)$ has a vertical asymptote at $x = 3$, but the vertical asymptote is *not* part of the graph (Figure 6-3). A graphing calculator may try to “connect the dots” between two consecutive points, one to the left of $x = 3$ and one to the right of $x = 3$. This creates a line that is nearly vertical and appears to be part of the graph.

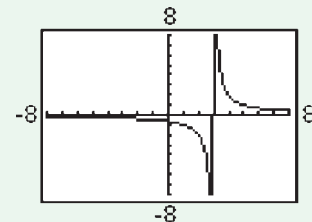


Figure 6-3

To show that this line is not part of the graph, we can graph the function in a *Dot Mode* (see the owner’s manual for your calculator). The graph of $g(x) = 2/(x-3)$ in dot mode indicates that the line $x = 3$ is not part of the function (Figure 6-4).

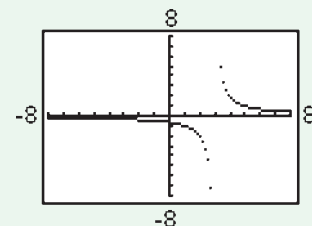


Figure 6-4

Skill Practice Answers

- 10a. $\frac{1}{3}$ b. $\frac{1}{2}$ c. 0
 d. Undefined
 e. $(-\infty, -6) \cup (-6, \infty)$

Section 6.1

Practice Exercises

Boost your **GRADE** at
mathzone.com!



- Practice Problems
- Self-Tests
- NetTutor
- e-Professors
- Videos

Study Skills Exercise

1. Define the key terms.
 - a. Rational expression
 - b. Rational function

Concept 1: Evaluating Rational Expressions

2. Evaluate the expression $\frac{5}{x+1}$ for $x = 0$, $x = 2$, $x = -1$, $x = -6$.
3. Evaluate the expression $\frac{x-4}{x+6}$ for $x = -6$, $x = -4$, $x = 0$, and $x = 4$.
4. Evaluate the expression $\frac{y-3}{y-2}$ for $y = 0$, $y = 2$, $y = 3$, $y = -4$.
5. Evaluate the expression $\frac{3a+1}{a^2+1}$ for $a = 1$, $a = 0$, $a = -\frac{1}{3}$, $a = -1$.
6. Evaluate the expression $\frac{2z-8}{z^2+9}$ for $z = 4$, $z = -4$, $z = 3$, $z = -3$.

Concept 2: Domain of a Rational Expression

For Exercises 7–16, determine the domain of the rational expression. Write the answer in set-builder notation.

7. $\frac{9}{y}$

8. $\frac{-10}{a}$

9. $\frac{v+1}{v-8}$

10. $\frac{t+9}{t+3}$

11. $\frac{3x-1}{2x-5}$

12. $\frac{6t+5}{3t+8}$

13. $\frac{q+1}{q^2+6q-27}$

14. $\frac{b^2}{2b^2+3b-5}$

15. $\frac{c}{c^2+25}$

16. $\frac{d}{d^2+16}$

Concept 3: Simplifying Rational Expressions to Lowest Terms

17. Given: $\frac{x^2+6x+8}{x^2+3x-4}$

- a. Factor the numerator and denominator.
- b. Write the domain in set-builder notation.
- c. Simplify the expression.

18. Given: $f(x) = \frac{x^2 - 6x}{2x^2 - 11x - 6}$
- Factor the numerator and denominator.
 - Write the domain in set-builder notation.
 - Simplify the expression.
19. Given: $p(x) = \frac{x^2 - 18x + 81}{x^2 - 81}$
- Factor the numerator and denominator.
 - Write the domain in set-builder notation.
 - Simplify the expression.
20. Given: $q(x) = \frac{x^2 + 14x + 49}{x^2 - 49}$
- Factor the numerator and denominator.
 - Write the domain in set-builder notation.
 - Simplify the expression.

For Exercises 21–42, simplify the rational expressions.

- | | | | |
|--|---|---|---------------------------------------|
| 21. $\frac{100x^3y^5}{36xy^8}$ | 22. $\frac{48ab^3c^2}{6a^7bc^0}$ | 23. $\frac{7w^{11}z^6}{14w^3z^3}$ | 24. $\frac{12r^9s^3}{24r^8s^4}$ |
| 25. $\frac{-3m^4n}{12m^6n^4}$ | 26. $\frac{-5x^3y^2}{20x^4y^2}$ | 27. $\frac{6a + 18}{9a + 27}$ | 28. $\frac{5y - 15}{3y - 9}$ |
| 29. $\frac{x - 5}{x^2 - 25}$ | 30. $\frac{3z - 6}{3z^2 - 12}$ | 31. $\frac{-7c}{21c^2 - 35c}$ | 32. $\frac{2p + 3}{2p^2 + 7p + 6}$ |
| 33. $\frac{2t^2 + 7t - 4}{-2t^2 - 5t + 3}$ | 34. $\frac{y^2 + 8y - 9}{y^2 - 5y + 4}$ | 35. $\frac{(p + 1)(2p - 1)^4}{(p + 1)^2(2p - 1)^2}$ | 36. $\frac{r(r - 3)^5}{r^3(r - 3)^2}$ |
| 37. $\frac{9 - z^2}{2z^2 + z - 15}$ | 38. $\frac{2c^2 + 2c - 12}{-8 + 2c + c^2}$ | 39. $\frac{2z^3 + 16}{-10 - 3z + z^2}$ | 40. $\frac{5p^2 - p - 4}{p^3 - 1}$ |
| 41. $\frac{10x^3 - 25x^2 + 4x - 10}{-4 - 10x^2}$ | 42. $\frac{8x^3 - 12x^2 + 6x - 9}{16x^4 - 9}$ | | |

Concept 4: Simplifying Ratios of -1

For Exercises 43–56, simplify the rational expressions.

- | | | | |
|-----------------------------|-----------------------------|----------------------------|-----------------------------|
| 43. $\frac{r + 6}{6 + r}$ | 44. $\frac{a + 2}{2 + a}$ | 45. $\frac{b + 8}{-b - 8}$ | 46. $\frac{7 + w}{-7 - w}$ |
| 47. $\frac{10 - x}{x - 10}$ | 48. $\frac{y - 14}{14 - y}$ | 49. $\frac{2t - 2}{1 - t}$ | 50. $\frac{5p - 10}{2 - p}$ |

51. $\frac{c+4}{c-4}$

52. $\frac{b+2}{b-2}$

53. $\frac{y-x}{12x^2-12y^2}$

54. $\frac{4w^2-49z^2}{14z-4w}$

55. $\frac{4x-2x^2}{5x-10}$

56. $\frac{2y-6}{3y^2-y^3}$

Mixed Exercises

For Exercises 57–70, simplify the rational expression.

57. $\frac{t^2-1}{t^2+7t+6}$

58. $\frac{x^2+4x+4}{x^2-4}$

59. $\frac{8p+8}{2p^2-4p-6}$

60. $\frac{15y-15}{3y^2+9y-12}$

61. $\frac{-16a^2bc^4}{8ab^2c^4}$

62. $\frac{-9x^3yz^2}{27x^4yz}$

63. $\frac{x^2-y^2}{8y-8x}$

64. $\frac{p^2-49}{14-2p}$

65. $\frac{b+4}{2b^2+5b-12}$

66. $\frac{c-6}{3c^2-17c-6}$

67. $\frac{-2x+34}{-4x+6}$

68. $\frac{-9w-3}{3w+12}$

69. $\frac{(a-2)^2(a-5)^3}{(a-2)^3(a-5)}$

70. $\frac{t^2(t-11)^4}{t^5(t-11)^2}$

Concept 5: Rational Functions

71. Let $h(x) = \frac{-3}{x-1}$. Find the function values $h(0)$, $h(1)$, $h(-3)$, $h(-1)$, $h(\frac{1}{2})$.

72. Let $k(x) = \frac{2}{x+1}$. Find the function values $k(0)$, $k(1)$, $k(-3)$, $k(-1)$, $k(\frac{1}{2})$.

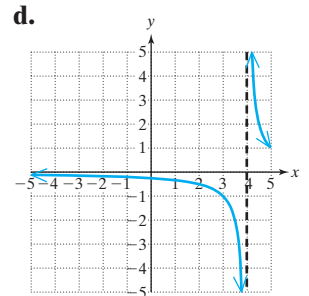
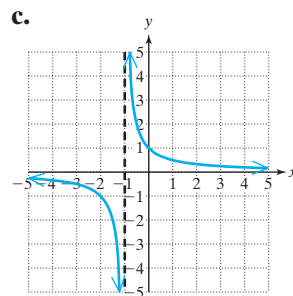
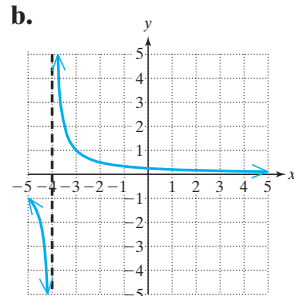
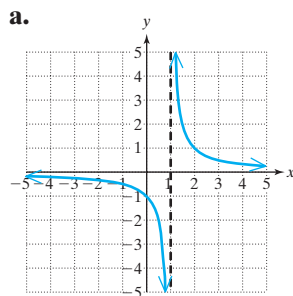
For Exercises 73–76, find the domain of each function and use that information to match the function with its graph.

73. $m(x) = \frac{1}{x+4}$

74. $n(x) = \frac{1}{x+1}$

75. $q(x) = \frac{1}{x-4}$

76. $p(x) = \frac{1}{x-1}$



For Exercises 77–86, write the domain of the function in interval notation.

77. $f(x) = \frac{1}{3x-1}$

78. $r(t) = \frac{1}{2t-1}$

79. $q(t) = \frac{t+2}{8}$

80. $p(x) = \frac{x-5}{3}$

81. $w(x) = \frac{x-2}{3x-6}$

82. $s(t) = \frac{-2}{4t+6}$

83. $m(x) = \frac{3}{6x^2 - 7x - 10}$

84. $n(x) = \frac{1}{2x^2 + 11x + 12}$

85. $r(x) = \frac{x + 1}{6x^3 - x^2 - 15x}$

86. $s(t) = \frac{t + 5}{t^3 - 8t^2 + 16t}$

Expanding Your Skills

87. Write a rational expression whose domain is $(-\infty, 2) \cup (2, \infty)$. (Answers may vary.)
88. Write a rational expression whose domain is $(-\infty, 3) \cup (3, \infty)$. (Answers may vary.)
89. Write a rational function whose domain is $(-\infty, -5) \cup (-5, \infty)$. (Answers may vary.)
90. Write a rational function whose domain is $(-\infty, -6) \cup (-6, \infty)$. (Answers may vary.)

Section 6.2**Multiplication and Division of Rational Expressions****Concepts**

1. Multiplication of Rational Expressions
2. Division of Rational Expressions

1. Multiplication of Rational Expressions

Recall that to multiply fractions, we multiply the numerators and multiply the denominators. The same is true for multiplying rational expressions.

Multiplication of Rational Expressions

Let p , q , r , and s represent polynomials, such that $q \neq 0$ and $s \neq 0$. Then

$$\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$$

For example:

Multiply the fractions

$$\frac{2}{3} \cdot \frac{5}{7} = \frac{10}{21}$$

Multiply the rational expressions

$$\frac{2x}{3y} \cdot \frac{5z}{7} = \frac{10xz}{21y}$$

Sometimes it is possible to simplify a ratio of common factors to 1 *before* multiplying. To do so, we must first factor the numerators and denominators of each fraction.

$$\frac{7}{10} \cdot \frac{15}{21} \xrightarrow{\text{Factor.}} \frac{\frac{1}{7}}{2 \cdot 5} \cdot \frac{\frac{1}{3} \cdot \frac{1}{3}}{3 \cdot 7} = \frac{1}{2}$$

The same process is also used to multiply rational expressions.

Multiplying Rational Expressions

1. Factor the numerators and denominators of all rational expressions.
2. Simplify the ratios of common factors to 1.
3. Multiply the remaining factors in the numerator, and multiply the remaining factors in the denominator.

Example 1 Multiplying Rational Expressions

Multiply.

$$\text{a. } \frac{5a - 5b}{10} \cdot \frac{2}{a^2 - b^2} \qquad \text{b. } \frac{4w - 20p}{2w^2 - 50p^2} \cdot \frac{2w^2 + 7wp - 15p^2}{3w + 9p}$$

Solution:

$$\text{a. } \frac{5a - 5b}{10} \cdot \frac{2}{a^2 - b^2}$$

$$= \frac{5(a - b)}{5 \cdot 2} \cdot \frac{2}{(a - b)(a + b)} \qquad \text{Factor numerator and denominator.}$$

$$= \frac{\overset{1}{\cancel{5}}(\overset{1}{\cancel{a - b}})}{5 \cdot 2} \cdot \frac{\overset{1}{\cancel{2}}}{(\overset{1}{\cancel{a - b}})(a + b)} \qquad \text{Simplify.}$$

$$= \frac{1}{a + b}$$

$$\text{b. } \frac{4w - 20p}{2w^2 - 50p^2} \cdot \frac{2w^2 + 7wp - 15p^2}{3w + 9p}$$

$$= \frac{4(w - 5p)}{2(w^2 - 25p^2)} \cdot \frac{(2w - 3p)(w + 5p)}{3(w + 3p)} \qquad \text{Factor numerator and denominator.}$$

$$= \frac{2 \cdot 2(w - 5p)}{2(w - 5p)(w + 5p)} \cdot \frac{(2w - 3p)(w + 5p)}{3(w + 3p)} \qquad \text{Factor further.}$$

$$= \frac{\overset{1}{\cancel{2}} \cdot 2(\overset{1}{\cancel{w - 5p}})}{2(\overset{1}{\cancel{w - 5p}})(\overset{1}{\cancel{w + 5p}})} \cdot \frac{(2w - 3p)(\overset{1}{\cancel{w + 5p}})}{3(w + 3p)} \qquad \text{Simplify common factors.}$$

$$= \frac{2(2w - 3p)}{3(w + 3p)}$$

Notice that the expression is left in factored form to show that it has been simplified to lowest terms.

Skill Practice Multiply.

$$\text{1. } \frac{3y - 6}{6y} \cdot \frac{y^2 + 3y + 2}{y^2 - 4} \qquad \text{2. } \frac{p^2 + 8p + 16}{10p + 10} \cdot \frac{2p + 6}{p^2 + 7p + 12}$$

Avoiding Mistakes:

If all factors in the numerator simplify to 1, do not forget to write the factor of 1 in the numerator.

2. Division of Rational Expressions

Recall that to divide fractions, multiply the first fraction by the reciprocal of the second fraction.

$$\text{Divide: } \frac{15}{14} \div \frac{10}{49} \xrightarrow[\text{of the second fraction.}]{\text{Multiply by the reciprocal}} \frac{15}{14} \cdot \frac{49}{10} = \frac{3 \cdot \overset{1}{\cancel{5}}}{2 \cdot \overset{1}{\cancel{7}}} \cdot \frac{\overset{1}{\cancel{7}} \cdot 7}{2 \cdot \overset{1}{\cancel{5}}} = \frac{21}{4}$$

The same process is used for dividing rational expressions.

Skill Practice Answers

$$\text{1. } \frac{y + 1}{2y} \qquad \text{2. } \frac{p + 4}{5(p + 1)}$$

Division of Rational Expressions

Let p , q , r , and s represent polynomials, such that $q \neq 0$, $r \neq 0$, $s \neq 0$. Then

$$\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \cdot \frac{s}{r} = \frac{ps}{qr}$$

Example 2 Dividing Rational Expressions

Divide. a. $\frac{8t^3 + 27}{9 - 4t^2} \div \frac{4t^2 - 6t + 9}{2t^2 - t - 3}$ b. $\frac{\frac{5c}{6d}}{\frac{10}{d^2}}$

Solution:

$$\begin{aligned} \text{a. } & \frac{8t^3 + 27}{9 - 4t^2} \div \frac{4t^2 - 6t + 9}{2t^2 - t - 3} \\ &= \frac{8t^3 + 27}{9 - 4t^2} \cdot \frac{2t^2 - t - 3}{4t^2 - 6t + 9} \\ &= \frac{(2t + 3)(4t^2 - 6t + 9)}{(3 - 2t)(3 + 2t)} \cdot \frac{(2t - 3)(t + 1)}{4t^2 - 6t + 9} \\ &= \frac{(2t + 3)(4t^2 - 6t + 9)}{(3 - 2t)(3 + 2t)} \cdot \frac{(2t - 3)(t + 1)}{4t^2 - 6t + 9} \\ &= (-1)(t + 1) \\ &= -(t + 1) \quad \text{or} \quad -t - 1 \end{aligned}$$

Multiply the first fraction by the reciprocal of the second.

Factor numerator and denominator. Notice $8t^3 + 27$ is a sum of cubes. Furthermore, $4t^2 - 6t + 9$ does not factor over the real numbers.

Simplify to lowest terms.

