5

Polynomials

In this chapter we study addition, subtraction, multiplication, and division of polynomials, along with an important operation called factoring.

The list of words represents key terms used in this chapter. Search for them in the puzzle and in the text throughout the chapter. By the end of this chapter, you should be familiar with all of these terms.

L
P K B
O Z Q V I
W F T K P R L
Z N G N C F A L A
D E G R E E S T K H I
B I N O M I A L G F A I M
H U P G I E C A G I K W P J O
Q U A D R A T I C E Q U A T I O N
C F R T T P Z M F M B U E D G H R J Y
W Y A H H G W O E F K B W X Y F S R J J L
H F B W Z I T N D S E T A G U J N O C C X X O
H F O J C L A I M O N O M C M M P X X C D Q S N P
Z U L Y H Z Y R M N F G C R J W E H L D F R R O T Z S
T Q A K I V I T F B S P E I V T C L Y M W S X I T F A N V

Key Terms

polynomial
coefficient
degree
monomial
binomial
GCF
quadratic equation
parabola
conjugates
trinomial

Section 5.1

Addition and Subtraction of Polynomials and Polynomial Functions

Concepts

1. Polynomials: Basic Definitions
2. Addition of Polynomials
3. Subtraction of Polynomials
4. Polynomial Functions

1. Polynomials: Basic Definitions

One commonly used algebraic expression is called a polynomial. A **polynomial** in x is defined as a finite sum of terms of the form ax^n, where a is a real number and the exponent n is a whole number. For each term, a is called the **coefficient**, and n is called the **degree of the term**. For example:

Term (Expressed in the Form ax^n)	Coefficient	Degree
$3x^5$	3	5
x^{14} ➛ rewrite as $1x^{14}$	1	14
7 ➛ rewrite as $7x^0$	7	0
$\frac{1}{2}p$ ➛ rewrite as $\frac{1}{2}p^1$	$\frac{1}{2}$	1

If a polynomial has exactly one term, it is categorized as a **monomial**. A two-term polynomial is called a **binomial**, and a three-term polynomial is called a **trinomial**. Usually the terms of a polynomial are written in descending order according to degree. In descending order, the highest-degree term is written first and is called the **leading term**. Its coefficient is called the **leading coefficient**. The **degree of a polynomial** is the largest degree of all its terms. Thus, the leading term determines the degree of the polynomial.

	Expression	Descending Order	Leading Coefficient	Degree of Polynomial
Monomials	$2x^9$	$2x^9$	2	9
	-49	-49	-49	0
Binomials	$10y - 7y^2$	$-7y^2 + 10y$	-7	2
	$6 - \frac{2}{3}b$	$-\frac{2}{3}b + 6$	$-\frac{2}{3}$	1
Trinomials	$w + 2w^3 + 9w^6$	$9w^6 + 2w^3 + w$	9	6
	$2.5a^4 - a^8 + 1.3a^3$	$-a^8 + 2.5a^4 + 1.3a^3$	-1	8

Polynomials may have more than one variable. In such a case, the degree of a term is the sum of the exponents of the variables contained in the term. For example, the term $2x^3y^4z$ has degree 8 because the exponents applied to x, y, and z are 3, 4, and 1, respectively.

The following polynomial has a degree of 12 because the highest degree of its terms is 12.

$$\underset{\substack{\uparrow \\ \text{degree} \\ 8}}{11x^4y^3z} \quad - \quad \underset{\substack{\uparrow \\ \text{degree} \\ 12}}{5x^3y^2z^7} \quad + \quad \underset{\substack{\uparrow \\ \text{degree} \\ 3}}{2x^2y} \quad + \quad \underset{\substack{\uparrow \\ \text{degree} \\ 0}}{7}$$

2. Addition of Polynomials

To add or subtract two polynomials, we combine *like* terms. Recall that two terms are ***like* terms** if they each have the same variables and the corresponding variables are raised to the same powers.

Example 1 Adding Polynomials

Add the polynomials.

a. $(3t^3 + 2t^2 - 5t) + (t^3 - 6t)$ **b.** $\left(\frac{2}{3}w^2 - w + \frac{1}{8}\right) + \left(\frac{4}{3}w^2 + 8w - \frac{1}{4}\right)$

c. $(a^2b + 7ab + 6) + (5a^2b - 2ab - 7)$

Solution:

a. $(3t^3 + 2t^2 - 5t) + (t^3 - 6t)$

$= 3t^3 + t^3 + 2t^2 + (-5t) + (-6t)$ Group *like* terms.

$= 4t^3 + 2t^2 - 11t$ Add *like* terms.

b. $\left(\frac{2}{3}w^2 - w + \frac{1}{8}\right) + \left(\frac{4}{3}w^2 + 8w - \frac{1}{4}\right)$

$= \frac{2}{3}w^2 + \frac{4}{3}w^2 + (-w) + 8w + \frac{1}{8} + \left(-\frac{1}{4}\right)$ Group *like* terms.

$= \frac{6}{3}w^2 + 7w + \left(\frac{1}{8} - \frac{2}{8}\right)$ Add fractions with common denominators.

$= 2w^2 + 7w - \frac{1}{8}$ Simplify.

c. $(a^2b + 7ab + 6) + (5a^2b - 2ab - 7)$

$= a^2b + 5a^2b + 7ab + (-2ab) + 6 + (-7)$ Group *like* terms.

$= 6a^2b + 5ab - 1$ Add *like* terms.

TIP: Addition of polynomials can be performed vertically by aligning *like* terms.

$$(a^2b + 7ab + 6) + (5a^2b - 2ab - 7) \longrightarrow \begin{array}{r} a^2b + 7ab + 6 \\ + \; 5a^2b - 2ab - 7 \\ \hline 6a^2b + 5ab - 1 \end{array}$$

Skill Practice Add the polynomials.

1. $(2x^2 + 5x - 2) + (6x^2 - 8x - 8)$
2. $\left(-\frac{1}{4}m^2 - 2m + \frac{1}{3}\right) + \left(\frac{3}{4}m^2 + 7m - \frac{1}{12}\right)$
3. $(-5a^2b - 6ab^2) + (2a^2b + ab^2)$

3. Subtraction of Polynomials

Subtraction of two polynomials is similar to subtracting real numbers. Add the opposite of the second polynomial to the first polynomial.

The opposite (or additive inverse) of a real number a is $-a$. Similarly, if A is a polynomial, then $-A$ is its opposite.

Skill Practice Answers

1. $8x^2 - 3x - 10$
2. $\frac{1}{2}m^2 + 5m + \frac{1}{4}$
3. $-3a^2b - 5ab^2$

Example 2 Finding the Opposite of a Polynomial

Find the opposite of the polynomials.

a. $4x$ **b.** $5a - 2b - c$ **c.** $5.5y^4 - 2.4y^3 + 1.1y - 3$

Solution:

a. The opposite of $4x$ is $-(4x)$, or $-4x$.

b. The opposite of $5a - 2b - c$ is $-(5a - 2b - c)$ or equivalently $-5a + 2b + c$.

c. The opposite of $5.5y^4 - 2.4y^3 + 1.1y - 3$ is $-(5.5y^4 - 2.4y^3 + 1.1y - 3)$ or equivalently $-5.5y^4 + 2.4y^3 - 1.1y + 3$.

TIP: Notice that the sign of each term is changed when finding the opposite of a polynomial.

Skill Practice Find the opposite of the polynomials.

4. $-7z$ **5.** $2p - 3q + r + 1$ **6.** $-3x^2 + x - 2.2$

Definition of Subtraction of Polynomials

If A and B are polynomials, then $A - B = A + (-B)$.

Example 3 Subtracting Polynomials

Subtract the polynomials.

a. $(3x^2 + 2x - 5) - (4x^2 - 7x + 2)$ **b.** $(6x^2y - 2xy + 5) - (x^2y - 3)$

Solution:

a. $(3x^2 + 2x - 5) - (4x^2 - 7x + 2)$

$= (3x^2 + 2x - 5) + (-4x^2 + 7x - 2)$ Add the opposite of the second polynomial.

$= 3x^2 + (-4x^2) + 2x + 7x + (-5) + (-2)$ Group *like* terms.

$= -x^2 + 9x - 7$ Combine *like* terms.

b. $(6x^2y - 2xy + 5) - (x^2y - 3)$

$= (6x^2y - 2xy + 5) + (-x^2y + 3)$ Add the opposite of the second polynomial.

$= 6x^2y + (-x^2y) + (-2xy) + 5 + 3$ Group *like* terms.

$= 5x^2y - 2xy + 8$ Combine *like* terms.

TIP: Subtraction of polynomials can be performed vertically by aligning *like* terms. Then add the opposite of the second polynomial. "Placeholders" (shown in bold) may be used to help line up *like* terms.

$$(6x^2y - 2xy + 5) - (x^2y - 3) \longrightarrow \begin{array}{r} 6x^2y - 2xy + 5 \\ \underline{-(x^2y + \mathbf{0xy} - 3)} \end{array} \xrightarrow[\text{opposite.}]{\text{Add the}} \begin{array}{r} 6x^2y - 2xy + 5 \\ \underline{+ -x^2y - \mathbf{0xy} + 3} \\ 5x^2y - 2xy + 8 \end{array}$$

Skill Practice Answers

4. $7z$ **5.** $-2p + 3q - r - 1$
6. $3x^2 - x + 2.2$

Skill Practice Subtract the polynomials.

7. $(6a^2b - 2ab) - (-3a^2b + 2ab + 3)$

8. $\left(\frac{1}{3}p^3 + \frac{3}{4}p^2 - p\right) - \left(\frac{1}{2}p^3 + \frac{1}{3}p^2 + \frac{1}{2}p\right)$

Example 4 Subtracting Polynomials

Subtract $\frac{1}{2}x^4 - \frac{3}{4}x^2 + \frac{1}{5}$ from $\frac{3}{2}x^4 + \frac{1}{2}x^2 - 4x$

Solution:

In general, to subtract a from b, we write $b - a$. Therefore, to subtract

$$\frac{1}{2}x^4 - \frac{3}{4}x^2 + \frac{1}{5} \quad \text{from} \quad \frac{3}{2}x^4 + \frac{1}{2}x^2 - 4x$$

we have

$$\left(\frac{3}{2}x^4 + \frac{1}{2}x^2 - 4x\right) - \left(\frac{1}{2}x^4 - \frac{3}{4}x^2 + \frac{1}{5}\right)$$

$$= \frac{3}{2}x^4 + \frac{1}{2}x^2 - 4x - \frac{1}{2}x^4 + \frac{3}{4}x^2 - \frac{1}{5}$$ Subtract the polynomials.

$$= \frac{3}{2}x^4 - \frac{1}{2}x^4 + \frac{1}{2}x^2 + \frac{3}{4}x^2 - 4x - \frac{1}{5}$$ Group *like* terms.

$$= \frac{3}{2}x^4 - \frac{1}{2}x^4 + \frac{2}{4}x^2 + \frac{3}{4}x^2 - 4x - \frac{1}{5}$$ Write *like* terms with a common denominator.

$$= \frac{2}{2}x^4 + \frac{5}{4}x^2 - 4x - \frac{1}{5}$$ Combine *like* terms.

$$= x^4 + \frac{5}{4}x^2 - 4x - \frac{1}{5}$$ Simplify.

Skill Practice

9. Subtract $(8t^2 - 4t - 3)$ from $(-6t^2 + t + 2)$.

4. Polynomial Functions

A **polynomial function** is a function defined by a finite sum of terms of the form ax^n, where a is a real number and n is a whole number. For example, the functions defined here are polynomial functions:

$$f(x) = 3x - 8$$

$$g(x) = 4x^5 - 2x^3 + 5x - 3$$

$$h(x) = -\frac{1}{2}x^4 + \frac{3}{5}x^3 - 4x^2 + \frac{5}{9}x - 1$$

$k(x) = 7$ ($7 = 7x^0$ which is of the form ax^n, where $n = 0$ is a whole number)

Skill Practice Answers

7. $9a^2b - 4ab - 3$

8. $-\frac{1}{6}p^3 + \frac{5}{12}p^2 - \frac{3}{2}p$

9. $-14t^2 + 5t + 5$

The following functions are *not* polynomial functions:

$m(x) = \frac{1}{x} - 8$ $\quad \left(\frac{1}{x} = x^{-1}\text{. The exponent } -1 \text{ is not a whole number.}\right)$

$q(x) = |x|$ $\quad (|x| \text{ is not of the form } ax^n)$

Example 5 Evaluating a Polynomial Function

Given $P(x) = x^3 + 2x^2 - x - 2$, find the function values.

a. $P(-3)$ **b.** $P(-1)$ **c.** $P(0)$ **d.** $P(2)$

Solution:

a. $P(x) = x^3 + 2x^2 - x - 2$

$$\begin{aligned} P(-3) &= (-3)^3 + 2(-3)^2 - (-3) - 2 \\ &= -27 + 2(9) + 3 - 2 \\ &= -27 + 18 + 3 - 2 \\ &= -8 \end{aligned}$$

b.
$$\begin{aligned} P(-1) &= (-1)^3 + 2(-1)^2 - (-1) - 2 \\ &= -1 + 2(1) + 1 - 2 \\ &= -1 + 2 + 1 - 2 \\ &= 0 \end{aligned}$$

c.
$$\begin{aligned} P(0) &= (0)^3 + 2(0)^2 - (0) - 2 \\ &= -2 \end{aligned}$$

d.
$$\begin{aligned} P(2) &= (2)^3 + 2(2)^2 - (2) - 2 \\ &= 8 + 2(4) - 2 - 2 \\ &= 8 + 8 - 2 - 2 \\ &= 12 \end{aligned}$$

The function values can be confirmed from the graph of $y = P(x)$ (Figure 5-1).

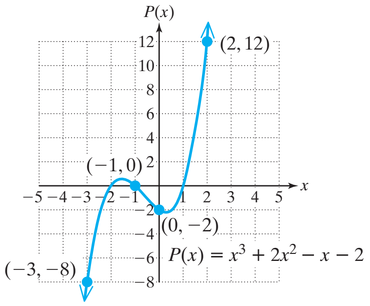


Figure 5-1

Skill Practice

10. Given: $P(x) = -2x^3 - 4x + 6$

a. Find $P(0)$. **b.** Find $P(-2)$.

Skill Practice Answers

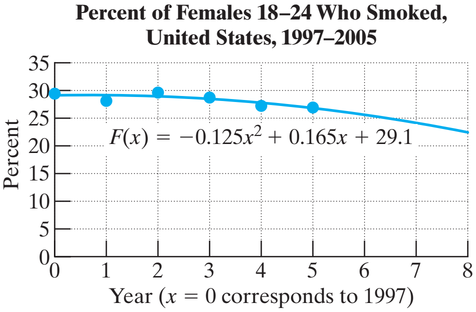
10a. $P(0) = 6$ **b.** $P(-2) = 30$

Example 6 Applying a Polynomial Function

The percent of females between the ages of 18 and 24 who smoked in the United States can be approximated by $F(x) = -0.125x^2 + 0.165x + 29.1$, where x is the number of years since 1997 and $F(x)$ is measured as a percent (Figure 5-2).

a. Evaluate $F(2)$ to 1 decimal place, and interpret the meaning in the context of this problem.

b. What percent of females between the ages of 18 and 24 smoked in the year 2005? Round to the nearest tenth of a percent.



(Source: Center for Disease Control.)

Figure 5-2

Solution:

a. $F(2) = -0.125(2)^2 + 0.165(2) + 29.1$ Substitute $x = 2$ into the function.

≈ 28.9

In the year 1999 ($x = 2$ years since 1997), approximately 28.9% of females between the ages of 18 and 24 smoked.

b. The year 2005 is 8 years since 1997. Substitute $x = 8$ into the function.

$F(8) = -0.125(8)^2 + 0.165(8) + 29.1$ Substitute $x = 8$ into the function.

$\approx 22.4\%$

Approximately 22.4% of females in the 18–24 age group smoked in 2005.

Skill Practice

11. The yearly cost of tuition at public two-year colleges from 1992 to 2006 can be approximated by $T(x) = -0.08x^2 + 61x + 1135$ for $0 \leq x \leq 14$, where x represents the number of years since 1992.

a. Find $T(13)$ and interpret the result.

b. Use the function T to approximate the cost of tuition in the year 1997.

Skill Practice Answers

11a. $T(13) \approx 1914$. In the year 2005, tuition for public two-year colleges averaged approximately \$1914.

b. \$1438

Section 5.1 Practice Exercises

Boost *your* GRADE at mathzone.com!

MathZone

- Practice Problems
- Self-Tests
- NetTutor
- e-Professors
- Videos

Study Skills Exercise

1. Define the key terms.

a. Polynomial
b. Coefficient
c. Degree of the term
d. Monomial
e. Binomial
f. Trinomial
g. Leading term
h. Leading coefficient
i. Degree of a polynomial
j. *Like* terms
k. Polynomial function

Concept 1: Polynomials: Basic Definitions

2. How many terms does the polynomial have? $2x^2y - 3xy + 5y^2 - 6$

For Exercises 3–8, write the polynomial in descending order. Then identify the leading coefficient and the degree.

3. $a^2 - 6a^3 - a$

4. $2b - b^4 + 5b^2$

5. $6x^2 - x + 3x^4 - 1$

6. $8 - 4y + y^5 - y^2$

7. $100 - t^2$

8. $-51 + s^2$

For Exercises 9–14, write a polynomial in one variable that is described by the following. (Answers may vary.)

9. A monomial of degree 5

10. A monomial of degree 4

11. A trinomial of degree 2

12. A trinomial of degree 3

13. A binomial of degree 4

14. A binomial of degree 2

Concept 2: Addition of Polynomials

For Exercises 15–24, add the polynomials and simplify.

15. $(-4m^2 + 4m) + (5m^2 + 6m)$

16. $(3n^3 + 5n) + (2n^3 - 2n)$

17. $(3x^4 - x^3 - x^2) + (3x^3 - 7x^2 + 2x)$

18. $(6x^3 - 2x^2 - 12) + (x^2 + 3x + 9)$

19. $\left(\frac{1}{2}w^3 + \frac{2}{9}w^2 - 1.8w\right) + \left(\frac{3}{2}w^3 - \frac{1}{9}w^2 + 2.7w\right)$

20. $\left(2.9t^4 - \frac{7}{8}t + \frac{5}{3}\right) + \left(-8.1t^4 - \frac{1}{8}t - \frac{1}{3}\right)$

21. Add $(9x^2 - 5x + 1)$ to $(8x^2 + x - 15)$.

22. Add $(-x^3 + 5x)$ to $(10x^3 + x^2 - 10)$.

23.
$$\begin{array}{r} 12x^3 \qquad\qquad + 6x - 8 \\ \underline{+\ (-3x^3 - 5x^2 - 4x)\quad} \end{array}$$

24.
$$\begin{array}{r} -8y^4 - 8y^3 - 6y^2 \qquad\qquad - 9 \\ \underline{+\ \ (4y^4 + 5y^3 \qquad\qquad - 10y - 3)} \end{array}$$

Concept 3: Subtraction of Polynomials

For Exercises 25–30, write the opposite of the given polynomial.

25. $-30y^3$

26. $-2x^2$

27. $4p^3 + 2p - 12$

28. $8t^2 - 4t - 3$

29. $-11ab^2 + a^2b$

30. $-23rs - 4r + 9s$

For Exercises 31–38, subtract the polynomials and simplify.

31. $(13z^5 - z^2) - (7z^5 + 5z^2)$

32. $(8w^4 + 3w^2) - (12w^4 - w^2)$

33. $(-3x^3 + 3x^2 - x + 6) - (-x^3 - x^2 - x + 1)$

34. $(-8x^3 + 6x + 7) - (-5x^3 - 2x - 4)$

35.
$$\begin{array}{r} 4t^3 - 6t^2 \qquad\qquad - 18 \\ \underline{-\ (3t^3 + 7t^2 + 9t - \ \ 5)} \end{array}$$

36.
$$\begin{array}{r} 5w^3 - 9w^2 + \ \ 6w + 13 \\ \underline{-\ (7w^3 \qquad\qquad - 10w - \ \ 8)} \end{array}$$

37. $\left(\frac{1}{5}a^2 - \frac{1}{2}ab + \frac{1}{10}b^2 + 3\right) - \left(-\frac{3}{10}a^2 + \frac{2}{5}ab - \frac{1}{2}b^2 - 5\right)$

38. $\left(\frac{4}{7}a^2 - \frac{1}{7}ab + \frac{1}{14}b^2 - 7\right) - \left(\frac{1}{2}a^2 - \frac{2}{7}ab - \frac{9}{14}b^2 + 1\right)$

39. Subtract $(9x^2 - 5x + 1)$ from $(8x^2 + x - 15)$.

40. Subtract $(-x^3 + 5x)$ from $(10x^3 + x^2 - 10)$.

41. Find the difference of $(3x^5 - 2x^3 + 4)$ and $(x^4 + 2x^3 - 7)$.

42. Find the difference of $(7x^{10} - 2x^4 - 3x)$ and $(-4x^3 - 5x^4 + x + 5)$.

Mixed Exercises

For Exercises 43–62, add or subtract as indicated. Write the answers in descending order, if possible.

43. $(8y^2 - 4y^3) - (3y^2 - 8y^3)$

44. $(-9y^2 - 8) - (4y^2 + 3)$

45. $(-2r - 6r^4) + (-r^4 - 9r)$

46. $(-8s^9 + 7s^2) + (7s^9 - s^2)$

47. $(5xy + 13x^2 + 3y) - (4x^2 - 8y)$

48. $(6p^2q - 2q) - (-2p^2q + 13)$

49. $(11ab - 23b^2) + (7ab - 19b^2)$

50. $(-4x^2y + 9) + (8x^2y - 12)$

51. $[2p - (3p + 5)] + (4p - 6) + 2$

52. $-(q - 2) - [4 - (2q - 3) + 5]$

53. $5 - [2m^2 - (4m^2 + 1)]$

54. $[4n^3 - (n^3 + 4)] + 3n^3$

55. $(6x^3 - 5) - (-3x^3 + 2x) - (2x^3 - 6x)$

56. $(9p^4 - 2) + (7p^4 + 1) - (8p^4 - 10)$

57. $(-ab + 5a^2b) + [7ab^2 - 2ab - (7a^2b + 2ab^2)]$

58. $(m^3n^2 + 4m^2n) - [-5m^3n^2 - 4mn - (7m^2n - 6mn)]$

59.
$$\begin{array}{rrrrrr} & -5x^4 & & -\ 11x^2 & & +\ 6 \\ \hline -\ (& -5x^4 & +\ 3x^3 & +\ 5x^2 & -\ 10x & +\ 5) \end{array}$$

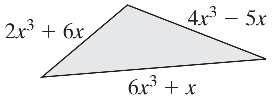
60.
$$\begin{array}{rrrrrr} & 9z^4 & & 2z^2 & & +\ 11 \\ \hline -\ (& 9z^4 & -\ 4z^3 & +\ 8z^2 & -\ 9z & -\ 4) \end{array}$$

61.
$$\begin{array}{rrrrrr} & -2.2p^5 & -\ 9.1p^4 & & +\ 5.3p^2 & -\ 7.9p \\ \hline + & & -\ 6.4p^4 & -\ 8.5p^3 & -\ 10.3p^2 & \end{array}$$

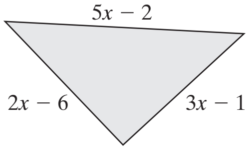
62.
$$\begin{array}{rrrrrr} & 5.5w^4 & & +\ 4.6w^2 & -\ 9.3w & -\ 8.3 \\ \hline + & 0.4w^4 & -\ 7.3w^3 & & -\ 5.8w & +\ 4.6 \end{array}$$

For Exercises 63–64, find the perimeter.

63.



64.



Concept 4: Polynomial Functions

For Exercises 65–72, determine whether the given function is a polynomial function. If it is a polynomial function, state the degree. If not, state the reason why.

65. $h(x) = \frac{2}{3}x^2 - 5$

66. $k(x) = -7x^4 - 0.3x + x^3$

67. $p(x) = 8x^3 + 2x^2 - \frac{3}{x}$

68. $q(x) = x^2 - 4x^{-3}$

69. $g(x) = -7$

70. $g(x) = 4x$

71. $M(x) = |x| + 5x$

72. $N(x) = x^2 + |x|$

73. Given $P(x) = -x^4 + 2x - 5$, find the function values.

a. $P(2)$ **b.** $P(-1)$ **c.** $P(0)$ **d.** $P(1)$

74. Given $N(x) = -x^2 + 5x$, find the function values.

a. $N(1)$ **b.** $N(-1)$ **c.** $N(2)$ **d.** $N(0)$

75. Given $H(x) = \frac{1}{2}x^3 - x + \frac{1}{4}$, find the function values.

a. $H(0)$ **b.** $H(2)$ **c.** $H(-2)$ **d.** $H(-1)$

76. Given $K(x) = \frac{2}{3}x^2 + \frac{1}{9}$, find the function values.

a. $K(0)$ **b.** $K(3)$ **c.** $K(-3)$ **d.** $K(-1)$

77. A rectangular garden is designed to be 3 ft longer than it is wide. Let x represent the width of the garden. Find a function P that represents the perimeter in terms of x.

78. A flowerbed is in the shape of a triangle with the larger side 3 times the middle side and the smallest side 2 ft shorter than the middle side. Let x represent the length of the middle side. Find a function P that represents the perimeter in terms of x.

79. The cost in dollars of producing x toy cars is $C(x) = 2.2x + 1$. The revenue received is $R(x) = 5.98x$. To calculate profit, subtract the cost from the revenue.

a. Write and simplify a function P that represents profit in terms of x.

b. Find the profit of producing 50 toy cars.

80. The cost in dollars of producing x lawn chairs is $C(x) = 2.5x + 10.1$. The revenue for selling x chairs is $R(x) = 6.99x$. To calculate profit, subtract the cost from the revenue.

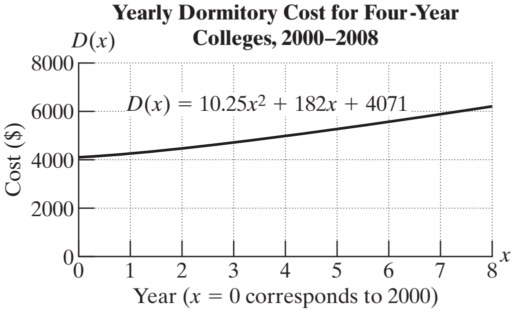
a. Write and simplify a function P that represents profit in terms of x.

b. Find the profit of producing 100 lawn chairs.

81. The function defined by $D(x) = 10.25x^2 + 182x + 4071$ approximates the yearly dormitory charges for private four-year colleges since the year 2000. $D(x)$ is measured in dollars, and $x = 0$ corresponds to the year 2000. Find the function values and interpret their meaning in the context of this problem.

a. $D(0)$ **b.** $D(2)$

c. $D(4)$ **d.** $D(6)$

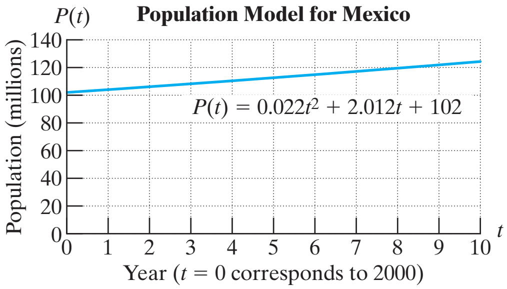

Yearly Dormitory Cost for Four-Year Colleges, 2000–2008

(Source: U.S. National Center for Education Statistics.)

82. The population of Mexico can be modeled by $P(t) = 0.022t^2 + 2.012t + 102$, where t is the number of years since 2000 and $P(t)$ is the number of people in millions.

a. Evaluate $P(0)$ and $P(6)$, and interpret their meaning in the context of this problem. Round to 1 decimal place if necessary.

b. If this trend continues, what will the population of Mexico be in the year 2010? Round to 1 decimal place if necessary.


Population Model for Mexico

83. The number of women, W, to be paid child support in the United States can be approximated by

$$W(t) = 143t + 6580$$

where t is the number of years after 2000, and $W(t)$ is the yearly total measured in thousands. (Source: U.S. Bureau of the Census.)

a. Evaluate $W(0)$, $W(5)$, and $W(10)$.

b. Interpret the meaning of the function value $W(10)$.

84. The total yearly amount of child support due (in billions of dollars) in the United States can be approximated by

$$D(t) = 0.925t + 4.625$$

where t is the number of years after 2000, and $D(t)$ is the amount due (in billions of dollars).

a. Evaluate $D(0)$, $D(4)$, and $D(8)$.

b. Interpret the meaning of the function value of $D(8)$.

