Systems of Linear Equations

3

In this chapter we solve systems of linear equations in two and three variables. Some new terms are introduced in the first section of this chapter. Unscramble each word to find a key word from this chapter. As a hint, there is a clue for each word. Complete the word scramble to familiarize yourself with the key terms.

1. NNEDPNTDIEE __ __ __ __ __ __ __ __ __ __ __ (A system of two linear equations representing more than one line)

2. NNCIEOSTINST __ __ __ __ __ __ __ __ __ __ __ __ (A system having no solution)

3. LOUINSTO __ __ __ __ __ __ __ __ (An ordered pair that satisfies both equations in a system of two equations)

4. INENSCOSTT __ __ __ __ __ __ __ __ __ __ (A system of equations that has one or more solutions)

5. DNEEDPETN __ __ __ __ __ __ __ __ __ (A system of two linear equations that represents only one line)

<table>
<tr><td>

Section 3.1

</td><td>

Solving Systems of Linear Equations by Graphing

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</table>

1. Solutions to Systems of Linear Equations

A linear equation in two variables has an infinite number of solutions that form a line in a rectangular coordinate system. Two or more linear equations form a **system of linear equations**. For example:

$$x - 3y = -5$$
$$2x + 4y = 10$$

A **solution to a system of linear equations** is an ordered pair that is a solution to *each* individual linear equation.

Example 1   Determining Solutions to a System of Linear Equations

Determine whether the ordered pairs are solutions to the system.

$$x + y = -6$$
$$3x - y = -2$$

a. $(-2, -4)$ **b.** $(0, -6)$

Solution:

a. Substitute the ordered pair $(-2, -4)$ into both equations:

$$x + y = -6 \longrightarrow \quad (-2) + (-4) = -6 \ ✔ \text{ True}$$
$$3x - y = -2 \longrightarrow 3(-2) - (-4) = -2 \ ✔ \text{ True}$$

Because the ordered pair $(-2, -4)$ is a solution to both equations, it is a solution to the *system* of equations.

b. Substitute the ordered pair $(0, -6)$ into both equations:

$$x + y = -6 \longrightarrow \quad (0) + (-6) = -6 \ ✔ \text{ True}$$
$$3x - y = -2 \longrightarrow 3(0) - (-6) \stackrel{?}{=} -2 \qquad \text{False}$$

Because the ordered pair $(0, -6)$ is not a solution to the second equation, it is *not* a solution to the system of equations.

Skill Practice

1. Determine whether the ordered pairs are solutions to the system.

$$3x + 2y = -8$$
$$y = 2x - 18$$

a. $(-2, -1)$ **b.** $(4, -10)$

Skill Practice Answers

1a. No   **b.** Yes

A solution to a system of two linear equations may be interpreted graphically as a point of intersection between the two lines. Notice that the lines intersect at $(-2, -4)$ (Figure 3-1).
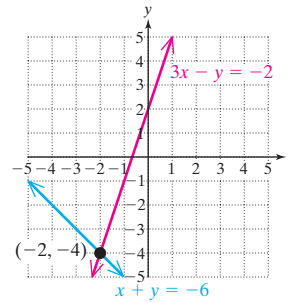

Figure 3-1

2. Dependent and Inconsistent Systems of Linear Equations

When two lines are drawn in a rectangular coordinate system, three geometric relationships are possible:

1. Two lines may intersect at *exactly one point*.

2. Two lines may intersect at *no point*. This occurs if the lines are parallel.

3. Two lines may intersect at *infinitely many points* along the line. This occurs if the equations represent the same line (the lines are coinciding).
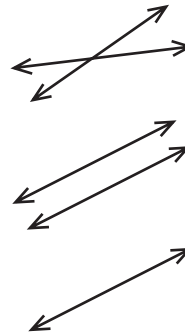
If a system of linear equations has one or more solutions, the system is said to be a **consistent system**. If a linear equation has no solution, it is said to be an **inconsistent system**.

If two equations represent the same line, then all points along the line are solutions to the system of equations. In such a case, the system is characterized as a **dependent system**. An **independent system** is one in which the two equations represent different lines.
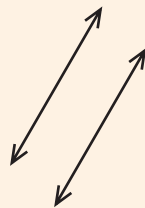
Solutions to Systems of Linear Equations in Two Variables

One unique solution   No solution   Infinitely many solutions

One point of intersection   Parallel lines   Coinciding lines

System is consistent.   System is inconsistent.   System is consistent.

System is independent.   System is independent.   System is dependent.

3. Solving Systems of Linear Equations by Graphing

Example 2 Solving a System of Linear Equations by Graphing

Solve the system by graphing both linear equations and finding the point(s) of intersection.

$$y = -\frac{3}{2}x + \frac{1}{2}$$

$$2x + 3y = -6$$

Solution:

To graph each equation, write the equation in slope-intercept form $y = mx + b$.

First equation: Second equation:

$y = -\frac{3}{2}x + \frac{1}{2}$ Slope: $-\frac{3}{2}$ $2x + 3y = -6$

$$3y = -2x - 6$$

$$\frac{3y}{3} = \frac{-2x}{3} - \frac{6}{3}$$

$$y = -\frac{2}{3}x - 2 \qquad \text{Slope: } -\frac{2}{3}$$

From their slope-intercept forms, we see that the lines have different slopes, indicating that the lines must intersect at exactly one point. Using the slope and y-intercept we can graph the lines to find the point of intersection (Figure 3-2).
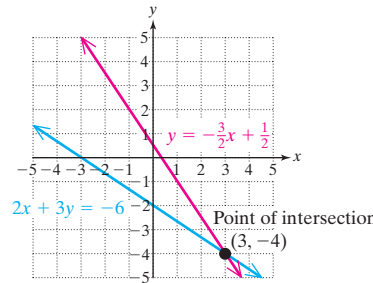

Figure 3-2

The point $(3, -4)$ appears to be the point of intersection. This can be confirmed by substituting $x = 3$ and $y = -4$ into both equations.

$$y = -\frac{3}{2}x + \frac{1}{2} \longrightarrow -4 \stackrel{?}{=} -\frac{3}{2}(3) + \frac{1}{2} \longrightarrow -4 = -\frac{9}{2} + \frac{1}{2} \checkmark \text{ True}$$

$$2x + 3y = -6 \longrightarrow 2(3) + 3(-4) \stackrel{?}{=} -6 \longrightarrow 6 - 12 = -6 \checkmark \text{ True}$$

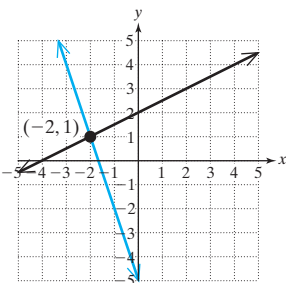The solution is $(3, -4)$.