TIP: In Example 2(a), the factors $(2t - 3)$ and $(3 - 2t)$ are opposites and form a ratio of -1 .

$$\frac{2t - 3}{3 - 2t} = -1$$

The factors $(2t + 3)$ and $(3 + 2t)$ are equal and form a ratio of 1 .

$$\frac{2t + 3}{3 + 2t} = 1$$

b. $\frac{\frac{5c}{6d}}{\frac{10}{d^2}}$ ← This fraction bar denotes division (\div).

$$\begin{aligned} &= \frac{5c}{6d} \div \frac{10}{d^2} \\ &= \frac{5c}{6d} \cdot \frac{d^2}{10} \end{aligned}$$

Multiply the first fraction by the reciprocal of the second.

$$\begin{aligned}
 &= \frac{1}{5c} \cdot \frac{1}{2} \cdot \frac{d}{d} \\
 &= \frac{cd}{12}
 \end{aligned}$$

Skill Practice Divide the rational expressions.

$$\begin{array}{ll}
 3. \frac{x^2 + x}{5x^3 - x^2} \div \frac{10x^2 + 12x + 2}{25x^2 - 1} & 4. \frac{\frac{4rs}{5t}}{\frac{2r}{t}}
 \end{array}$$

Skill Practice Answers

$$3. \frac{1}{2x} \quad 4. \frac{2s}{5}$$

Section 6.2

Practice Exercises

Boost your GRADE at
mathzone.com!



- Practice Problems
- Self-Tests
- NetTutor
- e-Professors
- Videos

Study Skills Exercise

- Write an example of how to multiply two fractions, divide two fractions, add two fractions, and subtract two fractions. For example: $\frac{5}{12} \cdot \frac{9}{10}$, $\frac{3}{4} \div \frac{15}{8}$, $\frac{5}{6} + \frac{7}{9}$, and $\frac{2}{7} - \frac{2}{7}$. Then as you learn about rational expressions, compare the operations on rational expressions to those on fractions. This is a great place to use 3×5 cards again. Write an example of an operation with fractions on one side and the same operation with rational expressions on the other side.

Review Exercises

For Exercises 2–9, simplify the rational expression to lowest terms.

$$2. \frac{3w^2 - 12}{w^2 + 5w + 6}$$

$$3. \frac{t^2 - 5t - 6}{t^2 - 7t + 6}$$

$$4. \frac{5y + 15}{5y^2 + 16y + 3}$$

$$5. \frac{2 - p}{p^2 - p - 2}$$

$$6. \frac{5x^2yz^3}{20xyz}$$

$$7. \frac{7x + 14}{7x^2 - 7x - 42}$$

$$8. \frac{25 - x^2}{x^2 - 10x + 25}$$

$$9. \frac{a^3b^2c^5}{2a^3bc^2}$$

Concept 1: Multiplication of Rational Expressions

For Exercises 10–21, multiply the rational expressions.

$$10. \frac{8w^2}{9} \cdot \frac{3}{2w^4}$$

$$11. \frac{16}{z^7} \cdot \frac{z^4}{8}$$

$$12. \frac{5p^2q^4}{12pq^3} \cdot \frac{6p^2}{20q^2}$$

$$13. \frac{27r^5}{7s} \cdot \frac{28rs^3}{9r^3s^2}$$

$$14. \frac{3z + 12}{8z^3} \cdot \frac{16z^3}{9z + 36}$$

$$15. \frac{x^2y}{x^2 - 4x - 5} \cdot \frac{2x^2 - 13x + 15}{xy^3}$$

16. $\frac{3y^2 + 18y + 15}{6y + 6} \cdot \frac{y - 5}{y^2 - 25}$

17. $\frac{10w - 8}{w + 2} \cdot \frac{3w^2 - w - 14}{25w^2 - 16}$

18. $\frac{x - 5y}{x^2 + xy} \cdot \frac{y^2 - x^2}{10y - 2x}$

19. $\frac{3x - 15}{4x^2 - 2x} \cdot \frac{10x - 20x^2}{5 - x}$

20. $x(x + 5)^2 \cdot \frac{2}{x^2 - 25}$

21. $y(y^2 - 4) \cdot \frac{y}{y + 2}$

Concept 2: Division of Rational Expressions

For Exercises 22–33, divide the rational expressions.

22. $\frac{6x^2y^2}{(x - 2)} \div \frac{3xy^2}{(x - 2)^2}$

23. $\frac{(r + 3)^2}{4r^3s} \div \frac{r + 3}{rs}$

24. $\frac{t^2 + 5t}{t + 1} \div (t + 5)$

25. $\frac{6p + 7}{p + 2} \div (36p^2 - 49)$

26. $\frac{a}{a - 10} \div \frac{a^3 + 6a^2 - 40a}{a^2 - 100}$

27. $\frac{b^2 - 6b + 9}{b^2 - b - 6} \div \frac{b^2 - 9}{4}$

28. $\frac{2x^2 + 5xy + 2y^2}{4x^2 - y^2} \div \frac{x^2 + xy - 2y^2}{2x^2 + xy - y^2}$

29. $\frac{6s^2 + st - 2t^2}{6s^2 - 5st + t^2} \div \frac{3s^2 + 17st + 10t^2}{6s^2 + 13st - 5t^2}$

30. $\frac{x^4 - x^3 + x^2 - x}{2x^3 + 2x^2 + x + 1} \div \frac{x^3 - 4x^2 + x - 4}{2x^3 - 8x^2 + x - 4}$

31. $\frac{a^3 + a + a^2 + 1}{a^3 + a^2 + ab^2 + b^2} \div \frac{a^3 + a + a^2b + b}{2a^2 + 2ab + ab^2 + b^3}$

32. $\frac{3y - y^2}{y^3 - 27} \div \frac{y}{y^2 + 3y + 9}$

33. $\frac{8x - 4x^2}{xy - 2y + 3x - 6} \div \frac{3x + 6}{y + 3}$

Mixed Exercises

For Exercises 34–45, perform the indicated operations.

34. $\frac{8a^4b^3}{3c} \div \frac{a^7b^2}{9c}$

35. $\frac{3x^5}{2x^2y^7} \div \frac{4x^3y}{6y^6}$

36. $\frac{2}{25x^2} \cdot \frac{5x}{12} \div \frac{2}{15x}$

37. $\frac{4y}{7} \div \frac{y^2}{14} \cdot \frac{3}{y}$

38. $\frac{10x^2 - 13xy - 3y^2}{8x^2 - 10xy - 3y^2} \cdot \frac{2y + 8x}{2x^2 + 2y^2}$

39. $\frac{6a^2 + ab - b^2}{10a^2 + 5ab} \cdot \frac{2a^3 + 4a^2b}{3a^2 + 5ab - 2b^2}$

40. $\frac{(a + b)^2}{a - b} \cdot \frac{a^3 - b^3}{a^2 - b^2} \div \frac{a^2 + ab + b^2}{(a - b)^2}$

41. $\frac{m^2 - n^2}{(m - n)^2} \div \frac{m^2 - 2mn + n^2}{m^2 - mn + n^2} \cdot \frac{(m - n)^4}{m^3 + n^3}$

42. $\frac{x^2 - 4y^2}{x + 2y} \div (x + 2y) \cdot \frac{2y}{x - 2y}$

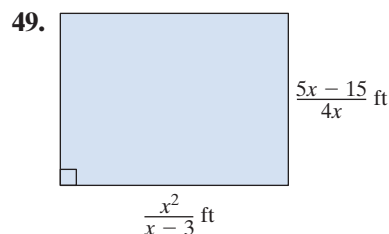
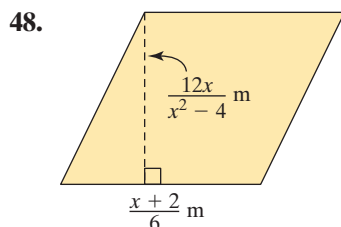
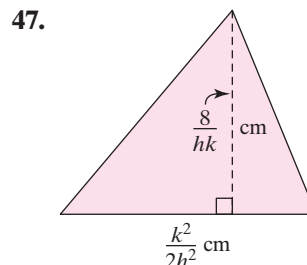
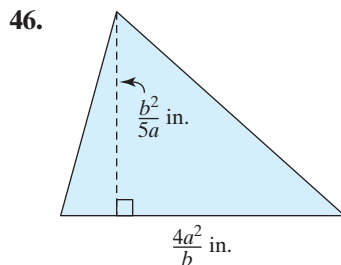
43. $\frac{x^2 - 6xy + 9y^2}{x^2 - 4y^2} \cdot \frac{x^2 - 5xy + 6y^2}{3y - x} \div \frac{x^2 - 9y^2}{x + 2y}$

44. $\frac{8x^3 - 27y^3}{4x^2 - 9y^2} \div \frac{8x^2 + 12xy + 18y^2}{2x + 3y}$

45. $\frac{25m^2 - 1}{125m^3 - 1} \div \frac{5m + 1}{25m^2 + 5m + 1}$

Expanding Your Skills

For Exercises 46–49, write an expression for the area of the figure and simplify.



Addition and Subtraction of Rational Expressions

Section 6.3

1. Addition and Subtraction of Rational Expressions with Like Denominators

To add or subtract rational expressions, the expressions must have the same denominator. As with fractions, we add or subtract rational expressions with the same denominator by combining the terms in the numerator and then writing the result over the common denominator. Then, if possible, we simplify the expression to lowest terms.

Addition and Subtraction of Rational Expressions

Let p , q , and r represent polynomials where $q \neq 0$. Then

$$1. \frac{p}{q} + \frac{r}{q} = \frac{p+r}{q} \qquad 2. \frac{p}{q} - \frac{r}{q} = \frac{p-r}{q}$$

Example 1 Adding and Subtracting Rational Expressions with Like Denominators

Add or subtract as indicated.

$$\text{a. } \frac{1}{8} + \frac{3}{8} \qquad \text{b. } \frac{5x}{2x-1} + \frac{3}{2x-1} \qquad \text{c. } \frac{x^2}{x-4} - \frac{x+12}{x-4}$$

Concepts

1. Addition and Subtraction of Rational Expressions with Like Denominators
2. Identifying the Least Common Denominator
3. Writing Equivalent Rational Expressions
4. Addition and Subtraction of Rational Expressions with Unlike Denominators

Solution:

$$\begin{aligned} \text{a. } \frac{1}{8} + \frac{3}{8} &= \frac{1+3}{8} \\ &= \frac{4}{8} \\ &= \frac{1}{2} \end{aligned}$$

Add the terms in the numerator.

Simplify the fraction.

$$\text{b. } \frac{5x}{2x-1} + \frac{3}{2x-1} = \frac{5x+3}{2x-1}$$

Add the terms in the numerator. The answer is already in lowest terms.

$$\begin{aligned} \text{c. } \frac{x^2}{x-4} - \frac{x+12}{x-4} \\ &= \frac{x^2 - (x+12)}{x-4} \end{aligned}$$

Combine the terms in the numerator. Use parentheses to group the terms in the numerator that follow the subtraction sign. This will help you remember to apply the distributive property.

$$= \frac{x^2 - x - 12}{x-4}$$

Apply the distributive property.

$$= \frac{(x-4)(x+3)}{(x-4)}$$

Factor the numerator and denominator.

$$= \frac{\cancel{(x-4)}(x+3)}{\cancel{(x-4)}}$$

Simplify the rational expression.

$$= x + 3$$

Skill Practice

Add or subtract as indicated.

1. $\frac{5}{12} - \frac{1}{12}$

2. $\frac{b^2+1}{b-2} + \frac{b-7}{b-2}$

3. $\frac{4t+1}{t+7} - \frac{t+3}{t+7}$

2. Identifying the Least Common Denominator

If two rational expressions have different denominators, each expression must be rewritten with a common denominator before adding or subtracting the expressions. The **least common denominator (LCD)** of two or more rational expressions is defined as the least common multiple of the denominators.

For example, consider the fractions $\frac{1}{20}$ and $\frac{1}{8}$. By inspection, you can probably see that the least common denominator is 40. To understand why, find the prime factorization of both denominators.

$$20 = 2^2 \cdot 5 \quad \text{and} \quad 8 = 2^3$$

Skill Practice Answers

1. $\frac{1}{3}$ 2. $b+3$ 3. $\frac{3t-2}{t+7}$

A common multiple of 20 and 8 must be a multiple of 5, a multiple of 2^2 , and a multiple of 2^3 . However, any number that is a multiple of $2^3 = 8$ is automatically a multiple of $2^2 = 4$. Therefore, it is sufficient to construct the least common denominator as the product of unique prime factors, where each factor is raised to its highest power.

The LCD of $\frac{1}{2^2 \cdot 5}$ and $\frac{1}{2^3}$ is $2^3 \cdot 5 = 40$.

Steps to Find the LCD of Two or More Rational Expressions

1. Factor all denominators completely.
2. The LCD is the product of unique prime factors from the denominators, where each factor is raised to the highest power to which it appears in any denominator.

Example 2 Finding the LCD of Rational Expressions

Find the LCD of the following rational expressions.

a. $\frac{1}{12}, \frac{5}{18}, \frac{7}{30}$

b. $\frac{1}{2x^3y}, \frac{5}{16xy^2z}$

c. $\frac{x+2}{2x-10}, \frac{x^2+3}{x^2-25}, \frac{6}{x^2+8x+16}$

d. $\frac{x+4}{x-3}, \frac{1}{3-x}$

Solution:

a. $\frac{1}{12}, \frac{5}{18}, \frac{7}{30}$

$$\frac{1}{2^2 \cdot 3}, \frac{5}{2 \cdot 3^2}, \frac{7}{2 \cdot 3 \cdot 5}$$

$$\text{LCD} = 2^2 \cdot 3^2 \cdot 5 = 180$$

Factor the denominators completely.

The LCD is the product of the factors 2, 3, and 5. Each factor is raised to its highest power.

b. $\frac{1}{2x^3y}, \frac{5}{16xy^2z}$

$$\frac{1}{2x^3y}, \frac{5}{2^4xy^2z}$$

$$\text{LCD} = 2^4x^3y^2z = 16x^3y^2z$$

Factor the denominators completely.

The LCD is the product of the factors 2, x , y , and z . Each factor is raised to its highest power.

c. $\frac{x+2}{2x-10}, \frac{x^2+3}{x^2-25}, \frac{6}{x^2+8x+16}$

$$\frac{x+2}{2(x-5)}, \frac{x^2+3}{(x-5)(x+5)}, \frac{6}{(x+4)^2}$$

$$\text{LCD} = 2(x-5)(x+5)(x+4)^2$$

Factor the denominators completely.

The LCD is the product of the factors 2, $(x-5)$, $(x+5)$, and $(x+4)$. Each factor is raised to its highest power.

d. $\frac{x+4}{x-3}; \frac{1}{3-x}$ The denominators are already factored.

Notice that $x-3$ and $3-x$ are opposite factors. If -1 is factored from either expression, the binomial factors will be the same.

$$\begin{array}{ccc} \frac{x+4}{x-3}; & \frac{1}{-1(-3+x)} & \frac{x+4}{-1(-x+3)}; \frac{1}{3-x} \\ \uparrow & \uparrow \text{Factor out } -1. & \uparrow \text{Factor out } -1. \\ \text{same binomial factors} & & \text{same binomial factors} \end{array}$$

$$\begin{array}{ccc} \text{LCD} = (x-3)(-1) & & \text{LCD} = (-1)(3-x) \\ = -x+3 & & = -3+x \\ = 3-x & & = x-3 \end{array}$$

The LCD can be taken as either $(3-x)$ or $(x-3)$.

Skill Practice Find the LCD of the rational expressions.

4. $\frac{7}{40}; \frac{1}{15}; \frac{5}{6}$ 5. $\frac{1}{9a^3b^2}; \frac{5}{18a^4b}$

6. $\frac{5}{x^2-4}; \frac{1}{2x+6}; \frac{6}{x^2+5x+6}$ 7. $\frac{6}{z-7}; \frac{1}{7-z}$

3. Writing Equivalent Rational Expressions

Rational expressions can be added if they have common denominators. Once the LCD has been determined, each rational expression must be converted to an **equivalent rational expression** with the indicated denominator.

Using the identity property of multiplication, we know that for $q \neq 0$ and $r \neq 0$,

$$\frac{p}{q} = \frac{p}{q} \cdot 1 = \frac{p}{q} \cdot \frac{r}{r} = \frac{pr}{qr}$$

This principle is used to convert a rational expression to an equivalent expression with a different denominator. For example, $\frac{1}{2}$ can be converted to an equivalent expression with a denominator of 12 as follows:

$$\frac{1}{2} = \frac{1}{2} \cdot \frac{6}{6} = \frac{1 \cdot 6}{2 \cdot 6} = \frac{6}{12}$$

In this example, we multiplied $\frac{1}{2}$ by a convenient form of 1. The ratio $\frac{6}{6}$ was chosen so that the product produced a new denominator of 12. Notice that multiplying $\frac{1}{2}$ by $\frac{6}{6}$ is equivalent to multiplying the numerator and denominator of the original expression by 6.