Expanding Your Skills

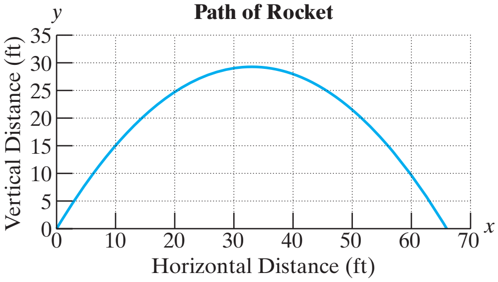
85. A toy rocket is shot from ground level at an angle of 60° from the horizontal. See the figure. The x- and y-positions of the rocket (measured in feet) vary with time t according to

$$x(t) = 25t$$

$$y(t) = -16t^2 + 43.3t$$

a. Evaluate $x(0)$ and $y(0)$, and write the values as an ordered pair. Interpret the meaning of these function values in the context of this problem. Match the ordered pair with a point on the graph.

b. Evaluate $x(1)$ and $y(1)$ and write the values as an ordered pair. Interpret the meaning of these function values in the context of this problem. Match the ordered pair with a point on the graph.



c. Evaluate $x(2)$ and $y(2)$, and write the values as an ordered pair. Match the ordered pair with a point on the graph.

Multiplication of Polynomials

Section 5.2

Concepts

1. Multiplying Polynomials
2. Special Case Products: Difference of Squares and Perfect Square Trinomials
3. Translations Involving Polynomials
4. Applications Involving a Product of Polynomials

1. Multiplying Polynomials

The properties of exponents covered in Section 1.8 can be used to simplify many algebraic expressions including the multiplication of monomials. To multiply monomials, first use the associative and commutative properties of multiplication to group coefficients and like bases. Then simplify the result by using the properties of exponents.

Example 1 Multiplying Monomials

Multiply the monomials.

a. $(3x^2y^7)(5x^3y)$ **b.** $(-3x^4y^3)(-2x^6yz^8)$

Solution:

a. $(3x^2y^7)(5x^3y)$

$= (3 \cdot 5)(x^2 \cdot x^3)(y^7 \cdot y)$ Group coefficients and like bases.

$= 15x^5y^8$ Add exponents and simplify.

b. $(-3x^4y^3)(-2x^6yz^8)$

$= [(-3)(-2)](x^4 \cdot x^6)(y^3 \cdot y)(z^8)$ Group coefficients and like bases.

$= 6x^{10}y^4z^8$ Add exponents and simplify.

Skill Practice Multiply the polynomials.

1. $(-8r^3s)(-4r^4s^4)$ **2.** $(-4ab)(7a^2)$

The distributive property is used to multiply polynomials: $a(b + c) = ab + ac$.

Example 2 Multiplying a Polynomial by a Monomial

Multiply the polynomials.

a. $5y^3(2y^2 - 7y + 6)$ **b.** $-4a^3b^7c\left(2ab^2c^4 - \frac{1}{2}a^5b\right)$

Solution:

a. $5y^3(2y^2 - 7y + 6)$

$= (5y^3)(2y^2) + (5y^3)(-7y) + (5y^3)(6)$ Apply the distributive property.

$= 10y^5 - 35y^4 + 30y^3$ Simplify each term.

b. $-4a^3b^7c\left(2ab^2c^4 - \frac{1}{2}a^5b\right)$

$= (-4a^3b^7c)(2ab^2c^4) + (-4a^3b^7c)\left(-\frac{1}{2}a^5b\right)$ Apply the distributive property.

$= -8a^4b^9c^5 + 2a^8b^8c$ Simplify each term.

Skill Practice Multiply the polynomials.

3. $-6b^2(2b^2 + 3b - 8)$ **4.** $8t^3\left(\frac{1}{2}t^3 - \frac{1}{4}t^2\right)$

Skill Practice Answers

1. $32r^7s^5$ **2.** $-28a^3b$

3. $-12b^4 - 18b^3 + 48b^2$

4. $4t^6 - 2t^5$

Thus far, we have illustrated polynomial multiplication involving monomials. Next, the distributive property will be used to multiply polynomials with more than one term. For example:

$$\begin{aligned}
\underbrace{(x+3)}(x+5) &= (x+3)x + (x+3)5 && \text{Apply the distributive property.}\\
&= (x+3)x + (x+3)5 && \text{Apply the distributive property again.}\\
&= x\cdot x + 3\cdot x + x\cdot 5 + 3\cdot 5\\
&= x^2 + 3x + 5x + 15\\
&= x^2 + 8x + 15 && \text{Combine } \textit{like} \text{ terms.}
\end{aligned}$$

Note: Using the distributive property results in multiplying each term of the first polynomial by each term of the second polynomial:

$$\begin{aligned}
(x+3)(x+5) &= x\cdot x + x\cdot 5 + 3\cdot x + 3\cdot 5\\
&= x^2 + 5x + 3x + 15\\
&= x^2 + 8x + 15
\end{aligned}$$

Example 3 Multiplying Polynomials

Multiply the polynomials.

a. $(2x^2 + 4)(3x^2 - x + 5)$ **b.** $(3y + 2)(7y - 6)$

Solution:

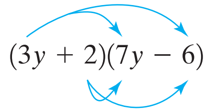
a. $(2x^2 + 4)(3x^2 - x + 5)$ Multiply each term in the first polynomial by each term in the second.

$$\begin{aligned}
&= (2x^2)(3x^2) + (2x^2)(-x) + (2x^2)(5) + (4)(3x^2) + (4)(-x) + (4)(5)\\
&= 6x^4 - 2x^3 + 10x^2 + 12x^2 - 4x + 20 && \text{Simplify each term.}\\
&= 6x^4 - 2x^3 + 22x^2 - 4x + 20 && \text{Combine } \textit{like} \text{ terms.}
\end{aligned}$$

TIP: Multiplication of polynomials can be performed vertically by a process similar to column multiplication of real numbers.

$$(2x^2 + 4)(3x^2 - x + 5) \longrightarrow \begin{array}{rrrrr}
 & & 3x^2 & -\ x & +\ 5\\
 & \times & 2x^2 & & +\ 4\\ \hline
 & & 12x^2 & -\ 4x & +\ 20\\
6x^4 & -\ 2x^3 & +\ 10x^2 & & \\ \hline
6x^4 & -\ 2x^3 & +\ 22x^2 & -\ 4x & +\ 20
\end{array}$$

Note: When multiplying by the column method, it is important to align *like* terms vertically before adding terms.



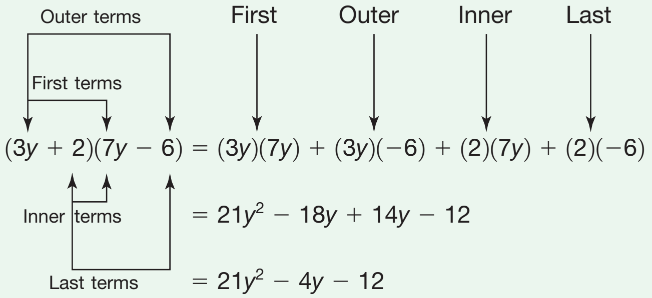
b. $(3y + 2)(7y - 6)$ — Multiply each term in the first polynomial by each term in the second.

$= (3y)(7y) + (3y)(-6) + (2)(7y) + (2)(-6)$ — Apply the distributive property.

$= 21y^2 - 18y + 14y - 12$ — Simplify each term.

$= 21y^2 - 4y - 12$ — Combine *like* terms.

TIP: The acronym, FOIL (**F**irst **O**uter **I**nner **L**ast) can be used as a memory device to multiply two binomials.



$$(3y + 2)(7y - 6) = (3y)(7y) + (3y)(-6) + (2)(7y) + (2)(-6)$$
$$= 21y^2 - 18y + 14y - 12$$
$$= 21y^2 - 4y - 12$$

Note: It is important to realize that the acronym FOIL may only be used when finding the product of two *binomials*.

Skill Practice Multiply the polynomials.

5. $(5y^2 - 6)(2y^2 - 8y - 1)$

6. $(4t + 5)(2t + 3)$

2. Special Case Products: Difference of Squares and Perfect Square Trinomials

In some cases the product of two binomials takes on a special pattern.

I. The first special case occurs when multiplying the sum and difference of the same two terms. For example:

$$\left.\begin{aligned}&(2x + 3)(2x - 3)\\ &\quad = 4x^2 - 6x + 6x - 9\\ &\quad = 4x^2 - 9\end{aligned}\right\}$$

Notice that the "middle terms" are opposites. This leaves only the difference between the square of the first term and the square of the second term. For this reason, the product is called a *difference of squares*.

Definition of Conjugates

The sum and difference of the same two terms are called **conjugates**. For example, we call $2x + 3$ the conjugate of $2x - 3$ and vice versa.

In general, $a + b$ and $a - b$ are conjugates of each other.

Skill Practice Answers

5. $10y^4 - 40y^3 - 17y^2 + 48y + 6$
6. $8t^2 + 22t + 15$

II. The second special case involves the square of a binomial. For example:

$$\left.\begin{array}{l} (3x+7)^2 \\ \quad = (3x+7)(3x+7) \\ \quad = 9x^2 + 21x + 21x + 49 \\ \quad = 9x^2 \quad + \quad 42x \quad + \quad 49 \\ \qquad \uparrow \qquad\quad \uparrow \qquad\quad\; \uparrow \\ \quad = (3x)^2 + 2(3x)(7) + (7)^2 \end{array}\right\}$$

When squaring a binomial, the product will be a trinomial called a *perfect square trinomial*. The first and third terms are formed by squaring the terms of the binomial. The middle term is twice the product of the terms in the binomial.

Note: The expression $(3x - 7)^2$ also results in a perfect square trinomial, but the middle term is negative.

$$(3x-7)(3x-7) = 9x^2 - 21x - 21x + 49 = 9x^2 - 42x + 49$$

The following table summarizes these special case products.

Special Case Product Formulas

1. $(a+b)(a-b) = a^2 - b^2$ The product is called a **difference of squares**.

2. $\left.\begin{array}{l} (a+b)^2 = a^2 + 2ab + b^2 \\ (a-b)^2 = a^2 - 2ab + b^2 \end{array}\right\}$ The product is called a **perfect square trinomial**.

It is advantageous for you to become familiar with these special case products because they will be presented again when we factor polynomials.

Example 4 Finding Special Products

Use the special product formulas to multiply the polynomials.

a. $(5x - 2)^2$ **b.** $(6c - 7d)(6c + 7d)$ **c.** $(4x^3 + 3y^2)^2$

Solution:

a. $(5x-2)^2$	$a = 5x, b = 2$
$= (5x)^2 - 2(5x)(2) + (2)^2$	Apply the formula $a^2 - 2ab + b^2$.
$= 25x^2 - 20x + 4$	Simplify each term.
b. $(6c - 7d)(6c + 7d)$	$a = 6c, b = 7d$
$= (6c)^2 - (7d)^2$	Apply the formula $a^2 - b^2$.
$= 36c^2 - 49d^2$	Simplify each term.
c. $(4x^3 + 3y^2)^2$	$a = 4x^3, b = 3y^2$
$= (4x^3)^2 + 2(4x^3)(3y^2) + (3y^2)^2$	Apply the formula $a^2 + 2ab + b^2$.
$= 16x^6 + 24x^3y^2 + 9y^4$	Simplify each term.

Skill Practice Multiply the polynomials.

7. $(c - 3)^2$ **8.** $(5x - 4)(5x + 4)$ **9.** $(7s^2 + 2t)^2$

The special case products can be used to simplify more complicated algebraic expressions.

Example 5 Using Special Products

Multiply the following expressions.

a. $(x + y)^3$ **b.** $[x + (y + z)][x - (y + z)]$

Solution:

a. $(x + y)^3$

$= (x + y)^2(x + y)$	Rewrite as the square of a binomial and another factor.
$= (x^2 + 2xy + y^2)(x + y)$	Expand $(x + y)^2$ by using the special case product formula.
$= (x^2)(x) + (x^2)(y) + (2xy)(x) + (2xy)(y) + (y^2)(x) + (y^2)(y)$	Apply the distributive property.
$= x^3 + x^2y + 2x^2y + 2xy^2 + xy^2 + y^3$	Simplify each term.
$= x^3 + 3x^2y + 3xy^2 + y^3$	Combine *like* terms.

b. $[x + (y + z)][x - (y + z)]$	This product is in the form $(a + b)(a - b)$, where $a = x$ and $b = (y + z)$.
$= (x)^2 - (y + z)^2$	Apply the formula $a^2 - b^2$.
$= (x)^2 - (y^2 + 2yz + z^2)$	Expand $(y + z)^2$ by using the special case product formula.
$= x^2 - y^2 - 2yz - z^2$	Apply the distributive property.

Skill Practice Multiply the polynomials.

10. $(b + 2)^3$ **11.** $[a + (b + 3)][a - (b + 3)]$

3. Translations Involving Polynomials

Example 6 Translating Between English Form and Algebraic Form

Complete the table.

English Form	Algebraic Form
The square of the sum of x and y	
	$x^2 + y^2$
The square of the product of 3 and x	

Skill Practice Answers

7. $c^2 - 6c + 9$ **8.** $25x^2 - 16$
9. $49s^4 + 28s^2t + 4t^2$
10. $b^3 + 6b^2 + 12b + 8$
11. $a^2 - b^2 - 6b - 9$

Solution:

English Form	Algebraic Form	Notes
The square of the sum of x and y	$(x + y)^2$	The *sum* is squared, not the individual terms.
The sum of the squares of x and y	$x^2 + y^2$	The individual terms x and y are squared first. Then the sum is taken.
The square of the product of 3 and x	$(3x)^2$	The product of 3 and x is taken. Then the result is squared.

Skill Practice Translate to algebraic form:

12. The square of the difference of a and b

13. The difference of the square of a and the square of b

14. Translate to English form: $a - b^2$.

4. Applications Involving a Product of Polynomials

Example 7 Applying a Product of Polynomials

A box is created from a sheet of cardboard 20 in. on a side by cutting a square from each corner and folding up the sides (Figures 5-3 and 5-4). Let x represent the length of the sides of the squares removed from each corner.

a. Find an expression for the volume of the box in terms of x.

b. Find the volume if a 4-in. square is removed.

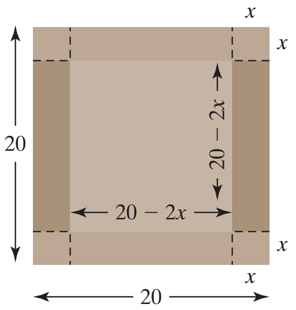


Figure 5-3

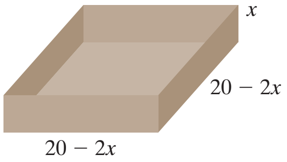


Figure 5-4

Solution:

a. The volume of a rectangular box is given by the formula $V = lwh$. The length and width can both be expressed as $20 - 2x$. The height of the box is x. Hence the volume is given by

$$\begin{aligned} V &= l \cdot w \cdot h \\ &= (20 - 2x)(20 - 2x)x \\ &= (20 - 2x)^2 x \\ &= (400 - 80x + 4x^2)x \\ &= 400x - 80x^2 + 4x^3 \\ &= 4x^3 - 80x^2 + 400x \end{aligned}$$

Skill Practice Answers

12. $(a - b)^2$ **13.** $a^2 - b^2$

14. The difference of a and the square of b

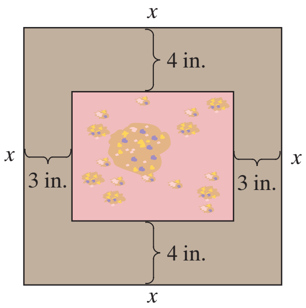
b. If a 4-in. square is removed from the corners of the box, we have $x = 4$ in. The volume is

$$\begin{aligned} V &= 4(4)^3 - 80(4)^2 + 400(4) \\ &= 4(64) - 80(16) + 400(4) \\ &= 256 - 1280 + 1600 \\ &= 576 \end{aligned}$$

The volume is 576 in.^{3}

Skill Practice

15. A rectangular photograph is mounted on a square piece of cardboard whose sides have length x. The border that surrounds the photo is 3 in. on each side and 4 in. on both top and bottom.



a. Write an expression for the area of the photograph and multiply.
b. Determine the area of the photograph if x is 12.

Skill Practice Answers

15a. $A = (x - 8)(x - 6)$;
$A = x^2 - 14x + 48$
b. 24 in.^{2}

Section 5.2 Practice Exercises

Boost *your* **GRADE** at mathzone.com!

MathZone

- Practice Problems
- Self-Tests
- NetTutor
- e-Professors
- Videos

Study Skills Exercise

1. Define the key terms.

a. Difference of squares **b. Conjugates** **c. Perfect square trinomial**

Review Exercises

2. Simplify. $(-4x^2y - 2xy + 3xy^2) - (2x^2y - 4xy^2) + (6x^2y + 5xy)$

3. Simplify. $(-2 - 3x) - [5 - (6x^2 + 4x + 1)]$

4. Given $f(x) = 4x^3 - 5$, find the function values.

a. $f(3)$ **b.** $f(0)$ **c.** $f(-2)$

5. Given $g(x) = x^4 - x^2 - 3$, find the function values.

a. $g(-1)$ **b.** $g(2)$ **c.** $g(0)$

For Exercises 6–7, perform the indicated operations.

6. $(3x^2 - 7x - 2) + (-x^2 + 3x - 5)$

7. $(3x^2 - 7x - 2) - (-x^2 + 3x - 5)$

8. Write the distributive property of multiplication over addition. Give an example of the distributive property. (Answers may vary.)

Concept 1: Multiplying Polynomials

For Exercises 9–46, multiply the polynomials by using the distributive property and the special product formulas.

9. $(7x^4y)(-6xy^5)$

10. $(-4a^3b^7)(-2ab^3)$

11. $\left(\frac{1}{4}tu^2\right)\left(8uv\right)$

12. $\left(-\frac{1}{5}mn^5\right)\left(-20np^3\right)$

13. $(2.2a^6b^4c^7)(5ab^4c^3)$

14. $(8.5c^4d^5e)(6cd^2e)$

15. $3ab(a + b)$

16. $2a(3 - a)$

17. $\frac{1}{5}(2a - 3)$

18. $\frac{1}{3}(6b + 4)$

19. $2m^3n^2(m^2n^3 - 3mn^2 + 4n)$

20. $3p^2q(p^3q^3 - pq^2 - 4p)$

21. $(x + y)(x - 2y)$

22. $(3a + 5)(a - 2)$

23. $(6x - 1)(5 + 2x)$

24. $(7 + 3x)(x - 8)$

25. $(4a - 9)(2a - 1)$

26. $(3b + 5)(b - 5)$

27. $(y^2 - 12)(2y^2 + 3)$

28. $(4p^2 - 1)(2p^2 + 5)$

29. $(5s + 3t)(5s - 2t)$

30. $(4a + 3b)(4a - b)$

31. $(n^2 + 10)(5n + 3)$

32. $(m^2 + 8)(3m + 7)$

33. $(1.3a - 4b)(2.5a + 7b)$

34. $(2.1x - 3.5y)(4.7x + 2y)$

35. $(2x + y)(3x^2 + 2xy + y^2)$

36. $(h - 5k)(h^2 - 2hk + 3k^2)$

37. $(x - 7)(x^2 + 7x + 49)$

38. $(x + 3)(x^2 - 3x + 9)$

39. $(4a - b)(a^3 - 4a^2b + ab^2 - b^3)$

40. $(3m + 2n)(m^3 + 2m^2n - mn^2 + 2n^3)$

41. $\left(\frac{1}{2}a - 2b + c\right)(a + 6b - c)$

42. $(x + y - 2z)(5x - y + z)$

43. $(-x^2 + 2x + 1)(3x - 5)$

44. $\left(\frac{1}{2}a^2 - 2ab + b^2\right)(2a + b)$

45. $\left(\frac{1}{5}y - 10\right)\left(\frac{1}{2}y - 15\right)$

46. $\left(\frac{2}{3}x + 6\right)\left(\frac{1}{2}x - 9\right)$

Concept 2: Special Case Products: Difference of Squares and Perfect Square Trinomials

For Exercises 47–66, multiply by using the special case products.

47. $(a - 8)(a + 8)$

48. $(b + 2)(b - 2)$

49. $(3p + 1)(3p - 1)$

50. $(5q - 3)(5q + 3)$

51. $\left(x - \frac{1}{3}\right)\left(x + \frac{1}{3}\right)$ **52.** $\left(\frac{1}{2}x + \frac{1}{3}\right)\left(\frac{1}{2}x - \frac{1}{3}\right)$ **53.** $(3h - k)(3h + k)$ **54.** $(x - 7y)(x + 7y)$

55. $(3h - k)^2$ **56.** $(x - 7y)^2$ **57.** $(t - 7)^2$ **58.** $(w + 9)^2$

59. $(u + 3v)^2$ **60.** $(a - 4b)^2$ **61.** $\left(h + \frac{1}{6}k\right)^2$ **62.** $\left(\frac{2}{5}x + 1\right)^2$

63. $(2z^2 - w^3)(2z^2 + w^3)$ **64.** $(a^4 - 2b^3)(a^4 + 2b^3)$ **65.** $(5x^2 - 3y)^2$ **66.** $(4p^3 - 2m)^2$

67. Multiply the expressions. Explain their similarities.

a. $(A - B)(A + B)$

b. $[(x + y) - B][(x + y) + B]$

68. Multiply the expressions. Explain their similarities.

a. $(A + B)(A - B)$

b. $[A + (3h + k)][A - (3h + k)]$

For Exercises 69–74, multiply the expressions.

69. $[(w + v) - 2][(w + v) + 2]$ **70.** $[(x + y) - 6][(x + y) + 6]$ **71.** $[2 - (x + y)][2 + (x + y)]$

72. $[a - (b + 1)][a + (b + 1)]$ **73.** $[(3a - 4) + b][(3a - 4) - b]$ **74.** $[(5p - 7) - q][(5p - 7) + q]$

75. Explain how to multiply $(x + y)^3$.

76. Explain how to multiply $(a - b)^3$.

For Exercises 77–80, multiply the expressions.

77. $(2x + y)^3$ **78.** $(x - 5y)^3$ **79.** $(4a - b)^3$ **80.** $(3a + 4b)^3$

81. Explain how you would multiply the binomials

$$(x - 2)(x + 6)(2x + 1)$$

82. Explain how you would multiply the binomials

$$(a + b)(a - b)(2a + b)(2a - b)$$

For Exercises 83–86, multiply the expressions containing more than two factors.

83. $2a^2(a + 5)(3a + 1)$ **84.** $-5y(2y - 3)(y + 3)$ **85.** $(x + 3)(x - 3)(x + 5)$ **86.** $(t + 2)(t - 3)(t + 1)$

Concept 3: Translations Involving Polynomials

For Exercises 87–90, translate from English form to algebraic form.

87. The square of the sum of r and t

88. The square of a plus the cube of b

89. The difference of x squared and y cubed

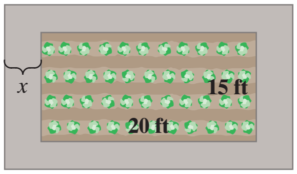
90. The square of the product of 3 and a

For Exercises 91–94, translate from algebraic form to English form.

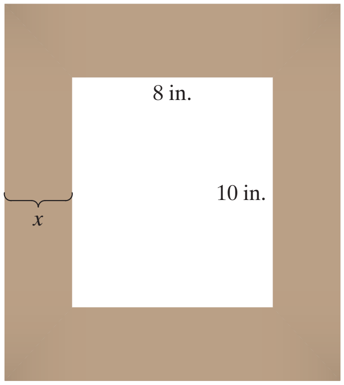
91. $p^3 + q^2$ **92.** $a^3 - b^3$ **93.** xy^2 **94.** $(c + d)^3$

Concept 4: Applications Involving a Product of Polynomials

95. A rectangular garden has a walk around it of width x. The garden is 20 ft by 15 ft. Find an expression representing the combined area A of the garden and walk. Simplify the result.



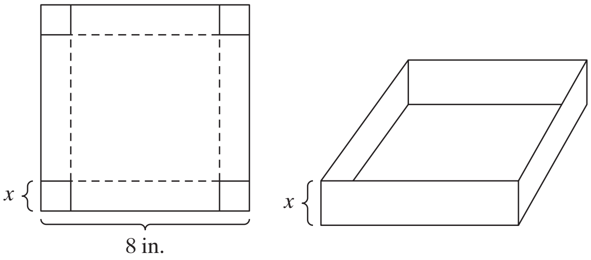
96. An 8-in. by 10-in. photograph is in a frame of width x. Find an expression that represents the area A of the frame alone. Simplify the result.



97. A box is created from a square piece of cardboard 8 in. on a side by cutting a square from each corner and folding up the sides. Let x represent the length of the sides of the squares removed from each corner.

a. Find an expression representing the volume of the box.

b. Find the volume if 1-in. squares are removed from the corners.



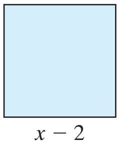
98. A box is created from a rectangular piece of metal with dimensions 12 in. by 9 in. by removing a square from each corner of the metal sheet and folding up the sides. Let x represent the length of the sides of the squares removed from each corner.

a. Find an expression representing the volume of the box.

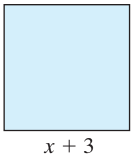
b. Find the volume if 2-in. squares are removed from the corners.

For Exercises 99–104, write an expression for the area and simplify your answer.

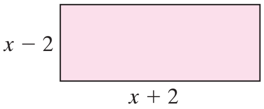
99. Square



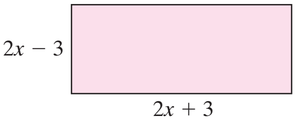
100. Square



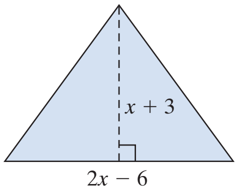
101. Rectangle



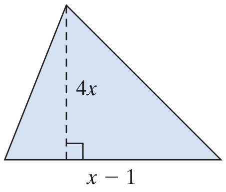
102. Rectangle



103. Triangle

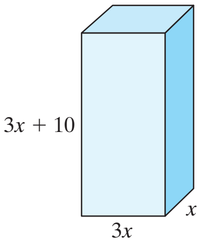


104. Triangle

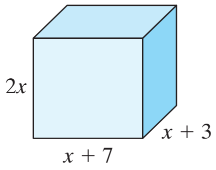


For Exercises 105–108, write an expression for the volume and simplify your answer.

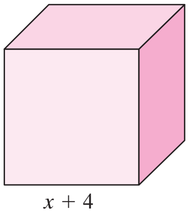
105.



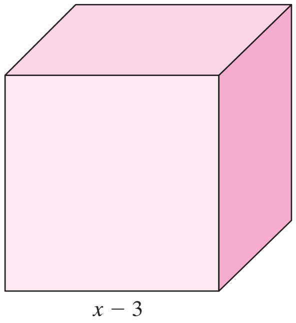
106.



107. Cube



108. Cube



Expanding Your Skills

109. Explain how to multiply $(x + 2)^4$.

110. Explain how to multiply $(y - 3)^4$.

111. $(2x - 3)$ multiplied by what binomial will result in the trinomial $10x^2 - 27x + 18$? Check your answer by multiplying the binomials.

112. $(4x + 1)$ multiplied by what binomial will result in the trinomial $12x^2 - 5x - 2$? Check your answer by multiplying the binomials.

113. $(4y + 3)$ multiplied by what binomial will result in the trinomial $8y^2 + 2y - 3$? Check your answer by multiplying the binomials.

114. $(3y - 2)$ multiplied by what binomial will result in the trinomial $3y^2 - 17y + 10$? Check your answer by multiplying the binomials.

Section 5.3 Division of Polynomials

Concepts

1. **Division by a Monomial**
2. **Long Division**
3. **Synthetic Division**

1. Division by a Monomial

Division of polynomials is presented in this section as two separate cases. The first case illustrates division by a monomial divisor. The second case illustrates division by a polynomial with two or more terms.

To divide a polynomial by a monomial, divide each individual term in the polynomial by the divisor and simplify the result.

To Divide a Polynomial by a Monomial

If a, b, and c are polynomials such that $c \neq 0$, then

$$\frac{a + b}{c} = \frac{a}{c} + \frac{b}{c} \qquad \text{Similarly,} \qquad \frac{a - b}{c} = \frac{a}{c} - \frac{b}{c}$$

Example 1 Dividing a Polynomial by a Monomial

Divide the polynomials.

a. $\dfrac{3x^4 - 6x^3 + 9x}{3x}$ **b.** $(10c^3d - 15c^2d^2 + 2cd^3) \div (5c^2d^2)$

Solution:

a. $\dfrac{3x^4 - 6x^3 + 9x}{3x}$

$= \dfrac{3x^4}{3x} - \dfrac{6x^3}{3x} + \dfrac{9x}{3x}$ Divide each term in the numerator by $3x$.

$= x^3 - 2x^2 + 3$ Simplify each term, using the properties of exponents.

b. $(10c^3d - 15c^2d^2 + 2cd^3) \div (5c^2d^2)$

$= \dfrac{10c^3d - 15c^2d^2 + 2cd^3}{5c^2d^2}$

$= \dfrac{10c^3d}{5c^2d^2} - \dfrac{15c^2d^2}{5c^2d^2} + \dfrac{2cd^3}{5c^2d^2}$ Divide each term in the numerator by $5c^2d^2$.

$= \dfrac{2c}{d} - 3 + \dfrac{2d}{5c}$ Simplify each term.

Skill Practice Divide.

1. $\dfrac{18y^3 - 6y^2 - 12y}{6y}$ **2.** $(-24a^3b^2 - 16a^2b^3 + 8ab) \div (-8ab)$

2. Long Division

If the divisor has two or more terms, a long division process similar to the division of real numbers is used.