Skill Practice

2. Solve by using the graphing method.

$$3x + y = -5$$

$$x - 2y = -4$$

Skill Practice Answers

2.

TIP: In Example 2, the lines could also have been graphed by using the *x*- and *y*-intercepts or by using a table of points. However, the advantage of writing the equations in slope-intercept form is that we can compare the slopes *and* *y*-intercepts of each line.

1. If the slopes differ, the lines are different and nonparallel and must cross in exactly one point.
2. If the slopes are the same and the *y*-intercepts are different, the lines are parallel and do not intersect.
3. If the slopes are the same and the *y*-intercepts are the same, the two equations represent the same line.

Example 3 Solving a System of Linear Equations by Graphing

Solve the system by graphing.

$$4x = 8$$

$$6y = -3x + 6$$

Solution:

The first equation $4x = 8$ can be written as $x = 2$. This is an equation of a vertical line. To graph the second equation, write the equation in slope-intercept form.

First equation: Second equation:

$4x = 8$ $6y = -3x + 6$

$x = 2$ $\dfrac{6y}{6} = \dfrac{-3x}{6} + \dfrac{6}{6}$

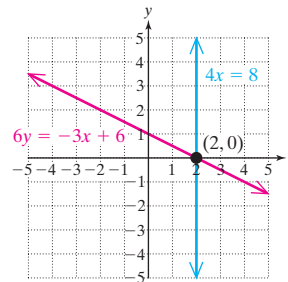 $y = -\dfrac{1}{2}x + 1$

Figure 3-3

The graphs of the lines are shown in Figure 3-3. The point of intersection is $(2, 0)$. This can be confirmed by substituting $(2, 0)$ into both equations.

$$4x = 8 \longrightarrow 4(2) = 8 \ \checkmark \text{ True}$$

$$6y = -3x + 6 \longrightarrow 6(0) = -3(2) + 6 \ \checkmark \text{ True}$$
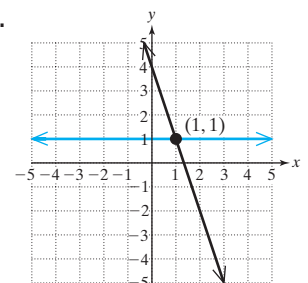
The solution is $(2, 0)$.

Skill Practice

3. Solve the system by graphing.

$$-4 = -4y$$

$$-3x - y = -4$$

Skill Practice Answers

3.

Example 4   **Solving a System of Equations by Graphing**

Solve the system by graphing.

$$-x + 3y = -6$$
$$6y = 2x + 6$$

Solution:

To graph the line, write each equation in slope-intercept form.

Figure 3-4

First equation: Second equation:

First equation:
$$-x + 3y = -6$$
$$3y = x - 6$$
$$\frac{3y}{3} = \frac{x}{3} - \frac{6}{3}$$
$$y = \frac{1}{3}x - 2$$

Second equation:
$$6y = 2x + 6$$
$$\frac{6y}{6} = \frac{2x}{6} + \frac{6}{6}$$
$$y = \frac{1}{3}x + 1$$
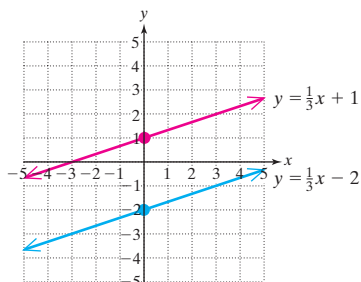
Because the lines have the same slope but different y-intercepts, they are parallel (Figure 3-4). Two parallel lines do not intersect, which implies that the system has no solution. The system is inconsistent.

Skill Practice

4. Solve the system by graphing.
$$2(y - x) = 0$$
$$-x + y = -3$$

Example 5   **Solving a System of Linear Equations by Graphing**

Solve the system by graphing.

$$x + 4y = 8$$
$$y = -\frac{1}{4}x + 2$$

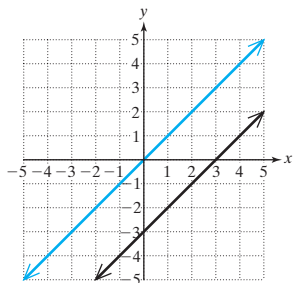**Solution:**

Write the first equation in slope-intercept form. The second equation is already in slope-intercept form.

Skill Practice Answers

4. No solution; inconsistent system

First equation: Second equation:

First equation:
$$x + 4y = 8$$
$$4y = -x + 8$$
$$\frac{4y}{4} = \frac{-x}{4} + \frac{8}{4}$$
$$y = -\frac{1}{4}x + 2$$

Second equation:
$$y = -\frac{1}{4}x + 2$$

Notice that the slope-intercept forms of the two lines are identical. Therefore, the equations represent the same line (Figure 3-5). The system is dependent, and the solution to the system of equations is the set of all points on the line.

Because not all the ordered pairs in the solution set can be listed, we can write the solution in set-builder notation. Furthermore, the equations $x + 4y = 8$ and $y = -\frac{1}{4}x + 2$ represent the same line. Therefore, the solution set may be written as $\{(x, y) \mid y = -\frac{1}{4}x + 2\}$ or $\{(x, y) \mid x + 4y = 8\}$.
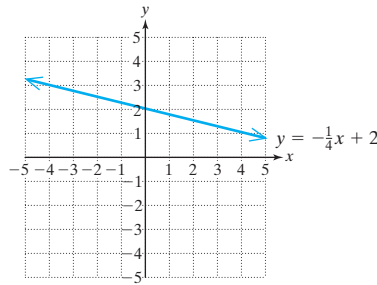

Figure 3-5

Skill Practice

5. Solve the system by graphing.

$$y = \frac{1}{2}x + 1$$

$$x - 2y = -2$$

Calculator Connections

The solution to a system of equations can be found by using either a *Trace* feature or an *Intersect* feature on a graphing calculator to find the point of intersection between two curves.