Example 3 Creating Equivalent Rational Expressions

Convert each expression to an equivalent rational expression with the indicated denominator.

a. $\frac{7}{5p^2} = \frac{\quad}{20p^6}$ b. $\frac{w}{w+5} = \frac{\quad}{w^2+3w-10}$

Skill Practice Answers

4. 120 5. $18a^4b^2$
 6. $2(x+2)(x-2)(x+3)$
 7. $z-7$ or $7-z$

Solution:

$$\text{a. } \frac{7}{5p^2} = \frac{\quad}{20p^6}$$

$$= \frac{\quad}{5p^2 \cdot 4p^4}$$

$20p^6$ can be written as $5p^2 \cdot 4p^4$.

$$\frac{7}{5p^2} = \frac{7}{5p^2} \cdot \frac{4p^4}{4p^4}$$

Multiply the numerator and denominator by the missing factor, $4p^4$.

$$= \frac{28p^4}{20p^6}$$

$$\text{b. } \frac{w}{w+5} = \frac{\quad}{w^2+3w-10}$$

$$= \frac{\quad}{(w+5)(w-2)}$$

Factor the denominator.

$$\frac{w}{w+5} = \frac{w}{w+5} \cdot \frac{w-2}{w-2}$$

Multiply the numerator and denominator by the missing factor $(w-2)$.

$$= \frac{w^2-2w}{(w+5)(w-2)}$$

Skill Practice

Fill in the blank to make an equivalent fraction with the given denominator.

$$8. \frac{1}{8xy} = \frac{\quad}{16x^3y^2}$$

$$9. \frac{5b}{b-3} = \frac{\quad}{b^2-9}$$

TIP: In the final answer in Example 3(b) we multiplied the polynomials in the numerator but left the denominator in factored form. This convention is followed because when we add and subtract rational expressions, the terms in the numerators must be combined.

4. Addition and Subtraction of Rational Expressions with Unlike Denominators

To add or subtract rational expressions with unlike denominators, we must convert each expression to an equivalent expression with the same denominator. For example, consider adding the expressions $\frac{3}{x-2} + \frac{5}{x+1}$. The LCD is $(x-2)(x+1)$. For each expression, identify the factors from the LCD that are missing in the denominator. Then multiply the numerator and denominator of the expression by the missing factor(s):

$$\frac{(3)}{(x-2)} \cdot \frac{(x+1)}{(x+1)} + \frac{(5)}{(x+1)} \cdot \frac{(x-2)}{(x-2)}$$

The rational expressions now have the same denominator and can be added.

Skill Practice Answers

8. $2x^2y$ 9. $5b^2 + 15b$

$$\begin{aligned}
 &= \frac{3(x+1) + 5(x-2)}{(x-2)(x+1)} && \text{Combine terms in the numerator.} \\
 &= \frac{3x+3+5x-10}{(x-2)(x+1)} && \text{Clear parentheses and simplify.} \\
 &= \frac{8x-7}{(x-2)(x+1)}
 \end{aligned}$$

Steps to Add or Subtract Rational Expressions

1. Factor the denominator of each rational expression.
2. Identify the LCD.
3. Rewrite each rational expression as an equivalent expression with the LCD as its denominator.
4. Add or subtract the numerators, and write the result over the common denominator.
5. Simplify if possible.

Example 4 Adding and Subtracting Rational Expressions with Unlike Denominators

Add or subtract as indicated.

$$\text{a. } \frac{3}{7b} + \frac{4}{b^2} \qquad \text{b. } \frac{3t-2}{t^2+4t-12} - \frac{5}{2t+12} \qquad \text{c. } \frac{2}{x} + \frac{x}{x+3} - \frac{3x+18}{x^2+3x}$$

Solution:

$$\begin{aligned}
 \text{a. } & \frac{3}{7b} + \frac{4}{b^2} && \text{Step 1: The denominators are already factored.} \\
 & && \text{Step 2: The LCD is } 7b^2. \\
 & = \frac{3}{7b} \cdot \frac{b}{b} + \frac{4}{b^2} \cdot \frac{7}{7} && \text{Step 3: Write each expression with the LCD.} \\
 & = \frac{3b}{7b^2} + \frac{28}{7b^2} && \text{Step 4: Add the numerators, and write the result over the LCD.} \\
 & = \frac{3b+28}{7b^2} && \text{Step 5: Simplify.}
 \end{aligned}$$

$$\begin{aligned}
 \text{b. } & \frac{3t-2}{t^2+4t-12} - \frac{5}{2t+12} && \text{Step 1: Factor the denominators.} \\
 & = \frac{3t-2}{(t+6)(t-2)} - \frac{5}{2(t+6)} && \text{Step 2: The LCD is } 2(t+6)(t-2). \\
 & = \frac{(2)}{(2)} \cdot \frac{(3t-2)}{(t+6)(t-2)} - \frac{5}{2(t+6)} \cdot \frac{(t-2)}{(t-2)} && \text{Step 3: Write each expression with the LCD.}
 \end{aligned}$$

$$= \frac{2(3t - 2) - 5(t - 2)}{2(t + 6)(t - 2)}$$

$$= \frac{6t - 4 - 5t + 10}{2(t + 6)(t - 2)}$$

$$= \frac{t + 6}{2(t + 6)(t - 2)}$$

$$= \frac{\cancel{t+6}}{2(\cancel{t+6})(t-2)}$$

$$= \frac{1}{2(t-2)}$$

c. $\frac{2}{x} + \frac{x}{x+3} - \frac{3x+18}{x^2+3x}$

$$= \frac{2}{x} + \frac{x}{x+3} - \frac{3x+18}{x(x+3)}$$

$$= \frac{2}{x} \cdot \frac{(x+3)}{(x+3)} + \frac{x}{(x+3)} \cdot \frac{x}{x} - \frac{3x+18}{x(x+3)}$$

$$= \frac{2(x+3) + x^2 - (3x+18)}{x(x+3)}$$

$$= \frac{2x + 6 + x^2 - 3x - 18}{x(x+3)}$$

$$= \frac{x^2 - x - 12}{x(x+3)}$$

$$= \frac{(x-4)(x+3)}{x(x+3)}$$

$$= \frac{(x-4)\cancel{(x+3)}}{x\cancel{(x+3)}}$$

$$= \frac{x-4}{x}$$

Step 4: Add the numerators and write the result over the LCD.

Step 5: Simplify.

Combine *like* terms.

Simplify.

Step 1: Factor the denominators.

Step 2: The LCD is $x(x+3)$.

Step 3: Write each expression with the LCD.

Step 4: Add the numerators, and write the result over the LCD.

Step 5: Simplify.

Combine *like* terms.

Factor the numerator.

Simplify to lowest terms.

Skill Practice

Add or subtract as indicated.

10. $\frac{4}{5y} + \frac{1}{3y^3}$

11. $\frac{2x+3}{x^2+x-2} - \frac{5}{3x-3}$

12. $\frac{a^2+a+24}{a^2-9} + \frac{5}{a+3}$

Skill Practice Answers

10. $\frac{12y^2+5}{15y^3}$ 11. $\frac{1}{3(x+2)}$

12. $\frac{a+3}{a-3}$

Example 5 Adding and Subtracting Rational Expressions with Unlike Denominators

Add or subtract as indicated.

a. $\frac{6}{w} + \frac{4}{-w}$ b. $\frac{x^2}{x-y} + \frac{y^2}{y-x}$

Solution:

a. $\frac{6}{w} + \frac{4}{-w}$

$$= \frac{6}{w} + \frac{4}{-w} \cdot \frac{(-1)}{(-1)}$$

$$= \frac{6}{w} + \frac{-4}{w}$$

$$= \frac{6 + (-4)}{w}$$

$$= \frac{2}{w}$$

b. $\frac{x^2}{x-y} + \frac{y^2}{y-x}$

$$= \frac{x^2}{(x-y)} + \frac{y^2}{(y-x)} \cdot \frac{(-1)}{(-1)}$$

$$= \frac{x^2}{x-y} + \frac{-y^2}{x-y}$$

$$= \frac{x^2 - y^2}{x-y}$$

$$= \frac{(x+y)(x-y)}{x-y}$$

$$= x + y$$

Step 1: The denominators are already factored.

Step 2: The denominators are opposites and differ by a factor of -1 . The LCD can either be taken as w or $-w$. We will use an LCD of w .

Step 3: Write each expression with the LCD. Note that $(-w)(-1) = w$.

Step 4: Add the numerators, and write the result over the LCD.

Step 5: Simplify.

Step 1: The denominators are already factored.

Step 2: The denominators are opposites and differ by a factor of -1 . The LCD can be taken as either $(x-y)$ or $(y-x)$. We will use an LCD of $(x-y)$.

Step 3: Write each expression with the LCD. Note that $(y-x)(-1) = -y+x = x-y$.

Step 4: Combine the numerators, and write the result over the LCD.

Step 5: Factor and simplify to lowest terms.

Skill Practice Add or subtract as indicated.

Skill Practice Answers

13. $\frac{-8}{y}$ or $\frac{8}{-y}$ 14. 3

13. $\frac{3}{-y} - \frac{5}{y}$ 14. $\frac{3a}{a-5} + \frac{15}{5-a}$

Section 6.3

Practice Exercises

Boost your GRADE at
mathzone.com!



- Practice Problems
- Self-Tests
- NetTutor
- e-Professors
- Videos

Study Skills Exercise

1. Define the key terms.

a. Least common denominator (LCD)

b. Equivalent rational expressions

Review Exercises

For Exercises 2–6, perform the indicated operation.

$$2. \frac{x}{x-y} \div \frac{x^2}{y-x}$$

$$3. \frac{9b+9}{4b+8} \cdot \frac{2b+4}{3b-3}$$

$$4. \frac{8a^2}{7b^3} \div \frac{4a}{21b}$$

$$5. \frac{(5-a)^2}{10a-2} \cdot \frac{25a^2-1}{a^2-10a+25}$$

$$6. \frac{x^2-z^2}{14x^2z^4} \div \frac{x^2+2xz+z^2}{3xz^3}$$

Concept 1: Addition and Subtraction of Rational Expressions with Like Denominators

For Exercises 7–16, add or subtract as indicated and simplify if possible.

$$7. \frac{3}{5x} + \frac{7}{5x}$$

$$8. \frac{1}{2x^2} - \frac{5}{2x^2}$$

$$9. \frac{x}{x^2-2x-3} - \frac{3}{x^2-2x-3}$$

$$10. \frac{x}{x^2+4x-12} + \frac{6}{x^2+4x-12}$$

$$11. \frac{5x-1}{(2x+9)(x-6)} - \frac{3x-6}{(2x+9)(x-6)}$$

$$12. \frac{4-x}{8x+1} - \frac{5x-6}{8x+1}$$

$$13. \frac{6}{x-5} + \frac{3}{5-x}$$

$$14. \frac{8}{2-x} + \frac{7}{x-2}$$

$$15. \frac{x-2}{x-6} - \frac{x+2}{6-x}$$

$$16. \frac{x-10}{x-8} - \frac{x+10}{8-x}$$

Concept 2: Identifying the Least Common Denominator

For Exercises 17–28, find the least common denominator (LCD).

$$17. \frac{5}{8}, \frac{3}{20x}$$

$$18. \frac{y}{15a}, \frac{y^2}{35}$$

$$19. \frac{-7}{24x}, \frac{5}{75x^2}$$

$$20. \frac{2}{7y^3}, \frac{-13}{5y^2}$$

$$21. \frac{-5}{6m^4}, \frac{1}{15mn^7}$$

$$22. \frac{13}{12cd^5}, \frac{9}{8c^3}$$

$$23. \frac{6}{(x-4)(x+2)}, \frac{-8}{(x-4)(x-6)}$$

$$24. \frac{x}{(2x-1)(x-7)}, \frac{2}{(2x-1)(x+1)}$$

25. $\frac{3}{x(x-1)(x+7)^2}; \frac{-1}{x^2(x+7)}$

26. $\frac{14}{(x-2)^2(x+9)}; \frac{41}{x(x-2)(x+9)}$

27. $\frac{5}{x-6}; \frac{x-5}{x^2-8x+12}$

28. $\frac{7a}{a+4}; \frac{a+12}{a^2-16}$

Concept 3: Writing Equivalent Rational Expressions

For Exercises 29–34, fill in the blank to make an equivalent fraction with the given denominator.

29. $\frac{5}{3x} = \frac{\quad}{9x^2y}$

30. $\frac{-5}{xy} = \frac{\quad}{4x^2y^3}$

31. $\frac{2x}{x-1} = \frac{\quad}{x(x-1)(x+2)}$

32. $\frac{5x}{2x-5} = \frac{\quad}{(2x-5)(x+8)}$

33. $\frac{y}{y+6} = \frac{\quad}{y^2+5y-6}$

34. $\frac{t^2}{t-8} = \frac{\quad}{t^2-6t-16}$

Concept 4: Addition and Subtraction of Rational Expressions with Unlike Denominators

For Exercises 35–66, add or subtract as indicated.

35. $\frac{4}{3p} - \frac{5}{2p^2}$

36. $\frac{6}{5a^2b} - \frac{1}{10ab}$

37. $\frac{x+3}{x^2} + \frac{x+5}{2x}$

38. $\frac{x+2}{5x^2} + \frac{x+4}{15x}$

39. $\frac{s-1}{s} - \frac{t+1}{t}$

40. $\frac{x+2}{x} - \frac{y-2}{y}$

41. $\frac{4a-2}{3a+12} - \frac{a-2}{a+4}$

42. $\frac{10}{b(b+5)} + \frac{2}{b}$

43. $\frac{6}{w(w-2)} + \frac{3}{w}$

44. $\frac{6y+5}{5y-25} - \frac{y+2}{y-5}$

45. $w+2 + \frac{1}{w-2}$

46. $h-3 + \frac{1}{h+3}$

47. $\frac{6b}{b-4} - \frac{1}{b+1}$

48. $\frac{a}{a-3} - \frac{5}{a+6}$

49. $\frac{t+5}{t-5} - \frac{10t-5}{t^2-25}$

50. $\frac{s+8}{s-8} - \frac{16s+64}{s^2-64}$

51. $\frac{x+2}{x^2-36} - \frac{x}{x^2+9x+18}$

52. $\frac{7}{x^2-x-2} + \frac{x}{x^2+4x+3}$

53. $\frac{9}{x^2-2x+1} - \frac{x-3}{x^2-x}$

54. $\frac{2}{4z^2-12z+9} - \frac{z+1}{2z^2-3z}$

55. $\frac{w}{6} + \frac{w+4}{-6}$

56. $\frac{4y}{3} + \frac{5}{-3}$

57. $\frac{n}{5-n} + \frac{2n-5}{n-5}$

58. $\frac{c}{7-c} + \frac{2c-7}{c-7}$

59. $\frac{2}{3x-15} + \frac{x}{25-x^2}$

60. $\frac{5}{9-x^2} - \frac{4}{x^2+4x+3}$

61. $\frac{k}{k+7} - \frac{2}{k^2+6k-7} + \frac{4k}{k-1}$

62. $\frac{t}{t-2} - \frac{1}{t^2-t-2} + \frac{3t}{t+1}$

63. $\frac{2x}{x^2-y^2} - \frac{1}{x-y} + \frac{1}{y-x}$

64. $\frac{3w-1}{2w^2+w-3} - \frac{2-w}{w-1} - \frac{w}{1-w}$

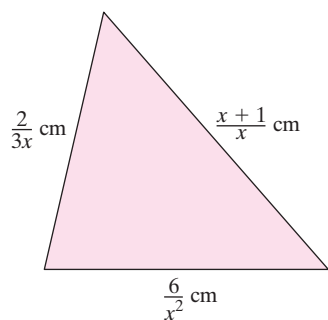
65. $\frac{3}{y} + \frac{2}{y-6}$

66. $\frac{-8}{p} + \frac{p}{p+5}$

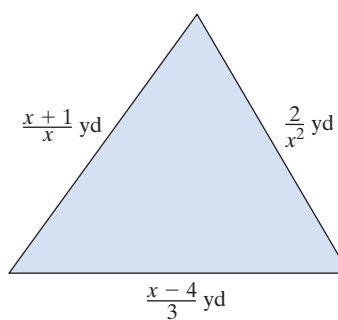
Expanding Your Skills

For Exercises 67–70, write an expression that represents the perimeter of the figure and simplify.

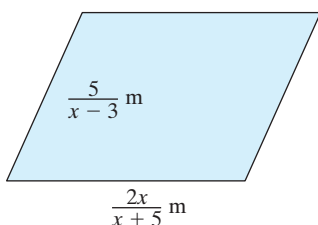
67.



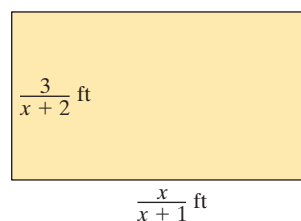
68.



69.



70.

**Complex Fractions****Section 6.4****1. Simplifying Complex Fractions by Method I**

A **complex fraction** is a fraction whose numerator or denominator contains one or more fractions. For example:

$$\frac{\frac{5x^2}{y}}{\frac{10x}{y^2}} \quad \text{and} \quad \frac{2 + \frac{1}{2} - \frac{1}{3}}{\frac{3}{4} + \frac{1}{6}}$$

are complex fractions.

Two methods will be presented to simplify complex fractions. The first method (Method I) follows the order of operations to simplify the numerator and denominator separately before dividing. The process is summarized as follows.