Example 2 Using Long Division to Divide Polynomials

Divide the polynomials by using long division.

$$(3x^2 - 14x - 10) \div (x - 2)$$

Solution:

$$x - 2\,\overline{)\,3x^2 - 14x - 10}$$

Divide the leading term in the dividend by the leading term in the divisor.

$\dfrac{3x^2}{x} = 3x$. This is the first term in the quotient.

$$\begin{array}{r} 3x\phantom{{}-14x-10} \\ x - 2\,\overline{)\,3x^2 - 14x - 10} \\ \underline{3x^2 - 6x}\phantom{{}-10} \end{array}$$

Multiply $3x$ by the divisor and record the result: $3x(x - 2) = 3x^2 - 6x$.

Skill Practice Answers

1. $3y^2 - y - 2$
2. $3a^2b + 2ab^2 - 1$

$$\begin{array}{r} 3x \\ x-2\,\overline{)\,3x^2-14x-10} \\ \underline{-3x^2+6x} \\ -8x \end{array}$$

Next, subtract the quantity $3x^2 - 6x$. To do this, add its opposite.

$$\begin{array}{r} 3x-8 \\ x-2\,\overline{)\,3x^2-14x-10} \\ \underline{-3x^2+6x} \\ -8x-10 \\ \underline{-8x+16} \end{array}$$

Bring down next column and repeat the process.

Divide the leading term by x: $\dfrac{-8x}{x} = -8$

Multiply the divisor by -8 and record the result: $-8(x - 2) = -8x + 16$.

$$\begin{array}{r} 3x-8 \\ x-2\,\overline{)\,3x^2-14x-10} \\ \underline{-3x^2+6x} \\ -8x-10 \\ \underline{+8x-16} \\ -26 \end{array}$$

Subtract the quantity $(-8x + 16)$ by adding its opposite.

The remainder is -26. We do not continue because the degree of the remainder is less than the degree of the divisor.

Summary:

The quotient is	$3x - 8$
The remainder is	-26
The divisor is	$x - 2$
The dividend is	$3x^2 - 14x - 10$

The solution to a long division problem is often written in the form: Quotient + remainder/divisor. Hence

$$(3x^2 - 14x - 10) \div (x - 2) = \mathbf{3x - 8 + \frac{-26}{x - 2}}$$

This answer can also be written as

$$3x - 8 - \frac{26}{x - 2}$$

The division of polynomials can be checked in the same fashion as the division of real numbers. To check, we know that

$$\begin{aligned} &\text{Dividend} = (\text{divisor})(\text{quotient}) + \text{remainder} \\ 3x^2 - 14x - 10 &\stackrel{?}{=} (x - 2)(3x - 8) + (-26) \\ &\stackrel{?}{=} 3x^2 - 8x - 6x + 16 + (-26) \\ &= 3x^2 - 14x - 10 \; ✔ \end{aligned}$$

Skill Practice Divide.

3. $(4x^2 + 6x - 8) \div (x + 3)$

Skill Practice Answers

3. $4x - 6 + \dfrac{10}{x + 3}$

Example 3 Using Long Division to Divide Polynomials

Divide the polynomials by using long division: $(-2x^3 - 10x^2 + 56) \div (2x - 4)$

Solution:

First note that the dividend has a missing power of x and can be written as $-2x^3 - 10x^2 + 0x + 56$. The term $0x$ is a placeholder for the missing term. It is helpful to use the placeholder to keep the powers of x lined up.

$$\begin{array}{rrrrr}
 & & -x^2 & & \\ \cline{2-5}
2x-4\big) & -2x^3 & -10x^2 & +0x & +56 \\
 & -2x^3 & +4x^2 & & \\ \cline{2-3}
\end{array}$$

Leave space for the missing power of x.

Divide $\dfrac{-2x^3}{2x} = -x^2$ to get the first term of the quotient.

$$\begin{array}{rrrrr}
 & & -x^2 & -7x & \\ \cline{2-5}
2x-4\big) & -2x^3 & -10x^2 & +0x & +56 \\
 & 2x^3 & -4x^2 & & \\ \cline{2-3}
 & & -14x^2 & +0x & \\
 & & -14x^2 & +28x & \\
\end{array}$$

Subtract by adding the opposite.

Bring down the next column.

Divide $\dfrac{-14x^2}{2x} = -7x$ to get the next term in the quotient.

$$\begin{array}{rrrrr}
 & & -x^2 & -7x & -14 \\ \cline{2-5}
2x-4\big) & -2x^3 & -10x^2 & +0x & +56 \\
 & 2x^3 & -4x^2 & & \\ \cline{2-3}
 & & -14x^2 & +0x & \\
 & & 14x^2 & -28x & \\ \cline{3-4}
 & & & -28x & +56 \\
 & & & -28x & +56 \\
\end{array}$$

Subtract by adding the opposite.

Bring down the next column.

Divide $\dfrac{-28x}{2x} = -14$ to get the next term in the quotient.

$$\begin{array}{rrrrr}
 & & -x^2 & -7x & -14 \\ \cline{2-5}
2x-4\big) & -2x^3 & -10x^2 & +0x & +56 \\
 & 2x^3 & -4x^2 & & \\ \cline{2-3}
 & & -14x^2 & +0x & \\
 & & 14x^2 & -28x & \\ \cline{3-4}
 & & & -28x & +56 \\
 & & & 28x & -56 \\ \cline{4-5}
 & & & & 0 \\
\end{array}$$

Subtract by adding the opposite.

The remainder is 0.

The solution is $-x^2 - 7x - 14$.

Skill Practice Divide.

4. $\dfrac{4y^3 - 2y + 7}{2y + 2}$

TIP: Both the divisor and dividend must be written in descending order before you do polynomial division.

Skill Practice Answers

4. $2y^2 - 2y + 1 + \dfrac{5}{2y + 2}$

In Example 3, the quotient is $-x^2 - 7x - 14$ and the remainder is 0.

Because the remainder is zero, $2x - 4$ divides *evenly* into $-2x^3 - 10x^2 + 56$. For this reason, the divisor and quotient are *factors* of $-2x^3 - 10x^2 + 56$. To check, we have

$$\text{Dividend} = (\text{divisor})\ (\text{quotient}) + \text{remainder}$$

$$\begin{aligned} -2x^3 - 10x^2 + 56 &\overset{?}{=} (2x - 4)(-x^2 - 7x - 14) + 0 \\ &\overset{?}{=} -2x^3 - 14x^2 - 28x + 4x^2 + 28x + 56 \\ &= -2x^3 - 10x^2 + 56 \ ✔ \end{aligned}$$

Example 4 Using Long Division to Divide Polynomials

Divide.

$$15x^3 - 4 + 6x^4 - 5x^2 \div (3x^2 - 4)$$

Solution:

Write the dividend in descending powers of x: $6x^4 + 15x^3 - 5x^2 - 4$. The dividend has a missing power of x and can be written as $6x^4 + 15x^3 - 5x^2 + 0x - 4$.

The divisor has a missing power of x and can be written as $3x^2 + 0x - 4$.

$$\begin{array}{r} 2x^2 \phantom{{}+15x^3-5x^2+0x-4} \\ 3x^2 + 0x - 4 \,\overline{)\,6x^4 + 15x^3 - 5x^2 + 0x - 4} \end{array}$$

Leave space for missing powers of x.

$$\begin{array}{rrrrrr} & & & 2x^2 & +\ 5x & +\ 1 \\ 3x^2 + 0x - 4 \,\big) & 6x^4 & +\ 15x^3 & -\ 5x^2 & +\ 0x & -\ 4 \\ & \underline{-6x^4} & \underline{-\ 0x^3} & \underline{+\ 8x^2} & & \\ & & 15x^3 & +\ 3x^2 & +\ 0x & \\ & & \underline{-15x^3} & \underline{-\ 0x^2} & \underline{+\ 20x} & \\ & & & 3x^2 & +\ 20x & -\ 4 \\ & & & -3x^2 & -\ 0x & +\ 4 \\ & & & & 20x & \end{array}$$

The remainder is $20x$. The degree of $20x$ is less than the degree of $3x^2 - 4$.

The solution is $(6x^4 + 15x^3 - 5x^2 - 4) \div (3x^2 - 4) = 2x^2 + 5x + 1 + \dfrac{20x}{3x^2 - 4}$

Skill Practice Divide.

5. $(x^3 + 1 + 2x^2) \div (x^2 + 1)$

Skill Practice Answers

5. $x + 2 + \dfrac{-x - 1}{x^2 + 1}$

3. Synthetic Division

In this section we introduced the process of long division to divide two polynomials. Next, we will learn another technique, called **synthetic division**, to divide two polynomials. Synthetic division may be used when dividing a polynomial by a first-degree divisor of the form $x - r$, where r is a constant. Synthetic division is considered a "shortcut" because it uses the coefficients of the divisor and dividend without writing the variables.

Consider dividing the polynomials $(3x^2 - 14x - 10) \div (x - 2)$.

$$\begin{array}{r} 3x - 8 \\ x - 2\,\overline{\smash{)}\,3x^2 - 14x - 10} \\ -\underline{(3x^2 - 6x)} \\ -8x - 10 \\ -\underline{(-8x + 16)} \\ -26 \end{array}$$

First note that the divisor $x - 2$ is in the form $x - r$, where $r = 2$. Hence synthetic division can also be used to find the quotient and remainder.

Step 1: Write the value of r in a box.

Step 2: Write the coefficients of the dividend to the right of the box.

$$\begin{array}{r|rrr} \underline{2} & 3 & -14 & -10 \\ & & & \\ \hline & 3 & & \end{array}$$

Step 3: Skip a line and draw a horizontal line below the list of coefficients.

Step 4: Bring down the leading coefficient from the dividend and write it below the line.

$$\begin{array}{r|rrr} \underline{2} & 3 & -14 & -10 \\ & & 6 & \\ \hline & 3 & -8 & \end{array}$$

Step 5: Multiply the value of r by the number below the line ($2 \times 3 = 6$). Write the result in the next column above the line.

Step 6: Add the numbers in the column above the line ($-14 + 6$), and write the result below the line.

Repeat steps 5 and 6 until all columns have been completed.

Step 7: To get the final result, we use the numbers below the line. The number in the last column is the remainder. The other numbers are the coefficients of the quotient.

$$\begin{array}{r|rrr} \underline{2} & 3 & -14 & -10 \\ & & 6 & -16 \\ \hline & 3 & -8 & \boxed{-26} \end{array}$$

A box is usually drawn around the remainder.

Quotient: $3x - 8$, remainder $= -26$

The degree of the quotient will always be 1 less than that of the dividend. Because the dividend is a second-degree polynomial, the quotient will be a first-degree polynomial. In this case, the quotient is $3x - 8$ and the remainder is -26.

Example 5 Using Synthetic Division to Divide Polynomials

Divide the polynomials $(5x + 4x^3 - 6 + x^4) \div (x + 3)$ by using synthetic division.

Solution:

As with long division, the terms of the dividend and divisor should be written in descending order. Furthermore, missing powers must be accounted for by using placeholders (shown here in bold). Hence,

$$5x + 4x^3 - 6 + x^4$$
$$= x^4 + 4x^3 + \mathbf{0x^2} + 5x - 6$$

To use synthetic division, the divisor must be in the form $(x - r)$. The divisor $x + 3$ can be written as $x - (-3)$. Hence, $r = -3$.

Step 1: Write the value of r in a box.

Step 2: Write the coefficients of the dividend to the right of the box.

$$\begin{array}{r|rrrrr} -3 & 1 & 4 & 0 & 5 & -6 \\ & & & & & \\ \hline & 1 & & & & \end{array}$$

Step 3: Skip a line and draw a horizontal line below the list of coefficients.

Step 4: Bring down the leading coefficient from the dividend and write it below the line.

Step 5: Multiply the value of r by the number below the line $(-3 \times 1 = -3)$. Write the result in the next column above the line.

$$\begin{array}{r|rrrrr} -3 & 1 & 4 & 0 & 5 & -6 \\ & & -3 & & & \\ \hline & 1 & 1 & & & \end{array}$$

Step 6: Add the numbers in the column above the line: $4 + (-3) = 1$.

Repeat steps 5 and 6:

$$\begin{array}{r|rrrrr} -3 & 1 & 4 & 0 & 5 & -6 \\ & & -3 & -3 & 9 & -42 \\ \hline & 1 & 1 & -3 & 14 & \boxed{-48} \end{array}$$

remainder

constant

x-term coefficient

x^2-term coefficient

x^3-term coefficient

The quotient is

$x^3 + x^2 - 3x + 14$.

The remainder is -48.

The solution is $x^3 + x^2 - 3x + 14 + \dfrac{-48}{x + 3}$

Skill Practice Divide the polynomials by using synthetic division. Identify the quotient and the remainder.

6. $(5y^2 - 4y + 2y^3 - 5) \div (y + 3)$

TIP: It is interesting to compare the long division process to the synthetic division process. For Example 5, long division is shown on the left, and synthetic division is shown on the right. Notice that the same pattern of coefficients used in long division appears in the synthetic division process.

$$\begin{array}{r} x^3 + x^2 - 3x + 14 \\ x + 3\,\overline{)\,x^4 + 4x^3 + 0x^2 + 5x - 6} \\ \underline{-(x^4 + 3x^3)} \qquad\qquad\qquad\quad\;\; \\ x^3 + 0x^2 \qquad\qquad\;\; \\ \underline{-(x^3 + 3x^2)} \qquad\qquad\;\; \\ -3x^2 + 5x \qquad\; \\ \underline{-(-3x^2 - 9x)} \qquad\; \\ 14x - 6 \\ \underline{-(14x + 42)} \\ -48 \end{array}$$

$$\begin{array}{r|rrrrr} -3 & 1 & 4 & 0 & 5 & -6 \\ & & -3 & -3 & 9 & -42 \\ \hline & 1 & 1 & -3 & 14 & \boxed{-48} \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & x^3 & x^2 & x & \text{constant} & \text{remainder} \end{array}$$

Quotient: $x^3 + x^2 - 3x + 14$
Remainder: -48

Example 6 Using Synthetic Division to Divide Polynomials

Divide the polynomials by using synthetic division. Identify the quotient and remainder.

a. $(2m^7 - 3m^5 + 4m^4 - m + 8) \div (m + 2)$

b. $(p^4 - 81) \div (p - 3)$

Solution:

a. Insert placeholders (bold) for missing powers of m.

$(2m^7 - 3m^5 + 4m^4 - m + 8) \div (m + 2)$

$(2m^7 + \mathbf{0m^6} - 3m^5 + 4m^4 + \mathbf{0m^3} + \mathbf{0m^2} - m + 8) \div (m + 2)$

Because $m + 2$ can be written as $m - (-2)$, $r = -2$.

$$\begin{array}{r|rrrrrrrr} -2 & 2 & 0 & -3 & 4 & 0 & 0 & -1 & 8 \\ & & -4 & 8 & -10 & 12 & -24 & 48 & -94 \\ \hline & 2 & -4 & 5 & -6 & 12 & -24 & 47 & \boxed{-86} \end{array}$$

Quotient: $2m^6 - 4m^5 + 5m^4 - 6m^3 + 12m^2 - 24m + 47$
Remainder: -86

The quotient is 1 degree less than dividend.

The solution is $2m^6 - 4m^5 + 5m^4 - 6m^3 + 12m^2 - 24m + 47 + \dfrac{-86}{m + 2}$.

Skill Practice Answers

6. Quotient: $2y^2 - y - 1$; remainder: -2

b. $(p^4 - 81) \div (p - 3)$

$(p^4 + \mathbf{0p^3} + \mathbf{0p^2} + \mathbf{0p} - 81) \div (p - 3)$ Insert placeholders (bold) for missing powers of p.

$$\begin{array}{r|rrrrr} 3 & 1 & 0 & 0 & 0 & -81 \\ & & 3 & 9 & 27 & 81 \\ \hline & 1 & 3 & 9 & 27 & \boxed{0} \end{array}$$

Quotient: $p^3 + 3p^2 + 9p + 27$

Remainder: 0

The solution is $p^3 + 3p^2 + 9p + 27$.

Skill Practice Divide the polynomials by using synthetic division. Identify the quotient and the remainder.

7. $(4c^4 - 3c^2 - 6c - 3) \div (c - 2)$ **8.** $(x^3 + 1) \div (x + 1)$

Skill Practice Answers

7. Quotient: $4c^3 + 8c^2 + 13c + 20$; remainder: 37

8. Quotient: $x^2 - x + 1$; remainder: 0

Section 5.3 Practice Exercises

Boost *your* GRADE at mathzone.com!

- Practice Problems
- Self-Tests
- NetTutor
- e-Professors
- Videos

Study Skills Exercise

1. Define the key term **synthetic division**.

Review Exercises

2. a. Add $(3x + 1) + (2x - 5)$.

b. Multiply $(3x + 1)(2x - 5)$.

3. a. Subtract $(a - 10b) - (5a + b)$.

b. Multiply $(a - 10b)(5a + b)$.

4. a. Subtract $(2y^2 + 1) - (y^2 - 5y + 1)$.

b. Multiply $(2y^2 + 1)(y^2 - 5y + 1)$.

5. a. Add $(x^2 - x) + (6x^2 + x + 2)$.

b. Multiply $(x^2 - x)(6x^2 + x + 2)$.

For Exercises 6–8, answers may vary.

6. Write an example of a product of two binomials and simplify.

7. Write an example of the square of a binomial and simplify.

8. Write an example of the product of conjugates and simplify.

Concept 1: Division by a Monomial

For Exercises 9–24, divide the polynomials. Check your answer by multiplication.

9. $\dfrac{16t^4 - 4t^2 + 20t}{-4t}$

10. $\dfrac{2x^3 + 8x^2 - 2x}{-2x}$

11. $(36y + 24y^2 + 6y^3) \div (3y)$

12. $(6p^2 - 18p^4 + 30p^5) \div (6p)$

13. $(4x^3y + 12x^2y^2 - 4xy^3) \div (4xy)$

14. $(25m^5n - 10m^4n + m^3n) \div (5m^3n)$

15. $(-8y^4 - 12y^3 + 32y^2) \div (-4y^2)$

16. $(12y^5 - 8y^6 + 16y^4 - 10y^3) \div (2y^3)$

17. $(3p^4 - 6p^3 + 2p^2 - p) \div (-6p)$

18. $(-4q^3 + 8q^2 - q) \div (-12q)$

19. $(a^3 + 5a^2 + a - 5) \div (a)$

20. $(2m^5 - 3m^4 + m^3 - m^2 + 9m) \div (m^2)$

21. $\dfrac{6s^3t^5 - 8s^2t^4 + 10st^2}{-2st^4}$

22. $\dfrac{-8r^4w^2 - 4r^3w + 2w^3}{-4r^3w}$

23. $(8p^4q^7 - 9p^5q^6 - 11p^3q - 4) \div (p^2q)$

24. $(20a^5b^5 - 20a^3b^2 + 5a^2b + 6) \div (a^2b)$

Concept 2: Long Division

25. **a.** Divide $(2x^3 - 7x^2 + 5x - 1) \div (x - 2)$, and identify the divisor, quotient, and remainder.

b. Explain how to check by using multiplication.

26. **a.** Divide $(x^3 + 4x^2 + 7x - 3) \div (x + 3)$, and identify the divisor, quotient, and remainder.

b. Explain how to check by using multiplication.

For Exercises 27–42, divide the polynomials by using long division. Check your answer by multiplication.

27. $(x^2 + 11x + 19) \div (x + 4)$

28. $(x^3 - 7x^2 + 13x + 3) \div (x - 2)$

29. $(3y^3 - 7y^2 - 4y + 3) \div (y - 3)$

30. $(z^3 - 2z^2 + 2z - 5) \div (z - 4)$

31. $(-12a^2 + 77a - 121) \div (3a - 11)$

32. $(28x^2 - 29x + 6) \div (4x - 3)$

33. $(18y^2 + 9y - 20) \div (3y + 4)$

34. $(-3y^2 + 2y + 1) \div (-y + 1)$

35. $(8a^3 + 1) \div (2a + 1)$

36. $(81x^4 - 1) \div (3x + 1)$

37. $(x^4 - x^3 - x^2 + 4x - 2) \div (x^2 + x - 1)$

38. $(2a^5 - 7a^4 + 11a^3 - 22a^2 + 29a - 10)$
$\div (2a^2 - 5a + 2)$

39. $(x^4 - 3x^2 + 10) \div (x^2 - 2)$

40. $(3y^4 - 25y^2 - 18) \div (y^2 - 3)$

41. $(n^4 - 16) \div (n - 2)$

42. $(m^3 + 27) \div (m + 3)$

Concept 3: Synthetic Division

43. Explain the conditions under which you may use synthetic division to divide polynomials.

44. Can synthetic division be used to divide $(4x^4 + 3x^3 - 7x + 9)$ by $(2x + 5)$? Explain why or why not.

45. Can synthetic division be used to divide $(6x^5 - 3x^2 + 2x - 14)$ by $(x^2 - 3)$? Explain why or why not.

46. Can synthetic division be used to divide $(3x^4 - x + 1)$ by $(x - 5)$? Explain why or why not.

47. Can synthetic division be used to divide $(2x^3 - 4x + 6)$ by $(x + 4)$? Explain why or why not.

48. The following table represents the result of a synthetic division.

$$\begin{array}{r|rrrr} 5 & 1 & -2 & -4 & 3 \\ & & 5 & 15 & 55 \\ \hline & 1 & 3 & 11 & \lfloor\underline{58} \end{array}$$

Use x as the variable.

a. Identify the divisor.

b. Identify the quotient.

c. Identify the remainder.

49. The following table represents the result of a synthetic division.

$$\begin{array}{r|rrrrr} -2 & 2 & 3 & 0 & -1 & 6 \\ & & -4 & 2 & -4 & 10 \\ \hline & 2 & -1 & 2 & -5 & \lfloor\underline{16} \end{array}$$

Use x as the variable.

a. Identify the divisor.

b. Identify the quotient.

c. Identify the remainder.

For Exercises 50–61, divide by using synthetic division. Check your answer by multiplication.

50. $(x^2 - 2x - 48) \div (x - 8)$

51. $(x^2 - 4x - 12) \div (x - 6)$

52. $(t^2 - 3t - 4) \div (t + 1)$

53. $(h^2 + 7h + 12) \div (h + 3)$

54. $(5y^2 + 5y + 1) \div (y - 1)$

55. $(3w^2 + w - 5) \div (w + 2)$

56. $(3 + 7y^2 - 4y + 3y^3) \div (y + 3)$

57. $(2z - 2z^2 + z^3 - 5) \div (z + 3)$

58. $(x^3 - 3x^2 + 4) \div (x - 2)$

59. $(3y^4 - 25y^2 - 18) \div (y - 3)$

60. $(4w^4 - w^2 + 6w - 3) \div \left(w - \frac{1}{2}\right)$

61. $(-12y^4 - 5y^3 - y^2 + y + 3) \div \left(y + \frac{3}{4}\right)$

Mixed Exercises

For Exercises 62–73, divide the polynomials by using an appropriate method.

62. $(-x^3 - 8x^2 - 3x - 2) \div (x + 4)$

63. $(8xy^2 - 9x^2y + 6x^2y^2) \div (x^2y^2)$

64. $(22x^2 - 11x + 33) \div (11x)$

65. $(2m^3 - 4m^2 + 5m - 33) \div (m - 3)$

66. $(12y^3 - 17y^2 + 30y - 10) \div (3y^2 - 2y + 5)$

67. $(90h^{12} - 63h^9 + 45h^8 - 36h^7) \div (9h^9)$

68. $(4x^4 + 6x^3 + 3x - 1) \div (2x^2 + 1)$

69. $(y^4 - 3y^3 - 5y^2 - 2y + 5) \div (y + 2)$

70. $(16k^{11} - 32k^{10} + 8k^8 - 40k^4) \div (8k^8)$

71. $(4m^3 - 18m^2 + 22m - 10) \div (2m^2 - 4m + 3)$

72. $(5x^3 + 9x^2 + 10x) \div (5x^2)$

73. $(15k^4 + 3k^3 + 4k^2 + 4) \div (3k^2 - 1)$

Expanding Your Skills

74. Given $P(x) = 4x^3 + 10x^2 - 8x - 20$,

a. Evaluate $P(-4)$.

b. Divide. $(4x^3 + 10x^2 - 8x - 20) \div (x + 4)$

c. Compare the value found in part (a) to the remainder found in part (b).

75. Given $P(x) = -3x^3 - 12x^2 + 5x - 8$,

a. Evaluate $P(-6)$.

b. Divide. $(-3x^3 - 12x^2 + 5x - 8) \div (x + 6)$

c. Compare the value found in part (a) to the remainder found in part (b).

76. Based on your solutions to Exercises 74–75, make a conjecture about the relationship between the value of a polynomial function, $P(x)$ at $x = r$ and the value of the remainder of $P(x) \div (x - r)$.

77. **a.** Use synthetic division to divide. $(7x^2 - 16x + 9) \div (x - 1)$

b. Based on your solution to part (a), is $x - 1$ a *factor* of $7x^2 - 16x + 9$?

78. **a.** Use synthetic division to divide. $(8x^2 + 13x + 5) \div (x + 1)$

b. Based on your solution to part (a), is $x + 1$ a *factor* of $8x^2 + 13x + 5$?

Chapter 5 Problem Recognition Exercises—Operations on Polynomials

Perform the indicated operations.

1. $(5t^2 - 6t + 2) - (3t^2 - 7t + 3)$

2. $-5x^2(3x^2 + x - 2)$

3. $(3x + 1)^2$

4. $\dfrac{24a^3 - 8a^2 + 16a}{8a}$

5. $(6z + 5)(6z - 5)$

6. $(6y^3 + 2y^2 + y - 2) + (3y^3 - 4y + 3)$

7. $(3b - 4)(2b - 1)$

8. $\dfrac{4x^2 + 6x + 1}{2x - 1}$

9. $(5a + 2)(2a^2 + 3a + 1)$

10. $(t^3 - 4t^2 + t - 9) + (t + 12) - (2t^2 - 6t)$

11. $(2b^3 - 3b - 10) \div (b - 2)$

12. $(p - 5)(p + 5) - (2p^2 + 3)$

13. $(k + 4)^2 + (-4k + 9)$

14. $(3x^4 - 11x^3 - 4x^2 - 5x + 20) \div (x - 4)$

15. $-2t(t^2 + 6t - 3) + t(3t + 2)(3t - 2)$

16. $\dfrac{7x^2y^3 - 14xy^2 - x^2}{-7xy}$

17. $\left(\frac{1}{4}p^3 - \frac{1}{6}p^2 + 5\right) - \left(-\frac{2}{3}p^3 + \frac{1}{3}p^2 - \frac{1}{5}p\right)$

18. $-6w^3(1.2w - 2.6w^2 + 5.1w^3)$

19. $(6a^2 - 4b)^2$

20. $\left(\frac{1}{2}z^2 - \frac{1}{3}\right)\left(\frac{1}{2}z^2 + \frac{1}{3}\right)$

21. $(m^2 - 6m + 7) - (2m^2 + 4m - 3)$

22. $\dfrac{15x^3 - 10x^2 - 5x}{-5x}$

23. $(m^2 - 6m + 7)(2m^2 + 4m - 3)$

24. $(x^3 - 64) \div (x - 4)$

25. $[5 - (a + b)]^2$

26. $[a - (x - y)][a + (x - y)]$

27. $(x + y)^2 - (x - y)^2$

28. $(a - 4)^3$

29. $\left(-\frac{1}{2}x + \frac{1}{3}\right)\left(\frac{1}{4}x - \frac{1}{2}\right)$

30. $-3x^2y^3z^4\left(\frac{1}{6}x^4yzw^3\right)$

Section 5.4 Greatest Common Factor and Factoring by Grouping

Concepts

1. Factoring Out the Greatest Common Factor
2. Factoring Out a Negative Factor
3. Factoring Out a Binomial Factor
4. Factoring by Grouping

1. Factoring Out the Greatest Common Factor

Sections 5.4 through 5.7 are devoted to a mathematical operation called factoring. To factor an integer means to write the integer as a product of two or more integers. To factor a polynomial means to express the polynomial as a product of two or more polynomials.

In the product $5 \cdot 7 = 35$, for example, 5 and 7 are factors of 35.

In the product $(2x + 1)(x - 6) = 2x^2 - 11x - 6$, the quantities $(2x + 1)$ and $(x - 6)$ are factors of $2x^2 - 11x - 6$.

The **greatest common factor (GCF)** of a polynomial is the greatest factor that divides each term of the polynomial evenly. For example, the greatest common factor of $9x^4 + 18x^3 - 6x^2$ is $3x^2$. To factor out the greatest common factor from a polynomial, follow these steps:

Steps to Remove the Greatest Common Factor

1. Identify the greatest common factor of all terms of the polynomial.
2. Write each term as the product of the GCF and another factor.
3. Use the distributive property to factor out the greatest common factor.

Note: To check the factorization, multiply the polynomials.

Example 1 Factoring Out the Greatest Common Factor

Factor out the greatest common factor.

a. $12x^3 + 30x^2$ **b.** $12c^2d^3 - 30c^3d^2 - 3cd$

Solution:

a. $12x^3 + 30x^2$ The GCF is $6x^2$.