For example, consider the system

$$-2x + y = 6$$

$$5x + y = -1$$

First graph the equations together on the same viewing window. Recall that to enter the equations into the calculator, the equations must be written with the *y*-variable isolated. That is, be sure to solve for *y* first.

Isolate *y.*

$$-2x + y = 6 \longrightarrow y = 2x + 6$$

$$5x + y = -1 \longrightarrow y = -5x - 1$$
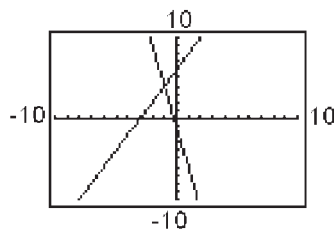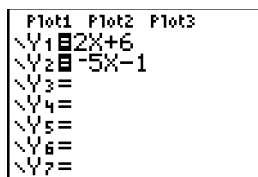
By inspection of the graph, it appears that the solution is $(-1, 4)$. The *Trace* option on the calculator may come close to $(-1, 4)$ but may not show the exact solution (Figure 3-6). However, an *Intersect* feature on a graphing calculator may provide the exact solution (Figure 3-7). See your user's manual for further details.

Skill Practice Answers

5. $\{(x, y) \mid y = \frac{1}{2}x + 1\}$; infinitely many solutions; dependent system

Using *Trace*

Figure 3-6

Using *Intersect*

Figure 3-7
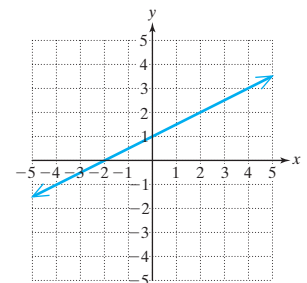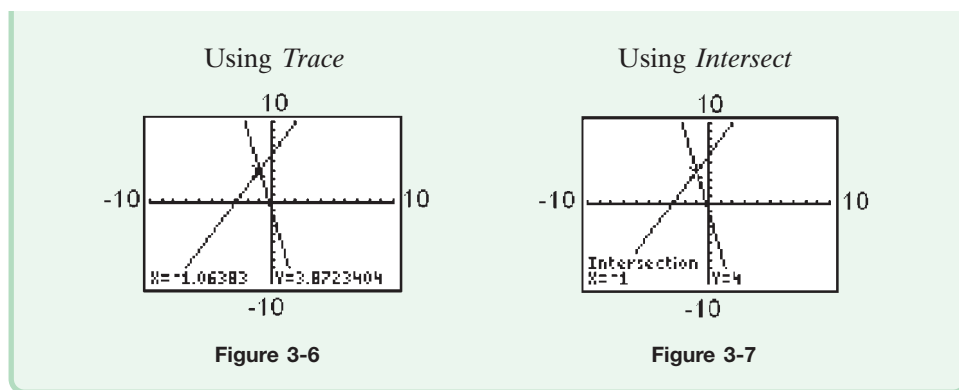
Section 3.1   Practice Exercises

Study Skills Exercises

1. Before you proceed further in Chapter 3, make your test corrections for the Chapter 2 test. See Exercise 1 of Section 2.1 for instructions.

2. Define the key terms.

a. System of linear equations   **b. Solution to a system of linear equations**

c. Consistent system   **d. Inconsistent system**

e. Dependent system   **f. Independent system**

Concept 1: Solutions to Systems of Linear Equations

For Exercises 3–8, determine which points are solutions to the given system.

3. $y = 8x - 5$
$y = 4x + 3$
$(-1, 13), (-1, 1), (2, 11)$

4. $y = -\dfrac{1}{2}x - 5$
$y = \dfrac{3}{4}x - 10$
$(4, -7), (0, -10), \left(3, -\dfrac{9}{2}\right)$

5. $2x - 7y = -30$
$y = 3x + 7$
$(0, -30), \left(\dfrac{3}{2}, 5\right), (-1, 4)$

6. $x + 2y = 4$
$y = -\dfrac{1}{2}x + 2$
$(-2, 3), (4, 0), \left(3, \dfrac{1}{2}\right)$

7. $x - y = 6$
$4x + 3y = -4$
$(4, -2), (6, 0), (2, 4)$

8. $x - 3y = 3$
$2x - 9y = 1$
$(0, 1), (4, -1), (9, 2)$

Concept 2: Dependent and Inconsistent Systems of Linear Equations

For Exercises 9–14, the graph of a system of linear equations is given.

   a. Identify whether the system is consistent or inconsistent.

   b. Identify whether the system is dependent or independent.

   c. Identify the number of solutions to the system.

9. $\quad y = x + 3$

$\quad\quad 3x + y = -1$
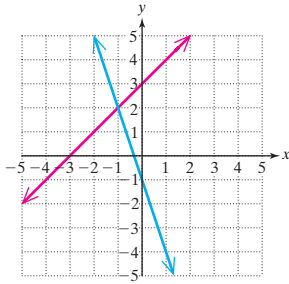
10. $5x - 3y = 6$

$\quad\quad 3y = 2x + 3$
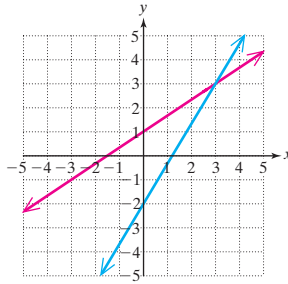
11. $2x = y + 4$

$\quad\quad -4x + 2y = 2$
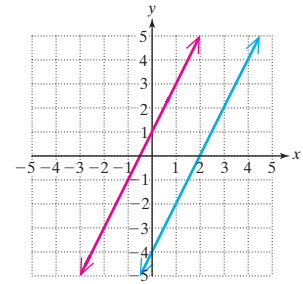
12. $\quad y = -2x - 3$