Concepts

1. Simplifying Complex Fractions by Method I
2. Simplifying Complex Fractions by Method II
3. Using Complex Fractions in Applications

Steps to Simplify a Complex Fraction—Method I

1. Add or subtract expressions in the numerator to form a single fraction. Add or subtract expressions in the denominator to form a single fraction.
2. Divide the rational expressions from step 1 by multiplying the numerator of the complex fraction by the reciprocal of the denominator of the complex fraction.
3. Simplify to lowest terms, if possible.

Example 1 Simplifying a Complex Fraction by Method I

Simplify the expression.

$$\frac{\frac{5x^2}{y}}{\frac{10x}{y^2}}$$

Solution:

$$\begin{aligned} & \frac{\frac{5x^2}{y}}{\frac{10x}{y^2}} \\ &= \frac{5x^2}{y} \cdot \frac{y^2}{10x} \\ &= \frac{5 \cdot x \cdot x}{y} \cdot \frac{y \cdot y}{2 \cdot 5 \cdot x} \\ &= \frac{\overset{1}{\cancel{5}} \cdot \overset{1}{\cancel{x}} \cdot x}{y} \cdot \frac{\overset{1}{\cancel{y}} \cdot y}{2 \cdot \overset{1}{\cancel{5}} \cdot \cancel{x}} \\ &= \frac{xy}{2} \end{aligned}$$

Step 1: The numerator and denominator of the complex fraction are already single fractions.

Step 2: Multiply the numerator of the complex fraction by the reciprocal of the denominator.

Factor the numerators and denominators.

Step 3: Simplify.

Skill Practice Simplify the expression.

$$1. \frac{\frac{18a^3}{b^2}}{\frac{6a^2}{b}}$$

Sometimes it is necessary to simplify the numerator and denominator of a complex fraction before the division is performed. This is illustrated in Example 2.

Skill Practice Answers

$$1. \frac{3a}{b}$$

Example 2 Simplifying a Complex Fraction by Method I

Simplify the expression.

$$\frac{2 + \frac{1}{2} - \frac{1}{3}}{\frac{3}{4} + \frac{1}{6}}$$

Solution:

$$\frac{2 + \frac{1}{2} - \frac{1}{3}}{\frac{3}{4} + \frac{1}{6}}$$

Step 1: Combine fractions in numerator and denominator separately.

$$= \frac{\frac{12}{6} + \frac{3}{6} - \frac{2}{6}}{\frac{9}{12} + \frac{2}{12}}$$

The LCD in the numerator is 6. The LCD in the denominator is 12.

$$= \frac{\frac{13}{6}}{\frac{11}{12}}$$

Form single fractions in the numerator and denominator.

$$= \frac{13}{6} \cdot \frac{12}{11}$$

Step 2: Multiply by the reciprocal of $\frac{11}{12}$, which is $\frac{12}{11}$.

$$= \frac{13}{\cancel{6}^2} \cdot \frac{\cancel{12}_2}{11}$$

$$= \frac{26}{11}$$

Step 3: Simplify.

Skill Practice Simplify the expression.

2.
$$\frac{\frac{1}{4} - \frac{5}{6}}{\frac{1}{2} + \frac{1}{3} + 1}$$

2. Simplifying Complex Fractions by Method II

We will now use a second method to simplify complex fractions—Method II. Recall that multiplying the numerator and denominator of a rational expression by the same quantity does not change the value of the expression. This is the basis for Method II.

Skill Practice Answers

2. $-\frac{7}{22}$

Steps to Simplify a Complex Fraction—Method II

1. Multiply the numerator and denominator of the complex fraction by the LCD of *all* individual fractions within the expression.
2. Apply the distributive property, and simplify the numerator and denominator.
3. Simplify to lowest terms, if possible.

Example 3 Simplifying Complex Fractions by Method II

Simplify by using Method II.

$$\frac{4 - \frac{6}{x}}{2 - \frac{3}{x}}$$

Solution:

$$\frac{4 - \frac{6}{x}}{2 - \frac{3}{x}}$$

The LCD of all individual terms is x .

$$= \frac{x \cdot \left(4 - \frac{6}{x}\right)}{x \cdot \left(2 - \frac{3}{x}\right)}$$

Step 1: Multiply the numerator and denominator of the complex fraction by the LCD, which is x .

$$= \frac{x \cdot (4) - x \cdot \left(\frac{6}{x}\right)}{x \cdot (2) - x \cdot \left(\frac{3}{x}\right)}$$

Step 2: Apply the distributive property.

$$= \frac{4x - 6}{2x - 3}$$

Step 3: Simplify numerator and denominator.

$$= \frac{2(2x - 3)}{2x - 3}$$

Factor and simplify to lowest terms.

$$= \frac{2\cancel{(2x - 3)}}{\cancel{2x - 3}}$$

$$= 2$$

Skill Practice Simplify the expression.

$$3. \frac{y - \frac{1}{y}}{1 - \frac{1}{y^2}}$$

Skill Practice Answers3. y

Example 4 Simplifying Complex Fractions by Method II

Simplify by using Method II. $\frac{x^{-1} - x^{-2}}{1 + 2x^{-1} - 3x^{-2}}$

Solution:

$$\frac{x^{-1} - x^{-2}}{1 + 2x^{-1} - 3x^{-2}}$$

$$= \frac{\frac{1}{x} - \frac{1}{x^2}}{1 + \frac{2}{x} - \frac{3}{x^2}}$$

$$= \frac{x^2 \cdot \left(\frac{1}{x} - \frac{1}{x^2}\right)}{x^2 \cdot \left(1 + \frac{2}{x} - \frac{3}{x^2}\right)}$$

$$= \frac{x^2\left(\frac{1}{x}\right) - x^2\left(\frac{1}{x^2}\right)}{x^2(1) + x^2\left(\frac{2}{x}\right) - x^2\left(\frac{3}{x^2}\right)}$$

$$= \frac{x - 1}{x^2 + 2x - 3}$$

$$= \frac{x - 1}{(x + 3)(x - 1)}$$

$$= \frac{\cancel{x} - 1}{(x + 3)(\cancel{x} - 1)}$$

$$= \frac{1}{x + 3}$$

Rewrite the expression with positive exponents. The LCD of all individual terms is x^2 .

Step 1: Multiply the numerator and denominator of the complex fraction by the LCD x^2 .

Step 2: Apply the distributive property.

Step 3: Factor and simplify to lowest terms.

Skill Practice Simplify the expression.

4. $\frac{c^{-1}b - b^{-1}c}{b^{-1} + c^{-1}}$

Example 5 Simplifying Complex Fractions by Method II

Simplify the expression by Method II. $\frac{\frac{1}{w+3} - \frac{1}{w-3}}{1 + \frac{9}{w^2-9}}$

Skill Practice Answers

4. $b - c$

Solution:

$$\frac{\frac{1}{w+3} - \frac{1}{w-3}}{1 + \frac{9}{w^2-9}}$$

$$= \frac{\frac{1}{w+3} - \frac{1}{w-3}}{1 + \frac{9}{(w+3)(w-3)}}$$

Factor all denominators to find the LCD.

The LCD of $\frac{1}{1}$, $\frac{1}{w+3}$, $\frac{1}{w-3}$, and $\frac{9}{(w+3)(w-3)}$ is $(w+3)(w-3)$.

$$= \frac{(w+3)(w-3)\left(\frac{1}{w+3} - \frac{1}{w-3}\right)}{(w+3)(w-3)\left[1 + \frac{9}{(w+3)(w-3)}\right]}$$

Step 1: Multiply numerator and denominator of the complex fraction by $(w+3)(w-3)$.

$$= \frac{\cancel{(w+3)}\cancel{(w-3)}\left(\frac{1}{\cancel{w+3}} - \cancel{(w+3)}\cancel{(w-3)}\left(\frac{1}{\cancel{w-3}}\right)\right)}{\cancel{(w+3)}\cancel{(w-3)}1 + \cancel{(w+3)}\cancel{(w-3)}\left[\frac{9}{\cancel{(w+3)}\cancel{(w-3)}}\right]}$$

Step 2: Distributive property.

$$= \frac{(w-3) - (w+3)}{(w+3)(w-3) + 9}$$

Step 3: Simplify.

$$= \frac{w-3-w-3}{w^2-9+9}$$

Apply the distributive property.

$$= \frac{-6}{w^2}$$

$$= -\frac{6}{w^2}$$

Skill Practice Simplify the expression.

$$5. \frac{\frac{2}{x+1} - \frac{1}{x-1}}{\frac{x}{x-1} - \frac{1}{x+1}}$$

3. Using Complex Fractions in Applications

Example 6 Finding the Slope of a Line

Find the slope of the line that passes through the given points.

$$\left(\frac{1}{2}, \frac{5}{6}\right) \quad \text{and} \quad \left(\frac{9}{8}, \frac{5}{4}\right)$$

Skill Practice Answers

$$5. \frac{x-3}{x^2+1}$$

Solution:

Let $(x_1, y_1) = \left(\frac{1}{2}, \frac{5}{6}\right)$ and $(x_2, y_2) = \left(\frac{9}{8}, \frac{5}{4}\right)$. Label the points.

$$\begin{aligned} \text{Slope} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{\frac{5}{4} - \frac{5}{6}}{\frac{9}{8} - \frac{1}{2}} \end{aligned}$$

Apply the slope formula.

Using Method II to simplify the complex fractions, we have

$$m = \frac{24\left(\frac{5}{4} - \frac{5}{6}\right)}{24\left(\frac{9}{8} - \frac{1}{2}\right)}$$

Multiply numerator and denominator by the LCD, 24.

$$= \frac{24\left(\frac{5}{4}\right) - 24\left(\frac{5}{6}\right)}{24\left(\frac{9}{8}\right) - 24\left(\frac{1}{2}\right)}$$

Apply the distributive property.

$$= \frac{30 - 20}{27 - 12}$$

Simplify.

$$= \frac{10}{15} = \frac{2}{3}$$

The slope is $\frac{2}{3}$.

Skill Practice

6. Find the slope of the line that contains the points

$$\left(-\frac{3}{5}, \frac{1}{4}\right) \text{ and } \left(\frac{7}{10}, -\frac{5}{2}\right).$$

Skill Practice Answers

6. $m = -\frac{55}{26}$

Section 6.4**Practice Exercises**

Boost your GRADE at
mathzone.com!



- Practice Problems
- Self-Tests
- NetTutor
- e-Professors
- Videos

Study Skills Exercise

1. Define the key term **complex fraction**.

Review Exercises

For Exercises 2–4, simplify to lowest terms.

2. $\frac{x^3 + y^3}{5x + 5y}$

3. $\frac{25a^3b^3c}{15a^4bc}$

4. $f(t) = \frac{6t^2 - 27t + 30}{12t - 30}$

For Exercises 5–8, perform the indicated operations.

5. $\frac{5}{x^2} + \frac{3}{2x}$

6. $\frac{2y-4}{y+1} \cdot \frac{y^2+3y+2}{y^2-4}$

7. $\frac{3}{a-5} - \frac{1}{a+1}$

8. $\frac{7}{12-6b} \div \frac{14b}{b^2+b-6}$

Concept 1: Simplifying Complex Fractions by Method I

For Exercises 9–16, simplify the complex fractions by using Method I.

9. $\frac{\frac{5x^2}{9y^2}}{\frac{3x}{y^2x}}$

10. $\frac{\frac{3w^2}{4rs}}{\frac{15wr}{s^2}}$

11. $\frac{\frac{x-6}{3x}}{\frac{3x-18}{9}}$

12. $\frac{\frac{a+4}{6}}{\frac{16-a^2}{3}}$

13. $\frac{\frac{2}{3} + \frac{1}{6}}{\frac{1}{2} - \frac{1}{4}}$

14. $\frac{\frac{7}{8} + \frac{3}{4}}{\frac{1}{3} - \frac{5}{6}}$

15. $\frac{2 - \frac{1}{y}}{4 + \frac{1}{y}}$

16. $\frac{\frac{1}{x} - 3}{\frac{1}{x} + 3}$

Concept 2: Simplifying Complex Fractions by Method II

For Exercises 17–44, simplify the complex fractions by using Method II.

17. $\frac{\frac{7y}{y+3}}{\frac{1}{4y+12}}$

18. $\frac{\frac{6x}{x-5}}{\frac{1}{4x-20}}$

19. $\frac{1 + \frac{1}{3}}{\frac{5}{6} - 1}$

20. $\frac{2 + \frac{4}{5}}{-1 + \frac{3}{10}}$

21. $\frac{\frac{3q}{p} - q}{q - \frac{q}{p}}$

22. $\frac{\frac{b}{a} + 3b}{b + \frac{2b}{a}}$

23. $\frac{\frac{2}{a} + \frac{3}{a^2}}{\frac{4}{a^2} - \frac{9}{a}}$

24. $\frac{\frac{2}{y^2} + \frac{1}{y}}{\frac{4}{y^2} - \frac{1}{y}}$

25. $\frac{t^{-1} - 1}{1 - t^{-2}}$

26. $\frac{d^{-2} - c^{-2}}{c^{-1} - d^{-1}}$

27. $\frac{-8}{\frac{6w}{w-1} - 4}$

28. $\frac{6}{2z - \frac{10}{z-4}}$

29. $\frac{\frac{y}{y+3}}{\frac{y}{y+3} + y}$

30. $\frac{\frac{4}{w-4}}{\frac{4}{w-4} - 1}$

31. $\frac{1 - \frac{1}{x} - \frac{6}{x^2}}{1 - \frac{4}{x} + \frac{3}{x^2}}$

32. $\frac{1 + \frac{1}{x} - \frac{12}{x^2}}{\frac{9}{x^2} + \frac{3}{x} - 2}$

33. $\frac{2 - \frac{2}{t+1}}{2 + \frac{2}{t}}$

34. $\frac{3 + \frac{3}{p-1}}{3 - \frac{3}{p}}$

35. $\frac{\frac{2}{a} - \frac{3}{a+1}}{\frac{2}{a+1} - \frac{3}{a}}$

36. $\frac{\frac{5}{b} + \frac{4}{b+1}}{\frac{4}{b} - \frac{5}{b+1}}$

37. $\frac{\frac{1}{y+2} + \frac{4}{y-3}}{\frac{2}{y-3} - \frac{7}{y+2}}$

38. $\frac{\frac{1}{t-4} + \frac{1}{t+5}}{\frac{6}{t+5} + \frac{2}{t-4}}$

39. $\frac{\frac{2}{x+h} - \frac{2}{x}}{h}$

40. $\frac{\frac{1}{2x+2h} - \frac{1}{2x}}{h}$

41. $\frac{x^{-2}}{x + 3x^{-1}}$

42. $\frac{x^{-1} + x^{-2}}{5x^{-2}}$

43. $\frac{2a^{-1} + 3b^{-2}}{a^{-1} - b^{-1}}$

44. $\frac{2m^{-1} + n^{-1}}{m^{-2} - 4n^{-1}}$

Concept 3: Using Complex Fractions in Applications

45. The slope formula is used to find the slope of the line passing through the points (x_1, y_1) and (x_2, y_2) . Write the slope formula from memory.

For Exercises 46–49, find the slope of the line that passes through the given points.

46. $\left(\frac{1}{2}, \frac{2}{5}\right), \left(\frac{1}{4}, -2\right)$ 47. $\left(-\frac{3}{7}, \frac{3}{5}\right), (-1, -3)$ 48. $\left(\frac{5}{8}, \frac{9}{10}\right), \left(-\frac{1}{16}, -\frac{1}{5}\right)$ 49. $\left(\frac{1}{4}, \frac{1}{3}\right), \left(\frac{1}{8}, \frac{1}{6}\right)$

Expanding Your Skills

50. Show that $(x + x^{-1})^{-1} = \frac{x}{x^2 + 1}$ by writing the expression on the left without negative exponents and simplifying.

51. Show that $(x^{-1} + y^{-1})^{-1} = \frac{xy}{x + y}$ by writing the expression on the left without negative exponents and simplifying.

52. Simplify. $\frac{x}{1 - \left(1 + \frac{1}{x}\right)^{-1}}$

53. Simplify. $\frac{x}{1 - \left(1 - \frac{1}{x}\right)^{-1}}$

Chapter 6**Problem Recognition Exercises—
Operations on Rational Expressions**

For Exercises 1–20, perform the indicated operation and simplify.