$= 6x^2(2x) + 6x^2(5)$ Write each term as the product of the GCF and another factor.

$= 6x^2(2x + 5)$ Factor out $6x^2$ by using the distributive property.

TIP: A factoring problem can be checked by multiplying the factors:

Check: $6x^2(2x + 5) = 12x^3 + 30x^2$ ✔

Avoiding Mistakes:

In Example 1(b), the GCF of $3cd$ is equal to one of the terms of the polynomial. In such a case, you must leave a 1 in place of that term after the GCF is factored out.

$3cd(4cd^2 - 10c^2d - 1)$

b. $12c^2d^3 - 30c^3d^2 - 3cd$ The GCF is $3cd$.

$= 3cd(4cd^2) - 3cd(10c^2d) - 3cd(1)$ Write each term as the product of the GCF and another factor.

$= 3cd(4cd^2 - 10c^2d - 1)$ Factor out $3cd$ by using the distributive property.

Check: $3cd(4cd^2 - 10c^2d - 1) = 12c^2d^3 - 30c^3d^2 - 3cd$ ✔

Skill Practice Factor out the greatest common factor.

1. $45y^5 - 15y^2 + 30y$ **2.** $16a^2b^5 + 12a^3b^3 + 4a^3b^2$

2. Factoring Out a Negative Factor

Sometimes it is advantageous to factor out the *opposite* of the GCF, particularly when the leading coefficient of the polynomial is negative. This is demonstrated in Example 2. Notice that this *changes the signs* of the remaining terms inside the parentheses.

Example 2 Factoring Out a Negative Factor

Factor out the quantity $-5a^2b$ from the polynomial $-5a^4b - 10a^3b^2 + 15a^2b^3$.

Solution:

$-5a^4b - 10a^3b^2 + 15a^2b^3$ The GCF is $5a^2b$. However, in this case we will factor out the opposite of the GCF, $-5a^2b$.

$= -5a^2b(a^2) + -5a^2b(2ab) + -5a^2b(-3b^2)$ Write each term as the product of $-5a^2b$ and another factor.

$= -5a^2b(a^2 + 2ab - 3b^2)$ Factor out $-5a^2b$ by using the distributive property.

Skill Practice

3. Factor out the quantity $-6xy$ from the polynomial $24x^4y^3 - 12x^2y + 18xy^2$.

3. Factoring Out a Binomial Factor

The distributive property may also be used to factor out a common factor that consists of more than one term. This is shown in Example 3.

Example 3 Factoring Out a Binomial Factor

Factor out the greatest common factor.

$$x^3(x + 2) - x(x + 2) - 9(x + 2)$$

Solution:

$x^3(x + 2) - x(x + 2) - 9(x + 2)$ The GCF is the quantity $(x + 2)$.

$= (x + 2)(x^3) - (x + 2)(x) - (x + 2)(9)$ Write each term as the product of $(x + 2)$ and another factor.

$= (x + 2)(x^3 - x - 9)$ Factor out $(x + 2)$ by using the distributive property.

Skill Practice

4. Factor out the greatest common factor.

$$a^2(b + 2) + 5(b + 2)$$

Skill Practice Answers

1. $15y(3y^4 - y + 2)$
2. $4a^2b^2(4b^3 + 3ab + a)$
3. $-6xy(-4x^3y^2 + 2x - 3y)$
4. $(b + 2)(a^2 + 5)$

4. Factoring by Grouping

When two binomials are multiplied, the product before simplifying contains four terms. For example:

$$\begin{aligned}(3a + 2)(2b - 7) &= (3a + 2)(2b) + (3a + 2)(-7)\\ &= (3a + 2)(2b) + (3a + 2)(-7)\\ &= 6ab + 4b - 21a - 14\end{aligned}$$

In Example 4, we learn how to reverse this process. That is, given a four-term polynomial, we will factor it as a product of two binomials. The process is called **factoring by grouping**.

Steps to Factor by Grouping

To factor a four-term polynomial by grouping:

1. Identify and factor out the GCF from all four terms.
2. Factor out the GCF from the first pair of terms. Factor out the GCF from the second pair of terms. (Sometimes it is necessary to factor out the *opposite* of the GCF.)
3. If the two terms share a common binomial factor, factor out the binomial factor.

Example 4 Factoring by Grouping

Factor by grouping.

$$6ab - 21a + 4b - 14$$

Solution:

$$6ab - 21a + 4b - 14$$

Step 1: Identify and factor out the GCF from all four terms. In this case the GCF is 1.

$$= 6ab - 21a \;\vdots\; + 4b - 14$$

Group the first pair of terms and the second pair of terms.

$$= 3a(2b - 7) + 2(2b - 7)$$

Step 2: Factor out the GCF from each pair of terms.

Note: The two terms now share a common binomial factor of $(2b - 7)$.

$$= (2b - 7)(3a + 2)$$

Step 3: Factor out the common binomial factor.

Check: $(2b - 7)(3a + 2) = 2b(3a) + 2b(2) - 7(3a) - 7(2)$

$$= 6ab + 4b - 21a - 14 ✔$$

Skill Practice Factor by grouping.

5. $7c^2 + cd + 14c + 2d$

Avoiding Mistakes:

In step 2, the expression $3a(2b - 7) + 2(2b - 7)$ is not yet factored because it is a *sum,* not a product. To factor the expression, you must carry it one step further.

$$3a(2b - 7) + 2(2b - 7)$$

$$= (2b - 7)(3a + 2)$$

The factored form must be represented as a product.

Skill Practice Answers

5. $(7c + d)(c + 2)$

Example 5 Factoring by Grouping

Factor by grouping.

$$x^3 + 3x^2 - 3x - 9$$

Solution:

$x^3 + 3x^2 - 3x - 9$

Step 1: Identify and factor out the GCF from all four terms. In this case the GCF is 1.

$= x^3 + 3x^2 \;\vdots\; -3x - 9$

Group the first pair of terms and the second pair of terms.

$= x^2(x + 3) - 3(x + 3)$

Step 2: Factor out x^2 from the first pair of terms.

Factor out -3 from the second pair of terms (this causes the signs to change in the second parentheses). The terms now contain a common binomial factor.

$= (x + 3)(x^2 - 3)$

Step 3: Factor out the common binomial $(x + 3)$.

TIP: One frequent question is, can the order be switched between factors? The answer is yes. Because multiplication is commutative, the order in which two or more factors are written does not matter. Thus, the following factorizations are equivalent:

$$(x + 3)(x^2 - 3) = (x^2 - 3)(x + 3)$$

Skill Practice Factor by grouping.

6. $a^3 - 4a^2 - 3a + 12$

Example 6 Factoring by Grouping

Factor by grouping.

$$24p^2q^2 - 18p^2q + 60pq^2 - 45pq$$

Solution:

$24p^2q^2 - 18p^2q + 60pq^2 - 45pq$

$= 3pq(8pq - 6p + 20q - 15)$

Step 1: Remove the GCF $3pq$ from all four terms.

$= 3pq(8pq - 6p \;\vdots\; + 20q - 15)$

Group the first pair of terms and the second pair of terms.

Skill Practice Answers

6. $(a^2 - 3)(a - 4)$

$= 3pq[2p(4q - 3) + 5(4q - 3)]$

Step 2: Factor out the GCF from each pair of terms. The terms share the binomial factor $(4q - 3)$.

$= 3pq(4q - 3)(2p + 5)$

Step 3: Factor out the common binomial $(4q - 3)$.

Skill Practice Factor the polynomial.

7. $24x^2y - 12x^2 + 20xy - 10x$

Notice that in step 3 of factoring by grouping, a common binomial is factored from the two terms. These binomials must be *exactly* the same in each term. If the two binomial factors differ, try rearranging the original four terms.

Example 7 Factoring by Grouping Where Rearranging Terms Is Necessary

Factor the polynomial.

$$4x + 6pa - 8a - 3px$$

Solution:

$4x + 6pa - 8a - 3px$

Step 1: Identify and factor out the GCF from all four terms. In this case the GCF is 1.

$= 4x + 6pa \;\vdots\; - 8a - 3px$

$= 2(2x + 3pa) - 1(8a + 3px)$

Step 2: The binomial factors in each term are different.

$= 4x - 8a \;\vdots\; - 3px + 6pa$

Try rearranging the original four terms in such a way that the first pair of coefficients is in the same ratio as the second pair of coefficients. Notice that the ratio 4 to 8 is the same as the ratio 3 to 6.

$= 4(x - 2a) - 3p(x - 2a)$

Step 2: Factor out 4 from the first pair of terms.

Factor out $-3p$ from the second pair of terms.

$= (x - 2a)(4 - 3p)$

Step 3: Factor out the common binomial factor.

Skill Practice Factor the polynomial.

8. $3ry + 2s + sy + 6r$

Skill Practice Answers

7. $2x(6x + 5)(2y - 1)$
8. $(3r + s)(2 + y)$

Section 5.4 Practice Exercises

Study Skills Exercise

1. Define the key terms.

a. Greatest common factor (GCF) **b. Factoring by grouping**

Review Exercises

For Exercises 2–8, perform the indicated operation.

2. $(-4a^3b^5c)(-2a^7c^2)$

3. $(7t^4 + 5t^3 - 9t) - (-2t^4 + 6t^2 - 3t)$

4. $(5x^3 - 9x + 5) + (4x^3 + 3x^2 - 2x + 1) - (6x^3 - 3x^2 + x + 1)$

5. $(5y^2 - 3)(y^2 + y + 2)$

6. $(a + 6b)^2$

7. $\dfrac{6v^3 - 12v^2 + 2v}{-2v}$

8. $\dfrac{3x^3 + 2x^2 - 4}{x + 2}$

Concept 1: Factoring Out the Greatest Common Factor

9. What is meant by a common factor in a polynomial? What is meant by the greatest common factor?

10. Explain how to find the greatest common factor of a polynomial.

For Exercises 11–26, factor out the greatest common factor.

11. $3x + 12$
12. $15x - 10$
13. $6z^2 + 4z$
14. $49y^3 - 35y^2$
15. $4p^6 - 4p$
16. $5q^2 - 5q$
17. $12x^4 - 36x^2$
18. $51w^4 - 34w^3$
19. $9st^2 + 27t$
20. $8a^2b^3 + 12a^2b$
21. $9a^2 + 27a + 18$
22. $3x^2 - 15x + 9$
23. $10x^2y + 15xy^2 - 35xy$
24. $12c^3d - 15c^2d + 3cd$
25. $13b^2 - 11a^2b - 12ab$
26. $6a^3 - 2a^2b + 5a^2$

Concept 2: Factoring Out a Negative Factor

For Exercises 27–32, factor out the indicated quantity.

27. $-x^2 - 10x + 7$: Factor out the quantity -1.

28. $-5y^2 + 10y + 3$: Factor out the quantity -1.

29. $12x^3y - 6x^2y - 3xy$: Factor out the quantity $-3xy$.

30. $32a^4b^2 + 24a^3b + 16a^2b$: Factor out the quantity $-8a^2b$.

31. $-2t^3 + 11t^2 - 3t$: Factor out the quantity $-t$.

32. $-7y^2z - 5yz - z$: Factor out the quantity $-z$.

Concept 3: Factoring Out a Binomial Factor

For Exercises 33–40, factor out the GCF.

33. $2a(3z - 2b) - 5(3z - 2b)$

34. $5x(3x + 4) + 2(3x + 4)$

35. $2x^2(2x - 3) + (2x - 3)$

36. $z(w - 9) + (w - 9)$

37. $y(2x + 1)^2 - 3(2x + 1)^2$

38. $a(b - 7)^2 + 5(b - 7)^2$

39. $3y(x - 2)^2 + 6(x - 2)^2$

40. $10z(z + 3)^2 - 2(z + 3)^2$

41. Solve the equation $U = Av + Acw$ for A by first factoring out A.

42. Solve the equation $S = rt + wt$ for t by first factoring out t.

43. Solve the equation $ay + bx = cy$ for y.

44. Solve the equation $cd + 2x = ac$ for c.

45. Construct a polynomial that has a greatest common factor of $3x^2$. (Answers may vary.)

46. Construct two different trinomials that have a greatest common factor of $5x^2y^3$. (Answers may vary.)

47. Construct a binomial that has a greatest common factor of $(c + d)$. (Answers may vary.)

Concept 4: Factoring by Grouping

48. If a polynomial has four terms, what technique would you use to factor it?

49. Factor the polynomials by grouping.

a. $2ax - ay + 6bx - 3by$

b. $10w^2 - 5w - 6bw + 3b$

c. Explain why you factored out $3b$ from the second pair of terms in part (a) but factored out the quantity $-3b$ from the second pair of terms in part (b).

50. Factor the polynomials by grouping.

a. $3xy + 2bx + 6by + 4b^2$

b. $15ac + 10ab - 6bc - 4b^2$

c. Explain why you factored out $2b$ from the second pair of terms in part (a) but factored out the quantity $-2b$ from the second pair of terms in part (b).

For Exercises 51–70, factor each polynomial by grouping (if possible).

51. $y^3 + 4y^2 + 3y + 12$

52. $ab + b + 2a + 2$

53. $6p - 42 + pq - 7q$

54. $2t - 8 + st - 4s$

55. $2mx + 2nx + 3my + 3ny$

56. $4x^2 + 6xy - 2xy - 3y^2$

57. $10ax - 15ay - 8bx + 12by$

58. $35a^2 - 15a + 14a - 6$

59. $x^3 - x^2 - 3x + 3$

60. $2rs + 4s - r - 2$

61. $6p^2q + 18pq - 30p^2 - 90p$

62. $5s^2t + 20st - 15s^2 - 60s$

63. $100x^3 - 300x^2 + 200x - 600$

64. $2x^5 - 10x^4 + 6x^3 - 30x^2$

65. $6ax - by + 2bx - 3ay$

66. $5pq - 12 - 4q + 15p$

67. $4a - 3b - ab + 12$

68. $x^2y + 6x - 3x^3 - 2y$

69. $7y^3 - 21y^2 + 5y - 10$

70. $5ax + 10bx - 2ac + 4bc$

71. Explain why the grouping method failed for Exercise 69.

72. Explain why the grouping method failed for Exercise 70.

73. The area of a rectangle of width w is given by $A = 2w^2 + w$. Factor the right-hand side of the equation to find an expression for the length of the rectangle.

74. The amount in a savings account bearing simple interest at an interest rate r for t years is given by

$A = P + Prt$ where P is the principal amount invested.

a. Solve the equation for P.

b. Compute the amount of principal originally invested if the account is worth \$12,705 after 3 years at a 7% interest rate.

Expanding Your Skills

For Exercises 75–82, factor out the greatest common factor and simplify.

75. $(a + 3)^4 + 6(a + 3)^5$

76. $(4 - b)^4 - 2(4 - b)^3$

77. $24(3x + 5)^3 - 30(3x + 5)^2$

78. $10(2y + 3)^2 + 15(2y + 3)^3$

79. $(t + 4)^2 - (t + 4)$

80. $(p + 6)^2 - (p + 6)$

81. $15w^2(2w - 1)^3 + 5w^3(2w - 1)^2$

82. $8z^4(3z - 2)^2 + 12z^3(3z - 2)^3$

Factoring Trinomials

Section 5.5

Concepts

1. Factoring Trinomials: AC-Method
2. Factoring Trinomials: Trial-and-Error Method
3. Factoring Trinomials with a Leading Coefficient of 1
4. Factoring Perfect Square Trinomials
5. Mixed Practice: Summary of Factoring Trinomials

1. Factoring Trinomials: AC-Method

In Section 5.4, we learned how to factor out the greatest common factor from a polynomial and how to factor a four-term polynomial by grouping. In this section we present two methods to factor trinomials. The first method is called the ac-method. The second method is called the trial-and-error method.

The product of two binomials results in a four-term expression that can sometimes be simplified to a trinomial. To factor the trinomial, we want to reverse the process.

$$\begin{aligned} \text{Multiply:}\quad (2x + 3)(x + 2) &\xrightarrow{\text{Multiply the binomials.}} 2x^2 + 4x + 3x + 6 \\ &\xrightarrow{\text{Add the middle terms.}} 2x^2 + 7x + 6 \end{aligned}$$

$$\begin{aligned} \text{Factor:}\quad 2x^2 + 7x + 6 &\xrightarrow{\substack{\text{Rewrite the middle term as}\\ \text{a sum or difference of terms.}}} 2x^2 + 4x + 3x + 6 \\ &\xrightarrow{\text{Factor by grouping.}} (2x + 3)(x + 2) \end{aligned}$$

To factor a trinomial $ax^2 + bx + c$ by the ac-method, we rewrite the middle term bx as a sum or difference of terms. The goal is to produce a four-term polynomial that can be factored by grouping. The process is outlined as follows.

The AC-Method to Factor $ax^2 + bx + c$ $(a \neq 0)$

1. Multiply the coefficients of the first and last terms, ac.
2. Find two integers whose product is ac and whose sum is b. (If no pair of integers can be found, then the trinomial cannot be factored further and is called a **prime polynomial**.)
3. Rewrite the middle term bx as the sum of two terms whose coefficients are the integers found in step 2.
4. Factor by grouping.

The ac-method for factoring trinomials is illustrated in Example 1. Before we begin, however, keep these two important guidelines in mind.

- For any factoring problem you encounter, always factor out the GCF from all terms first.
- To factor a trinomial, write the trinomial in the form $ax^2 + bx + c$.

Example 1 Factoring a Trinomial by the AC-Method

Factor. $12x^2 - 5x - 2$

Solution:

$12x^2 - 5x - 2$

The GCF is 1.

$a = 12 \qquad b = -5 \qquad c = -2$

Step 1: The expression is written in the form $ax^2 + bx + c$. Find the product $ac = 12(-2) = -24$.

Factors of –24	Factors of –24
(1)(−24)	(−1)(24)
(2)(−12)	(−2)(12)
(3)(−8)	(−3)(8)
(4)(−6)	(−4)(6)

Step 2: List all the factors of −24, and find the pair whose sum equals −5.

The numbers 3 and −8 produce a product of −24 and a sum of −5.

$$12x^2 - 5x - 2$$

$$= 12x^2 + 3x - 8x - 2$$

Step 3: Write the middle term of the trinomial as two terms whose coefficients are the selected numbers 3 and −8.

$$= 12x^2 + 3x \;\vdots\; - 8x - 2$$

Step 4: Factor by grouping.

$$= 3x(4x + 1) - 2(4x + 1)$$

$$= (4x + 1)(3x - 2)$$

The check is left for the reader.

Skill Practice

1. Factor $10x^2 + x - 3$.

Skill Practice Answers

1. $(5x + 3)(2x - 1)$

TIP: One frequently asked question is whether the order matters when we rewrite the middle term of the trinomial as two terms (step 3). The answer is no. From Example 1, the two middle terms in step 3 could have been reversed.

$$\begin{aligned} 12x^2 - 5x - 2 &= 12x^2 - 8x + 3x - 2 \\ &= 4x(3x - 2) + 1(3x - 2) \\ &= (3x - 2)(4x + 1) \end{aligned}$$

This example also shows that the order in which two factors are written does not matter. The expression $(3x - 2)(4x + 1)$ is equivalent to $(4x + 1)(3x - 2)$ because multiplication is a commutative operation.

Example 2 Factoring a Trinomial by the AC-Method

Factor the trinomial by using the ac-method. $-20c^3 + 34c^2d - 6cd^2$

Solution:

$-20c^3 + 34c^2d - 6cd^2$

$= -2c(10c^2 - 17cd + 3d^2)$ Factor out $-2c$.

Step 1: Find the product $a \cdot c = (10)(3) = 30$

Factors of 30	Factors of 30
$1 \cdot 30$	$(-1)(-30)$
$2 \cdot 15$	$(-2)(-15)$
$5 \cdot 6$	$(-5)(-6)$

Step 2: The numbers -2 and -15 form a product of 30 and a sum of -17.

$= -2c(10c^2 - 17cd + 3d^2)$

Step 3: Write the middle term of the trinomial as two terms whose coefficients are -2 and -15.

$= -2c(10c^2 - 2cd \mid - 15cd + 3d^2)$

Step 4: Factor by grouping.

$= -2c[2c(5c - d) - 3d(5c - d)]$

$= -2c(5c - d)(2c - 3d)$

Skill Practice Factor by the ac-method.

2. $-4wz^3 - 2w^2z^2 + 20w^3z$

TIP: In Example 2, removing the GCF from the original trinomial produced a new trinomial with smaller coefficients. This makes the factoring process simpler because the product ac is smaller.

Original trinomial	With the GCF factored out
$-20c^3 + 34c^2d - 6cd^2$	$-2c(10c^2 - 17cd + 3d^2)$
$ac = (-20)(-6) = 120$	$ac = (10)(3) = 30$

Skill Practice Answers

2. $-2wz(2z + 5w)(z - 2w)$

2. Factoring Trinomials: Trial-and-Error Method

Another method that is widely used to factor trinomials of the form $ax^2 + bx + c$ is the trial-and-error method. To understand how the trial-and-error method works, first consider the multiplication of two binomials:

$$(2x + 3)(1x + 2) = \overset{\text{Product of } 2 \cdot 1}{2x^2} \underbrace{+\, 4x + 3x}_{\substack{\text{sum of products}\\ \text{of inner terms}\\ \text{and outer terms}}} + \overset{\text{Product of } 3 \cdot 2}{6} = 2x^2 + 7x + 6$$

To factor the trinomial $2x^2 + 7x + 6$, this operation is reversed. Hence

Factors of 2

$$2x^2 + 7x + 6 = (\square x \quad \square)(\square x \quad \square)$$

Factors of 6

We need to fill in the blanks so that the product of the first terms in the binomials is $2x^2$ and the product of the last terms in the binomials is 6. Furthermore, the factors of $2x^2$ and 6 must be chosen so that the sum of the products of the inner terms and outer terms equals $7x$.

To produce the product $2x^2$, we might try the factors $2x$ and x within the binomials.

$$(2x \quad \square)(x \quad \square)$$

To produce a product of 6, the remaining terms in the binomials must either both be positive or both be negative. To produce a positive middle term, we will try positive factors of 6 in the remaining blanks until the correct product is found. The possibilities are $1 \cdot 6, 2 \cdot 3, 3 \cdot 2$, and $6 \cdot 1$.

$(2x + 1)(x + 6) = 2x^2 + 12x + 1x + 6 = 2x^2 + 13x + 6$	Wrong middle term
$(2x + 2)(x + 3) = 2x^2 + 6x + 2x + 6 = 2x^2 + 8x + 6$	Wrong middle term
$\mathbf{(2x + 3)(x + 2)} = 2x^2 + 4x + 3x + 6 = \mathbf{2x^2 + 7x + 6}$	Correct!
$(2x + 6)(x + 1) = 2x^2 + 2x + 6x + 6 = 2x^2 + 8x + 6$	Wrong middle term

The correct factorization of $2x^2 + 7x + 6$ is $(2x + 3)(x + 2)$. ✔

As this example shows, we factor a trinomial of the form $ax^2 + bx + c$ by shuffling the factors of a and c within the binomials until the correct product is obtained. However, sometimes it is not necessary to test all the possible combinations of factors. In this example, the GCF of the original trinomial is 1. Therefore, any binomial factor that shares a common factor *greater than 1* does not need to be considered. In this case the possibilities $(2x + 2)(x + 3)$ and $(2x + 6)(x + 1)$ cannot work.

$$\underbrace{(2x + 2)}_{\substack{\text{Common}\\ \text{factor of 2}}}(x + 3) \qquad \underbrace{(2x + 6)}_{\substack{\text{Common}\\ \text{factor of 2}}}(x + 1)$$

The steps to factor a trinomial by the trial-and-error method are outlined as follows.

The Trial-and-Error Method to Factor $ax^2 + bx + c$

1. Factor out the greatest common factor.
2. List all pairs of positive factors of a and pairs of positive factors of c. Consider the reverse order for either list of factors.
3. Construct two binomials of the form

Factors of a

$$(\square x \quad \square)(\square x \quad \square)$$

Factors of c

Test each combination of factors and signs until the correct product is found. If no combination of factors produces the correct product, the trinomial cannot be factored further and is a **prime polynomial**.

Example 3 Factoring a Trinomial by the Trial-and-Error Method

Factor the trinomial by the trial-and-error method. $10x^2 - 9x - 1$

Solution:

$10x^2 - 9x - 1$ **Step 1:** Factor out the GCF from all terms. The GCF is 1. The trinomial is written in the form $ax^2 + bx + c$.

To factor $10x^2 - 9x - 1$, two binomials must be constructed in the form

Factors of 10

$$(\square x \quad \square)(\square x \quad \square)$$

Factors of -1

Step 2: To produce the product $10x^2$, we might try $5x$ and $2x$ or $10x$ and $1x$. To produce a product of -1, we will try the factors $1(-1)$ and $-1(1)$.

Step 3: Construct all possible binomial factors, using different combinations of the factors of $10x^2$ and -1.

$(5x + 1)(2x - 1) = 10x^2 - 5x + 2x - 1 = 10x^2 - 3x - 1$ Wrong middle term

$(5x - 1)(2x + 1) = 10x^2 + 5x - 2x - 1 = 10x^2 + 3x - 1$ Wrong middle term

The numbers 1 and -1 did not produce the correct trinomial when coupled with $5x$ and $2x$, so we try $10x$ and $1x$.

$(10x - 1)(1x + 1) = 10x^2 + 10x - 1x - 1 = 10x^2 + 9x - 1$ Wrong middle term

$\mathbf{(10x + 1)(1x - 1)} = 10x^2 - 10x + 1x - 1 = \mathbf{10x^2 - 9x - 1}$ Correct!

Hence $10x^2 - 9x - 1 = (10x + 1)(x - 1)$

Skill Practice Factor by trial and error.

3. $5y^2 - 9y + 4$

Skill Practice Answers

3. $(5y - 4)(y - 1)$

In Example 3, the factors of -1 must have opposite signs to produce a negative product. Therefore, one binomial factor is a sum and one is a difference. Determining the correct signs is an important aspect of factoring trinomials. We suggest the following guidelines:

TIP: Given the trinomial $ax^2 + bx + c$ $(a > 0)$, the signs can be determined as follows:

1. If c is *positive*, then the signs in the binomials must be the same (either both positive or both negative). The correct choice is determined by the middle term. If the middle term is positive, then both signs must be positive. If the middle term is negative, then both signs must be negative.

Example: $20x^2 + 43x + \overset{\substack{c \text{ is positive.} \\ \downarrow}}{21}$ $\quad (4x \underbrace{+ \ 3)(5x \ +}_{\text{same signs}} 7)$

Example: $20x^2 - 43x + \overset{\substack{c \text{ is positive.} \\ \downarrow}}{21}$ $\quad (4x \underbrace{- \ 3)(5x \ -}_{\text{same signs}} 7)$

2. If c is *negative*, then the signs in the binomials must be different. The middle term in the trinomial determines which factor gets the positive sign and which factor gets the negative sign.

Example: $x^2 + 3x - \overset{\substack{c \text{ is negative.} \\ \downarrow}}{28}$ $\quad (x \underbrace{+ \ 7)(x \ -}_{\text{different signs}} 4)$

Example: $x^2 - 3x - \overset{\substack{c \text{ is negative.} \\ \downarrow}}{28}$ $\quad (x \underbrace{- \ 7)(x \ +}_{\text{different signs}} 4)$

Example 4 Factoring a Trinomial

Factor the trinomial by the trial-and-error method. $8y^2 + 13y - 6$

Solution:

$8y^2 + 13y - 6$

Step 1: The GCF is 1.

$(\square y \ \ \square)(\square y \ \ \square)$

Factors of 8	Factors of 6
$1 \cdot 8$	$1 \cdot 6$
$2 \cdot 4$	$2 \cdot 3$
	$3 \cdot 2$ (reverse order)
	$6 \cdot 1$ (reverse order)

Step 2: List the positive factors of 8 and positive factors of 6. Consider the reverse order in one list of factors.

Step 3: Construct all possible binomial factors by using different combinations of the factors of 8 and 6.

$(2y \quad 1)(4y \quad 6)$
$(2y \quad 2)(4y \quad 3)$
$(2y \quad 3)(4y \quad 2)$
$(2y \quad 6)(4y \quad 1)$
$(1y \quad 1)(8y \quad 6)$
$(1y \quad 3)(8y \quad 2)$

Without regard to signs, these factorizations cannot work because the terms in the binomial share a common factor greater than 1.

Test the remaining factorizations. Keep in mind that to produce a product of -6, the signs within the parentheses must be opposite (one positive and one negative). Also, the sum of the products of the inner terms and outer terms must be combined to form $13y$.

$(1y \quad 6)(8y \quad 1)$ *Incorrect.* Wrong middle term.

Regardless of signs, the product of inner terms $48y$ and the product of outer terms $1y$ cannot be combined to form the middle term $13y$.

$(1y \quad 2)(8y \quad 3)$ *Correct.* The terms $16y$ and $3y$ can be combined to form the middle term $13y$, provided the signs are applied correctly. We require $+16y$ and $-3y$.

Hence, the correct factorization of $8y^2 + 13y - 6$ is $(y + 2)(8y - 3)$.