$\quad\quad -4x - 2y = 0$
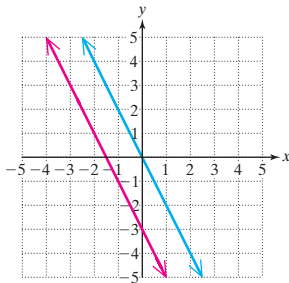
13. $\quad y = \dfrac{1}{3}x + 2$

$\quad\quad -x + 3y = 6$
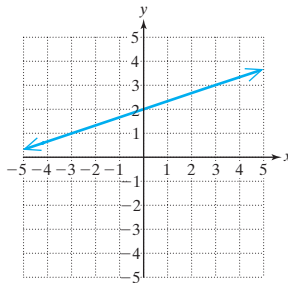
14. $\quad y = -\dfrac{2}{3}x - 1$

$\quad\quad -4x - 6y = 6$
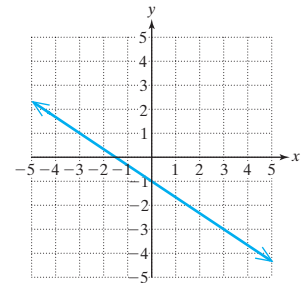
Concept 3: Solving Systems of Linear Equations by Graphing

For Exercises 15–32, solve the systems of equations by graphing.

15. $2x + \ \ y = 4$

$\quad\quad x + 2y = -1$
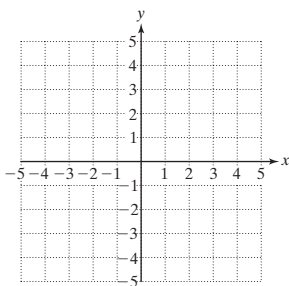
16. $4x - 3y = 12$

$\quad\quad 3x + 4y = -16$
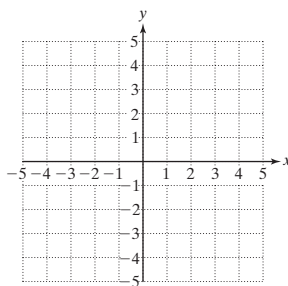
17. $y = -2x + 3$

$\quad\quad y = \ \ \ 5x - 4$
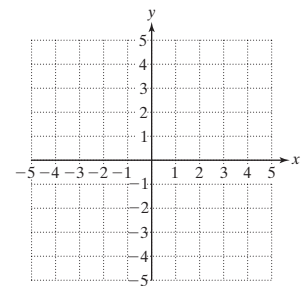
18. $y = 2x + 5$

  $y = -x + 2$
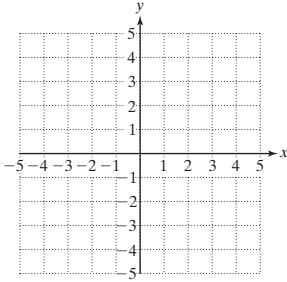
19. $y = \dfrac{1}{3}x - 5$

  $y = -\dfrac{2}{3}x - 2$
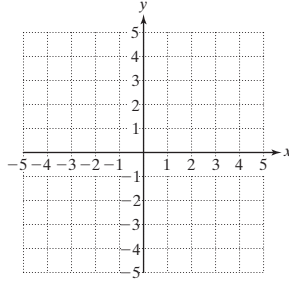
20. $y = \dfrac{1}{2}x + 2$

  $y = \dfrac{5}{2}x - 2$
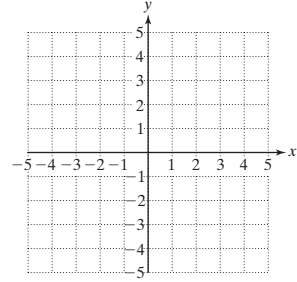
21. $x = 4$

  $y = 2x - 3$
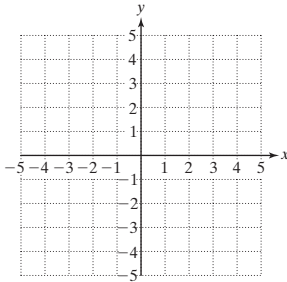
22. $3x + 2y = 6$

  $y = -3$
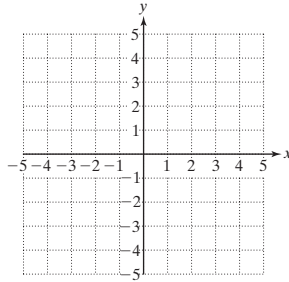
23. $y = -2x + 3$

  $-2x = y + 1$
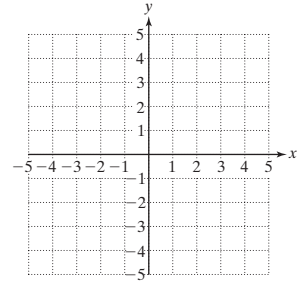
24. $y = \dfrac{1}{3}x - 2$

  $x = 3y - 9$
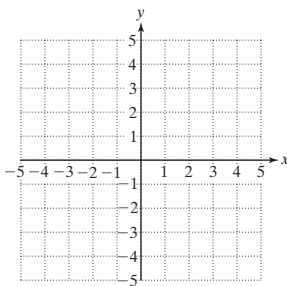
25. $y = \dfrac{2}{3}x - 1$