1. $\frac{2}{2y-3} - \frac{3}{2y} + 1$ 2. $(x+5) + \left(\frac{7}{x-4}\right)$

3. $\frac{5x^2 - 6x + 1}{x^2 - 1} \div \frac{16x^2 - 9}{4x^2 + 7x + 3}$

4. $\frac{a^2 - 25}{3a^2 + 3ab} \cdot \frac{a^2 + 4a + ab + 4b}{a^2 + 9a + 20}$

5. $\frac{4}{y+1} + \frac{y+2}{y^2-1} - \frac{3}{y-1}$

6. $\frac{8w^2}{w^3 - 16w} - \frac{4w}{w^2 - 4w}$

7. $\frac{a^2 - 16}{2x + 6} \cdot \frac{x + 3}{a - 4}$

8. $\frac{t^2 - 9}{t} \div \frac{t + 3}{t + 2}$

9. $\frac{2 + \frac{1}{a}}{4 - \frac{1}{a^2}}$

10. $\frac{\frac{6x^2y}{5}}{\frac{3x}{y}}$

11. $\frac{6xy}{x^2 - y^2} + \frac{x + y}{y - x}$

12. $(x^2 - 6x + 8) \cdot \left(\frac{3}{x-2}\right)$

13. $\frac{1}{w-1} - \frac{w+2}{3w-3}$

14. $\frac{3y+6}{y^2-3y-10} \div \frac{27}{y-5}$

15. $\frac{y + \frac{2}{y} - 3}{1 - \frac{2}{y}}$

16. $\frac{2}{t-3} - \frac{3}{t+2} + 5$

$$17. \frac{4x^2 + 22x + 24}{4x + 4} \cdot \frac{6x + 6}{4x^2 - 9}$$

$$18. \frac{12x^3y^5z}{5x^4} \div \frac{16xy^7}{10z^2}$$

$$19. (y + 2) \cdot \frac{2y + 1}{y^2 - 4} - \frac{y - 2}{y + 3}$$

$$20. \frac{a^2}{a - 10} - \frac{20a - 100}{a - 10}$$

Section 6.5

Rational Equations

Concepts

1. Solving Rational Equations
2. Formulas Involving Rational Equations

1. Solving Rational Equations

Thus far we have studied two types of equations in one variable: linear equations and quadratic equations. In this section, we will study another type of equation called a rational equation.

Definition of a Rational Equation

An equation with one or more rational expressions is called a **rational equation**.

The following equations are rational equations:

$$\frac{1}{2}x + \frac{1}{3} = \frac{1}{4}x \qquad \frac{3}{5} + \frac{1}{x} = \frac{2}{3} \qquad 3 - \frac{6w}{w + 1} = \frac{6}{w + 1}$$

To understand the process of solving a rational equation, first review the procedure of clearing fractions from Section 1.4.

Example 1 Solving a Rational Equation

Solve the equation. $\frac{1}{2}x + \frac{1}{3} = \frac{1}{4}x$

Solution:

$$\begin{aligned} \frac{1}{2}x + \frac{1}{3} &= \frac{1}{4}x \\ 12\left(\frac{1}{2}x + \frac{1}{3}\right) &= 12\left(\frac{1}{4}x\right) \\ 12 \cdot \frac{1}{2}x + 12 \cdot \frac{1}{3} &= 12 \cdot \frac{1}{4}x \\ 6x + 4 &= 3x \\ 3x &= -4 \\ x &= -\frac{4}{3} \end{aligned}$$

The LCD of all terms in the equation is 12.

Multiply both sides by 12 to clear fractions.

Apply the distributive property.

Solve the resulting equation.

Check:

$$\begin{aligned} \frac{1}{2}x + \frac{1}{3} &= \frac{1}{4}x \\ \frac{1}{2}\left(-\frac{4}{3}\right) + \frac{1}{3} &\stackrel{?}{=} \frac{1}{4}\left(-\frac{4}{3}\right) \\ -\frac{2}{3} + \frac{1}{3} &\stackrel{?}{=} -\frac{1}{3} \\ -\frac{1}{3} &= -\frac{1}{3} \checkmark \end{aligned}$$

Skill Practice Solve the equation.

$$1. \frac{1}{2}x + \frac{21}{20} = \frac{1}{5}x$$

The same process of clearing fractions is used to solve rational equations when variables are present in the denominator.

Example 2 Solving a Rational Equation

Solve the equation. $\frac{3}{5} + \frac{1}{x} = \frac{2}{3}$

Solution:

$$\frac{3}{5} + \frac{1}{x} = \frac{2}{3}$$

The LCD of all terms in the equation is $15x$.

$$15x\left(\frac{3}{5} + \frac{1}{x}\right) = 15x\left(\frac{2}{3}\right)$$

Multiply by $15x$ to clear fractions.

$$15x \cdot \frac{3}{5} + 15x \cdot \frac{1}{x} = 15x \cdot \frac{2}{3}$$

Apply the distributive property.

$$9x + 15 = 10x$$

Solve the resulting equation.

$$15 = x$$

Check: $x = 15$ $\frac{3}{5} + \frac{1}{x} = \frac{2}{3}$

$$\frac{3}{5} + \frac{1}{(15)} \stackrel{?}{=} \frac{2}{3}$$

$$\frac{9}{15} + \frac{1}{15} \stackrel{?}{=} \frac{2}{3}$$

$$\frac{10}{15} = \frac{2}{3} \checkmark$$

Skill Practice

2. Solve. $\frac{3}{y} + \frac{4}{3} = -1$

Example 3 Solving a Rational Equation

Solve the equation. $3 - \frac{6w}{w+1} = \frac{6}{w+1}$

Skill Practice Answers

1. $x = -\frac{7}{2}$ 2. $y = -\frac{9}{7}$

Solution:

$$3 - \frac{6w}{w+1} = \frac{6}{w+1}$$

$$(w+1)(3) - (w+1)\left(\frac{6w}{w+1}\right) = (w+1)\left(\frac{6}{w+1}\right)$$

$$(w+1)(3) - \cancel{(w+1)}\left(\frac{6w}{\cancel{w+1}}\right) = \cancel{(w+1)}\left(\frac{6}{\cancel{w+1}}\right)$$

$$3w + 3 - 6w = 6$$

$$-3w = 3$$

$$w = -1$$

Check: $3 - \frac{6w}{w+1} = \frac{6}{w+1}$

$$3 - \frac{6(-1)}{(-1)+1} \stackrel{?}{=} \frac{6}{(-1)+1}$$

The denominator is 0
for the value of $w = -1$.

Because the value $w = -1$ makes the denominator zero in one (or more) of the rational expressions within the equation, the equation is *undefined* for $w = -1$.

No other potential solutions exist, so the equation $3 - \frac{6w}{w+1} = \frac{6}{w+1}$ has no solution.

Skill Practice Solve the equation.

$$3. \frac{4}{b+2} + \frac{2}{b-2} = \frac{8}{b^2-4}$$

Examples 1–3 show that the steps for solving a rational equation mirror the process of clearing fractions from Section 1.4. However, there is one significant difference. The solutions of a rational equation must be defined in each rational expression in the equation. When $w = -1$ is substituted into the expression $\frac{6w}{w+1}$ or $\frac{6}{w+1}$, the denominator is zero and the expression is undefined. Hence $w = -1$ cannot be a solution to the equation

$$3 - \frac{6w}{w+1} = \frac{6}{w+1}$$

The steps for solving a rational equation are summarized as follows.

The LCD of all terms in the equation is $w + 1$.

Multiply by $(w + 1)$ on both sides to clear fractions.

Apply the distributive property.

Solve the resulting equation.

Skill Practice Answers

3. No solution ($b = 2$ does not check.)

Steps for Solving a Rational Equation

1. Factor the denominators of all rational expressions. Identify any values of the variable for which any expression is undefined.
2. Identify the LCD of all terms in the equation.
3. Multiply both sides of the equation by the LCD.
4. Solve the resulting equation.
5. Check the potential solutions in the original equation. Note that any value from step 1 for which the equation is undefined cannot be a solution to the equation.

Example 4 Solving Rational Equations

Solve the equations.

a. $1 + \frac{3}{x} = \frac{28}{x^2}$ b. $\frac{36}{p^2 - 9} = \frac{2p}{p + 3} - 1$

Solution:

a. $1 + \frac{3}{x} = \frac{28}{x^2}$ The LCD of all terms in the equation is x^2 .
Expressions will be undefined for $x = 0$.

$x^2\left(1 + \frac{3}{x}\right) = x^2\left(\frac{28}{x^2}\right)$ Multiply both sides by x^2 to clear fractions.

$x^2 \cdot 1 + x^2 \cdot \frac{3}{x} = x^2 \cdot \frac{28}{x^2}$ Apply the distributive property.

$x^2 + 3x = 28$ The resulting equation is quadratic.

$x^2 + 3x - 28 = 0$ Set the equation equal to zero and factor.

$(x + 7)(x - 4) = 0$

$x = -7$ or $x = 4$

Check: $x = -7$

$$1 + \frac{3}{x} = \frac{28}{x^2}$$

$$1 + \frac{3}{-7} \stackrel{?}{=} \frac{28}{(-7)^2}$$

$$\frac{49}{49} - \frac{21}{49} \stackrel{?}{=} \frac{28}{49}$$

$$\frac{28}{49} = \frac{28}{49} \checkmark$$

Check: $x = 4$

$$1 + \frac{3}{x} = \frac{28}{x^2}$$

$$1 + \frac{3}{4} \stackrel{?}{=} \frac{28}{(4)^2}$$

$$\frac{16}{16} + \frac{12}{16} \stackrel{?}{=} \frac{28}{16}$$

$$\frac{28}{16} = \frac{28}{16} \checkmark$$

Both solutions check.

$$\text{b. } \frac{36}{p^2 - 9} = \frac{2p}{p + 3} - 1$$

$$\frac{36}{(p + 3)(p - 3)} = \frac{2p}{p + 3} - 1$$

The LCD is $(p + 3)(p - 3)$.

Expressions will be undefined for $p = 3$ and $p = -3$.

Multiply both sides by the LCD to clear fractions.

$$(p + 3)(p - 3) \left[\frac{36}{(p + 3)(p - 3)} \right] = (p + 3)(p - 3) \left(\frac{2p}{p + 3} \right) - (p + 3)(p - 3)(1)$$

$$\cancel{(p + 3)(p - 3)} \left[\frac{36}{\cancel{(p + 3)(p - 3)}} \right] = \cancel{(p + 3)(p - 3)} \left(\frac{2p}{\cancel{p + 3}} \right) - (p + 3)(p - 3)(1)$$

$$36 = 2p(p - 3) - (p + 3)(p - 3) \quad \text{Solve the resulting equation.}$$

$$36 = 2p^2 - 6p - (p^2 - 9) \quad \text{The equation is quadratic.}$$

$$36 = 2p^2 - 6p - p^2 + 9$$

$$36 = p^2 - 6p + 9$$

$$0 = p^2 - 6p - 27$$

Set the equation equal to zero and factor.

$$0 = (p - 9)(p + 3)$$

$$p = 9 \quad \text{or} \quad p = -3$$

$p = -3$ is not in the domain of the original expressions. Therefore, it cannot be a solution. This can be verified when we check.

Check: $p = 9$

$$\frac{36}{p^2 - 9} = \frac{2p}{p + 3} - 1$$

$$\frac{36}{(9)^2 - 9} \stackrel{?}{=} \frac{2(9)}{(9) + 3} - 1$$

$$\frac{36}{72} \stackrel{?}{=} \frac{18}{12} - 1$$

$$\frac{1}{2} \stackrel{?}{=} \frac{3}{2} - 1$$

$$\frac{1}{2} = \frac{1}{2} \checkmark$$

The solution is $p = 9$.

Check: $p = -3$

$$\frac{36}{p^2 - 9} = \frac{2p}{p + 3} - 1$$

$$\frac{36}{(-3)^2 - 9} \stackrel{?}{=} \frac{2(-3)}{(-3) + 3} - 1$$

$$\frac{36}{0} \stackrel{?}{=} \frac{-6}{0} - 1$$

Denominator is zero.

Skill Practice Solve the equations.

$$4. \frac{y}{4} - \frac{1}{2} = \frac{2}{y}$$

$$5. \frac{6}{x + 2} - \frac{20x}{x^2 - x - 6} = \frac{x}{x + 2}$$

Skill Practice Answers

4. $y = 4; y = -2$

5. $x = -9$ ($x = -2$ does not check.)

2. Formulas Involving Rational Equations

Example 5 Solving Literal Equations Involving Rational Expressions

Solve for the indicated variable.

a. $P = \frac{A}{1 + rt}$ for r b. $V = \frac{mv}{m + M}$ for m

Solution:

a. $P = \frac{A}{1 + rt}$ for r

$$(1 + rt)P = (1 + rt)\left(\frac{A}{1 + rt}\right) \quad \text{Multiply both sides by the LCD} = (1 + rt).$$

$$(1 + rt)P = \cancel{(1 + rt)}\left(\frac{A}{\cancel{1 + rt}}\right) \quad \text{Clear fractions.}$$

$$P + Prt = A \quad \text{Apply the distributive property to clear parentheses.}$$

$$Prt = A - P \quad \text{Isolate the } r\text{-term on one side.}$$

$$\frac{Prt}{Pt} = \frac{A - P}{Pt} \quad \text{Divide both sides by } Pt.$$

$$r = \frac{A - P}{Pt}$$

b. $V = \frac{mv}{m + M}$ for m

$$V(m + M) = \left(\frac{mv}{m + M}\right)\cancel{(m + M)} \quad \text{Multiply by the LCD and clear fractions.}$$

$$V(m + M) = mv$$

$$Vm + VM = mv$$

$$Vm - mv = -VM$$

$$m(V - v) = -VM$$

$$\frac{m\cancel{(V - v)}}{\cancel{(V - v)}} = \frac{-VM}{(V - v)} \quad \text{Use the distributive property to clear parentheses.}$$

$$m = \frac{-VM}{V - v} \quad \text{Collect all } m \text{ terms on one side.}$$

$$m = \frac{-VM}{V - v} \quad \text{Factor out } m.$$

$$m = \frac{-VM}{V - v} \quad \text{Divide by } (V - v).$$

Avoiding Mistakes:

Variables in algebra are case-sensitive. Therefore, V and v are different variables, and M and m are different variables.

TIP: The factor of -1 that appears in the numerator may be written in the denominator or out in front of the expression. The following expressions are equivalent:

$$m = \frac{-VM}{V - v}$$

$$\text{or } m = -\frac{VM}{V - v}$$

$$\text{or } m = \frac{VM}{-(V - v)} = \frac{VM}{v - V}$$

Skill Practice

6. Solve the equation for q .

$$\frac{1}{q} + \frac{1}{r} = \frac{1}{s}$$

7. Solve the equation for x .

$$y = \frac{ax + b}{x + d}$$

Skill Practice Answers

6. $q = \frac{rs}{r-s}$ or $q = \frac{-rs}{s-r}$

7. $x = \frac{b-yd}{y-a}$ or $x = \frac{yd-b}{a-y}$

Section 6.5

Practice Exercises

Boost your GRADE at
mathzone.com!

- Practice Problems
- Self-Tests
- NetTutor
- e-Professors
- Videos

Study Skills Exercise

1. Define the key term **rational equation**.

Review Exercises

For Exercises 2–8, perform the indicated operations.

2. $\frac{1}{x^2 - 16} + \frac{1}{x^2 + 8x + 16}$

3. $\frac{3}{y^2 - 1} - \frac{2}{y^2 - 2y + 1}$

4. $\frac{m^2 - 9}{m^2 - 3m} \div (m^2 - m - 12)$

5. $\frac{2t^2 + 7t + 3}{4t^2 - 1} \div (t + 3)$

6. $\frac{1 + x^{-1}}{1 - x^{-2}}$

7. $\frac{x + y}{x^{-1} + y^{-1}}$

8. $\frac{5x + 1}{x - 5} \cdot \frac{2x - 10}{x + 1}$

Concept 1: Solving Rational Equations

For Exercises 9–40, solve the rational equations.

9. $\frac{x + 2}{3} - \frac{x - 4}{4} = \frac{1}{2}$

10. $\frac{x + 6}{3} - \frac{x + 8}{5} = 0$

11. $\frac{3y}{4} - 6 = \frac{y}{4}$

12. $\frac{2w}{5} - 8 = \frac{4w}{5}$

13. $\frac{5}{4p} - \frac{7}{6} + 3 = 0$

14. $\frac{7}{15w} - \frac{3}{10} - 2 = 0$

15. $\frac{1}{2} - \frac{3}{2x} = \frac{4}{x} - \frac{5}{12}$

16. $\frac{2}{3x} + \frac{1}{4} = \frac{11}{6x} - \frac{1}{3}$

17. $\frac{3}{x - 4} + 2 = \frac{5}{x - 4}$

18. $\frac{5}{x + 3} - 2 = \frac{7}{x + 3}$

19. $\frac{1}{3} + \frac{2}{w - 3} = 1$

20. $\frac{3}{5} + \frac{7}{p + 2} = 2$

21. $\frac{12}{x} - \frac{12}{x - 5} = \frac{2}{x}$

22. $\frac{25}{y} - \frac{25}{y - 2} = \frac{2}{y}$

23. $\frac{1}{4}a - 4a^{-1} = 0$

24. $\frac{1}{3}t - 12t^{-1} = 0$

25. $3a^{-2} - 4a^{-1} = -1$

26. $-3w^{-1} = 2 + w^{-1}$

27. $8t^{-1} + 2 = 3t^{-1}$

28. $6z^{-2} - 5z^{-1} = 0$

29.
$$\frac{4}{t-2} - \frac{8}{t^2-2t} = -2$$

31.
$$\frac{6}{5y+10} - \frac{1}{y-5} = \frac{4}{y^2-3y-10}$$

33.
$$\frac{x}{x-5} + \frac{1}{5} = \frac{5}{x-5}$$

35.
$$\frac{6}{x^2-4x+3} - \frac{1}{x-3} = \frac{1}{4x-4}$$

37.
$$\frac{3}{k-2} - \frac{5}{k+2} + \frac{6k}{4-k^2} = 0$$

39.
$$\frac{5}{x^2-7x+12} = \frac{2}{x-3} + \frac{5}{x-4}$$

30.
$$\frac{x}{x+6} = \frac{72}{x^2-36} + 4$$

32.
$$\frac{-3}{x^2-7x+12} - \frac{2}{x^2+x-12} = \frac{10}{x^2-16}$$

34.
$$\frac{x}{x-2} + \frac{2}{3} = \frac{2}{x-2}$$

36.
$$\frac{1}{4x^2-36} - \frac{5}{x+3} + \frac{2}{x-3} = 0$$

38.
$$\frac{h}{2} - \frac{h}{h-4} = -\frac{4}{h-4}$$

40.
$$\frac{9}{x^2+7x+10} = \frac{5}{x+2} - \frac{3}{x+5}$$

Concept 2: Formulas Involving Rational Equations

For Exercises 41–58, solve the formula for the indicated variable.