Skill Practice Factor by trial-and-error.

4. $4t^2 + 5t - 6$

Example 5 Factoring a Trinomial by the Trial-and-Error Method

Factor the trinomial by the trial-and-error method.

$$-80x^3y + 208x^2y^2 - 20xy^3$$

Solution:

$-80x^3y + 208x^2y^2 - 20xy^3$

$= -4xy(20x^2 - 52xy + 5y^2)$ **Step 1:** Factor out $-4xy$.

$= -4xy(\square x \quad \square y)(\square x \quad \square y)$

Factors of 20	**Factors of 5**
$1 \cdot 20$	$1 \cdot 5$
$2 \cdot 10$	$5 \cdot 1$
$4 \cdot 5$	

Step 2: List the positive factors of 20 and positive factors of 5. Consider the reverse order in one list of factors.

Step 3: Construct all possible binomial factors by using different combinations of the factors of 20 and factors of 5. The signs in the parentheses must both be negative.

$$\left.\begin{aligned} &-4xy(1x - 1y)(20x - 5y) \\ &-4xy(2x - 1y)(10x - 5y) \\ &-4xy(4x - 1y)(5x - 5y) \end{aligned}\right\}$$

Incorrect. These binomials contain a common factor.

Skill Practice Answers

4. $(4t - 3)(t + 2)$

$-4xy(1x - 5y)(20x - 1y)$	*Incorrect.*	Wrong middle term. $-4xy(x - 5y)(20x - 1y)$ $= -4xy(20x^2 - 101xy + 5y^2)$
$-4xy(2x - 5y)(10x - 1y)$	***Correct.***	$-4xy(2x - 5y)(10x - 1y)$ $= \mathbf{-4xy(20x^2 - 52xy + 5y^2)}$ $= -80x^3y + 208x^2y^2 - 20xy^3$
$-4xy(4x - 5y)(5x - 1y)$	*Incorrect.*	Wrong middle term. $-4xy(4x - 5y)(5x - 1y)$ $= -4xy(20x^2 - 29x + 5y^2)$

The correct factorization of $-80x^3y + 208x^2y^2 - 20xy^3$ is $-4xy(2x - 5y)(10x - y)$.

Skill Practice Factor by the trial-and-error method.

5. $-4z^3 - 22z^2 - 30z$

3. Factoring Trinomials with a Leading Coefficient of 1

If a trinomial has a leading coefficient of 1, the factoring process simplifies significantly. Consider the trinomial $x^2 + bx + c$. To produce a leading term of x^2, we can construct binomials of the form $(x + \square)(x + \square)$. The remaining terms may be satisfied by two numbers p and q whose product is c and whose sum is b:

$$(x + \overbrace{p)(x + q}^{\text{Factors of } c}) = x^2 + qx + px + pq = x^2 + \underbrace{(p + q)}_{\text{Sum } = b}x + \underbrace{pq}_{\text{Product } = c}$$

This process is demonstrated in Example 6.

Example 6 Factoring a Trinomial with a Leading Coefficient of 1

Factor the trinomial.

$$x^2 - 10x + 16$$

Solution:

$x^2 - 10x + 16$	Factor out the GCF from all terms. In this case, the GCF is 1.
$= (x \quad \square)(x \quad \square)$	The trinomial is written in the form $x^2 + bx + c$. To form the product x^2, use the factors x and x.

Skill Practice Answers

5. $-2z(2z + 5)(z + 3)$

Next, look for two numbers whose product is 16 and whose sum is -10. Because the middle term is negative, we will consider only the negative factors of 16.

Factors of 16	**Sum**
$-1(-16)$	$-1 + (-16) = -17$
$\mathbf{-2(-8)}$	$\mathbf{-2 + (-8) = -10}$
$-4(-4)$	$-4 + (-4) = -8$

The numbers are -2 and -8.

Hence $x^2 - 10x + 16 = (x - 2)(x - 8)$

Skill Practice Factor.

6. $c^2 + 6c - 27$

4. Factoring Perfect Square Trinomials

Recall from Section 5.2 that the square of a binomial always results in a **perfect square trinomial**.

$$(a + b)^2 = (a + b)(a + b) = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = (a - b)(a - b) = a^2 - ab - ab + b^2 = a^2 - 2ab + b^2$$

For example, $(\underset{\substack{\swarrow \\ a\,=\,2x}}{2x} + \underset{\substack{\downarrow \\ b\,=\,7}}{7})^2 = (2x)^2 + 2(2x)(7) + (7)^2 = \underset{\substack{\downarrow \\ a^2}}{4x^2} + \underset{\substack{\downarrow \\ 2ab}}{28x} + \underset{\substack{\downarrow \\ b^2}}{49}$

To factor the trinomial $4x^2 + 28x + 49$, the ac-method or the trial-and-error method can be used. However, recognizing that the trinomial is a perfect square trinomial, we can use one of the following patterns to reach a quick solution.

Factored Form of a Perfect Square Trinomial

$$a^2 + 2ab + b^2 = (a + b)^2$$

$$a^2 - 2ab + b^2 = (a - b)^2$$

TIP: To determine if a trinomial is a perfect square trinomial, follow these steps:

1. Check if the first and third terms are both perfect squares with positive coefficients.
2. If this is the case, identify *a* and *b*, and determine if the middle term equals 2*ab*.

Example 7 Factoring Perfect Square Trinomials

Factor the trinomials completely.

a. $x^2 + 12x + 36$ **b.** $4x^2 - 36xy + 81y^2$

Skill Practice Answers

6. $(c + 9)(c - 3)$

Solution:

a. $x^2 + 12x + 36$

The GCF is 1.

Perfect squares

$$= \underbrace{x^2}_{} + 12x + \underbrace{36}_{}$$

- The first and third terms are positive.
- The first term is a perfect square: $x^2 = (x)^2$
- The third term is a perfect square: $36 = (6)^2$
- The middle term is twice the product of x and 6:

$$12x = 2(x)(6)$$

$$= (x)^2 + 2(x)(6) + (6)^2$$

Hence the trinomial is in the form $a^2 + 2ab + b^2$, where $a = x$ and $b = 6$.

$$= (x + 6)^2$$

Factor as $(a + b)^2$.

b. $4x^2 - 36xy + 81y^2$

The GCF is 1.

Perfect squares

$$= 4x^2 - 36xy + 81y^2$$

- The first and third terms are positive.
- The first term is a perfect square: $4x^2 = (2x)^2$.
- The third term is a perfect square: $81y^2 = (9y)^2$.
- The middle term:

$$-36xy = -2(2x)(9y)$$

$$= (2x)^2 - 2(2x)(9y) + (9y)^2$$

The trinomial is in the form $a^2 - 2ab + b^2$, where $a = 2x$ and $b = 9y$.

$$= (2x - 9y)^2$$

Factor as $(a - b)^2$.

Skill Practice Factor completely.

7. $x^2 + 2x + 1$

8. $9y^2 - 12yz + 4z^2$

5. Mixed Practice: Summary of Factoring Trinomials

Summary: Factoring Trinomials of the Form $ax^2 + bx + c$ $(a \neq 0)$

When factoring trinomials, the following guidelines should be considered:

1. Factor out the greatest common factor.
2. Check to see if the trinomial is a perfect square trinomial. If so, factor it as either $(a + b)^2$ or $(a - b)^2$. (With a perfect square trinomial, you do not need to use the ac-method or trial-and-error method.)
3. If the trinomial is not a perfect square, use either the ac-method or the trial-and-error method to factor.
4. Check the factorization by multiplication.

Skill Practice Answers

7. $(x + 1)^2$ **8.** $(3y - 2z)^2$

Example 8 Factoring Trinomials

Factor the trinomials completely.

a. $80s^3t + 80s^2t^2 + 20st^3$ **b.** $5w^2 + 50w + 45$ **c.** $2p^2 + 9p + 14$

Solution:

a. $80s^3t + 80s^2t^2 + 20st^3$

$= 20st(4s^2 + 4st + t^2)$ The GCF is $20st$.

Perfect squares

- The first and third terms are positive.
- The first and third terms are perfect squares: $4s^2 = (2s)^2$ and $t^2 = (t)^2$
- Because $4st = 2(2s)(t)$, the trinomial is in the form $a^2 + 2ab + b^2$, where $a = 2s$ and $b = t$.

$= 20st(4s^2 + 4st + t^2)$

$= 20st(2s + t)^2$ Factor as $(a + b)^2$.

b. $5w^2 + 50w + 45$

$= 5(w^2 + 10w + 9)$ The GCF is 5.

Perfect squares

The first and third terms are perfect squares: $w^2 = (w)^2$ and $9 = (3)^2$.

$= 5(w^2 + 10w + 9)$ However, the middle term $10w \neq 2(w)(3)$. Therefore, this is *not* a perfect square trinomial.

$= 5(w + 9)(w + 1)$ To factor, use either the ac-method or the trial-and-error method.

c. $2p^2 + 9p + 14$ The GCF is 1. The trinomial is not a perfect square trinomial because neither 2 nor 14 is a perfect square. Therefore, try factoring by either the ac-method or the trial-and-error method. We use the trial-and-error method here.

Factors of 2	**Factors of 14**
$2 \cdot 1$	$1 \cdot 14$
	$14 \cdot 1$
	$2 \cdot 7$
	$7 \cdot 2$

After constructing all factors of 2 and 14, we see that no combination of factors will produce the correct result.

$(2p + 14)(p + 1)$ *Incorrect*: $(2p + 14)$ contains a common factor of 2.

$(2p + 2)(p + 7)$ *Incorrect*: $(2p + 2)$ contains a common factor of 2.

$(2p + 1)(p + 14) = 2p^2 + 28p + p + 14 \longrightarrow 2p^2 + 29p + 14$ *Incorrect*

(wrong middle term)

$(2p + 7)(p + 2) = 2p^2 + 4p + 7p + 14 \longrightarrow 2p^2 + 11p + 14$ *Incorrect*

(wrong middle term)

Because none of the combinations of factors results in the correct product, we say that the trinomial $2p^2 + 9p + 14$ is prime. This polynomial cannot be factored by the techniques presented here.

Skill Practice Factor completely.

9. $-x^2 + 6x - 9$ **10.** $6v^2 - 12v - 18$ **11.** $6r^2 - 13rs + 10s^2$

Skill Practice Answers

9. $-(x - 3)^2$
10. $6(v + 1)(v - 3)$
11. Prime

Section 5.5 Practice Exercises

Boost *your* GRADE at mathzone.com!

MathZone

- Practice Problems
- Self-Tests
- NetTutor
- e-Professors
- Videos

Study Skills Exercise

1. Define the key terms:

a. prime polynomial **b. perfect square trinomial**

Review Exercises

2. Explain how to check a factoring problem.

For Exercises 3–8, factor the polynomial completely.

3. $36c^2d^7e^{11} + 12c^3d^5e^{15} - 6c^2d^4e^7$

4. $5x^3y^3 + 15x^4y^2 - 35x^2y^4$

5. $2x(3a - b) - (3a - b)$

6. $6(v - 8) - 3u(v - 8)$

7. $wz^2 + 2wz - 33az - 66a$

8. $3a^2x + 9ab - abx - 3b^2$

Concepts 1–3: Factoring Trinomials

In Exercises 9–46, factor the trinomial completely by using any method. Remember to look for a common factor first.

9. $b^2 - 12b + 32$

10. $a^2 - 12a + 27$

11. $y^2 + 10y - 24$

12. $w^2 + 3w - 54$

13. $x^2 + 13x + 30$

14. $t^2 + 9t + 8$

15. $c^2 - 6c - 16$

16. $z^2 - 3z - 28$

17. $2x^2 - 7x - 15$

18. $2y^2 - 13y + 15$

19. $a + 6a^2 - 5$

20. $10b^2 - 3 - 29b$

21. $s^2 + st - 6t^2$

22. $p^2 - pq - 20q^2$

23. $3x^2 - 60x + 108$

24. $4c^2 + 12c - 72$

25. $2c^2 - 2c - 24$

26. $3x^2 + 12x - 15$

27. $2x^2 + 8xy - 10y^2$

28. $20z^2 + 26zw - 28w^2$

29. $33t^2 - 18t + 2$

30. $5p^2 - 10p + 7$

31. $3x^2 + 14xy + 15y^2$

32. $2a^2 + 15ab - 27b^2$

33. $5u^3v - 30u^2v^2 + 45uv^3$

34. $3a^3 + 30a^2b + 75ab^2$

35. $x^3 - 5x^2 - 14x$

36. $p^3 + 2p^2 - 24p$

37. $-23z - 5 + 10z^2$

38. $3 + 16y^2 + 14y$

39. $b^2 + 2b + 15$

40. $x^2 - x - 1$

41. $-2t^2 + 12t + 80$

42. $-3c^2 + 33c - 72$

43. $14a^2 + 13a - 12$

44. $12x^2 - 16x + 5$

45. $6a^2b + 22ab + 12b$

46. $6cd^2 + 9cd - 42c$

Concept 4: Factoring Perfect Square Trinomials

47. **a.** Multiply the binomials $(x + 5)(x + 5)$.
  b. How do you factor $x^2 + 10x + 25$?

48. **a.** Multiply the binomials $(2w - 5)(2w - 5)$.
  b. How do you factor $4w^2 - 20w + 25$?

49. **a.** Multiply the binomials $(3x - 2y)^2$.
  b. How do you factor $9x^2 - 12xy + 4y^2$?

50. **a.** Multiply the binomials $(x + 7y)^2$.
  b. How do you factor $x^2 + 14xy + 49y^2$?

For Exercises 51–56, fill in the blank to make the trinomial a perfect square trinomial.

51. $9x^2 +$ (______) $+ 25$

52. $16x^4 -$ (______) $+ 1$

53. $b^2 - 12b +$ (______)

54. $4w^2 + 28w +$ (______)

55. (______)$z^2 + 16z + 1$

56. (______)$x^2 - 42x + 49$

For Exercises 57–66, factor out the greatest common factor. Then determine if the polynomial is a perfect square trinomial. If it is, factor it.

57. $y^2 - 8y + 16$

58. $x^2 + 10x + 25$

59. $64m^2 + 80m + 25$

60. $100c^2 - 140c + 49$

61. $w^2 - 5w + 9$

62. $2a^2 + 14a + 98$

63. $9a^2 - 30ab + 25b^2$

64. $16x^4 - 48x^2y + 9y^2$

65. $16t^2 - 80tv + 20v^2$

66. $12x^2 - 12xy + 3y^2$

Concept 5: Mixed Practice: Summary of Factoring Trinomials

For Exercises 67–88, factor completely by using an appropriate method. (Be sure to note the number of terms in the polynomial.)

67. $3x^3 - 9x^2 + 5x - 15$

68. $ay + ax - 5cy - 5cx$

69. $a^2 + 12a + 36$

70. $9 - 6b + b^2$

71. $81w^2 + 90w + 25$

72. $49a^2 - 28ab + 4b^2$

73. $3x(a + b) - 6(a + b)$

74. $4p(t - 8) + 2(t - 8)$

75. $12a^2bc^2 + 4ab^2c^2 - 6abc^3$

76. $18x^2z - 6xyz + 30xz^2$

77. $-20x^3 + 74x^2 - 60x$

78. $-24y^3 + 90y^2 - 75y$

79. $2y^2 - 9y - 4$

80. $3w^2 - 12w + 4$

81. $p^3q - p^2q^2 - 12pq^3$

82. $c^3d - 19c^2d^2 + 90cd^3$

83. $1 - 4d + 3d^2$

84. $2 - 5a + 2a^2$

85. $ax - 5a^2 + 2bx - 10ab$

86. $my + y^2 - 3xm - 3xy$

87. $8z^2 + 24zw - 224w^2$

88. $9x^2 - 18xy - 135y^2$

For Exercises 89–96, factor the expressions that define each function.

89. $f(x) = 2x^2 + 13x - 7$

90. $g(x) = 3x^2 + 14x + 8$

91. $m(t) = t^2 - 22t + 121$

92. $n(t) = t^2 + 20t + 100$

93. $P(x) = x^3 + 4x^2 + 3x$

94. $Q(x) = x^4 + 6x^3 + 8x^2$

95. $h(a) = a^3 + 5a^2 - 6a - 30$

96. $k(a) = a^3 - 4a^2 + 2a - 8$

Expanding Your Skills

97. A student factored $4y^2 - 10y + 4$ as $(2y - 1)(2y - 4)$ on her factoring test. Why did her professor deduct several points, even though $(2y - 1)(2y - 4)$ does multiply out to $4y^2 - 10y + 4$?

98. A student factored $9w^2 + 36w + 36$ as $(3w + 6)^2$ on his factoring test. Why did his instructor deduct several points, even though $(3w + 6)^2$ does multiply out to $9w^2 + 36w + 36$?

Section 5.6 Factoring Binomials

Concepts

1. Difference of Squares
2. Using a Difference of Squares in Grouping
3. Sum and Difference of Cubes
4. Summary of Factoring Binomials
5. Factoring Binomials of the Form $x^6 - y^6$

1. Difference of Squares

Up to this point we have learned to

- Factor out the greatest common factor from a polynomial.
- Factor a four-term polynomial by grouping.
- Recognize and factor perfect square trinomials.
- Factor trinomials by the ac-method and by the trial-and-error method.

Next, we will learn how to factor binomials that fit the pattern of a difference of squares. Recall from Section 5.2 that the product of two conjugates results in a **difference of squares**

$$(a + b)(a - b) = a^2 - b^2$$

Therefore, to factor a difference of squares, the process is reversed. Identify a and b and construct the conjugate factors.

Factored Form of a Difference of Squares

$$a^2 - b^2 = (a + b)(a - b)$$

Example 1 Factoring the Difference of Squares

Factor the binomials completely.

a. $16x^2 - 9$ **b.** $98c^2d - 50d^3$ **c.** $z^4 - 81$

Solution:

a. $16x^2 - 9$	The GCF is 1. The binomial is a difference of squares.
$= (4x)^2 - (3)^2$	Write in the form $a^2 - b^2$, where $a = 4x$ and $b = 3$.
$= (4x + 3)(4x - 3)$	Factor as $(a + b)(a - b)$.
b. $98c^2d - 50d^3$	
$= 2d(49c^2 - 25d^2)$	The GCF is $2d$. The resulting binomial is a difference of squares.
$= 2d[(7c)^2 - (5d)^2]$	Write in the form $a^2 - b^2$, where $a = 7c$ and $b = 5d$.
$= 2d(7c + 5d)(7c - 5d)$	Factor as $(a + b)(a - b)$.
c. $z^4 - 81$	The GCF is 1. The binomial is a difference of squares.
$= (z^2)^2 - (9)^2$	Write in the form $a^2 - b^2$, where $a = z^2$ and $b = 9$.
$= (z^2 + 9)(z^2 - 9)$	Factor as $(a + b)(a - b)$.
	$z^2 - 9$ is also a difference of squares.
$= (z^2 + 9)\overbrace{(z + 3)(z - 3)}$	

Skill Practice Factor completely.

1. $4z^2 - 1$ **2.** $7y^3z - 63yz^3$ **3.** $b^4 - 16$

The difference of squares $a^2 - b^2$ factors as $(a - b)(a + b)$. However, the *sum* of squares is not factorable.

Sum of Squares

Suppose a and b have no common factors. Then the sum of squares $a^2 + b^2$ is *not* factorable over the real numbers.

That is, $a^2 + b^2$ is prime over the real numbers.

To see why $a^2 + b^2$ is not factorable, consider the product of binomials:

$$(a \quad b)(a \quad b) \overset{?}{=} a^2 + b^2$$

If all possible combinations of signs are considered, none produces the correct product.

$(a + b)(a - b) = a^2 - b^2$	Wrong sign
$(a + b)(a + b) = a^2 + 2ab + b^2$	Wrong middle term
$(a - b)(a - b) = a^2 - 2ab + b^2$	Wrong middle term

After exhausting all possibilities, we see that if a and b share no common factors, then the sum of squares $a^2 + b^2$ is a prime polynomial.

Skill Practice Answers

1. $(2z - 1)(2z + 1)$
2. $7yz(y + 3z)(y - 3z)$
3. $(b^2 + 4)(b - 2)(b + 2)$

2. Using a Difference of Squares in Grouping

Sometimes a difference of squares can be used along with other factoring techniques.

Example 2 Using a Difference of Squares in Grouping

Factor completely. $y^3 - 6y^2 - 4y + 24$

Solution:

$$y^3 - 6y^2 - 4y + 24$$

The GCF is 1.

$$= y^3 - 6y^2 \;\vdots\; - 4y + 24$$

The polynomial has four terms. Factor by grouping.

$$= y^2(y - 6) - 4(y - 6)$$

$$= (y - 6)(y^2 - 4)$$

$y^2 - 4$ is a difference of squares.

$$= (y - 6)\overbrace{(y + 2)(y - 2)}$$

Skill Practice Factor completely.

4. $a^3 + 5a^2 - 9a - 45$

3. Sum and Difference of Cubes

For binomials that represent the sum or difference of cubes, factor by using the following formulas.

Factoring a Sum and Difference of Cubes

Sum of cubes: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

Difference of cubes: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Multiplication can be used to confirm the formulas for factoring a sum or difference of cubes.

$$(a + b)(a^2 - ab + b^2) = a^3 - \cancel{a^2b} + \cancel{ab^2} + \cancel{a^2b} - \cancel{ab^2} + b^3 = a^3 + b^3 \checkmark$$

$$(a - b)(a^2 + ab + b^2) = a^3 + \cancel{a^2b} + \cancel{ab^2} - \cancel{a^2b} - \cancel{ab^2} - b^3 = a^3 - b^3 \checkmark$$

TIP: To help remember the placement of the signs in factoring the sum or difference of cubes, remember SOAP: Same sign, Opposite signs, Always Positive.

To help you remember the formulas for factoring a sum or difference of cubes, keep the following guidelines in mind.

- The factored form is the product of a binomial and a trinomial.
- The first and third terms in the trinomial are the squares of the terms within the binomial factor.
- Without regard to sign, the middle term in the trinomial is the product of terms in the binomial factor.

Square the first term of the binomial.

Product of terms in the binomial

$$x^3 + 8 = (x)^3 + (2)^3 = (x + 2)[(x)^2 - (x)(2) + (2)^2]$$

Square the last

Skill Practice Answers

4. $(a + 5)(a - 3)(a + 3)$

- The sign within the binomial factor is the same as the sign of the original binomial.
- The first and third terms in the trinomial are always positive.
- The sign of the middle term in the trinomial is opposite the sign within the binomial.

Same sign Positive

$$x^3 + 8 = (x)^3 + (2)^3 = (x + 2)[(x)^2 - (x)(2) + (2)^2]$$

Opposite signs

Example 3 Factoring a Difference of Cubes

Factor. $8x^3 - 27$

Solution:

$8x^3 - 27$	$8x^3$ and 27 are perfect cubes.
$= (2x)^3 - (3)^3$	Write as $a^3 - b^3$, where $a = 2x$ and $b = 3$.
$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$	
$(2x)^3 - (3)^3 = (2x - 3)[(2x)^2 + (2x)(3) + (3)^2]$	Apply the difference of cubes formula.
$= (2x - 3)(4x^2 + 6x + 9)$	Simplify.

Skill Practice Factor completely.

5. $125p^3 - 8$

Example 4 Factoring the Sum of Cubes

Factor. $125t^3 + 64z^6$

Solution:

$125t^3 + 64z^6$	$125t^3$ and $64z^6$ are perfect cubes.
$= (5t)^3 + (4z^2)^3$	Write as $a^3 + b^3$, where $a = 5t$ and $b = 4z^2$.
$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$	Apply the sum of cubes formula.
$(5t)^3 + (4z^2)^3 = [(5t) + (4z^2)][(5t)^2 - (5t)(4z^2) + (4z^2)^2]$	
$= (5t + 4z^2)(25t^2 - 20tz^2 + 16z^4)$	Simplify.

Skill Practice Factor completely.

6. $x^3 + 1000$

Skill Practice Answers

5. $(5p - 2)(25p^2 + 10p + 4)$
6. $(x + 10)(x^2 - 10x + 100)$

4. Summary of Factoring Binomials

After factoring out the greatest common factor, the next step in any factoring problem is to recognize what type of pattern it follows. Exponents that are divisible by 2 are perfect squares, and those divisible by 3 are perfect cubes. The formulas for factoring binomials are summarized here.

Factoring Binomials

1. Difference of squares: $a^2 - b^2 = (a + b)(a - b)$
2. Difference of cubes: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
3. Sum of cubes: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

Example 5 Review of Factoring Binomials

Factor the binomials.

a. $m^3 - \frac{1}{8}$ **b.** $9k^2 + 24m^2$ **c.** $128y^6 + 54x^3$ **d.** $50y^6 - 8x^2$

Solution:

a. $m^3 - \dfrac{1}{8}$ — m^3 is a perfect cube: $m^3 = (m)^3$. $\frac{1}{8}$ is a perfect cube: $\frac{1}{8} = (\frac{1}{2})^3$.

$= (m)^3 - \left(\dfrac{1}{2}\right)^3$ — This is a difference of cubes, where $a = m$ and $b = \frac{1}{2}$: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

$= \left(m - \dfrac{1}{2}\right)\left(m^2 + \dfrac{1}{2}m + \dfrac{1}{4}\right)$ — Factor.

b. $9k^2 + 24m^2$ — Factor out the GCF.

$= 3(3k^2 + 8m^2)$ — The resulting binomial is not a difference of squares or a sum or difference of cubes. It cannot be factored further over the real numbers.

c. $128y^6 + 54x^3$ — Factor out the GCF.

$= 2(64y^6 + 27x^3)$ — Both 64 and 27 are perfect cubes, and the exponents of both x and y are multiples of 3. This is a sum of cubes, where $a = 4y^2$ and $b = 3x$.

$= 2[(4y^2)^3 + (3x)^3]$ — $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

$= 2(4y^2 + 3x)(16y^4 - 12xy^2 + 9x^2)$ — Factor.

d. $50y^6 - 8x^2$ Factor out the GCF.

$= 2(25y^6 - 4x^2)$ Both 25 and 4 are perfect squares. The exponents of both x and y are multiples of 2. This is a difference of squares, where $a = 5y^3$ and $b = 2x$.

$= 2[(5y^3)^2 - (2x)^2]$ $a^2 - b^2 = (a + b)(a - b)$.

$= 2(5y^3 + 2x)(5y^3 - 2x)$

Skill Practice Factor the binomials.

7. $x^2 - \frac{1}{25}$

8. $16y^3 + 4y$

9. $24a^4 - 3a$

10. $18p^4 - 50t^2$

5. Factoring Binomials of the Form $x^6 - y^6$

Example 6 Factoring Binomials

Factor the binomial $x^6 - y^6$ as

a. A difference of cubes

b. A difference of squares

Solution:

Notice that the expressions x^6 and y^6 are both perfect squares and perfect cubes because the exponents are both multiples of 2 and of 3. Consequently, $x^6 - y^6$ can be factored initially as either a difference of cubes or a difference of squares.

a. $x^6 - y^6$

Difference of cubes

$= (x^2)^3 - (y^2)^3$ Write as $a^3 - b^3$, where $a = x^2$ and $b = y^2$.

$= (x^2 - y^2)[(x^2)^2 + (x^2)(y^2) + (y^2)^2]$ Apply the formula $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

$= (x^2 - y^2)(x^4 + x^2y^2 + y^4)$ Factor $x^2 - y^2$ as a difference of squares.

$= \overbrace{(x + y)(x - y)}(x^4 + x^2y^2 + y^4)$ The expression $x^4 + x^2y^2 + y^4$ cannot be factored by using the skills learned thus far.

Skill Practice Answers

7. $\left(x + \frac{1}{5}\right)\left(x - \frac{1}{5}\right)$

8. $4y(4y^2 + 1)$

9. $3a(2a - 1)(4a^2 + 2a + 1)$

10. $2(3p^2 + 5t)(3p^2 - 5t)$

b. $x^6 - y^6$

Difference of squares

$= (x^3)^2 - (y^3)^2$ Write as $a^2 - b^2$, where $a = x^3$ and $b = y^3$.

$= (x^3 + y^3)(x^3 - y^3)$ Apply the formula $a^2 - b^2 = (a + b)(a - b)$.

Sum of cubes Difference of cubes

Factor $x^3 + y^3$ as a sum of cubes. Factor $x^3 - y^3$ as a difference of cubes.

$$= \overbrace{(x + y)(x^2 - xy + y^2)}\overbrace{(x - y)(x^2 + xy + y^2)}$$

TIP: If given a choice between factoring a binomial as a difference of squares or as a difference of cubes, it is recommended that you factor initially as a difference of squares. As Example 6 illustrates, factoring as a difference of squares leads to a more complete factorization. Hence,

$$a^6 - b^6 = (a - b)(a^2 + ab + b^2)(a + b)(a^2 - ab + b^2)$$

Skill Practice Factor completely.

11. $a^6 - 64$

Skill Practice Answers

11. $(a - 2)(a + 2)(a^2 + 2a + 4)(a^2 - 2a + 4)$

Section 5.6 Practice Exercises

Boost *your* GRADE at mathzone.com!