  $2x = 3y + 3$
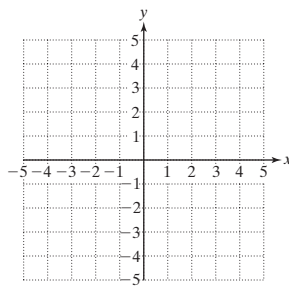
26. $4x = 16 - 8y$

  $y = -\dfrac{1}{2}x + 2$
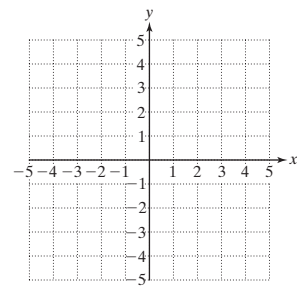
27. $2x = 4$

$\dfrac{1}{2}y = -1$
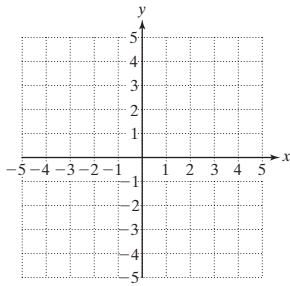
28. $y + 7 = 6$

$-5 = 2x$
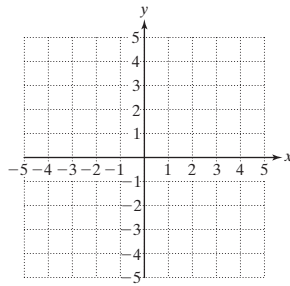
29. $-x + 3y = 6$

$6y = 2x + 12$
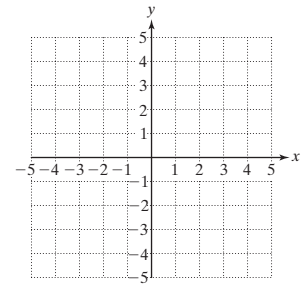
30. $3x = 2y - 4$

$-4y = -6x - 8$
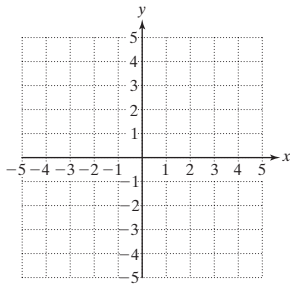
31. $2x - y = 4$

$4x + 2 = 2y$
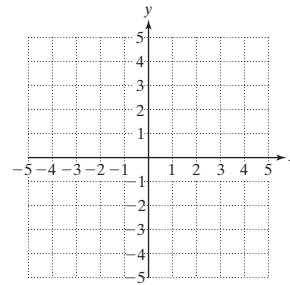
32. $x = 4y + 4$

$-2x + 8y = -16$
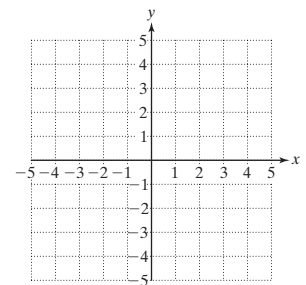
For Exercises 33–36, identify each statement as true or false.

33. A consistent system is a system that always has a unique solution.

34. A dependent system is a system that has no solution.

35. If two lines coincide, the system is dependent.

36. If two lines are parallel, the system is independent.

Graphing Calculator Exercises

For Exercises 37–42, use a graphing calculator to graph each linear equation on the same viewing window. Use a *Trace* or *Intersect* feature to find the point(s) of intersection.

37. $y = 2x - 3$
$y = -4x + 9$

38. $y = -\dfrac{1}{2}x + 2$

$y = \dfrac{1}{3}x - 3$

39. $x + y = 4$

   $-2x + y = -5$

40. $x - 2y = -2$

   $-3x + 2y = 6$

41. $-x + 3y = -6$

   $6y = 2x + 6$

42. $x + 4y = 8$

   $y = -\dfrac{1}{4}x + 2$

Section 3.2	**Solving Systems of Equations by Using the Substitution Method**

Concepts

1. **The Substitution Method**
2. **Solving Inconsistent Systems and Dependent Systems**

1. The Substitution Method

Graphing a system of equations is one method to find the solution of the system. In this section and Section 3.3, we will present two algebraic methods to solve a system of equations. The first is called the *substitution method*. This technique is particularly important because it can be used to solve more advanced problems including nonlinear systems of equations.

The first step in the substitution process is to isolate one of the variables from one of the equations. Consider the system

$$x + y = 16$$
$$x - y = 4$$

Solving the first equation for x yields $x = 16 - y$. Then, because x is equal to $16 - y$, the expression $16 - y$ can replace x in the second equation. This leaves the second equation in terms of y only.

First equation: $x + y = 16 \xrightarrow{\text{Solve for } x.} x = \underline{16 - y}$

Second equation: $(16 - y) - y = 4$ Substitute $x = 16 - y$.

$16 - 2y = 4$ Solve for y.

$-2y = -12$

$y = 6$

$x = 16 - y$ To find x, substitute $y = 6$ back into the equation $x = 16 - y$.

$x = 16 - (6)$

$x = 10$ The solution is $(10, 6)$.

> ### Solving a System of Equations by the Substitution Method
>
> **1.** Isolate one of the variables from one equation.
>
> **2.** Substitute the quantity found in step 1 into the *other* equation.
>
> **3.** Solve the resulting equation.
>
> **4.** Substitute the value found in step 3 back into the equation in step 1 to find the value of the remaining variable.
>
> **5.** Check the solution in both equations, and write the answer as an ordered pair.

Example 1 Using the Substitution Method to Solve a Linear Equation

Solve the system by using the substitution method. $-3x + 4y = 9$

$$x = -\frac{1}{3}y + 2$$

Solution:

$$-3x + 4y = 9$$

$$x = -\frac{1}{3}y + 2$$ **Step 1:** In the second equation, x is already isolated.

$$-3\left(-\frac{1}{3}y + 2\right) + 4y = 9$$ **Step 2:** Substitute the quantity $-\frac{1}{3}y + 2$ for x in the *other* equation.

$$y - 6 + 4y = 9$$ **Step 3:** Solve for y.

$$5y = 15$$

$$y = 3$$

Now use the known value of y to solve for the remaining variable x.

$$x = -\frac{1}{3}y + 2$$

$$x = -\frac{1}{3}(3) + 2$$ **Step 4:** Substitute $y = 3$ into the equation $x = -\frac{1}{3}y + 2$.

$$x = -1 + 2$$

$$x = 1$$

Step 5: Check the ordered pair $(1, 3)$ in each original equation.

$$-3x + 4y = 9 \qquad\qquad x = -\frac{1}{3}y + 2$$

$$-3(1) + 4(3) \stackrel{?}{=} 9 \qquad\qquad 1 \stackrel{?}{=} -\frac{1}{3}(3) + 2$$

$$-3 + 12 = 9 \;\checkmark\; \text{True} \qquad\qquad 1 = -1 + 2 \;\checkmark\; \text{True}$$

The solution is $(1, 3)$.