41. $K = \frac{ma}{F}$ for m

42. $K = \frac{ma}{F}$ for a

43. $K = \frac{IR}{E}$ for E

44. $K = \frac{IR}{E}$ for R

45. $I = \frac{E}{R+r}$ for R

46. $I = \frac{E}{R+r}$ for r

47. $h = \frac{2A}{B+b}$ for B

48. $\frac{V}{\pi h} = r^2$ for h

49. $x = \frac{at+b}{t}$ for t

50. $\frac{T+mf}{m} = g$ for m

51. $\frac{x-y}{xy} = z$ for x

52. $\frac{w-n}{wn} = P$ for w

53. $a+b = \frac{2A}{h}$ for h

54. $1+rt = \frac{A}{P}$ for P

55. $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$ for R

56. $\frac{b+a}{ab} = \frac{1}{f}$ for b

57. $v = \frac{s_2-s_1}{t_2-t_1}$ for t_2

58. $a = \frac{v_2-v_1}{t_2-t_1}$ for v_1

Mixed Review

59. a. Simplify.
$$\frac{3}{w-5} + \frac{10}{w^2-25} - \frac{1}{w+5}$$

b. Solve.
$$\frac{3}{w-5} + \frac{10}{w^2-25} - \frac{1}{w+5} = 0$$

c. What is the difference in the type of problem given in parts (a) and (b)?

60. a. Simplify.
$$\frac{x}{2x+4} + \frac{2}{3x+6} - 1$$

b. Solve.
$$\frac{x}{2x+4} + \frac{2}{3x+6} = 1$$

c. What is the difference in the type of problem given in parts (a) and (b)?

For Exercises 61–70, perform the indicated operation and simplify, or solve the equation for the variable.

$$61. \frac{2}{a^2 + 4a + 3} + \frac{1}{a + 3}$$

$$62. \frac{1}{c + 6} + \frac{4}{c^2 + 8c + 12}$$

$$63. \frac{7}{y^2 - y - 2} + \frac{1}{y + 1} - \frac{3}{y - 2} = 0$$

$$64. \frac{3}{b + 2} - \frac{1}{b - 1} - \frac{5}{b^2 + b - 2} = 0$$

$$65. \frac{x}{x - 1} - \frac{12}{x^2 - x}$$

$$66. \frac{3}{5t - 20} + \frac{4}{t - 4}$$

$$67. \frac{3}{w} - 5 = \frac{7}{w} - 1$$

$$68. \frac{-3}{y^2} - \frac{1}{y} = -2$$

$$69. \frac{4p + 1}{8p - 12} + \frac{p - 3}{2p - 3}$$

$$70. \frac{x + 1}{2x + 4} - \frac{x^2}{x + 2}$$

Expanding Your Skills

71. If 5 is added to the reciprocal of a number, the result is $\frac{16}{3}$. Find the number.
72. If $\frac{2}{3}$ is added to the reciprocal of a number, the result is $\frac{17}{3}$. Find the number.
73. If 7 is decreased by the reciprocal of a number, the result is $\frac{9}{2}$. Find the number.
74. If a number is added to its reciprocal, the result is $\frac{13}{6}$. Find the number.

Section 6.6

Applications of Rational Equations and Proportions

Concepts

1. Solving Proportions
2. Applications of Proportions
3. Similar Triangles
4. Applications of Rational Equations

1. Solving Proportions

A proportion is a rational equation that equates two ratios.

Definition of Ratio and Proportion

1. The **ratio** of a to b is $\frac{a}{b}$ ($b \neq 0$) and can also be expressed as $a:b$ or $a \div b$.
2. An equation that equates two ratios or rates is called a **proportion**.
Therefore, if $b \neq 0$ and $d \neq 0$, then $\frac{a}{b} = \frac{c}{d}$ is a proportion.

The process for solving rational equations can be used to solve proportions.

Example 1 Solving a Proportion

Solve the proportion. $\frac{5}{19} = \frac{95}{y}$

Solution:

$$\frac{5}{19} = \frac{95}{y} \quad \text{The LCD is } 19y. \text{ Note that } y \neq 0.$$

$$19y \left(\frac{5}{19} \right) = 19y \left(\frac{95}{y} \right) \quad \text{Multiply both sides by the LCD.}$$

$$19y \left(\frac{5}{19} \right) = 19y \left(\frac{95}{y} \right) \quad \text{Clear fractions.}$$

$$5y = 1805 \quad \text{Solve the resulting equation.}$$

$$\frac{5y}{5} = \frac{1805}{5}$$

$$y = 361 \quad \text{The solution checks in the original equation.}$$

Skill Practice Solve the proportion.

1. $\frac{8}{5} = \frac{12}{x}$

TIP: For any proportion

$$\frac{a}{b} = \frac{c}{d} \quad b \neq 0, d \neq 0$$

the cross products of terms are equal. Hence, $ad = bc$. Finding the cross product is a quick way to clear fractions in a proportion.* Consider Example 1:

$$\frac{5}{19} = \frac{95}{y}$$

$$5y = (19)(95) \quad \text{Equate the cross products.}$$

$$5y = 1805$$

$$y = 361$$

*It is important to realize that this method is only valid for proportions.

2. Applications of Proportions

Example 2 Solving a Proportion

In the U.S. Senate in 2006 the ratio of Democrats to Republicans was 4 to 5. If there were 44 Democrats, how many Republicans were there?

Skill Practice Answers

1. $x = \frac{15}{2}$ or 7.5

Solution:

One method of solving this problem is to set up a proportion. Write two equivalent ratios depicting the number of Democrats to the number of Republicans. Let x represent the unknown number of Republicans in the Senate.

$$\begin{array}{c} \boxed{\text{Given}} \\ \text{ratio} \end{array} \begin{array}{c} \longrightarrow \\ \longrightarrow \end{array} \frac{4}{5} = \frac{44}{x} \begin{array}{c} \longleftarrow \\ \longleftarrow \end{array} \begin{array}{c} \boxed{\text{Number of Democrats}} \\ \boxed{\text{Number of Republicans}} \end{array}$$

$$5x \left(\frac{4}{5} \right) = 5x \left(\frac{44}{x} \right) \quad \text{Multiply both sides by the LCD } 5x.$$

$$4x = 5(44) \quad \text{Clear fractions.}$$

$$4x = 220$$

$$x = 55$$

There were 55 Republicans in the U.S. Senate in 2006.

Skill Practice

2. The ratio of cats to dogs at an animal rescue facility is 8 to 5. How many dogs are in the facility if there are 400 cats?

Example 3 Solving a Proportion

The ratio of male to female police officers in a certain town is 11:3. If the total number of officers is 112, how many are men and how many are women?

Solution:

Let x represent the number of male police officers.

Then $112 - x$ represents the number of female police officers.

$$\begin{array}{c} \boxed{\text{Male}} \\ \boxed{\text{Female}} \end{array} \begin{array}{c} \longrightarrow \\ \longrightarrow \end{array} \frac{11}{3} = \frac{x}{112 - x} \begin{array}{c} \longleftarrow \\ \longleftarrow \end{array} \begin{array}{c} \boxed{\text{Number of males}} \\ \boxed{\text{Number of females}} \end{array}$$

$$3(112 - x) \left(\frac{11}{3} \right) = 3(112 - x) \left(\frac{x}{112 - x} \right) \quad \text{Multiply both sides by } 3(112 - x).$$

$$11(112 - x) = 3x$$

The resulting equation is linear.

$$1232 - 11x = 3x$$

$$1232 = 14x$$

$$\frac{1232}{14} = \frac{14x}{14}$$

$$x = 88$$

The number of male police officers is $x = 88$.

The number of female officers is $112 - x = 112 - 88 = 24$.

Skill Practice

3. Professor Wolfe has a ratio of passing students to failing students of 5 to 4. One semester he had a total of 207 students. How many students passed and how many failed?

Skill Practice Answers

2. 250 dogs
3. 115 passed and 92 failed.

3. Similar Triangles

Proportions are used in geometry with **similar triangles**. Two triangles are said to be similar if their corresponding angles are equal. In such a case, the lengths of the corresponding sides are proportional. The triangles in Figure 6-5 are similar. Therefore, the following ratios are equivalent.

$$\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$$

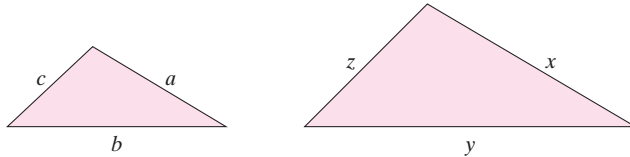


Figure 6-5

Example 4 Using Similar Triangles in an Application

The shadow cast by a yardstick is 2 ft long. The shadow cast by a tree is 11 ft long. Find the height of the tree.

Solution:

Let x represent the height of the tree.

We will assume that the measurements were taken at the same time of day. Therefore, the angle of the sun is the same on both objects, and we can set up similar triangles (Figure 6-6).

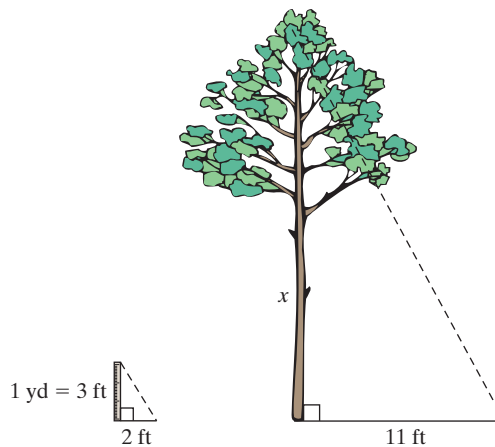


Figure 6-6

Height of yardstick	→	$\frac{3 \text{ ft}}{2 \text{ ft}} = \frac{x}{11 \text{ ft}}$	←	Height of tree
Length of yardstick's shadow	→		←	Length of tree's shadow

$$\frac{3}{2} = \frac{x}{11}$$

Write an equation.

$$22 \cdot \left(\frac{3}{2}\right) = 22 \cdot \left(\frac{x}{11}\right)$$

Multiply by the LCD.

$$33 = 2x$$

Solve the equation.

$$\frac{33}{2} = \frac{2x}{2}$$

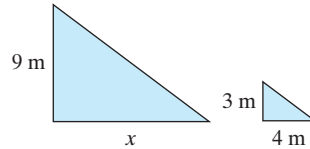
$$16.5 = x$$

Interpret the results.

The tree is 16.5 ft high.

Skill Practice

4. Solve for x , given that the triangles are similar.



4. Applications of Rational Equations

Example 5 Solving a Business Application of Rational Expressions

To produce mountain bikes, the manufacturer has a fixed cost of \$56,000 plus a variable cost of \$140 per bike. The cost in dollars to produce x bikes is given by

$C(x) = 56,000 + 140x$. The average cost per bike

$\bar{C}(x)$ is defined as $\bar{C}(x) = \frac{C(x)}{x}$ or

$$\bar{C}(x) = \frac{56,000 + 140x}{x}$$



Find the number of bikes the manufacturer must produce so that the average cost per bike is \$180.

Solution:

Given

$$\bar{C}(x) = \frac{56,000 + 140x}{x}$$

$$180 = \frac{56,000 + 140x}{x}$$

Replace $\bar{C}(x)$ by 180 and solve for x .

$$x(180) = x \cdot \frac{56,000 + 140x}{x}$$

Multiply by the LCD x .

$$180x = 56,000 + 140x$$

The resulting equation is linear.

$$40x = 56,000$$

Solve for x .

$$x = \frac{56,000}{40}$$

$$= 1400$$

The manufacturer must produce 1400 bikes for the average cost to be \$180.

Skill Practice

5. To manufacture ski equipment, a company has a fixed cost of \$48,000 plus a variable cost of \$200 per pair of skis. The cost to produce x pairs of skis is $C(x) = 48,000 + 200x$. The average cost per pair of skis is

$$\bar{C}(x) = \frac{48,000 + 200x}{x}$$

Find the number of sets of skis the manufacturer must produce so that the average cost per set is \$250.

Skill Practice Answers

4. $x = 12$ m
5. 960 pairs of skis

Example 6 Solving an Application Involving Distance, Rate, and Time

An athlete's average speed on her bike is 14 mph faster than her average speed running. She can bike 31.5 mi in the same time that it takes her to run 10.5 mi. Find her speed running and her speed biking.

Solution:

Because the speed biking is given in terms of the speed running, let x represent the running speed.

Let x represent the speed running.

Then $x + 14$ represents the speed biking.

Organize the given information in a chart.

	Distance	Rate	Time
<i>Running</i>	10.5	x	$\frac{10.5}{x}$
<i>Biking</i>	31.5	$x + 14$	$\frac{31.5}{x + 14}$

\uparrow
 Because $d = rt$, then $t = \frac{d}{r}$

The time required to run 10.5 mi is the same as the time required to bike 31.5 mi, so we can equate the two expressions for time:

$$\frac{10.5}{x} = \frac{31.5}{x + 14}$$

The LCD is $x(x + 14)$.

$$x(x + 14)\left(\frac{10.5}{x}\right) = x(x + 14)\left(\frac{31.5}{x + 14}\right)$$

Multiply by $x(x + 14)$ to clear fractions.

$$10.5(x + 14) = 31.5x$$

The resulting equation is linear.

$$10.5x + 147 = 31.5x$$

Solve for x .

$$-21x = -147$$

$$x = 7$$

The athlete's speed running is 7 mph.

The speed biking is $x + 14$ or $7 + 14 = 21$ mph.

Skill Practice

6. Devon can cross-country ski 5 km/hr faster than his sister, Shanelle. Devon skis 45 km in the same amount of time that Shanelle skis 30 km. Find their speeds.

Skill Practice Answers

6. Shanelle skis 10 km/hr and Devon skis 15 km/hr.

Example 7 Solving an Application Involving “Work”

JoAn can wallpaper a bathroom in 3 hr. Bonnie can wallpaper the same bathroom in 5 hr. How long would it take them if they worked together?

Solution:

Let x represent the time required for both people working together to complete the job.

One method to approach this problem is to determine the portion of the job that each person can complete in 1 hr and extend that rate to the portion of the job completed in x hr.

- JoAn can perform the job in 3 hr. Therefore, she completes $\frac{1}{3}$ of the job in 1 hr and $\frac{1}{3}x$ jobs in x hr.
- Bonnie can perform the job in 5 hr. Therefore, she completes $\frac{1}{5}$ of the job in 1 hr and $\frac{1}{5}x$ jobs in x hr.

	Work Rate	Time	Portion of Job Completed
<i>JoAn</i>	$\frac{1}{3}$ job/hr	x hr	$\frac{1}{3}x$ jobs
<i>Bonnie</i>	$\frac{1}{5}$ job/hr	x hr	$\frac{1}{5}x$ jobs

The sum of the portions of the job completed by each person must equal one whole job:

$$\left(\begin{array}{c} \text{Portion of job} \\ \text{completed} \\ \text{by JoAn} \end{array} \right) + \left(\begin{array}{c} \text{portion of job} \\ \text{completed} \\ \text{by Bonnie} \end{array} \right) = \left(\begin{array}{c} 1 \\ \text{whole} \\ \text{job} \end{array} \right)$$

$$\frac{1}{3}x + \frac{1}{5}x = 1 \quad \text{The LCD is 15.}$$

$$15 \left(\frac{1}{3}x + \frac{1}{5}x \right) = 15(1) \quad \text{Multiply by the LCD.}$$

$$15 \cdot \frac{1}{3}x + 15 \cdot \frac{1}{5}x = 15 \cdot 1 \quad \text{Apply the distributive property.}$$

$$5x + 3x = 15 \quad \text{Solve the resulting linear equation.}$$

$$8x = 15$$

$$x = \frac{15}{8} \quad \text{or} \quad x = 1\frac{7}{8}$$

Together JoAn and Bonnie can wallpaper the bathroom in $1\frac{7}{8}$ hr.

Skill Practice

7. Antonio can install a new roof in 4 days. Carlos can install the same size roof in 6 days. How long will it take them to install a roof if they work together?