MathZone

- Practice Problems
- Self-Tests
- NetTutor
- e-Professors
- Videos

Study Skills Exercises

1. Multiplying polynomials and factoring polynomials are inverse operations. That is, to check a factoring problem you can multiply, and to check a multiplication problem you can factor. To practice both operations, write a factored polynomial on one side of a 3 × 5 card with the directions, *Multiply*. On the other side of the card, write the expanded form of the polynomial with the directions, *Factor*. Now you can mix up the cards and get a good sense of what is meant by the directions: *Factor* and *Multiply*.

2. Define the key terms.

a. Difference of squares **b. Sum of cubes** **c. Difference of cubes**

Review Exercises

For Exercises 3–10, factor completely.

3. $4x^2 - 20x + 25$

4. $9t^2 - 42t + 49$

5. $10x + 6xy + 5 + 3y$

6. $21a + 7ab - 3b - b^2$

7. $32p^2 - 28p - 4$

8. $6q^2 + 37q - 35$

9. $45a^2 - 9ac$

10. $11xy^2 - 55y^3$

Concept 1: Difference of Squares

11. Explain how to identify and factor a difference of squares.

12. Can you factor $25x^2 + 4$?

For Exercises 13–22, factor the binomials. Identify the binomials that are prime.

13. $x^2 - 9$

14. $y^2 - 25$

15. $16 - w^2$

16. $81 - b^2$

17. $8a^2 - 162b^2$

18. $50c^2 - 72d^2$

19. $25u^2 + 1$

20. $w^2 + 4$

21. $2a^4 - 32$

22. $5y^4 - 5$

Concept 2: Using the Difference of Squares in Grouping

For Exercises 23–30, use the difference of squares along with factoring by grouping.

23. $x^3 - x^2 - 16x + 16$

24. $x^3 + 5x^2 - x - 5$

25. $4x^3 + 12x^2 - x - 3$

26. $5x^3 - x^2 - 45x + 9$

27. $4y^3 + 12y^2 - y - 3$

28. $9z^3 - 5z^2 - 36z + 20$

29. $x^2 - y^2 - ax - ay$

30. $5m - 5n + m^2 - n^2$

Concept 3: Sum and Difference of Cubes

31. Explain how to identify and factor a sum of cubes.

32. Explain how to identify and factor a difference of cubes.

For Exercises 33–42, factor the sum or difference of cubes.

33. $8x^3 - 1$ (Check by multiplying.)

34. $y^3 + 64$ (Check by multiplying.)

35. $125c^3 + 27$

36. $216u^3 - v^3$

37. $x^3 - 1000$

38. $8y^3 - 27$

39. $64t^3 + 1$

40. $125r^3 + 1$

41. $2000y^6 + 2x^3$

42. $16z^4 - 54z$

Concept 4: Summary of Factoring Binomials

For Exercises 43–70, factor completely.

43. $36y^2 - \frac{1}{25}$

44. $16p^2 - \frac{1}{9}$

45. $18d^{12} - 32$

46. $3z^8 - 12$

47. $242v^2 + 32$

48. $8p^2 + 200$

49. $4x^2 - 16$

50. $9m^2 - 81n^2$

51. $25 - 49q^2$

52. $1 - 25p^2$

53. $(t + 2s)^2 - 36$

54. $(5x + 4)^2 - y^2$

55. $27 - t^3$

56. $8 + y^3$

57. $27a^3 + \frac{1}{8}$

58. $b^3 + \frac{27}{125}$

59. $2m^3 + 16$

60. $3x^3 - 375$

61. $x^4 - y^4$

62. $81u^4 - 16v^4$

63. $a^9 + b^9$

64. $27m^9 - 8n^9$

65. $\frac{1}{8}p^3 - \frac{1}{125}$

66. $1 - \frac{1}{27}d^3$

67. $4w^2 + 25$

68. $64 + a^2$

69. $\frac{1}{25}x^2 - \frac{1}{4}y^2$

70. $\frac{1}{100}a^2 - \frac{4}{49}b^2$

Concept 5: Factoring Binomials of the Form $x^6 - y^6$

For Exercises 71–78, factor completely.

71. $a^6 - b^6$ (*Hint:* First factor as a difference of squares.)

72. $64x^6 - y^6$

73. $64 - y^6$

74. $1 - p^6$

75. $h^6 + k^6$ (*Hint:* Factor as a sum of cubes.)

76. $27q^6 + 125p^6$

77. $8x^6 + 125$

78. $t^6 + 1$

Mixed Exercises

79. Find a difference of squares that has $(2x + 3)$ as one of its factors.

80. Find a difference of squares that has $(4 - p)$ as one of its factors.

81. Find a difference of cubes that has $(4a^2 + 6a + 9)$ as its trinomial factor.

82. Find a sum of cubes that has $(25c^2 - 10cd + 4d^2)$ as its trinomial factor.

83. Find a sum of cubes that has $(4x^2 + y)$ as its binomial factor.

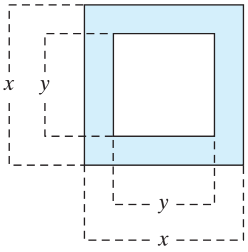
84. Find a difference of cubes that has $(3t - r^2)$ as its binomial factor.

85. Consider the shaded region:

a. Find an expression that represents the area of the shaded region.

b. Factor the expression found in part (a).

c. Find the area of the shaded region if $x = 6$ in. and $y = 4$ in.

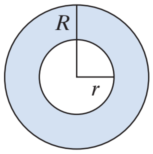


86. A manufacturer needs to know the area of a metal washer. The outer radius of the washer is R and the inner radius is r.

a. Find an expression that represents the area of the washer.

b. Factor the expression found in part (a).

c. Find the area of the washer if $R = \frac{1}{2}$ in. and $r = \frac{1}{4}$ in. (Round to the nearest 0.01 in.^{2})



Expanding Your Skills

For Exercises 87–90, factor the polynomials by using the difference of squares, sum of cubes, or difference of cubes with grouping.

87. $x^2 - y^2 + x + y$

88. $64m^2 - 25n^2 + 8m + 5n$

89. $x^3 + y^3 + x + y$

90. $4pu^3 - 4pv^3 - 7yu^3 + 7yv^3$

Additional Factoring Strategies

Section 5.7

Concepts

1. General Factoring Review
2. Additional Factoring Strategies
3. Factoring Using Substitution

1. General Factoring Review

We now review the techniques of factoring presented thus far along with a general strategy for factoring polynomials.

Factoring Strategy

1. Factor out the greatest common factor (Section 5.4).
2. Identify whether the polynomial has two terms, three terms, or more than three terms.
3. If the polynomial has more than three terms, try factoring by grouping (Section 5.4 and Section 5.6).
4. If the polynomial has three terms, check first for a perfect square trinomial. Otherwise, factor the trinomial with the ac-method or the trial-and-error method (Section 5.5).
5. If the polynomial has two terms, determine if it fits the pattern for a difference of squares, difference of cubes, or sum of cubes. Remember, a sum of squares is not factorable over the real numbers (Section 5.6).
6. Be sure to factor the polynomial completely.
7. Check by multiplying.

Example 1 Factoring Polynomials

Factor out the GCF and identify the number of terms and type of factoring pattern represented by the polynomial. Then factor the polynomial completely.

a. $abx^2 - 3ax + 5bx - 15$ **b.** $20y^2 - 110y - 210$

c. $4p^3 + 20p^2 + 25p$ **d.** $w^3 + 1000$ **e.** $d^4 - \frac{1}{16}$

Solution:

a. $abx^2 - 3ax + 5bx - 15$

$abx^2 - 3ax \;\vdots\; + 5bx - 15$

$= ax(bx - 3) + 5(bx - 3)$

$= (bx - 3)(ax + 5)$

The GCF is 1. The polynomial has four terms. Therefore, factor by grouping.

b. $20y^2 - 110y - 210$

$= 10(2y^2 - 11y - 21)$

$= 10(2y + 3)(y - 7)$

The GCF is 10. The polynomial has three terms. The trinomial is not a perfect square trinomial. Use either the ac-method or the trial-and-error method.

c. $4p^3 + 20p^2 + 25p$

$= p(4p^2 + 20p + 25)$

The GCF is p. The polynomial has three terms and is a perfect square trinomial, $a^2 + 2ab + b^2$, where $a = 2p$ and $b = 5$.

$= p(2p + 5)^2$

Apply the formula $a^2 + 2ab + b^2 = (a + b)^2$.

d. $w^3 + 1000$

$$= (w)^3 + (10)^3$$

The GCF is 1. The polynomial has two terms. The binomial is a sum of cubes, $a^3 + b^3$, where $a = w$ and $b = 10$.

$$= (w + 10)(w^2 - 10w + 100)$$

Apply the formula $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

e. $d^4 - \dfrac{1}{16}$

$$= (d^2)^2 - \left(\frac{1}{4}\right)^2$$

d^4 and $\frac{1}{16}$ are perfect squares.

$$= \left(d^2 + \frac{1}{4}\right)\left(d^2 - \frac{1}{4}\right)$$

Factor as a difference of squares.

$$= \left(d^2 + \frac{1}{4}\right)\overbrace{\left(d - \frac{1}{2}\right)\left(d + \frac{1}{2}\right)}$$

The binomial $d^2 - \frac{1}{4}$ is also a difference of squares.

Avoiding Mistakes:

Remember that a sum of squares such as $d^2 + \frac{1}{4}$ cannot be factored over the real numbers.

Skill Practice Factor completely.

1. $2cx - 5cy + 2dx - 5dy$
2. $-30y^2 + 35y + 15$
3. $9w^3 - 12w^2 + 4w$
4. $8x^3 + 125y^3$
5. $\dfrac{1}{81}x^4 - 1$

2. Additional Factoring Strategies

Some factoring problems may require more than one type of factoring. We also may encounter polynomials that require slight variations on the factoring techniques already learned. These are demonstrated in Examples 2–5.

Example 2 Factoring a Trinomial Involving Fractional Coefficients

Factor completely. $\dfrac{1}{9}x^2 + \dfrac{1}{3}x + \dfrac{1}{4}$

Solution:

$$\frac{1}{9}x^2 + \frac{1}{3}x + \frac{1}{4}$$

$$= \left(\frac{1}{3}x\right)^2 + 2\left(\frac{1}{3}x\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2$$

$$= \left(\frac{1}{3}x + \frac{1}{2}\right)^2$$

The fractions may make this polynomial look difficult to factor. However, notice that both $\frac{1}{9}x^2$ and $\frac{1}{4}$ are perfect squares. Furthermore, the middle term $\frac{1}{3}x = 2(\frac{1}{3}x)(\frac{1}{2})$. Therefore, the trinomial is a perfect square trinomial.

Skill Practice Factor completely.

6. $\dfrac{1}{16}y^2 - \dfrac{1}{10}y + \dfrac{1}{25}$

Skill Practice Answers

1. $(c + d)(2x - 5y)$
2. $-5(3y + 1)(2y - 3)$
3. $w(3w - 2)^2$
4. $(2x + 5y)(4x^2 - 10xy + 25y^2)$
5. $\left(\frac{1}{3}x - 1\right)\left(\frac{1}{3}x + 1\right)\left(\frac{1}{9}x^2 + 1\right)$
6. $\left(\frac{1}{4}y - \frac{1}{5}\right)^2$

3. Factoring Using Substitution

Sometimes it is convenient to use substitution to convert a polynomial into a simpler form before factoring.

Example 3 Using Substitution to Factor a Polynomial

Factor by using substitution. $(2x - 7)^2 - 3(2x - 7) - 40$

Solution:

$\underset{\downarrow}{(2x - 7)^2} - 3\underset{\downarrow}{(2x - 7)} - 40$

$= \boldsymbol{u}^2 - 3\boldsymbol{u} - 40$ Substitute $\boldsymbol{u} = 2x - 7$. The trinomial is simpler in form.

$= (\boldsymbol{u} - 8)(\boldsymbol{u} + 5)$ Factor the trinomial.

$= [(2x - 7) - 8][(2x - 7) + 5]$ Reverse substitute. Replace $\boldsymbol{u}$ by $2x - 7$.

$= (2x - 7 - 8)(2x - 7 + 5)$ Simplify.

$= (2x - 15)(2x - 2)$ The second binomial has a GCF of 2.

$= (2x - 15)(2)(x - 1)$ Factor out the GCF from the second binomial.

$= 2(2x - 15)(x - 1)$

Skill Practice Factor by using substitution.

7. $(3x + 1)^2 + 2(3x + 1) - 15$

Example 4 Using Substitution to Factor a Polynomial

Factor by using substitution. $6y^6 - 5y^3 - 4$

Solution:

$6y^6 - 5y^3 - 4$ Let $\boldsymbol{u} = y^3$.

$= 6\boldsymbol{u}^2 - 5\boldsymbol{u} - 4$ Substitute $\boldsymbol{u}$ for y^3 in the trinomial.

$= (2\boldsymbol{u} + 1)(3\boldsymbol{u} - 4)$ Factor the trinomial.

$= (2y^3 + 1)(3y^3 - 4)$ Reverse substitute. Replace $\boldsymbol{u}$ with y^3.

The factored form of $6y^6 - 5y^3 - 4$ is $(2y^3 + 1)(3y^3 - 4)$.

Skill Practice Factor by using substitution.

8. $2x^4 + 7x^2 + 3$

Skill Practice Answers

7. $3(3x - 2)(x + 2)$
8. $(2x^2 + 1)(x^2 + 3)$

Example 5 Factoring a Four-Term Polynomial by Grouping Three Terms

Factor completely. $x^2 - y^2 - 6y - 9$

Solution:

Grouping "2 by 2" will not work to factor this polynomial. However, if we factor out -1 from the last three terms, the resulting trinomial will be a perfect square trinomial.

$x^2 - y^2 - 6y - 9$	Group the last three terms.
$= x^2 - 1(y^2 + 6y + 9)$	Factor out -1 from the last three terms.
$= x^2 - (y + 3)^2$	Factor the perfect square trinomial $y^2 + 6y + 9$ as $(y + 3)^2$.
	The quantity $x^2 - (y + 3)^2$ is a difference of squares, $a^2 - b^2$, where $a = x$ and $b = (y + 3)$.
$= [x - (y + 3)][x + (y + 3)]$	Factor as $a^2 - b^2 = (a + b)(a - b)$.
$= (x - y - 3)(x + y + 3)$	Apply the distributive property to clear the inner parentheses.

Skill Practice Factor completely.

9. $x^2 + 10x + 25 - y^2$

Avoiding Mistakes:

When factoring the expression $x^2 - (y + 3)^2$ as a difference of squares, be sure to use parentheses around the quantity $(y + 3)$. This will help you remember to "distribute the negative" in the expression $[x - (y + 3)]$.

$$[x - (y + 3)] = (x - y - 3)$$

TIP: From Example 5, the expression $x^2 - (y + 3)^2$ can also be factored by using substitution. Let $u = y + 3$.

$x^2 - (y + 3)^2$	
$= x^2 - u^2$	Substitution $u = y + 3$.
$= (x - u)(x + u)$	Factor as a difference of squares.
$= [x - (y + 3)][x + (y + 3)]$	Substitute back.
$= (x - y - 3)(x + y + 3)$	Apply the distributive property.

Skill Practice Answers

9. $(x + 5 - y)(x + 5 + y)$

Section 5.7 Practice Exercises

Boost *your* GRADE at mathzone.com!

MathZone

- Practice Problems
- Self-Tests
- NetTutor
- e-Professors
- Videos

Review Exercises

1. What is meant by a prime factor?

2. What is the first step in factoring any polynomial?

3. When factoring a binomial, what patterns do you look for?

4. When factoring a trinomial, what pattern do you look for first?

5. What do you look for when factoring a perfect square trinomial?

6. What do you look for when factoring a four-term polynomial?

Concept 1: General Factoring Review

For Exercises 7–66,

a. Identify the category in which the polynomial best fits (you may need to factor out the GCF first). Choose from

- difference of squares
- sum of squares
- difference of cubes
- sum of cubes
- perfect square trinomial
- trinomial (ac-method or trial-and-error)
- four terms—grouping
- none of these

b. Factor the polynomial completely.

7. $6x^2 - 21x - 45$

8. $8m^3 - 10m^2 - 3m$

9. $8a^2 - 50$

10. $ab + ay - b^2 - by$

11. $14u^2 - 11uv + 2v^2$

12. $9p^2 - 12pq + 4q^2$

13. $16x^3 - 2$

14. $9m^2 + 16n^2$

15. $27y^3 + 125$

16. $3x^2 - 16$

17. $128p^6 + 54q^3$

18. $5b^2 - 30b + 45$

19. $16a^4 - 1$

20. $81u^2 - 90uv + 25v^2$

21. $p^2 - 12p + 36 - c^2$

22. $4x^2 + 16$

23. $12ax - 6ay + 4bx - 2by$

24. $125y^3 - 8$

25. $5y^2 + 14y - 3$

26. $2m^4 - 128$

27. $t^2 - 100$

28. $4m^2 - 49n^2$

29. $y^3 + 27$

30. $x^3 + 1$

31. $d^2 + 3d - 28$

32. $c^2 + 5c - 24$

33. $x^2 - 12x + 36$

34. $p^2 + 16p + 64$

35. $2ax^2 - 5ax + 2bx - 5b$

36. $8x^2 - 4bx + 2ax - ab$

37. $10y^2 + 3y - 4$

38. $12z^2 + 11z + 2$

39. $10p^2 - 640$

40. $50a^2 - 72$

41. $z^4 - 64z$

42. $t^4 - 8t$

43. $b^3 - 4b^2 - 45b$

44. $y^3 - 14y^2 + 40y$

45. $9w^2 + 24wx + 16x^2$

46. $4k^2 - 20kp + 25p^2$

47. $60x^2 - 20x + 30ax - 10a$

48. $50x^2 - 200x + 10cx - 40c$

49. $w^4 - 16$

50. $k^4 - 81$

51. $t^6 - 8$

52. $p^6 + 27$

53. $8p^2 - 22p + 5$

54. $9m^2 - 3m - 20$

55. $36y^2 - 12y + 1$

56. $9a^2 + 42a + 49$

57. $2x^2 + 50$

58. $4y^2 + 64$

59. $12r^2s^2 + 7rs^2 - 10s^2$

60. $7z^2w^2 - 10zw^2 - 8w^2$

61. $x^2 + 8xy - 33y^2$

62. $s^2 - 9st - 36t^2$

63. $m^6 + n^3$

64. $a^3 - b^6$

65. $x^2 - 4x$

66. $y^2 - 9y$

Concept 2: Additional Factoring Strategies

For Exercises 67–70, factor the polynomial in part (a). Then use substitution to help factor the polynomials in parts (b) and (c).

67. **a.** $u^2 - 10u + 25$
   b. $x^4 - 10x^2 + 25$
   c. $(a + 1)^2 - 10(a + 1) + 25$

68. **a.** $u^2 + 12u + 36$
   b. $y^4 + 12y^2 + 36$
   c. $(b - 2)^2 + 12(b - 2) + 36$

69. **a.** $u^2 + 11u - 26$
   b. $w^6 + 11w^3 - 26$
   c. $(y - 4)^2 + 11(y - 4) - 26$

70. **a.** $u^2 + 17u + 30$
   b. $z^6 + 17z^3 + 30$
   c. $(x + 3)^2 + 17(x + 3) + 30$

For Exercises 71–80, factor by using substitution.

71. $3y^6 + 11y^3 + 6$

72. $3x^4 - 5x^2 - 12$

73. $4p^4 + 5p^2 + 1$

74. $t^4 + 3t^2 + 2$

75. $x^4 + 15x^2 + 36$

76. $t^6 - 16t^3 + 63$

77. $(3x - 1)^2 - (3x - 1) - 6$

78. $(2x + 5)^2 - (2x + 5) - 12$

79. $2(x - 5)^2 + 9(x - 5) + 4$

80. $4(x - 3)^2 + 7(x - 3) + 3$

For Exercises 81–114, factor completely using the strategy found on page 371 and any additional techniques of factoring illustrated in Examples 2–5.

81. $x^2(x + y) - y^2(x + y)$

82. $u^2(u - v) - v^2(u - v)$

83. $(a + 3)^4 + 6(a + 3)^5$

84. $(4 - b)^4 - 2(4 - b)^3$

85. $24(3x + 5)^3 - 30(3x + 5)^2$

86. $10(2y + 3)^2 + 15(2y + 3)^3$

87. $\frac{1}{100}x^2 + \frac{1}{35}x + \frac{1}{49}$

88. $\frac{1}{25}a^2 + \frac{1}{15}a + \frac{1}{36}$

89. $(5x^2 - 1)^2 - 4(5x^2 - 1) - 5$

90. $(x^3 + 4)^2 - 10(x^3 + 4) + 24$

91. $16p^4 - q^4$

92. $s^4t^4 - 81$

93. $y^3 + \frac{1}{64}$

94. $z^3 + \frac{1}{125}$

95. $6a^3 + a^2b - 6ab^2 - b^3$

96. $4p^3 + 12p^2q - pq^2 - 3q^3$

97. $\frac{1}{9}t^2 + \frac{1}{6}t + \frac{1}{16}$

98. $\frac{1}{25}y^2 + \frac{1}{5}y + \frac{1}{4}$

99. $x^2 + 12x + 36 - a^2$

100. $a^2 + 10a + 25 - b^2$

101. $p^2 + 2pq + q^2 - 81$

102. $m^2 - 2mn + n^2 - 9$

103. $b^2 - (x^2 + 4x + 4)$

104. $p^2 - (y^2 - 6y + 9)$

105. $4 - u^2 + 2uv - v^2$

106. $25 - a^2 - 2ab - b^2$

107. $6ax - by + 2bx - 3ay$

108. $5pq - 12 - 4q + 15p$

109. $u^6 - 64$ [*Hint:* Factor first as a difference of squares, $(u^3)^2 - (8)^2$.]

110. $1 - v^6$

111. $x^8 - 1$

112. $y^8 - 256$

113. $2(3w - 5)^2 - 19(3w - 5) + 35$

114. $3(2y + 3)^2 + 23(2y + 3) - 8$

Expanding Your Skills

For Exercises 115–118, factor completely. Then check by multiplying.

115. $a^2 - b^2 + a + b$

116. $25c^2 - 9d^2 + 5c - 3d$

117. $5wx^3 + 5wy^3 - 2zx^3 - 2zy^3$

118. $3xu^3 - 3xv^3 - 5yu^3 + 5yv^3$

Solving Equations by Using the Zero Product Rule

Section 5.8

Concepts

1. Solving Equations by Using the Zero Product Rule
2. Applications of Quadratic Equations
3. Definition of a Quadratic Function
4. Applications of Quadratic Functions

1. Solving Equations by Using the Zero Product Rule

In Section 1.4 we defined a linear equation in one variable as an equation of the form $ax + b = 0$ $(a \neq 0)$. A linear equation in one variable is sometimes called a first-degree polynomial equation because the highest degree of all its terms is 1. A second-degree polynomial equation is called a quadratic equation.

Definition of a Quadratic Equation in One Variable

If a, b, and c are real numbers such that $a \neq 0$, then a **quadratic equation** is an equation that can be written in the form

$$ax^2 + bx + c = 0$$

The following equations are quadratic because they can each be written in the form $ax^2 + bx + c = 0 (a \neq 0)$.

$$\begin{array}{rrr} -4x^2 + 4x = 1 & x(x - 2) = 3 & (x - 4)(x + 4) = 9 \\ -4x^2 + 4x - 1 = 0 & x^2 - 2x = 3 & x^2 - 16 = 9 \\ & x^2 - 2x - 3 = 0 & x^2 - 25 = 0 \\ & & x^2 + 0x - 25 = 0 \end{array}$$

One method to solve a quadratic equation is to factor the equation and apply the zero product rule. The **zero product rule** states that if the product of two factors is zero, then one or both of its factors is equal to zero.

The Zero Product Rule

If $ab = 0$, then $a = 0$ or $b = 0$.

For example, the quadratic equation $x^2 - x - 12 = 0$ can be written in factored form as $(x - 4)(x + 3) = 0$. By the zero product rule, one or both factors must be zero. Hence, either $x - 4 = 0$ or $x + 3 = 0$. Therefore, to solve the quadratic equation, set each factor to zero and solve for x.

$(x - 4)(x + 3) = 0$ Apply the zero product rule.

$\swarrow \qquad \searrow$

$x - 4 = 0 \quad \text{or} \quad x + 3 = 0$ Set each factor to zero.

$x = 4 \quad \text{or} \quad x = -3$ Solve each equation for x.

Quadratic equations, like linear equations, arise in many applications of mathematics, science, and business. The following steps summarize the factoring method to solve a quadratic equation.

Steps to Solve a Quadratic Equation by Factoring

1. Write the equation in the form $ax^2 + bx + c = 0$.
2. Factor the equation completely.
3. Apply the zero product rule. That is, set each factor equal to zero and solve the resulting equations.*

*The solution(s) found in step 3 may be checked by substitution in the original equation.

Example 1 Solving Quadratic Equations

Solve.

a. $2x^2 - 5x = 12$

b. $\frac{1}{2}x^2 + \frac{2}{3}x = 0$

c. $9x(4x + 2) - 10x = 8x + 25$

d. $2x(x + 5) + 3 = 2x^2 - 5x + 1$

Solution:

a. $2x^2 - 5x = 12$

$2x^2 - 5x - 12 = 0$ Write the equation in the form $ax^2 + bx + c = 0$.

$(2x + 3)(x - 4) = 0$ Factor the polynomial completely.

$\swarrow \qquad \searrow$

$2x + 3 = 0 \quad \text{or} \quad x - 4 = 0$ Set each factor equal to zero.

$2x = -3 \quad \text{or} \quad x = 4$ Solve each equation.

$x = -\frac{3}{2} \quad \text{or} \quad x = 4$

Check: $x = -\frac{3}{2}$

$$\begin{aligned} 2x^2 - 5x &= 12 \\ 2\left(-\frac{3}{2}\right)^2 - 5\left(-\frac{3}{2}\right) &\stackrel{?}{=} 12 \\ 2\left(\frac{9}{4}\right) + \frac{15}{2} &\stackrel{?}{=} 12 \\ \frac{18}{4} + \frac{30}{4} &\stackrel{?}{=} 12 \\ \frac{48}{4} &= 12 \checkmark \end{aligned}$$

Check: $x = 4$

$$\begin{aligned} 2x^2 - 5x &= 12 \\ 2(4)^2 - 5(4) &\stackrel{?}{=} 12 \\ 2(16) - 20 &\stackrel{?}{=} 12 \\ 32 - 20 &= 12 \checkmark \end{aligned}$$

b. $\frac{1}{2}x^2 + \frac{2}{3}x = 0$ The equation is already in the form $ax^2 + bx + c = 0$. (*Note:* $c = 0$.)

$6\left(\frac{1}{2}x^2 + \frac{2}{3}x\right) = 6(0)$ Clear fractions.

$3x^2 + 4x = 0$

$x(3x + 4) = 0$ Factor completely.

$x = 0$ or $3x + 4 = 0$ Set each factor equal to zero.

$x = 0$ or $x = -\frac{4}{3}$ Solve each equation for x.

Check: $x = 0$

$$\begin{aligned} \frac{1}{2}x^2 + \frac{2}{3}x &= 0 \\ \frac{1}{2}(0)^2 + \frac{2}{3}(0) &\stackrel{?}{=} 0 \\ 0 &= 0 \checkmark \end{aligned}$$

Check: $x = -\frac{4}{3}$

$$\begin{aligned} \frac{1}{2}x^2 + \frac{2}{3}x &= 0 \\ \frac{1}{2}\left(-\frac{4}{3}\right)^2 + \frac{2}{3}\left(-\frac{4}{3}\right) &\stackrel{?}{=} 0 \\ \frac{1}{2}\left(\frac{16}{9}\right) - \frac{8}{9} &\stackrel{?}{=} 0 \\ \frac{8}{9} - \frac{8}{9} &= 0 \checkmark \end{aligned}$$

c. $9x(4x + 2) - 10x = 8x + 25$

$36x^2 + 18x - 10x = 8x + 25$ Clear parentheses.

$36x^2 + 8x = 8x + 25$ Combine *like* terms.

$36x^2 - 25 = 0$ Make one side of the equation equal to zero. The equation is in the form $ax^2 + bx + c = 0$. (*Note:* $b = 0$.)

$(6x - 5)(6x + 5) = 0$ Factor completely.

$6x - 5 = 0$ or $6x + 5 = 0$ Set each factor equal to zero.

$6x = 5$ or $6x = -5$ Solve each equation.

$$\frac{6x}{6} = \frac{5}{6} \quad \text{or} \quad \frac{6x}{6} = \frac{-5}{6}$$

$$x = \frac{5}{6} \quad \text{or} \quad x = -\frac{5}{6}$$ The check is left to the reader.

d. $2x(x + 5) + 3 = 2x^2 - 5x + 1$

$2x^2 + 10x + 3 = 2x^2 - 5x + 1$ Clear parentheses.

$15x + 2 = 0$ Make one side of the equation equal to zero. The equation is not quadratic. It is in the form $ax + b = 0$, which is linear. Solve by using the method for linear equations.

$15x = -2$

$$x = \frac{-2}{15}$$

The check is left to the reader.

Skill Practice Solve.