Skill Practice

1. Solve by using the substitution method.

$$x = 2y + 3$$
$$4x - 2y = 0$$

Example 2 Using the Substitution Method to Solve a Linear System

Solve the system by using the substitution method.

$$3x - 2y = -7$$
$$6x + y = 6$$

Solution:

The y variable in the second equation is the easiest variable to isolate because its coefficient is 1.

$$3x - 2y = -7$$
$$6x + y = 6 \longrightarrow y = -6x + 6$$
Step 1: Solve the second equation for y.

$$3x - 2(-6x + 6) = -7$$
Step 2: Substitute the quantity $-6x + 6$ for y in the *other* equation.
$$3x + 12x - 12 = -7$$
$$15x - 12 = -7$$
$$15x = 5$$
Step 3: Solve for x.
$$\frac{15x}{15} = \frac{5}{15}$$
$$x = \frac{1}{3}$$

$$y = -6x + 6$$
Step 4: Substitute $x = \frac{1}{3}$ into the equation $y = -6x + 6$.
$$y = -6\left(\frac{1}{3}\right) + 6$$
$$y = -2 + 6$$
$$y = 4$$

Avoiding Mistakes:

Do not substitute $y = -6x + 6$ into the same equation from which it came. This mistake will result in an identity:

$$6x + y = 6$$
$$6x + (-6x + 6) = 6$$
$$6x - 6x + 6 = 6$$
$$6 = 6$$

$$3x - 2y = -7 \qquad 6x + y = 6$$
$$3\left(\frac{1}{3}\right) - 2(4) \stackrel{?}{=} -7 \qquad 6\left(\frac{1}{3}\right) + 4 \stackrel{?}{=} 6$$
$$1 - 8 = -7 ✔ \qquad 2 + 4 = 6 ✔$$

Step 5: Check the ordered pair $\left(\frac{1}{3}, 4\right)$ in each original equation.

The solution is $\left(\frac{1}{3}, 4\right)$.

Skill Practice Answers

1. $(-1, -2)$

2. Solve by the substitution method.

$$3x + y = 8$$
$$x - 2y = 12$$

2. Solving Inconsistent Systems and Dependent Systems

Example 3 Using the Substitution Method to Solve a Linear System

Solve the system by using the substitution method.

$$x = 2y - 4$$
$$-2x + 4y = 6$$

Solution:

$$x = 2y - 4$$ **Step 1:** The x variable is already isolated.

$$-2x + 4y = 6$$

$$-2(2y - 4) + 4y = 6$$ **Step 2:** Substitute the quantity $x = 2y - 4$ into the *other* equation.

$$-4y + 8 + 4y = 6$$ **Step 3:** Solve for y.

$$8 = 6$$ The equation reduces to a contradiction, indicating that the system has no solution. The lines never intersect and must be parallel. The system is *inconsistent*.

There is no solution.
The system is inconsistent.

TIP: The answer to Example 3 can be verified by writing each equation in slope-intercept form and graphing the equations.

Equation 1   Equation 2

$$x = 2y - 4$$ $$-2x + 4y = 6$$

$$2y = x + 4$$ $$4y = 2x + 6$$

$$\frac{2y}{2} = \frac{x}{2} + \frac{4}{2}$$ $$\frac{4y}{4} = \frac{2x}{4} + \frac{6}{4}$$

$$y = \frac{1}{2}x + 2$$ $$y = \frac{1}{2}x + \frac{3}{2}$$
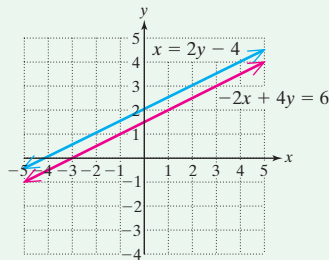
Notice that the equations have the same slope, but different y-intercepts; therefore, the lines must be parallel. There is no solution to this system of equations.

3. Solve by the substitution method.

$$8x - 16y = 3$$
$$y = \frac{1}{2}x + 1$$

Skill Practice Answers

2. $(4, -4)$
3. No solution; Inconsistent system

Example 4 Solving a Dependent System

Solve by using the substitution method.

$$4x - 2y = -6$$
$$y - 3 = 2x$$

Solution:

$$4x - 2y = -6$$

$$y - 3 = 2x \longrightarrow y = \underline{2x + 3} \qquad \textbf{Step 1:} \text{ Solve for one of the variables.}$$

$$4x - 2(2x + 3) = -6 \qquad \textbf{Step 2:} \text{ Substitute the quantity } 2x + 3 \text{ for } y \text{ in the } other \text{ equation.}$$

$$4x - 4x - 6 = -6 \qquad \textbf{Step 3:} \text{ Solve for } x.$$
$$-6 = -6$$

The system reduces to the identity $-6 = -6$. Therefore, the original two equations are equivalent, and the system is dependent. The solution consists of all points on the common line. Because the equations $4x - 2y = -6$ and $y - 3 = 2x$ represent the same line, the solution may be written as

$$\{(x, y)\,|\,4x - 2y = -6\} \qquad \text{or} \qquad \{(x, y)\,|\,y - 3 = 2x\}$$

Skill Practice

4. Solve the system by using substitution.

$$3x + 6y = 12$$
$$2y = -x + 4$$

TIP: We can confirm the results of Example 4 by writing each equation in slope-intercept form. The slope-intercept forms are identical, indicating that the lines are the same.

slope-intercept form

$$4x - 2y = -6 \longrightarrow -2y = -4x - 6 \longrightarrow y = 2x + 3$$

$$y - 3 = 2x \longrightarrow y = 2x + 3$$

Skill Practice Answers

4. Infinitely many solutions;
$\{(x, y)\,|\,3x + 6y = 12\}$;
Dependent system

Section 3.2   **Practice Exercises**

Study Skills Exercise

1. Check your progress by answering these questions.

Yes ____ No ____ Did you have sufficient time to study for the test on Chapter 2? If not, what could you have done to create more time for studying?