Skill Practice Answers

7. $\frac{12}{5}$ or $2\frac{2}{5}$ days

Section 6.6

Practice Exercises

Boost your GRADE at
mathzone.com!



- Practice Problems
- Self-Tests
- NetTutor
- e-Professors
- Videos

Study Skills Exercise

1. Define the key terms.

a. Ratio

b. Proportion

c. Similar triangles

Review Exercises

For Exercises 2–10, perform the indicated operation and simplify, or solve the equation for the variable.

$$2. 3 - \frac{6}{x} = x + 8$$

$$3. 2 + \frac{6}{x} = x + 7$$

$$4. \frac{5}{3x-6} - \frac{3}{4x-8}$$

$$5. \frac{4}{5t-1} + \frac{1}{10t-2}$$

$$6. \frac{2}{y-1} - \frac{5}{4} = \frac{-1}{y+1}$$

$$7. \frac{5}{w-2} = 7 - \frac{10}{w+2}$$

$$8. \frac{5}{x^2-y^2} + \frac{3x}{x^3+x^2y}$$

$$9. \frac{7a}{a^2+2ab+b^2} + \frac{4}{a^2+ab}$$

$$10. \frac{3t}{t+6} = t + \frac{2t}{t+6}$$

Concept 1: Solving Proportions

For Exercises 11–24, solve the proportions.

$$11. \frac{y}{6} = \frac{20}{15}$$

$$12. \frac{12}{18} = \frac{14}{x}$$

$$13. \frac{9}{75} = \frac{m}{50}$$

$$14. \frac{n}{15} = \frac{12}{45}$$

$$15. \frac{p-1}{4} = \frac{p+3}{3}$$

$$16. \frac{q-5}{2} = \frac{q+2}{3}$$

$$17. \frac{x+1}{5} = \frac{4}{15}$$

$$18. \frac{t-1}{7} = \frac{2}{21}$$

$$19. \frac{5-2x}{x} = \frac{1}{4}$$

$$20. \frac{2y+3}{y} = \frac{3}{2}$$

$$21. \frac{2}{y-1} = \frac{y-3}{4}$$

$$22. \frac{1}{x-5} = \frac{x-3}{3}$$

$$23. \frac{1}{49w} = \frac{w}{9}$$

$$24. \frac{1}{4z} = \frac{z}{25}$$

Concept 2: Applications of Proportions

25. A preschool advertises that it has a 3-to-1 ratio of children to adults. If 18 children are enrolled, how many adults must be on the staff?
26. An after-school care facility tries to maintain a 4-to-1 ratio of children to adults. If the facility hired five adults, what is the maximum number of children that can enroll?
27. A 3.5-oz box of candy has a total of 21.0 g of fat. How many grams of fat would a 14-oz box of candy contain?
28. A 6-oz box of candy has 350 calories. How many calories would a 10-oz box contain?



29. A fisherman in the North Atlantic catches eight swordfish for a total of 1840 lb. Approximately how many swordfish were caught if a commercial fishing boat arrives in port with 230,000 lb of swordfish?
30. If a 64-oz bottle of laundry detergent costs \$4.00, how much would an 80-oz bottle cost?
31. Pam drives her Toyota Prius 243 mi in city driving on 4.5 gal of gas. At this rate how many gallons of gas are required to drive 621 mi?
32. On a map, the distance from Sacramento, California, to San Francisco, California, is 8 cm. The legend gives the actual distance as 96 mi. On the same map, Fatima measured 7 cm from Sacramento to Modesto, California. What is the actual distance?
33. Yellowstone National Park in Wyoming has the largest population of free-roaming bison. To approximate the number of bison, 200 are captured and tagged and then left free to roam. Later, a sample of 120 bison is observed and 6 have tags. Approximate the population of bison in the park.
34. Laws have been instituted in Florida to help save the manatee. To establish the number of manatees in Florida, 150 manatees were tagged. A new sample was taken later, and among the 40 manatees in the sample, 3 were marked. Approximate the number of manatees in Florida.
35. The ratio of men to women accountants in a large accounting firm is 2 to 1. If the total number of accountants is 81, how many are men and how many are women?
36. The ratio of Hank's income spent on rent to his income spent on car payments is 3 to 1. If he spends a total of \$1640 per month on the rent and car payment, how much does he spend on each item?
37. The ratio of single men in their 20s to single women in their 20s is 119 to 100 (Source: U.S. Census). In a random group of 1095 single college students in their 20s, how many are men and how many are women?
38. A chemist mixes water and alcohol in a 7 to 8 ratio. If she makes a 450-L solution, how much is water and how much is alcohol?

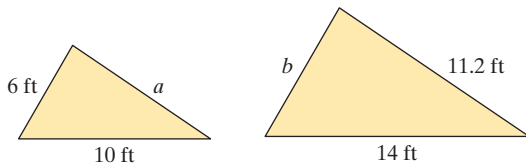


Concept 3: Similar Triangles

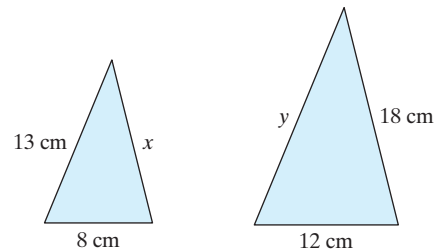
For Exercises 39–42, the triangles shown here are similar. Find the lengths of the missing sides.



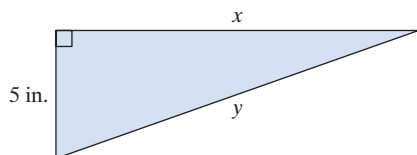
39.



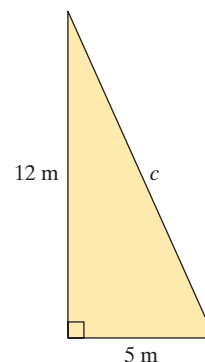
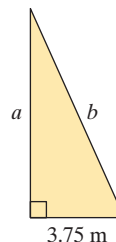
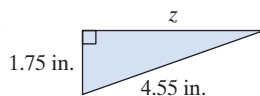
40.



41.



42.

**Concept 4: Applications of Rational Equations**

43. The profit in dollars for selling x mountain bikes is given by $P(x) = 360x - 56,000$. The average profit per bike is defined as

$$\bar{P}(x) = \frac{P(x)}{x} \quad \text{or} \quad \bar{P}(x) = \frac{360x - 56,000}{x}$$

Find the number of bikes the manufacturer must produce so that the average profit per bike is \$110.

44. The profit in dollars when x ballpoint pens are sold is given by $P(x) = 0.47x - 100$. The average profit per unit is defined as

$$\bar{P}(x) = \frac{P(x)}{x} \quad \text{or} \quad \bar{P}(x) = \frac{0.47x - 100}{x}$$

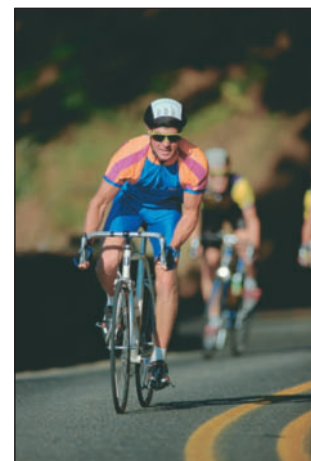
Find the number of pens that must be produced so that the average profit per pen is \$0.27.

45. A motorist travels 80 mi while driving in a bad rainstorm. In sunny weather, the motorist drives 20 mph faster and covers 120 mi in the same amount of time. Find the speed of the motorist in the rainstorm and the speed in sunny weather.
46. Brooke walks 2 km/hr slower than her older sister Adrianna. If Brooke can walk 12 km in the same amount of time that Adrianna can walk 18 km, find their speeds.
47. The current in a stream is 2 mph. Find the speed of a boat in still water if it goes 26 mi downstream (with the current) in the same amount of time it takes to go 18 mi upstream (against the current).
48. A bus leaves a terminal at 9:00. A car leaves 3 hr later and averages a speed 21 mph faster than that of the bus. If the car overtakes the bus after 196 mi, find the average speed of the bus and the average speed of the car.
49. A bicyclist rides 24 mi against a wind and returns 24 mi with the wind. His average speed for the return trip is 8 mph faster. If x represents the bicyclist's speed going out against the wind, then the total time required for the round trip is given by

$$t(x) = \frac{24}{x} + \frac{24}{x + 8}, \quad \text{where } x > 0$$

How fast did the cyclist ride against the wind if the total time of the trip was 3.2 hr?

50. A boat travels 45 mi to an island and 45 mi back again. Changes in the wind and tide made the average speed on the return trip 3 mph slower than the speed on the way out. If the total time of the trip took 6 hr 45 min (6.75 hr), find the speed going to the island and the speed of the return trip.



51. One painter can paint a room in 6 hr. Another painter can paint the same room in 8 hr. How long would it take them working together?
52. One carpenter can complete a kitchen in 8 days. With the help of another carpenter, they can do the job together in 4 days. How long would it take the second carpenter if he worked alone?
53. Gus works twice as fast as Sid. Together they can dig a garden in 4 hr. How long would it take each person working alone?
54. It takes a child 3 times longer to vacuum a house than an adult. If it takes 1 hr for one adult and one child working together to vacuum a house, how long would it take each person working alone?

Expanding Your Skills

55. Find the value of y so that the slope of the line between the points $(3, 1)$ and $(11, y)$ is $\frac{1}{2}$.
56. Find the value of x so that the slope of the line between the points $(-2, -5)$ and $(x, 10)$ is 3.
57. Find the value of x so that the slope of the line between the points $(4, -2)$ and $(x, 2)$ is 4.
58. Find the value of y so that the slope of the line between the points $(3, 2)$ and $(-1, y)$ is $-\frac{3}{4}$.

Chapter 6

SUMMARY

Section 6.1

Rational Expressions and Rational Functions

Key Concepts

A **rational expression** is in the form $\frac{p}{q}$ where p and q are polynomials and $q \neq 0$.

A **rational function** is a function of the form $\frac{p(x)}{q(x)}$, where p and q are polynomial functions and $q(x) \neq 0$.

The **domain** of a rational expression or a rational function excludes the values for which the denominator is zero.

To simplify a rational expression to lowest terms, factor the numerator and denominator completely. Then simplify factors whose ratio is 1 or -1 . A rational expression written in lowest terms will still have the same restrictions on the domain as the original expression.

Examples

Example 1

Find the domain of the function

$$f(x) = \frac{x - 3}{(x + 4)(2x - 1)}$$

$$\text{Domain: } \left\{ x \mid x \text{ is a real number and } x \neq -4, x \neq \frac{1}{2} \right\}$$

Example 2

Simplify to lowest terms.

$$\frac{t^2 - 6t - 16}{5t + 10}$$

$$\frac{(t - 8)(t + 2)}{5(t + 2)} = \frac{t - 8}{5} \quad \text{provided } t \neq -2$$

Section 6.2

Multiplication and Division
of Rational Expressions

Key Concepts

To multiply rational expressions, factor the numerators and denominators completely. Then simplify factors whose ratio is 1 or -1 .

To divide rational expressions, multiply by the reciprocal of the divisor.

Examples

Example 1

$$\begin{aligned} & \frac{b^2 - a^2}{a^2 - 2ab + b^2} \cdot \frac{a^2 - 3ab + 2b^2}{2a + 2b} \\ &= \frac{\overset{-1}{(b-a)}\overset{1}{(b+a)}}{\overset{-1}{(a-b)}\overset{1}{b}} \cdot \frac{\overset{1}{(a-2b)}\overset{1}{(a-b)}}{2\overset{1}{(a+b)}} && \text{Factor.} \\ &= -\frac{a-2b}{2} \quad \text{or} \quad \frac{2b-a}{2} && \text{Simplify.} \end{aligned}$$

Example 2

$$\begin{aligned} & \frac{9x+3}{x^2-4} \div \frac{3x+1}{4x+8} \\ &= \frac{3\overset{1}{(3x+1)}}{(x-2)\overset{1}{(x+2)}} \cdot \frac{4\overset{1}{(x+2)}}{\overset{1}{3x+1}} && \text{Factor. Multiply by} \\ & && \text{the reciprocal.} \\ &= \frac{12}{x-2} && \text{Simplify.} \end{aligned}$$

Section 6.3

Addition and Subtraction
of Rational Expressions

Key Concepts

To add or subtract rational expressions, the expressions must have the same denominator.

The **least common denominator (LCD)** is the product of unique factors from the denominators, in which each factor is raised to its highest power.

Examples

Example 1

$$\begin{aligned} & \text{For } \frac{1}{3(x-1)^3(x+2)} \quad \text{and} \quad \frac{-5}{6(x-1)(x+7)^2} \\ & \text{LCD} = 6(x-1)^3(x+2)(x+7)^2 \end{aligned}$$

Steps to Add or Subtract Rational Expressions

1. Factor the denominator of each rational expression.
2. Identify the LCD.
3. Rewrite each rational expression as an equivalent expression with the LCD as its denominator. (This is accomplished by multiplying the numerator and denominator of each rational expression by the missing factor(s) from the LCD.)
4. Add or subtract the numerators, and write the result over the common denominator.
5. Simplify if possible.

Example 2

$$\frac{c}{c^2 - c - 12} - \frac{1}{2c - 8}$$

$$= \frac{c}{(c - 4)(c + 3)} - \frac{1}{2(c - 4)}$$

Factor the denominators.

The LCD is $2(c - 4)(c + 3)$

$$\frac{2}{2} \cdot \frac{c}{(c - 4)(c + 3)} - \frac{1}{2(c - 4)} \cdot \frac{(c + 3)}{(c + 3)}$$

Write equivalent fractions with LCD.

$$= \frac{2c - (c + 3)}{2(c - 4)(c + 3)}$$

Subtract.

$$= \frac{2c - c - 3}{2(c - 4)(c + 3)}$$

Simplify.

$$= \frac{c - 3}{2(c - 4)(c + 3)}$$

Section 6.4**Complex Fractions****Key Concepts**

Complex fractions can be simplified by using Method I or Method II.

Method I uses the order of operations to simplify the numerator and denominator separately before multiplying by the reciprocal of the denominator of the complex fraction.

Examples**Example 1**

Simplify by using Method I.

$$\frac{\frac{5}{x}}{\frac{10}{x^2}}$$

$$= \frac{5}{x} \cdot \frac{x^2}{10}$$

$$= \frac{x}{2}$$

To use Method II, multiply the numerator and denominator of the complex fraction by the LCD of all the individual fractions. Then simplify the result.

Example 2

Simplify by using Method II.

$$\frac{1 - \frac{4}{w^2}}{1 - \frac{1}{w} - \frac{6}{w^2}} \quad \text{The LCD is } w^2.$$

$$= \frac{w^2 \left(1 - \frac{4}{w^2}\right)}{w^2 \left(1 - \frac{1}{w} - \frac{6}{w^2}\right)}$$

$$= \frac{w^2 - 4}{w^2 - w - 6}$$

$$= \frac{(w - 2)(\cancel{w + 2})}{(w - 3)(\cancel{w + 2})}$$

$$= \frac{w - 2}{w - 3}$$

Section 6.5 Rational Equations

Key Concepts**Steps to Solve a Rational Equation**

1. Factor the denominators of all rational expressions. Identify any restrictions on the variable.
2. Identify the LCD of all expressions in the equation.
3. Multiply both sides of the equation by the LCD.
4. Solve the resulting equation.
5. Check each potential solution.

Examples**Example 1**

$$\frac{1}{w} - \frac{1}{2w - 1} = \frac{-2w}{2w - 1}$$

The LCD is $w(2w - 1)$.

$$\frac{1}{\cancel{w}(2w - 1)} \cdot \frac{1}{w} - \frac{1}{w(\cancel{2w - 1})} \cdot \frac{1}{\cancel{2w - 1}}$$

$$= \frac{-2w}{w(\cancel{2w - 1})} \cdot \frac{-2w}{\cancel{2w - 1}}$$

$$(2w - 1)1 - w(1) = w(-2w)$$

$$2w - 1 - w = -2w^2 \quad (\text{quadratic equation})$$

$$2w^2 + w - 1 = 0$$

$$(2w - 1)(w + 1) = 0$$

$$w = \frac{1}{2} \quad \text{or} \quad w = -1$$

$$w = -1 \left(w = \frac{1}{2} \text{ does not check} \right)$$

Section 6.6

Applications of Rational Equations and Proportions

Key Concepts

An equation that equates two **ratios** is called a **proportion**:

$$\frac{a}{b} = \frac{c}{d} \text{ provided } b \neq 0, d \neq 0$$

Proportions can be used to solve many types of applications including similar triangles, work problems, and distance, rate, and time problems.