1. $y^2 - 2y = 35$

2. $3x^2 = 7x$

3. $5a(2a - 3) + 4(a + 1) = 3a(3a - 2)$

4. $t^2 - 3t + 1 = t^2 + 2t + 11$

The zero product rule can be used to solve higher-degree polynomial equations provided one side of the equation is zero and the other is written in factored form.

Example 2 Solving Higher-Degree Polynomial Equations

Solve the equations.

a. $-2(y + 7)(y - 1)(10y + 3) = 0$

b. $z^3 + 3z^2 - 4z - 12 = 0$

Solution:

a. $-2(y + 7)(y - 1)(10y + 3) = 0$

One side of the equation is zero, and the other side is already factored.

$$-2 = 0 \quad \text{or} \quad y + 7 = 0 \quad \text{or} \quad y - 1 = 0 \quad \text{or} \quad 10y + 3 = 0$$ Set each factor equal to zero.

$$\text{No solution} \qquad y = -7 \quad \text{or} \quad y = 1 \quad \text{or} \quad y = -\frac{3}{10}$$ Solve each equation for y.

Notice that when the constant factor is set to zero, the result is the contradiction $-2 = 0$. The constant factor does not produce a solution to the equation. Therefore, the only solutions are $y = -7$, $y = 1$, and $y = -\frac{3}{10}$. Each solution can be checked in the original equation.

b. $z^3 + 3z^2 - 4z - 12 = 0$ This is a higher-degree polynomial equation.

$z^3 + 3z^2 \;\vdots\; - 4z - 12 = 0$ One side of the equation is zero. Now factor. Because there are four terms, try factoring by grouping.

$z^2(z + 3) - 4(z + 3) = 0$

Skill Practice Answers

1. $y = 7$ or $y = -5$

2. $x = 0$ or $x = \frac{7}{3}$

3. $a = 4$ or $a = 1$

4. $t = -2$

$(z + 3)(z^2 - 4) = 0$ $z^2 - 4$ can be factored further as a difference of squares.

$(z + 3)(z - 2)(z + 2) = 0$

$z + 3 = 0$ or $z - 2 = 0$ or $z + 2 = 0$ Set each factor equal to zero.

$z = -3$ or $z = 2$ or $z = -2$ Solve each equation.

Skill Practice Solve the equations

5. $3(w + 2)(2w + 1)(w - 8) = 0$ **6.** $x^3 + x^2 - 9x - 9 = 0$

2. Applications of Quadratic Equations

Example 3 Application of a Quadratic Equation

The product of two consecutive odd integers is 20 more than the smaller integer. Find the integers.

Solution:

Let x represent the smaller odd integer and $x + 2$ represent the next consecutive odd integer. The equation representing their product is

$x(x + 2) = x + 20$

$x^2 + 2x = x + 20$ Clear parentheses.

$x^2 + x - 20 = 0$ Make the equation equal to zero.

$(x + 5)(x - 4) = 0$ Factor.

$x + 5 = 0$ or $x - 4 = 0$ Set each factor equal to zero.

$x = -5$ or $x = 4$ Solve each equation.

Since we are looking for consecutive *odd* integers, $x = 4$ is not a solution. Since $x = -5$ and $x + 2 = -3$, the integers are -5 and -3.

Skill Practice

7. The product of two consecutive even integers is 40 more than 5 times the smaller integer. Find the integers.

Example 4 Application of a Quadratic Equation

The length of a basketball court is 6 ft less than 2 times the width. If the total area is 4700 ft^2, find the dimensions of the court.

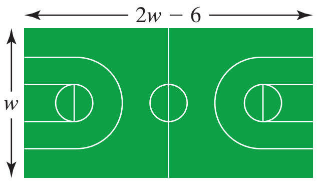


Figure 5-5

Solution:

If the width of the court is represented by w, then the length can be represented by $2w - 6$ (Figure 5-5).

Skill Practice Answers

5. $w = -2$ or $w = -\frac{1}{2}$ or $w = 8$
6. $x = -1$ or $x = 3$ or $x = -3$
7. 8 and 10

$$A = (\text{length})(\text{width})$$ Area of a rectangle

$$4700 = (2w - 6)w$$ Mathematical equation

$$4700 = 2w^2 - 6w$$

$$2w^2 - 6w - 4700 = 0$$ Set the equation equal to zero and factor.

$$2(w^2 - 3w - 2350) = 0$$ Factor out the GCF.

$$2(w - 50)(w + 47) = 0$$ Factor the trinomial.

$$2 \cancel{=} 0 \quad \text{or} \quad w - 50 = 0 \quad \text{or} \quad w + 47 = 0$$ Set each factor equal to zero.

contradiction

$$w = 50 \quad \text{or} \quad w \cancel{=} -47$$ A negative width is not possible.

The width is 50 ft.

The length is $2w - 6 = 2(50) - 6 = 94$ ft.

Skill Practice

8. The width of a rectangle is 5 in. less than 3 times the length. The area is 2 in.^{2} Find the length and width.

Example 5 Application of a Quadratic Equation

A region of coastline off Biscayne Bay is approximately in the shape of a right angle. The corresponding triangular area has sandbars and is marked off on navigational charts as being shallow water. If one leg of the triangle is 0.5 mi shorter than the other leg, and the hypotenuse is 2.5 mi, find the lengths of the legs of the triangle (Figure 5-6).

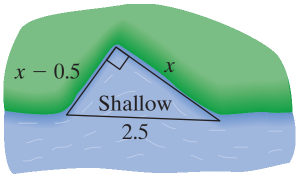


Figure 5-6

Solution:

Let x represent the longer leg.

Then $x - 0.5$ represents the shorter leg.

$$a^2 + b^2 = c^2$$ Pythagorean theorem

$$x^2 + (x - 0.5)^2 = (2.5)^2$$

$$x^2 + \overbrace{(x)^2 - 2(x)(0.5) + (0.5)^2} = 6.25$$

$$x^2 + x^2 - x + 0.25 = 6.25$$

TIP: Recall that the square of a binomial results in a perfect square trinomial.

$$(a - b)^2 = a^2 - 2ab + b^2$$
$$(x - 0.5)^2 = (x)^2 - 2(x)(0.5) + (0.5)^2$$
$$= x^2 - x + 0.25$$

Skill Practice Answers

8. Width: 1 in.; length: 2 in.

$2x^2 - x - 6 = 0$ Write the equation in the form $ax^2 + bx + c = 0$.

$(2x + 3)(x - 2) = 0$ Factor.

$2x + 3 = 0$ or $x - 2 = 0$ Set both factors to zero.

$x \xcancel{=} -\frac{3}{2}$ or $x = 2$ Solve both equations for x.

The side of a triangle cannot be negative, so we reject the solution $x = -\frac{3}{2}$.

Therefore, one leg of the triangle is 2 mi.

The other leg is $x - 0.5 = 2 - 0.5 = 1.5$ mi.

Skill Practice

9. The longer leg of a right triangle measures 7 ft more than the shorter leg. The hypotenuse is 8 ft longer than the shorter leg. Find the lengths of the sides of the triangle.

3. Definition of a Quadratic Function

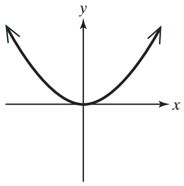
In Section 4.3, we graphed several basic functions by plotting points, including $f(x) = x^2$. This function is called a quadratic function, and its graph is in the shape of a **parabola**. In general, any second-degree polynomial function is a quadratic function.

Definition of a Quadratic Function

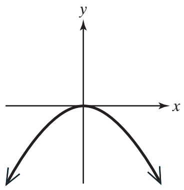
Let a, b, and c represent real numbers such that $a \neq 0$. Then a function in the form $f(x) = ax^2 + bx + c$ is called a **quadratic function**.

The graph of a quadratic function is a parabola that opens up or down. The leading coefficient a determines the direction of the parabola. For the quadratic function defined by $f(x) = ax^2 + bx + c$:

If $a > 0$, the parabola opens up. For example, $f(x) = x^2$



If $a < 0$, the parabola opens down. For example, $g(x) = -x^2$



Recall from Section 4.3 that the x-intercepts of a function $y = f(x)$ are the real solutions to the equation $f(x) = 0$. The y-intercept is found by evaluating $f(0)$.

Skill Practice Answers

9. The sides are 5, 12, and 13 ft.

Example 6 Finding the x- and y-Intercepts of a Quadratic Function

Find the x- and y-intercepts.

$$f(x) = x^2 - x - 12$$

Solution:

To find the x-intercept, substitute $f(x) = 0$.

$f(x) = x^2 - x - 12$	
$0 = x^2 - x - 12$	Substitute 0 for $f(x)$. The result is a quadratic equation.
$0 = (x - 4)(x + 3)$	Factor.
$x - 4 = 0 \quad \text{or} \quad x + 3 = 0$	Set each factor equal to zero.
$x = 4 \quad \text{or} \quad x = -3$	Solve each equation.

The x-intercepts are $(4, 0)$ and $(-3, 0)$.

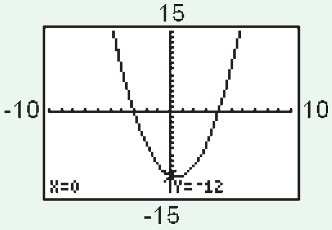
To find the y-intercept, find $f(0)$.

$f(x) = x^2 - x - 12$	
$f(0) = (0)^2 - (0) - 12$	Substitute $x = 0$.
$= -12$	

The y-intercept is $(0, -12)$.

Calculator Connections

The graph of $f(x) = x^2 - x - 12$ supports the solution to Example 6. The graph appears to cross the x-axis at -3 and 4. The y-intercept is given as $(0, -12)$.



Skill Practice

10. Find the x- and y-intercepts of the function defined by $f(x) = x^2 + 8x + 12$.

4. Applications of Quadratic Functions

Example 7 Application of a Quadratic Function

A model rocket is shot vertically upward with an initial velocity of 288 ft/sec. The function given by $h(t) = -16t^2 + 288t$ relates the rocket's height h (in feet) to the time t after launch (in seconds).

Skill Practice Answers

10. x-intercepts: $(-6, 0)$ and $(-2, 0)$; y-intercept: $(0, 12)$

a. Find $h(0)$, $h(5)$, $h(10)$, and $h(15)$, and interpret the meaning of these function values in the context of the rocket's height and time after launch.

b. Find the t-intercepts of the function, and interpret their meaning in the context of the rocket's height and time after launch.

c. Find the time(s) at which the rocket is at a height of 1152 ft.

Solution:

a. $h(t) = -16t^2 + 288t$

$h(0) = -16(0)^2 + 288(0) = 0$

$h(5) = -16(5)^2 + 288(5) = 1040$

$h(10) = -16(10)^2 + 288(10) = 1280$

$h(15) = -16(15)^2 + 288(15) = 720$

$h(0) = 0$ indicates that at $t = 0$ sec, the height of the rocket is 0 ft.

$h(5) = 1040$ indicates that 5 sec after launch, the height of the rocket is 1040 ft.

$h(10) = 1280$ indicates that 10 sec after launch, the height of the rocket is 1280 ft.

$h(15) = 720$ indicates that 15 sec after launch, the height of the rocket is 720 ft.

b. The t-intercepts of the function are represented by the real solutions of the equation $h(t) = 0$.

$-16t^2 + 288t = 0$ Set $h(t) = 0$.

$-16t(t - 18) = 0$ Factor.

$-16t = 0$ or $t - 18 = 0$ Apply the zero product rule.

$t = 0$ or $t = 18$

The rocket is at ground level initially (at $t = 0$ sec) and then again after 18 sec when it hits the ground.

c. Set $h(t) = 1152$ and solve for t.

$h(t) = -16t^2 + 288t$

$1152 = -16t^2 + 288t$ Substitute 1152 for $h(t)$.

$16t^2 - 288t + 1152 = 0$ Set the equation equal to zero.

$16(t^2 - 18t + 72) = 0$ Factor out the GCF.

$16(t - 6)(t - 12) = 0$ Factor.

$t = 6$ or $t = 12$

The rocket will reach a height of 1152 ft after 6 sec (on the way up) and after 12 sec (on the way down). (See Figure 5-7.)

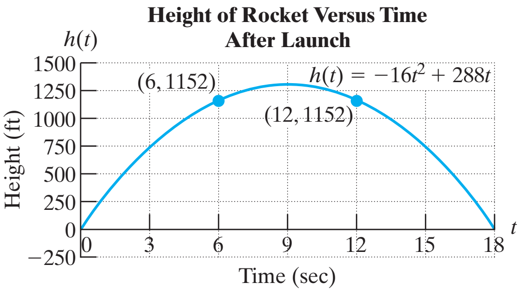


Figure 5-7

Skill Practice

11. An object is dropped from the top of a building that is 144 ft high. The function given by $h(t) = -16t^2 + 144$ relates the height h of the object (in feet) to the time t in seconds after it is dropped.

a. Find $h(0)$ and interpret the meaning of the function value in the context of this problem.

b. Find the t-intercept(s) and interpret the meaning in the context of this problem.

Skill Practice Answers

11a. $h(0) = 144$, which is the initial height of the object (after 0 sec).

b. The t-intercept is (3, 0) which means the object is at ground level (0 ft high) after 3 sec. The intercept (−3, 0) does not make sense for this problem since time cannot be negative.

Section 5.8 Practice Exercises

Study Skills Exercise

1. Define the key terms.

a. Quadratic equation **b. Zero product rule** **c. Quadratic function** **d. Parabola**

Review Exercises

2. Write the factored form for each binomial, if possible.

a. $x^2 - y^2$ **b.** $x^2 + y^2$ **c.** $x^3 - y^3$ **d.** $x^3 + y^3$

For Exercises 3–8, factor completely.

3. $10x^2 + 3x$

4. $7x^2 - 28$

5. $2p^2 - 9p - 5$

6. $3q^2 - 4q - 4$

7. $t^3 - 1$

8. $z^2 - 11z + 30$

Concept 1: Solving Equations by Using the Zero Product Rule

9. What conditions are necessary to solve an equation by using the zero product rule?

10. State the zero product rule.

For Exercises 11–16, determine which of the equations are written in the correct form to apply the zero product rule directly. If an equation is not in the correct form, explain what is wrong.

11. $2x(x - 3) = 0$

12. $(u + 1)(u - 3) = 10$

13. $3p^2 - 7p + 4 = 0$

14. $t^2 - t - 12 = 0$

15. $a(a + 3)^2 = 5$

16. $\left(\frac{2}{3}x - 5\right)\left(x + \frac{1}{2}\right) = 0$

For Exercises 17–50, solve the equation.

17. $(x + 3)(x + 5) = 0$

18. $(x + 7)(x - 4) = 0$

19. $(2w + 9)(5w - 1) = 0$

20. $(3a + 1)(4a - 5) = 0$

21. $x(x + 4)(10x - 3) = 0$

22. $t(t - 6)(3t - 11) = 0$

23. $0 = 5(y - 0.4)(y + 2.1)$

24. $0 = -4(z - 7.5)(z - 9.3)$

25. $x^2 + 6x - 27 = 0$

26. $2x^2 + x - 15 = 0$

27. $2x^2 + 5x = 3$

28. $-11x = 3x^2 - 4$

29. $10x^2 = 15x$

30. $5x^2 = 7x$

31. $6(y - 2) - 3(y + 1) = 8$

32. $4x + 3(x - 9) = 6x + 1$

33. $-9 = y(y + 6)$

34. $-62 = t(t - 16) + 2$

35. $9p^2 - 15p - 6 = 0$

36. $6y^2 + 2y = 48$

37. $(x + 1)(2x - 1)(x - 3) = 0$

38. $2x(x - 4)^2(4x + 3) = 0$

39. $(y - 3)(y + 4) = 8$

40. $(t + 10)(t + 5) = 6$

41. $(2a - 1)(a - 1) = 6$

42. $w(6w + 1) = 2$

43. $p^2 + (p + 7)^2 = 169$

44. $x^2 + (x + 2)^2 = 100$

45. $3t(t + 5) - t^2 = 2t^2 + 4t - 1$

46. $a^2 - 4a - 2 = (a + 3)(a - 5)$

47. $2x^3 - 8x^2 - 24x = 0$

48. $2p^3 + 20p^2 + 42p = 0$

49. $w^3 = 16w$

50. $12x^3 = 27x$

Concept 2: Applications of Quadratic Equations

51. If 5 is added to the square of a number, the result is 30. Find all such numbers.

52. Four less than the square of a number is 77. Find all such numbers.

53. The square of a number is equal to 12 more than the number. Find all such numbers.

54. The square of a number is equal to 20 more than the number. Find all such numbers.

55. The product of two consecutive integers is 42. Find the integers.

56. The product of two consecutive integers is 110. Find the integers.

57. The product of two consecutive odd integers is 63. Find the integers.

58. The product of two consecutive even integers is 120. Find the integers.

59. A rectangular pen is to contain 35 ft^2 of area. If the width is 2 ft less than the length, find the dimensions of the pen.

60. The length of a rectangular photograph is 7 in. more than the width. If the area is 78 in.^2, what are the dimensions of the photograph?

61. The length of a rectangular room is 5 yd more than the width. If the area is 300 yd^2, find the length and the width of the room.

62. The top of a rectangular dining room table is twice as long as it is wide. Find the dimensions of the table if the area is 18 ft^2.

63. The height of a triangle is 1 in. more than the base. If the height is increased by 2 in. while the base remains the same, the new area becomes 20 in.^2

a. Find the base and height of the original triangle.

b. Find the area of the original triangle.

64. The base of a triangle is 2 cm more than the height. If the base is increased by 4 cm while the height remains the same, the new area is 56 cm^2.

a. Find the base and height of the original triangle.

b. Find the area of the original triangle.

65. The area of a triangular garden is 25 ft^2. The base is twice the height. Find the base and the height of the triangle.

66. The height of a triangle is 1 in. more than twice the base. If the area is 18 $in.^2$, find the base and height of the triangle.

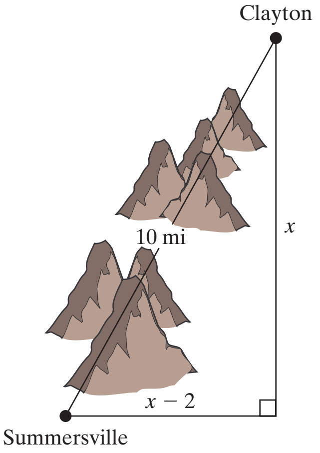
67. The sum of the squares of two consecutive positive integers is 41. Find the integers.

68. The sum of the squares of two consecutive, positive even integers is 164. Find the integers.

69. Justin must travel from Summersville to Clayton. He can drive 10 mi through the mountains at 40 mph. Or he can drive east and then north on superhighways at 60 mph. The alternative route forms a right angle as shown in the diagram. The eastern leg is 2 mi less than the northern leg.

a. Find the total distance Justin would travel in going the alternative route.

b. If Justin wants to minimize the time of the trip, which route should he take?



70. A 17-ft ladder is standing up against a wall. The distance between the base of the ladder and the wall is 7 ft less than the distance between the top of the ladder and the base of the wall. Find the distance between the base of the ladder and the wall.

71. A right triangle has side lengths represented by three consecutive even integers. Find the lengths of the three sides, measured in meters.

72. The hypotenuse of a right triangle is 3 m more than twice the short leg. The longer leg is 2 m more than twice the shorter leg. Find the lengths of the sides.

Concept 3: Definition of a Quadratic Function

For Exercises 73–76,

a. Find the values of x for which $f(x) = 0$. **b.** Find $f(0)$.

73. $f(x) = x^2 - 3x$ **74.** $f(x) = 4x^2 + 2x$ **75.** $f(x) = 5(x - 7)$ **76.** $f(x) = 4(x + 5)$

For Exercises 77–80, find the x- and y-intercepts for the functions defined by $y = f(x)$.

77. $f(x) = \frac{1}{2}(x - 2)(x + 1)(2x)$ **78.** $f(x) = (x + 1)(x - 2)(x + 3)^2$

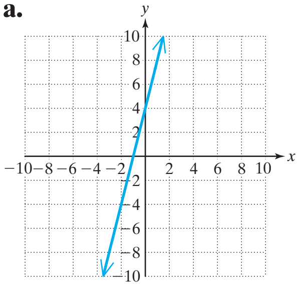
79. $f(x) = x^2 - 2x + 1$ **80.** $f(x) = x^2 + 4x + 4$

For Exercises 81–84, find the x-intercepts of each function and use that information to match the function with its graph.

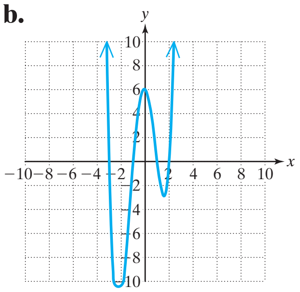
81. $g(x) = (x + 3)(x - 3)$ **82.** $h(x) = x(x - 2)(x + 4)$ **83.** $f(x) = 4(x + 1)$

84. $k(x) = (x + 1)(x + 3)(x - 2)(x - 1)$

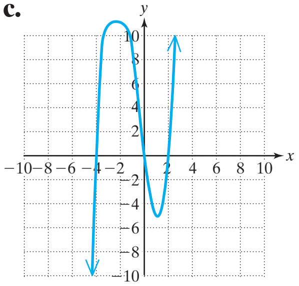
a.



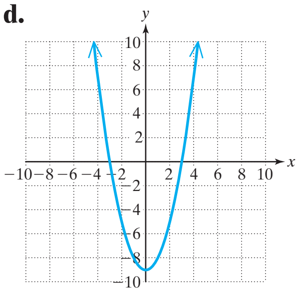
b.



c.



d.



Concept 4: Applications of Quadratic Functions

85. A rocket is fired upward from ground level with an initial velocity of 490 m/sec. The height of the rocket $s(t)$ in meters is a function of the time t in seconds after launch.

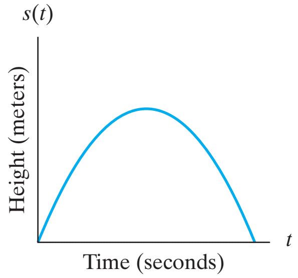
$$s(t) = -4.9t^2 + 490t$$

a. What characteristics of s indicate that it is a quadratic function?

b. Find the t-intercepts of the function.

c. What do the t-intercepts mean in the context of this problem?

d. At what times is the rocket at a height of 485.1 m?



86. A certain company makes water purification systems. The factory can produce x water systems per year. The profit $P(x)$ the company makes is a function of the number of systems x it produces.

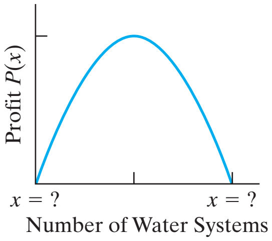
$$P(x) = -2x^2 + 1000x$$

a. Is this function linear or quadratic?

b. Find the number of water systems x that would produce a zero profit.

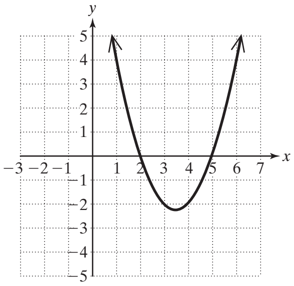
c. What points on the graph do the answers in part (b) represent?

d. Find the number of systems for which the profit is \$80,000.

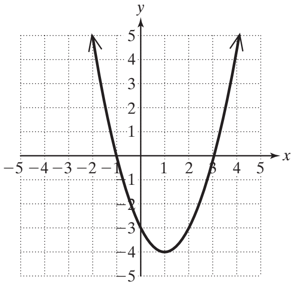


For Exercises 87–90, factor the functions represented by $f(x)$. Explain how the factored form relates to the graph of the function. Can the graph of the function help you determine the factors of the function?

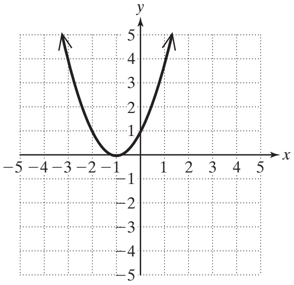
87. $f(x) = x^2 - 7x + 10$



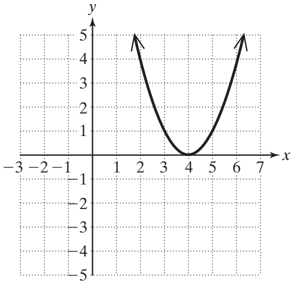
88. $f(x) = x^2 - 2x - 3$



89. $f(x) = x^2 + 2x + 1$



90. $f(x) = x^2 - 8x + 16$



Expanding Your Skills

For Exercises 91–94, find an equation that has the given solutions. For example, 2 and -1 are solutions to $(x - 2)(x + 1) = 0$ or $x^2 - x - 2 = 0$. In general, x_1 and x_2 are solutions to the equation $a(x - x_1)(x - x_2) = 0$, where a can be any nonzero real number. For each problem, there is more than one correct answer depending on your choice of a.

91. $x = -3$ and $x = 1$

92. $x = 2$ and $x = -2$

93. $x = 0$ and $x = -5$

94. $x = 0$ and $x = -3$

Graphing Calculator Exercises

For Exercises 95–98, graph Y_1. Use the *Zoom* and *Trace* features to approximate the x-intercepts. Then solve $Y_1 = 0$ and compare the solutions to the x-intercepts.

95. $Y_1 = -x^2 + x + 2$

96. $Y_1 = -x^2 - x + 20$

97. $Y_1 = x^2 - 6x + 9$

98. $Y_1 = x^2 + 4x + 4$

Chapter 5 SUMMARY

Section 5.1 Addition and Subtraction of Polynomials and Polynomial Functions

Key Concepts

A **polynomial** in x is defined by a finite sum of terms of the form ax^n, where a is a real number and n is a whole number.

- a is the **coefficient** of the term.
- n is the **degree of the term**.

The **degree of a polynomial** is the largest degree of its terms.

The term of a polynomial with the largest degree is the **leading term**. Its coefficient is the **leading coefficient**.

A one-term polynomial is a **monomial**.

A two-term polynomial is a **binomial**.

A three-term polynomial is a **trinomial**.

To add or subtract polynomials, add or subtract *like* terms.

Examples

Example 1

$7y^4 - 2y^2 + 3y + 8$

is a polynomial with leading coefficient 7 and degree 4.

Example 2

$f(x) = 4x^3 - 6x - 11$

f is a polynomial function with leading term $4x^3$ and leading coefficient 4. The degree of f is 3.

Example 3

For $f(x) = 4x^3 - 6x - 11$, find $f(-1)$.

$$\begin{aligned} f(-1) &= 4(-1)^3 - 6(-1) - 11 \\ &= -9 \end{aligned}$$

Example 4

$$\begin{aligned} &(-4x^3y + 3x^2y^2) - (7x^3y - 5x^2y^2) \\ &= -4x^3y + 3x^2y^2 - 7x^3y + 5x^2y^2 \\ &= -11x^3y + 8x^2y^2 \end{aligned}$$

Section 5.2 Multiplication of Polynomials

Key Concepts

To multiply polynomials, multiply each term in the first polynomial by each term in the second polynomial.

Special Products

1. Multiplication of **conjugates**

   $(x + y)(x - y) = x^2 - y^2$

   The product is called a **difference of squares**.

2. Square of a binomial

   $(x + y)^2 = x^2 + 2xy + y^2$

   $(x - y)^2 = x^2 - 2xy + y^2$

   The product is called a **perfect square trinomial**.

Examples

Example 1

$$\begin{aligned} &(x - 2)(3x^2 - 4x + 11) \\ &= 3x^3 - 4x^2 + 11x - 6x^2 + 8x - 22 \\ &= 3x^3 - 10x^2 + 19x - 22 \end{aligned}$$

Example 2

$$\begin{aligned} &(3x + 5)(3x - 5) \\ &= (3x)^2 - (5)^2 \\ &= 9x^2 - 25 \end{aligned}$$

Example 3

$$\begin{aligned} &(4y + 3)^2 \\ &= (4y)^2 + (2)(4y)(3) + (3)^2 \\ &= 16y^2 + 24y + 9 \end{aligned}$$

Section 5.3 Division of Polynomials

Key Concepts

Division of polynomials:

1. For division by a monomial, use the properties

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c} \qquad \text{and} \qquad \frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}$$

for $c \neq 0$.

2. If the divisor has more than one term, use long division.

3. **Synthetic division** may be used to divide a polynomial by a binomial in the form $x - r$, where r is a constant.

Examples

Example 1

$$\frac{-12a^2 - 6a + 9}{-3a}$$

$$= \frac{-12a^2}{-3a} - \frac{6a}{-3a} + \frac{9}{-3a}$$

$$= 4a + 2 - \frac{3}{a}$$

Example 2

$(3x^2 - 5x + 1) \div (x + 2)$

$$\begin{array}{r@{}l}
 & 3x - 11 \\
x + 2 & \overline{)3x^2 - 5x + 1} \\
 & \underline{-(3x^2 + 6x)} \\
 & \qquad -11x + 1 \\
 & \qquad \underline{-(-11x - 22)} \\
 & \qquad\qquad\quad 23
\end{array}$$

Answer: $3x - 11 + \dfrac{23}{x + 2}$

Example 3

$(3x^2 - 5x + 1) \div (x + 2)$

$$\begin{array}{r|rrr}
-2 & 3 & -5 & 1 \\
 & & -6 & 22 \\
\hline
 & 3 & -11 & \boxed{23}
\end{array}$$

Answer: $3x - 11 + \dfrac{23}{x + 2}$

Section 5.4 Greatest Common Factor and Factoring by Grouping

Key Concepts

The **greatest common factor (GCF)** is the largest factor common to all terms of a polynomial. To factor out the GCF from a polynomial, use the distributive property.