Yes ___ No ___ Did you work all of the assigned homework problems in Chapter 2?

Yes ___ No ___ If you encountered difficulty, did you see your instructor or tutor for help?

Yes ___ No ___ Have you taken advantage of the textbook supplements such as the *Student Solutions Manual* and MathZone?

Review Exercises

For Exercises 2–5, using the slope-intercept form of the lines, **a.** determine whether the system is consistent or inconsistent and **b.** determine whether the system is dependent or independent.

2. $y = 8x - 1$
$2x - 16y = 3$

3. $4x + 6y = 1$
$10x + 15y = \dfrac{5}{2}$

4. $2x - 4y = 0$
$x - 2y = 9$

5. $6x + 3y = 8$
$8x + 4y = -1$

For Exercises 6–7, solve the system by graphing.

6. $x - y = 4$
$3x + 4y = 12$
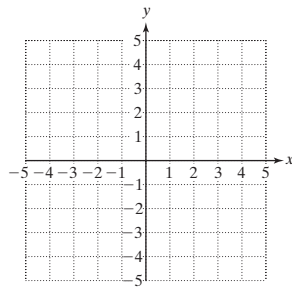
7. $y = 2x + 3$
$6x + 3y = 9$
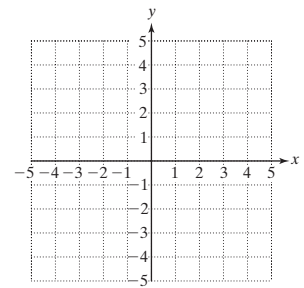
Concept 1: The Substitution Method

For Exercises 8–17, solve by using the substitution method.

8. $4x + 12y = 4$
$y = 5x + 11$

9. $y = -3x - 1$
$2x - 3y = -8$

10. $x = 10y + 34$
$-7x + y = -31$

11. $-3x + 8y = -1$
$4x - y = 11$

12. $12x - 2y = 0$
$-7x + y = -1$

13. $3x + 12y = 24$
$x - 5y = 17$

14. $x - 3y = -4$
$2x + 3y = -5$

15. $x - y = 8$
$3x + 2y = 9$

16. $5x - 2y = 10$
$y = x - 1$

17. $2x - y = -1$
$y = -2x$

18. Describe the process of solving a system of linear equations by using substitution.

Concept 2: Solving Inconsistent Systems and Dependent Systems

For Exercises 19–26, solve the systems.

19. $2x - 6y = -2$
$x = 3y - 1$

20. $-2x + 4y = 22$
$x = 2y - 11$

21. $y = \dfrac{1}{7}x + 3$
$x - 7y = -4$

22. $x = -\dfrac{3}{2}y + \dfrac{1}{2}$

   $4x + 6y = 7$

23. $5x - y = 10$

   $2y = 10x - 5$

24. $x + 4y = 8$

   $3x = 3 - 12y$

25. $3x - y = 7$

   $-14 + 6x = 2y$

26. $x = 4y + 1$

   $-12y = -3x + 3$

27. When using the substitution method, explain how to determine whether a system of linear equations is dependent.

28. When using the substitution method, explain how to determine whether a system of linear equations is inconsistent.

Mixed Exercises

For Exercises 29–50, solve the system by using the substitution method.

29. $x = 1.3y + 1.5$

   $y = 1.2x - 4.6$

30. $y = 0.8x - 1.8$

   $1.1x = -y + 9.6$

31. $y = \dfrac{2}{3}x - \dfrac{1}{3}$

   $x = \dfrac{1}{4}y + \dfrac{17}{4}$

32. $x = \dfrac{1}{6}y - \dfrac{5}{3}$

   $y = \dfrac{1}{5}x + \dfrac{21}{5}$

33. $-2x + y = 4$

   $-\dfrac{1}{4}x + \dfrac{1}{8}y = \dfrac{1}{4}$

34. $8x - y = 8$

   $\dfrac{1}{3}x - \dfrac{1}{24}y = \dfrac{1}{2}$

35. $3x + 2y = 6$

   $y = x + 3$

36. $-x + 4y = -4$

   $y = x - 1$

37. $-300x - 125y = 1350$

   $y + 2 = 8$

38. $200y = 150x$

   $y - 4 = 1$

39. $2x - y = 6$

   $\dfrac{1}{6}x - \dfrac{1}{12}y = \dfrac{1}{2}$

40. $x - 4y = 8$

   $\dfrac{1}{16}x - \dfrac{1}{4}y = \dfrac{1}{2}$

41. $y = 200x - 320$

   $y = -150x + 1080$

42. $y = -54x + 300$

   $y = 20x - 70$

43. $y = -2.7x - 5.1$

   $y = 3.1x - 63.1$

44. $y = 6.8x + 2.3$

   $y = -4.1x + 56.8$

45. $4x + 4y = 5$

   $x - 4y = -\dfrac{5}{2}$

46. $-2x + y = -6$

   $6x - 13y = -12$

47. $2(x + 2y) = 12$

   $-6x = 5y - 8$

48. $5x - 2y = -25$

   $10x = 3(y - 10)$

49. $5(3x - y) = 10$

   $4y = 7x - 3$

50. $2x = -3(y + 3)$

   $3x - 4y = -22$

<div style="background:#2ba8e0">**Solving Systems of Equations by Using the Addition Method**</div>

<div style="background:#2ba8e0">Section 3.3</div>

1. The Addition Method

The next method we present to solve systems of linear equations is the *addition method* (sometimes called the elimination method). With the addition method, begin by writing both equations in standard form $Ax + By = C$. Then we create an equivalent system by multiplying one or both equations by appropriate constants to create opposite coefficients on either the x- or the y-variable. Next the equations can be added to eliminate the variable having opposite coefficients. This process is demonstrated in Example 1.

<div style="background:#a5228f">**Concepts**</div>

1. The Addition Method
2. Solving Inconsistent Systems and Dependent Systems

Example 1 Solving a System by the Addition Method

Solve the system by using the addition method.

$$3x - 4y = 2$$
$$4x + y = 9$$

Solution:

$$3x - 4y = 2 \qquad\qquad 3x - 4y = 2$$
$$4x + y = 9 \xrightarrow{\text{Multiply by 4.}} 16x + 4y = 36$$

Multiply the second equation by 4. This makes the coefficients of the y-variables *opposite*.

$$3x - 4y = 2$$
$$\underline{16x + 4y = 36}$$
$$19x \qquad = 38$$
$$x = 2$$

Now if the equations are added, the y-variable will be eliminated.