Examples

Example 1

A sample of 85 g of a particular ice cream contains 17 g of fat. How much fat does 324 g of the same ice cream contain?

$$\frac{17 \text{ g fat}}{85 \text{ g ice cream}} = \frac{x \text{ g fat}}{324 \text{ g ice cream}}$$

$$\frac{17}{85} = \frac{x}{324} \quad \text{Multiply by the LCD.}$$

$$\left(\frac{1}{85} \cdot 324\right) \cdot \frac{17}{85} = \left(85 \cdot \frac{1}{324}\right) \cdot \frac{x}{324}$$

$$5508 = 85x$$

$$x = 64.8 \text{ g}$$

Chapter 6

Review Exercises

Section 6.1

- For the rational expression $\frac{t-2}{t+9}$
 - Evaluate the expression (if possible) for $t = 0, 1, 2, -3, -9$
 - Write the domain of the expression in set-builder notation.
- For the rational expression $\frac{k+1}{k-5}$
 - Evaluate the expression (if possible) for $k = 0, 1, 5, -1, -2$
 - Write the domain of the expression in set-builder notation.
- Let $k(y) = \frac{y}{y^2-1}$.
 - Find the function values (if they exist): $k(2)$, $k(0)$, $k(1)$, $k(-1)$, $k\left(\frac{1}{2}\right)$.
 - Identify the domain for k . Write the answer in interval notation.

$$4. \text{ Let } h(x) = \frac{x}{x^2+1}.$$

- Find the function values (if they exist): $h(1)$, $h(0)$, $h(-1)$, $h(-3)$, $h\left(\frac{1}{2}\right)$.
- Identify the domain for h . Write the answer in interval notation.

For Exercises 5–12, simplify the rational expressions.

- $\frac{28a^3b^3}{14a^2b^3}$
- $\frac{25x^2yz^3}{125xyz}$
- $\frac{x^2-4x+3}{x-3}$
- $\frac{k^2+3k-10}{k^2-5k+6}$
- $\frac{x^3-27}{9-x^2}$
- $\frac{a^4-81}{3-a}$
- $\frac{2t^2+3t-5}{7-6t-t^2}$
- $\frac{y^3-4y}{y^2-5y+6}$

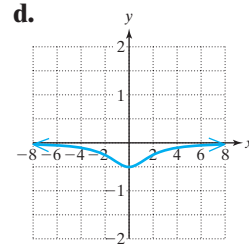
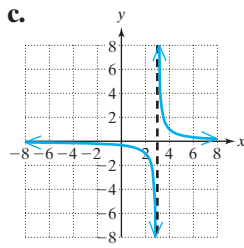
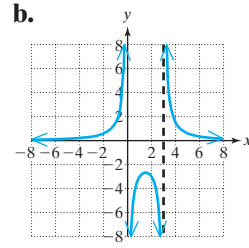
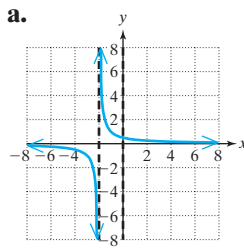
For Exercises 13–16, write the domain of each function in set-builder notation. Use that information to match the function with its graph.

13. $f(x) = \frac{1}{x-3}$

14. $m(x) = \frac{1}{x+2}$

15. $k(x) = \frac{6}{x^2-3x}$

16. $p(x) = \frac{-2}{x^2+4}$



Section 6.2

For Exercises 17–28, multiply or divide as indicated.

17. $\frac{3a+9}{a^2} \cdot \frac{a^3}{6a+18}$

18. $\frac{4-y}{5} \div \frac{2y-8}{15}$

19. $\frac{x-4y}{x^2+xy} \div \frac{20y-5x}{x^2-y^2}$

20. $(x^2+5x-24)\left(\frac{x+8}{x-3}\right)$

21. $\frac{7k+28}{2k+4} \cdot \frac{k^2-2k-8}{k^2+2k-8}$

22. $\frac{ab+2a+b+2}{ab-3b+2a-6} \cdot \frac{ab-3b+4a-12}{ab-b+4a-4}$

23. $\frac{x^2+8x-20}{x^2+6x-16} \div \frac{x^2+6x-40}{x^2+3x-40}$

24. $\frac{2b-b^2}{b^3-8} \cdot \frac{b^2+2b+4}{b^2}$

25. $\frac{2w}{21} \div \frac{3w^2}{7} \cdot \frac{4}{w}$

26. $\frac{5y^2-20}{y^3+2y^2+y+2} \div \frac{7y}{y^3+y}$

27. $\frac{x^2+x-20}{x^2-4x+4} \cdot \frac{x^2+x-6}{12+x-x^2} \div \frac{2x+10}{10-5x}$

28. $(9k^2-25) \cdot \left(\frac{k+5}{3k-5}\right)$

Section 6.3

For Exercises 29–40, add or subtract as indicated.

29. $\frac{1}{x} + \frac{1}{x^2} - \frac{1}{x^3}$

30. $\frac{1}{x+2} + \frac{5}{x-2}$

31. $\frac{y}{2y-1} + \frac{3}{1-2y}$

32. $\frac{a+2}{2a+6} - \frac{3}{a+3}$

33. $\frac{-2}{y^2-1} + \frac{y}{y-1}$

34. $4x+3 - \frac{2x+1}{x+4}$

35. $\frac{w^2}{w-4} + \frac{8w-16}{4-w}$

36. $\frac{6}{x^2+4x+3} + \frac{7}{x^2+5x+6}$

37. $\frac{2}{3x-5} - 8$

38. $\frac{7}{4k^2-k-3} + \frac{1}{4k^2-7k+3}$

39. $\frac{1}{t} - \frac{4}{2t^2+3t} + \frac{2t-5}{6t+9}$

40. $\frac{y}{y-3} - \frac{2y-5}{y+2} - 4$

Section 6.4

For Exercises 41–48, simplify the complex fraction.

41. $\frac{\frac{2x}{3x^2-3}}{\frac{4x}{6x-6}}$

42. $\frac{\frac{k+2}{3}}{\frac{5}{k-2}}$

43. $\frac{\frac{2}{x} + \frac{1}{xy}}{\frac{4}{x^2}}$

44. $\frac{\frac{4}{y} - 1}{\frac{1}{y} - \frac{4}{y^2}}$

$$45. \frac{\frac{1}{a-1} + 1}{\frac{1}{a+1} - 1}$$

$$46. \frac{\frac{3}{x-1} - \frac{1}{1-x}}{\frac{2}{x-1} - \frac{2}{x}}$$

$$47. \frac{1 + xy^{-1}}{x^2y^{-2} - 1}$$

$$48. \frac{5a^{-1} + (ab)^{-1}}{3a^{-2}}$$

For Exercises 49–50, find the slope of the line containing the two points.

$$49. \left(\frac{2}{3}, -\frac{7}{4}\right) \text{ and } \left(\frac{13}{6}, -\frac{5}{3}\right)$$

$$50. \left(\frac{8}{15}, -\frac{1}{3}\right) \text{ and } \left(\frac{13}{10}, \frac{9}{5}\right)$$

Section 6.5

For Exercises 51–56, solve the equation.

$$51. \frac{x+3}{x^2-x} - \frac{8}{x^2-1} = 0$$

$$52. \frac{y}{y+3} + \frac{3}{y-3} = \frac{18}{y^2-9}$$

$$53. x - 9 = \frac{72}{x-8} \quad 54. \frac{3x+1}{x+5} = \frac{x-1}{x+1} + 2$$

$$55. 5y^{-2} + 1 = 6y^{-1} \quad 56. 1 + \frac{7}{6}m^{-1} = \frac{13}{6}m^{-1}$$

$$57. \text{Solve for } x. \quad c = \frac{ax+b}{x}$$

$$58. \text{Solve for } P. \quad \frac{A}{rt} = P + \frac{P}{rt}$$

Section 6.6

For Exercises 59–62, solve the proportions.

$$59. \frac{5}{4} = \frac{x}{6}$$

$$60. \frac{x}{36} = \frac{6}{7}$$

$$61. \frac{x+2}{3} = \frac{5(x+1)}{4}$$

$$62. \frac{x}{x+2} = \frac{-3}{5}$$

63. In one game Peyton Manning completed 34 passes for 357 yd. At this rate how many yards would be gained for 22 passes?

64. Erik bought \$136 Canadian with \$100 American. At this rate, how many Canadian dollars can he buy with \$235 American?

65. The average cost to produce x units is given by

$$\bar{C}(x) = \frac{1.5x + 4200}{x}$$

How many units need to be produced to have an average cost per unit of \$54?

66. Stephen drove his car 45 mi. He ran out of gas and had to walk 3 mi to a gas station. His speed driving is 15 times his speed walking. If the total time for the drive and walk was $1\frac{1}{2}$ hr, what was his speed driving?

67. Two pipes can fill a tank in 6 hr. The larger pipe works twice as fast as the smaller pipe. How long would it take each pipe to fill the tank if they worked separately?

Chapter 6

Test

1. For the expression $\frac{2x+6}{x^2-x-12}$

- Write the domain in set-builder notation.
- Simplify the expression to lowest terms.

2. For the function $h(x) = \frac{2x-14}{x^2-49}$

- Evaluate $h(0)$, $h(5)$, $h(7)$, and $h(-7)$, if possible.

b. Write the domain of h in set-builder notation.

3. Write the domain of $k(x) = \frac{5x-3}{7}$ in interval notation.

For Exercises 4–5, simplify to lowest terms.

$$4. \frac{12m^3n^7}{18mn^8}$$

$$5. \frac{9x^2-9}{3x^2+2x-5}$$

6. Find the slope of the line containing the points $\left(\frac{1}{12}, -\frac{3}{4}\right)$ and $\left(\frac{5}{6}, -\frac{8}{3}\right)$.

For Exercises 7–13, simplify.

7. $\frac{2x - 5}{25 - 4x^2} \cdot (2x^2 - x - 15)$

8. $\frac{x^2}{x - 4} - \frac{8x - 16}{x - 4}$

9. $\frac{4x}{x + 1} + x + \frac{2}{x + 1}$

10. $\frac{3 + \frac{3}{k}}{4 + \frac{4}{k}}$

11. $\frac{2u^{-1} + 2v^{-1}}{4u^{-3} + 4v^{-3}}$

12. $\frac{ax + bx + 2a + 2b}{ax - 3a + bx - 3b} \cdot \frac{x - 3}{x - 5} \div \frac{x + 2}{ax - 5a}$

13. $\frac{3}{x^2 + 8x + 15} - \frac{1}{x^2 + 7x + 12} - \frac{1}{x^2 + 9x + 20}$

For Exercises 14–16, solve the equation.

14. $\frac{7}{z + 1} - \frac{z - 5}{z^2 - 1} = \frac{6}{z}$

15. $\frac{3}{y^2 - 9} + \frac{4}{y + 3} = 1$

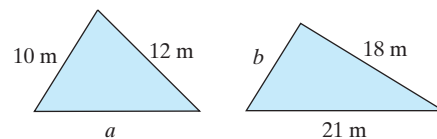
16. $\frac{4x}{x - 4} = 3 + \frac{16}{x - 4}$

17. Solve for T . $\frac{1 + Tv}{T} = p$

18. Solve for m_1 . $F = \frac{Gm_1m_2}{r^2}$

19. If the reciprocal of a number is added to 3 times the number, the result is $\frac{13}{2}$. Find the number.

20. The following triangles are similar. Find the lengths of the missing sides.



21. On a certain map, the distance between New York and Los Angeles is 8.2 in., and the actual distance is 2820 mi. What is the distance between two cities that are 5.7 in. apart on the same map? Round to the nearest mile.

22. The average profit per unit for a manufacturer to sell x units is given by

$$\bar{P}(x) = \frac{1.75x - 1500}{x}$$

How many units must be sold to obtain an average profit of \$0.25 per unit?

23. Lance can ride 48 mi on his bike against the wind. With the wind at his back, he rides 4 mph faster and can ride 60 mi in the same amount of time. Find his speed riding against the wind and his speed riding with the wind.

24. Gail can type a chapter in a book in 4 hr. Jack can type a chapter in a book in 10 hr. How long would it take them to type a chapter if they worked together?

Chapters 1–6

Cumulative Review Exercises

1. Check the sets to which each number belongs.

Set \ Number	-22	π	6	$-\sqrt{2}$
Real numbers				
Irrational numbers				
Rational numbers				
Integers				
Whole numbers				
Natural numbers				

2. At the age of 30, tennis player Steffi Graf announced her retirement from tennis. After 17 years on the tour, her total prize money amounted to $\$2.1839777 \times 10^7$.

- a. Write the amount of prize money in expanded form.
 b. What was Steffi Graf's average winnings per year? Round to the nearest dollar.

3. Perform the indicated operations.

$$(2x - 3)(x - 4) - (x - 5)^2$$

4. The area of a trapezoid is given by

$$A = \frac{1}{2}h(b_1 + b_2).$$

- a. Solve for b_1 .

- b. Find b_1 when $h = 4$ cm, $b_2 = 6$ cm, and $A = 32$ cm².

5. The dimensions of a rectangular swimming pool are such that the length is 10 m less than twice the width. If the total perimeter is 160 m, find the length and width.

6. Solve the system of equations.

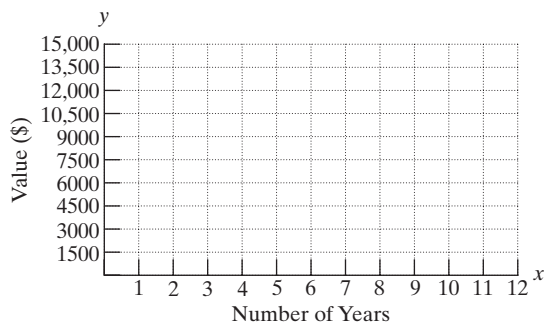
$$\begin{aligned} x - 3y + z &= 1 \\ 2x - y - 2z &= 2 \\ x + 2y - 3z &= -1 \end{aligned}$$

7. Find an equation of the line through $(-3, 5)$ that is perpendicular to the line $y = 3x$. Write the answer in slope-intercept form.

8. The value $V(x)$ of a car (in dollars) decreases with age according to $V(x) = 15,000 - 1250x$, $0 \leq x \leq 12$, where x represents the age of the car in years.

- a. Is this function constant, linear, quadratic, or other?

- b. Sketch the function over its domain.



- c. What is the y -intercept? What does the y -intercept mean in the context of this problem?

- d. What is the x -intercept? What does the x -intercept mean in the context of this problem?

- e. What is the slope? Interpret the slope in the context of this problem.

- f. What is the value of $V(5)$? Interpret the value of $V(5)$ in the context of the problem.

- g. After how many years will the car be worth \$5625?

9. The speed of a car varies inversely as the time to travel a fixed distance. A car traveling the speed limit of 60 mph travels between two points in 10 sec. How fast is a car moving if it takes only 8 sec to cover the same distance?

10. Find the x -intercepts. $f(x) = -12x^3 + 17x^2 - 6x$

11. Factor $64y^3 - 8z^6$ completely over the real numbers.

12. Write the domain of the functions f and g .

a. $f(x) = \frac{x + 7}{2x - 3}$ b. $g(x) = \frac{x + 3}{x^2 - x - 12}$

13. Perform the indicated operations.

$$\frac{2x^2 + 11x - 21}{4x^2 - 10x + 6} \div \frac{2x^2 - 98}{x^2 - x + xa - a}$$

14. Reduce to lowest terms.

$$\frac{x^2 - 6x + 8}{20 - 5x}$$

15. Perform the indicated operations.

$$\frac{x}{x^2 + 5x - 50} - \frac{1}{x^2 - 7x + 10} + \frac{1}{x^2 + 8x - 20}$$

16. Simplify the complex fraction.

$$\frac{1 - \frac{49}{c^2}}{\frac{7}{c} + 1}$$

17. Solve the equation.

$$\frac{4y}{y + 2} - \frac{y}{y - 1} = \frac{9}{y^2 + y - 2}$$

18. Max knows that the distance between Roanoke, Virginia, and Washington, D.C., is 195 mi. On a certain map, the distance between the two cities is 6.5 in. On the same map, the distance between Roanoke and Cincinnati, Ohio, is 9.25 in. Find the distance in miles between Roanoke and Cincinnati. Round to the nearest mile.

19. Determine whether the equation represents a horizontal or vertical line. Then identify the slope of the line.

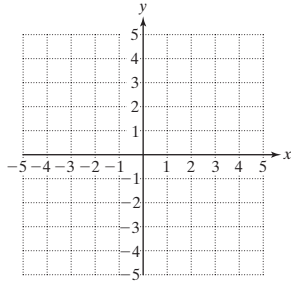
a. $x = -5$

b. $2y = 8$

20. Simplify. $\left(\frac{2x^{-4}y}{3z^2}\right)^{-1}(4xy^7)$

21. Find the slope and y-intercept of $2x - y = 3$.

22. Graph the line $2x - y = 3$.



23. Solve the system: $y = -x + 2$
 $3x - 2y = -4$

24. Add: $(x^3 + 5x^2 - 4) + (x^2 + 1)$

25. Subtract: $(x^3 + 5x^2 - 4) - (x^2 + 1)$

26. Multiply: $(x^3 + 5x^2 - 4)(x^2 + 1)$

27. Is $(4, 5)$ a solution to the system $-x + 2y = 6$
 $-2x + y = 13$

28. For $h(x) = \frac{-x^2 + 2}{x}$ find $h(-2)$, $h(-1)$, $h(0)$, $h(1)$, and $h(2)$.

29. Factor $64 - y^2$.

30. A motorcycle travels 6 mph faster than a sports car. If the car can go 90 miles in the same time as the motorcycle can go 99 miles, find the speed of the motorcycle.