A four-term polynomial may be **factored by grouping**.

Steps to Factor by Grouping

1. Identify and factor out the GCF from all four terms.
2. Factor out the GCF from the first pair of terms. Factor out the GCF from the second pair of terms. (Sometimes it is necessary to factor out the *opposite* of the GCF.)
3. If the two pairs of terms share a common binomial factor, factor out the binomial factor.

Examples

Example 1

$3x^2(a + b) - 6x(a + b)$

$= 3x(a + b)x - 3x(a + b)(2)$

$= 3x(a + b)(x - 2)$

Example 2

$60xa - 30xb - 80ya + 40yb$

$= 10[6xa - 3xb \mid - 8ya + 4yb]$

$= 10[3x(2a - b) - 4y(2a - b)]$

$= 10(2a - b)(3x - 4y)$

Section 5.5 Factoring Trinomials

Key Concepts

AC-Method

To factor trinomials of the form $ax^2 + bx + c$:

1. Factor out the GCF.
2. Find the product ac.
3. Find two integers whose product is ac and whose sum is b. (If no pair of numbers can be found, then the trinomial is prime.)
4. Rewrite the middle term bx as the sum of two terms whose coefficients are the numbers found in step 3.
5. Factor the polynomial by grouping.

Examples

Example 1

$10y^2 + 35y - 20 = 5(2y^2 + 7y - 4)$

$ac = (2)(-4) = -8$

Find two integers whose product is -8 and whose sum is 7. The numbers are 8 and -1.

$5[2y^2 + 8y \mid - 1y - 4]$

$= 5[2y(y + 4) - 1(y + 4)]$

$= 5(y + 4)(2y - 1)$

Trial-and-Error Method

To factor trinomials in the form $ax^2 + bx + c$:

1. Factor out the GCF.
2. List the pairs of factors of a and the pairs of factors of c. Consider the reverse order in either list.
3. Construct two binomials of the form

Factors of a

$$(\square x \quad \square)(\square x \quad \square)$$

Factors of c

4. Test each combination of factors until the product of the outer terms and the product of inner terms add to the middle term.
5. If no combination of factors works, the polynomial is prime.

The factored form of a **perfect square trinomial** is the square of a binomial:

$$a^2 + 2ab + b^2 = (a + b)^2$$

$$a^2 - 2ab + b^2 = (a - b)^2$$

Example 2

$$10y^2 + 35y - 20 = 5(2y^2 + 7y - 4)$$

The pairs of factors of 2 are $2 \cdot 1$.
The pairs of factors of -4 are

$$\begin{array}{ll} -1 \cdot 4 & 1 \cdot (-4) \\ -2 \cdot 2 & 2 \cdot (-2) \\ -4 \cdot 1 & 4 \cdot (-1) \end{array}$$

$(2y - 2)(y + 2) = 2y^2 + 2y - 4$ No

$(2y - 4)(y + 1) = 2y^2 - 2y - 4$ No

$(2y + 1)(y - 4) = 2y^2 - 7y - 4$ No

$(2y + 2)(y - 2) = 2y^2 - 2y - 4$ No

$(2y + 4)(y - 1) = 2y^2 + 2y - 4$ No

$(2y - 1)(y + 4) = 2y^2 + 7y - 4$ Yes

Therefore, $10y^2 + 35y - 20$ factors as $5(2y - 1)(y + 4)$.

Example 3

$$9w^2 - 30wz + 25z^2$$

$$= (3w)^2 - 2(3w)(5z) + (5z)^2$$

$$= (3w - 5z)^2$$

Section 5.6 Factoring Binomials

Key Concepts

Factoring Binomials: Summary

Difference of squares:

$$a^2 - b^2 = (a + b)(a - b)$$

Difference of cubes:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Sum of cubes:

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

Examples

Example 1

$$25u^2 - 9v^4 = (5u + 3v^2)(5u - 3v^2)$$

Example 2

$$8c^3 - d^6 = (2c - d^2)(4c^2 + 2cd^2 + d^4)$$

Example 3

$$27w^9 + 64x^3$$

$$= (3w^3 + 4x)(9w^6 - 12w^3x + 16x^2)$$

Section 5.7 Additional Factoring Strategies

Key Concepts

1. Factor out the GCF (Section 5.4).
2. Identify whether the polynomial has two terms, three terms, or more than three terms.
3. If the polynomial has more than three terms, try factoring by grouping (Section 5.4).
4. If the polynomial has three terms, check first for a perfect square trinomial. Otherwise, factor by using the ac-method or trial-and-error method (Section 5.5).
5. If the polynomial has two terms, determine if it fits the pattern for a difference of squares, difference of cubes, or sum of cubes. Remember, a sum of squares is not factorable over the real numbers (Section 5.6).
6. Be sure to factor the polynomial completely.
7. Check by multiplying.

Examples

Example 1

$9x^2 - 4x + 9x^3$

$= x(9x - 4 + 9x^2)$ Factor out the GCF.

$= x(9x^2 + 9x - 4)$ Descending order.

$= x(3x + 4)(3x - 1)$ Factor the trinomial.

Example 2

$4a^2 - 12ab + 9b^2 - c^2$

$= 4a^2 - 12ab + 9b^2 \mid - c^2$ Group 3 by 1.

$= (2a - 3b)^2 - c^2$ Perfect square trinomial.

$= (2a - 3b - c)(2a - 3b + c)$ Difference of squares.

Section 5.8 Solving Equations by Using the Zero Product Rule

Key Concepts

An equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$, is a **quadratic equation**.

The **zero product rule** states that if $a \cdot b = 0$, then $a = 0$ or $b = 0$. The zero product rule can be used to solve a quadratic equation or higher-degree polynomial equation that is factored and equal to zero.

$f(x) = ax^2 + bx + c\ (a \neq 0)$ defines a **quadratic function**. The x-intercepts of a function defined by $y = f(x)$ are determined by finding the real solutions to the equation $f(x) = 0$. The y-intercept of a function $y = f(x)$ is at $f(0)$.

Examples

Example 1

$0 = x(2x - 3)(x + 4)$

$x = 0$ or $2x - 3 = 0$ or $x + 4 = 0$

$x = \frac{3}{2}$ or $x = -4$

Example 2

Find the x-intercepts.

$f(x) = 3x^2 - 8x + 5$

$0 = 3x^2 - 8x + 5$

$0 = (3x - 5)(x - 1)$

$3x - 5 = 0$ or $x - 1 = 0$

$x = \frac{5}{3}$ or $x = 1$

The x-intercepts are $(\frac{5}{3}, 0)$ and $(1, 0)$.

Find the y-intercept.

$f(x) = 3x^2 - 8x + 5$

$f(0) = 3(0)^2 - 8(0) + 5$

$f(0) = 5$

The y-intercept is $(0, 5)$.

Chapter 5 Review Exercises

Section 5.1

For Exercises 1–2, identify the polynomial as a monomial, binomial, or trinomial; then give the degree of the polynomial.

1. $6x^4 + 10x - 1$ **2.** 18

3. Given the polynomial function defined by $g(x) = 4x - 7$, find the function values.

a. $g(0)$ **b.** $g(-4)$ **c.** $g(3)$

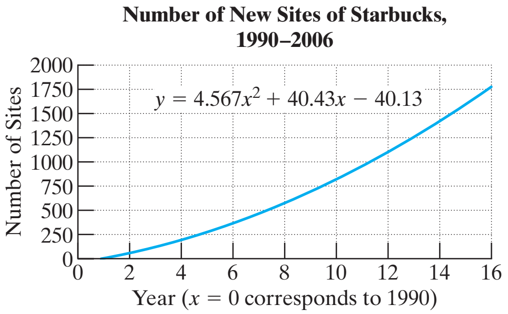
4. Given the polynomial function defined by $p(x) = -x^4 - x + 12$, find the function values.

a. $p(0)$ **b.** $p(1)$ **c.** $p(-2)$

5. The number of new sites established by Starbucks in the years from 1990 to 2006 can be approximated by the function $S(x) = 4.567x^2 + 40.43x - 40.13$, where $x = 0$ represents the year 1990.

a. Evaluate $S(5)$ and $S(13)$ to the nearest whole unit. Match the function values with points on the graph (see the figure).

b. Interpret the meaning of the function value for $S(13)$.



For Exercises 6–13, add or subtract the polynomials as indicated.

6. $(x^2 - 2x - 3xy - 7) + (-3x^2 - x + 2xy + 6)$

7. $(7xy - 3xz + 5yz) + (13xy - 15xz - 8yz)$

8. $(8a^2 - 4a^3 - 3a) - (3a^2 - 9a - 7a^3)$

9. $(3a^2 - 2a - a^3) - (5a^2 - a^3 - 8a)$

10. $\left(\frac{5}{8}x^4 - \frac{1}{4}x^2 - \frac{1}{2}\right) - \left(-\frac{3}{8}x^4 + \frac{3}{4}x^2 + \frac{1}{2}\right)$

11. $\left(\frac{5}{6}x^4 + \frac{1}{2}x^2 - \frac{1}{3}\right) - \left(-\frac{1}{6}x^4 - \frac{1}{4}x^2 - \frac{1}{3}\right)$

12. $(7x - y) - [-(2x + y) - (-3x - 6y)]$

13. $-(4x - 4y) - [(4x + 2y) - (3x + 7y)]$

14. Add $-4x + 6$ to $-7x - 5$.

15. Add $2x^2 - 4x$ to $2x^2 - 7x$.

16. Subtract $-4x + 6$ from $-7x - 5$.

17. Subtract $2x^2 - 4x$ from $2x^2 - 7x$.

Section 5.2

For Exercises 18–35, multiply the polynomials.

18. $2x(x^2 - 7x - 4)$ **19.** $-3x(6x^2 - 5x + 4)$

20. $(x + 6)(x - 7)$ **21.** $(x - 2)(x - 9)$

22. $\left(\frac{1}{2}x + 1\right)\left(\frac{1}{2}x - 5\right)$ **23.** $\left(-\frac{1}{5} + 2y\right)\left(\frac{1}{5} + y\right)$

24. $(3x + 5)(9x^2 - 15x + 25)$

25. $(x - y)(x^2 + xy + y^2)$

26. $(2x - 5)^2$ **27.** $\left(\frac{1}{2}x + 4\right)^2$

28. $(3y - 11)(3y + 11)$ **29.** $(6w - 1)(6w + 1)$

30. $\left(\frac{2}{3}t + 4\right)\left(\frac{2}{3}t - 4\right)$ **31.** $\left(z + \frac{1}{4}\right)\left(z - \frac{1}{4}\right)$

32. $[(x + 2) - b][(x + 2) + b]$

33. $[c - (w + 3)][c + (w + 3)]$

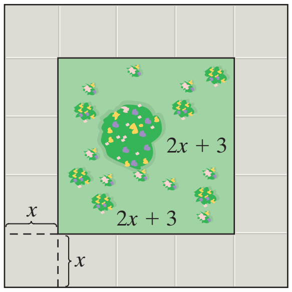
34. $(2x + 1)^3$ **35.** $(y^2 - 3)^3$

36. A square garden is surrounded by a walkway of uniform width x. If the sides of the garden are given by the expression $2x + 3$, find and simplify a polynomial that represents

a. The area of the garden.

b. The area of the walkway and garden.

c. The area of the walkway only.



37. The length of a rectangle is 2 ft more than 3 times the width. Let x represent the width of the rectangle.

a. Write a function P that represents the perimeter of the rectangle.

b. Write a function A that represents the area of the rectangle.

38. In parts (a) and (b), one of the statements is true and the other is false. Identify the true statement and explain why the false statement is incorrect.

a. $2x^2 + 5x = 7x^3$ $\quad$ $(2x^2)(5x) = 10x^3$

b. $4x - 7x = -3x$ $\quad$ $4x - 7x = -3$

Section 5.3

For Exercises 39–40, divide the polynomials.

39. $(6x^3 + 12x^2 - 9x) \div (3x)$

40. $(10x^4 + 15x^3 - 20x^2) \div (-5x^2)$

41. **a.** Divide $(9y^4 + 14y^2 - 8) \div (3y + 2)$.

b. Identify the quotient and the remainder.

c. Explain how you can check your answer.

For Exercises 42–45, divide the polynomials by using long division.

42. $(x^2 + 7x + 10) \div (x + 5)$

43. $(x^2 + 8x - 16) \div (x + 4)$

44. $(2x^5 - 4x^4 + 2x^3 - 4) \div (x^2 - 3x)$

45. $(2x^5 + 3x^3 + x^2 - 4) \div (x^2 + x)$

46. Explain the conditions under which you may use synthetic division.

47. The following table is the result of a synthetic division.

3	2	5	−2	6	1
		6	33	93	297
	2	11	31	99	298

Use x as the variable.

a. Identify the divisor.

b. Identify the quotient.

c. Identify the remainder.

For Exercises 48–52, divide the polynomials by using synthetic division.

48. $(t^3 - 3t^2 + 8t - 12) \div (t - 2)$

49. $(x^2 + 7x + 14) \div (x + 5)$

50. $(x^2 + 8x + 20) \div (x + 4)$

51. $(w^3 - 6w^2 + 8) \div (w - 3)$

52. $(p^4 - 16) \div (p - 2)$

Section 5.4

For Exercises 53–57, factor by removing the greatest common factor.

53. $-x^3 - 4x^2 + 11x$

54. $21w^3 - 7w + 14$

55. $5x(x - 7) - 2(x - 7)$

56. $3t(t + 4) + 5(t + 4)$

57. $2x^2 - 26x$

For Exercises 58–61, factor by grouping (remember to take out the GCF first).

58. $m^3 - 8m^2 + m - 8$

59. $24x^3 - 36x^2 + 72x - 108$

60. $4ax^2 + 2bx^2 - 6ax - 3xb$

61. $y^3 - 6y^2 + y - 6$

Section 5.5

62. What characteristics determine a perfect square trinomial?

For Exercises 63–72, factor the polynomials by using any method.

63. $18x^2 + 27xy + 10y^2$

64. $2 + 7k + 6k^2$

65. $60a^2 + 65a^3 - 20a^4$

66. $8b^2 - 40b + 50$

67. $n^2 + 10n + 25$

68. $2x^2 + 5x + 12$

69. $y^3 - y(10 - 3y)$

70. $m + 18 - m(m - 2)$

71. $9x^2 - 12x + 4$

72. $25q^2 + 30q + 9$

Section 5.6

For Exercises 73–79, factor the binomials.

73. $25 - y^2$

74. $x^3 - \frac{1}{27}$

75. $b^2 + 64$

76. $a^3 + 64$

77. $h^3 + 9h$

78. $k^4 - 16$

79. $9y^3 - 4y$

For Exercises 80–81, factor by grouping and by using the difference of squares.

80. $x^2 - 8xy + 16y^2 - 9$ (*Hint*: Group three terms that constitute a perfect square trinomial, then factor as a difference of squares.)

81. $a^2 + 12a + 36 - b^2$

Section 5.7

For Exercises 82–95, factor completely using the factoring strategy found on page 371.

82. $12s^3t - 45s^2t^2 - 12st^3$

83. $5p^4q - 20q^3$

84. $4d^2(3 + d) - (3 + d)$

85. $(y - 4)^3 + 4(y - 4)^2$

86. $49x^2 + 36 - 84x$

87. $80z + 32 + 50z^2$

88. $18a^2 + 39a - 15$

89. $w^4 + w^3 - 56w^2$

90. $8n + n^4$

91. $14m^3 - 14$

92. $b^2 + 16b + 64 - 25c^2$

93. $a^2 - 6a + 9 - 16x^2$

94. $(9w + 2)^2 + 4(9w + 2) - 5$

95. $(4x + 3)^2 - 12(4x + 3) + 36$

Section 5.8

96. How do you determine if an equation is quadratic?

97. What shape is the graph of a quadratic function?

For Exercises 98–101, label the equation as quadratic or linear.

98. $x^2 + 6x = 7$

99. $(x - 3)(x + 4) = 9$

100. $2x - 5 = 3$

101. $x + 3 = 5x^2$

For Exercises 102–105, use the zero product rule to solve the equations.

102. $x^2 - 2x - 15 = 0$

103. $8x^2 = 59x - 21$

104. $2t(t + 5) + 1 = 3t - 3 - t^2$

105. $3(x - 1)(x + 5)(2x - 9) = 0$

For Exercises 106–109, find the x- and y-intercepts of the function. Then match the function with its graph.

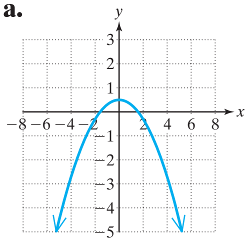
106. $f(x) = -4x^2 + 4$

107. $g(x) = 2x^2 - 2$

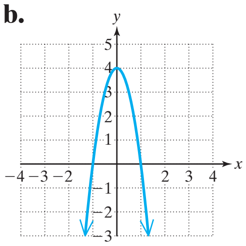
108. $h(x) = 5x^3 - 10x^2 - 20x + 40$

109. $k(x) = -\frac{1}{8}x^2 + \frac{1}{2}$

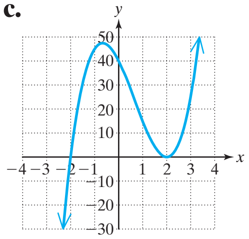
a.



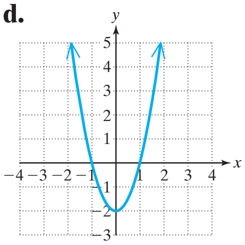
b.



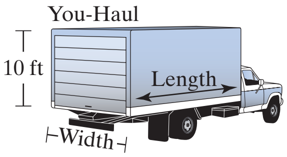
c.



d.



110. A moving van has the capacity to hold 1200 ft^3 in volume. If the van is 10 ft high and the length is 1 ft less than twice the width, find the dimensions of the van.



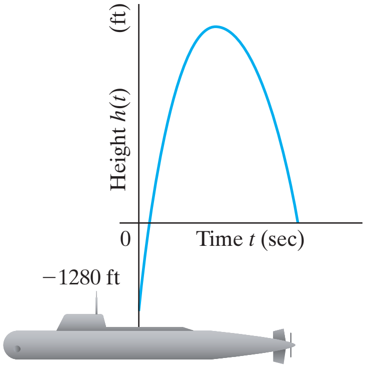
111. A missile is shot upward from a submarine 1280 ft below sea level. The initial velocity of the missile is 672 ft/sec. A function that approximates the height of the missile (relative to sea level) is given by

$$h(t) = -16t^2 + 672t - 1280$$

where $h(t)$ is the height in feet and t is the time in seconds.

a. Complete the table to determine the height of the missile for the given values of t.

Time t (sec)	Height $h(t)$ (ft)
0	
1	
3	
10	
20	
30	
42	



b. Interpret the meaning of a negative value of $h(t)$.

c. Factor the function to find the time required for the missile to emerge from the water and the time required for the missile to reenter the water. (*Hint*: The height of the missile will be zero at sea level.)

Chapter 5 Test

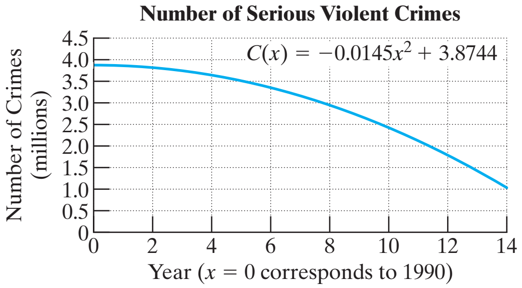
1. For the function defined by $F(x) = 5x^3 - 2x^2 + 8$, find the function values $F(-1)$, $F(2)$, and $F(0)$.

2. The number of serious violent crimes in the United States for the years 1990–2003 can be approximated by the function

$C(x) = -0.0145x^2 + 3.8744$, where $x = 0$ corresponds to the year 1990 and $C(x)$ is in millions.

a. Evaluate $C(2)$, $C(6)$, and $C(12)$. Match the function values with points on the graph (see the figure).

b. Interpret the meaning of the function value for $C(12)$.



(Source: Bureau of Justice Statistics.)

3. Perform the indicated operations. Write the answer in descending order.

$$(5x^2 - 7x + 3) - (x^2 + 5x - 25) + (4x^2 + 4x - 20)$$

For Exercises 4–6, multiply the polynomials. Write the answer in descending order.

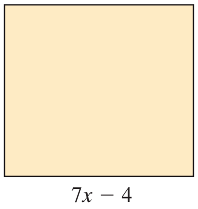
4. $(2a - 5)(a^2 - 4a - 9)$

5. $\left(\frac{1}{3}x - \frac{3}{2}\right)(6x + 4)$

6. $(5x - 4y^2)(5x + 4y^2)$

7. Explain why $(5x + 7)^2 \neq 25x^2 + 49$.

8. Write and simplify an expression that describes the area of the square.



9. Divide the polynomials.

$$(2x^3y^4 + 5x^2y^2 - 6xy^3 - xy) \div (2xy)$$

10. Divide the polynomials.

$$(10p^3 + 13p^2 - p + 3) \div (2p + 3)$$

11. Divide the polynomials by using synthetic division. $(y^4 - 2y + 5) \div (y - 2)$

12. Explain the strategy for factoring a polynomial expression.

13. Explain the process to solve a polynomial equation by the zero product rule.

For Exercises 14–26, factor completely.

14. $3a^2 + 27ab + 54b^2$

15. $c^4 - 1$

16. $xy - 7x + 3y - 21$

17. $49 + p^2$

18. $-10u^2 + 30u - 20$

19. $12t^2 - 75$

20. $5y^2 - 50y + 125$

21. $21q^2 + 14q$

22. $2x^3 + x^2 - 8x - 4$

23. $y^3 - 125$

24. $x^2 + 8x + 16 - y^2$

25. $r^6 - 256r^2$

26. $12a - 6ac + 2b - bc$

For Exercises 27–32, solve the equation.

27. $(2x - 3)(x + 5) = 0$

28. $x^2 - 7x = 0$

29. $x^2 - 6x = 16$

30. $x(5x + 4) = 1$

31. $4x - 64x^3 = 0$

32. $x^2 + \frac{1}{2}x + \frac{1}{16} = 0$

For Exercises 33–36, find the x- and y-intercepts of the function. Then match the function with its graph.

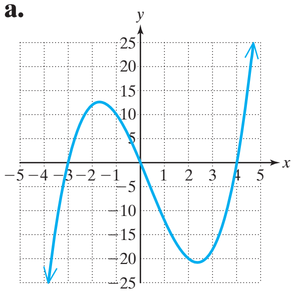
33. $f(x) = x^2 - 6x + 8$

34. $k(x) = x^3 + 4x^2 - 9x - 36$

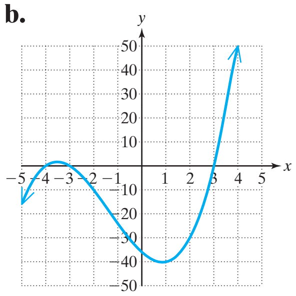
35. $p(x) = -2x^2 - 8x - 6$

36. $q(x) = x^3 - x^2 - 12x$

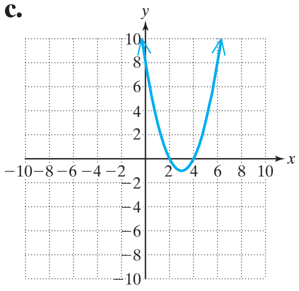
a.



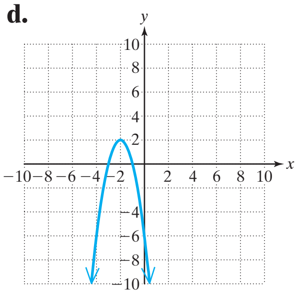
b.



c.



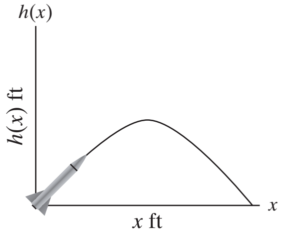
d.



37. A child launches a toy rocket from the ground. The height of the rocket h can be determined by its horizontal distance from the launch pad x by

$$h(x) = -\frac{x^2}{256} + x$$

where x and h are in feet and $x \geq 0$ and $h \geq 0$.



How many feet from the launch pad will the rocket hit the ground?

38. The recent population, P (in millions) of Japan can be approximated by:

$$P(t) = -0.01t^2 - 0.062t + 127.7,$$

where $t = 0$ represents the year 2000.

a. Approximate the number of people in Japan in the year 2006.

b. If the trend continues, predict the population of Japan in the year 2015.

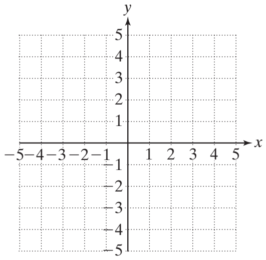
Chapters 1–5 Cumulative Review Exercises

1. Graph the inequality and express the set in interval notation: All real numbers at least 5, but not more than 12

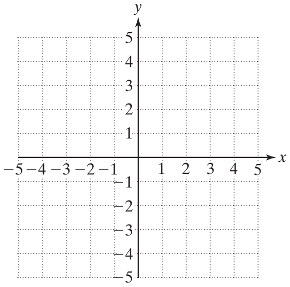
2. Simplify the expression $3x^2 - 5x + 2 - 4(x^2 + 3)$.

3. Graph from memory.

a. $y = x^2$



b. $y = |x|$

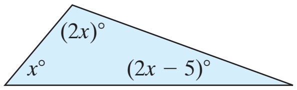


4. Simplify the expression $(\frac{1}{3})^{-2} - (\frac{1}{2})^3$.

5. In 1998, the population of Mexico was approximately 9.85×10^7. At the current growth rate of 1.7%, this number is expected to double after 42 years. How many people does this represent? Express your answer in scientific notation.

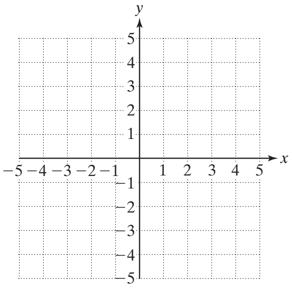
6. In the 2006 Orange Bowl football championship, Penn State scored 3 points more than Florida State in a three overtime thriller. The total number of points scored was 49. Find the number of points scored by each team.

7. Find the value of each angle in the triangle.



8. Divide $(x^3 + 64) \div (x + 4)$.

9. Determine the slope and y-intercept of the line $4x - 3y = -9$, and graph the line.



10. If y varies directly with x and inversely with z, and $y = 6$ when $x = 9$ and $z = \frac{1}{2}$, find y when $x = 3$ and $z = 4$.

11. Simplify the expression.

$$\left(\frac{36a^{-2}b^{4}}{18b^{-6}}\right)^{-3}$$

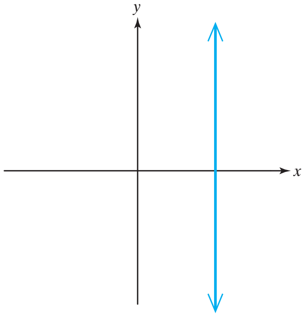
12. Solve the system.

$$\begin{aligned} 2x - y + 2z &= 1 \\ -3x + 5y - 2z &= 11 \\ x + y - 2z &= -1 \end{aligned}$$

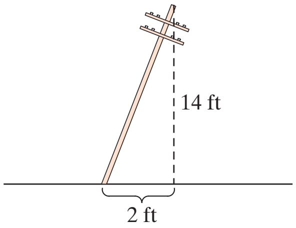
13. Determine whether the relation is a function.

a. $\{(2, 1), (3, 1), (-8, 1), (5, 1)\}$

b.



14. A telephone pole is leaning after a storm (see figure). What is the slope of the pole?



15. Given $P(x) = \frac{1}{6}x^2 + x - 5$, find the function value $P(6)$.

16. Solve for x: $\frac{1}{3}x - \frac{1}{6} = \frac{1}{2}(x - 3)$.

17. Given $3x - 2y = 5$, solve for y.

18. A student scores 76, 85, and 92 on her first three algebra tests.

a. Is it possible for her to score high enough on the fourth test to bring her test average up to 90? Assume that each test is weighted equally and that the maximum score on a test is 100 points.

b. What is the range of values required for the fourth test so that the student's test average will be between 80 and 89, inclusive?

19. How many liters of a 40% acid solution and how many liters of a 15% acid solution must be mixed to obtain 25 L of a 30% acid solution?

20. Multiply the polynomials $(4b - 3)(2b^2 + 1)$.

21. Add the polynomials.

$$(5a^2 + 3a - 1) + (3a^3 - 5a + 6)$$

22. Divide the polynomials $(6w^3 - 5w^2 - 2w) \div (2w^2)$

For Exercises 23–25, solve the equations.

23. $y^2 - 5y = 14$

24. $25x^2 = 36$

25. $a^3 + 9a^2 + 20a = 0$