Solve for x.

$$3x - 4y = 2$$
$$3(2) - 4y = 2$$
$$6 - 4y = 2$$
$$-4y = -4$$
$$y = 1$$

Substitute $x = 2$ back into one of the original equations and solve for y.

Check the ordered pair $(2, 1)$ in each original equation:

$$3x - 4y = 2 \longrightarrow 3(2) - 4(1) \stackrel{?}{=} 2 \longrightarrow 6 - 4 = 2 ✔$$
$$4x + y = 9 \longrightarrow 4(2) + (1) \stackrel{?}{=} 9 \longrightarrow 8 + 1 = 9 ✔$$

The solution is $(2, 1)$.

TIP: Substituting $x = 2$ into the other equation, $4x + y = 9$, produces the same value for y.

$$4x + y = 9$$
$$4(2) + y = 9$$
$$8 + y = 9$$
$$y = 1$$

Skill Practice

1. Solve by the addition method.

$$2x - 3y = 13$$
$$x + 2y = 3$$

Skill Practice Answers

1. $(5, -1)$

The steps to solve a system of linear equations in two variables by the addition method is outlined in the following box.

Solving a System of Equations by the Addition Method

1. Write both equations in standard form: $Ax + By = C$

2. Clear fractions or decimals (optional).

3. Multiply one or both equations by nonzero constants to create opposite coefficients for one of the variables.

4. Add the equations from step 3 to eliminate one variable.

5. Solve for the remaining variable.

6. Substitute the known value found in step 5 into one of the original equations to solve for the other variable.

7. Check the ordered pair in *both* equations.

Example 2 Solving a System by the Addition Method

Solve the system by using the addition method.

$$4x + 5y = 2$$
$$3x = 1 - 4y$$

Solution:

$$4x + 5y = 2 \longrightarrow 4x + 5y = 2$$
$$3x = 1 - 4y \longrightarrow 3x + 4y = 1$$

Step 1: Write both equations in standard form. There are no fractions or decimals.

We may choose to eliminate either variable. To eliminate x, change the coefficients to 12 and -12.

$$4x + 5y = 2 \xrightarrow{\text{Multiply by 3.}} 12x + 15y = 6$$
$$3x + 4y = 1 \xrightarrow{\text{Multiply by } -4.} -12x - 16y = -4$$

Step 3: Multiply the first equation by 3.

Multiply the second equation by -4.

$$12x + 15y = 6$$
$$\underline{-12x - 16y = -4}$$
$$-y = 2$$
$$y = -2$$

Step 4: Add the equations.

Step 5: Solve for y.

$$4x + 5y = 2$$
$$4x + 5(-2) = 2$$
$$4x - 10 = 2$$
$$4x = 12$$
$$x = 3$$

The solution is $(3, -2)$.

Step 6: Substitute $y = -2$ back into one of the original equations and solve for x.

Step 7: The ordered pair $(3, -2)$ checks in both original equations.

2. Solve by the addition method.

$$2y = 5x - 4$$
$$3x - 4y = 1$$

TIP: To eliminate the x variable in Example 2, both equations were multiplied by appropriate constants to create $12x$ and $-12x$. We chose 12 because it is the *least common multiple* of 4 and 3.

　　We could have solved the system by eliminating the y-variable. To eliminate y, we would multiply the top equation by 4 and the bottom equation by -5. This would make the coefficients of the y-variable 20 and -20, respectively.

$$4x + 5y = 2 \xrightarrow{\text{Multiply by 4.}} 16x + 20y = 8$$
$$3x + 4y = 1 \xrightarrow{\text{Multiply by } -5.} -15x - 20y = -5$$

Example 3 | **Solving a System of Equations by the Addition Method**

Solve the system by using the addition method.

$$x - 2y = 6 + y$$
$$0.05y = 0.02x - 0.10$$

Solution:

$$x - 2y = 6 + y \longrightarrow x - 3y = 6$$
$$0.05y = 0.02x - 0.10 \longrightarrow -0.02x + 0.05y = -0.10$$

Step 1: Write both equations in standard form.

$$x \quad - 3y = 6$$
$$-0.02x + 0.05y = -0.10 \xrightarrow{\text{Multiply by 100.}} -2x + 5y = -10$$

Step 2: Clear decimals.

$$x - 3y = 6 \xrightarrow{\text{Multiply by 2.}} 2x - 6y = 12$$
$$-2x + 5y = -10 \longrightarrow \underline{-2x + 5y = -10}$$

Step 3: Create opposite coefficients.

$$-y = 2$$

Step 4: Add the equations.

$$y = -2$$

Step 5: Solve for y.

$$x - 2y = 6 + y$$
$$x - 2(-2) = 6 + (-2)$$
$$x + 4 = 4$$
$$x = 0$$

Step 6: To solve for x, substitute $y = -2$ into one of the original equations.

Step 7: Check the ordered pair $(0, -2)$ in each original equation.

$$x - 2y = 6 + y \longrightarrow (0) - 2(-2) \overset{?}{=} 6 + (-2) \longrightarrow 4 = 4 \checkmark$$

$$0.05y = 0.02x - 0.10 \longrightarrow 0.05(-2) \overset{?}{=} 0.02(0) - 0.10 \longrightarrow -0.10 = -0.10 \checkmark$$

The solution is $(0, -2)$.

Skill Practice

3. Solve by the addition method.

$$0.2x + 0.3y = 1.5$$
$$5x + 3y = 20 - y$$

2. Solving Inconsistent Systems and Dependent Systems

Example 4   **Solving a System of Equations by the Addition Method**

Solve the system by using the addition method.

$$\frac{1}{5}x - \frac{1}{2}y = 1$$
$$-4x + 10y = -20$$

Solution:

$$\frac{1}{5}x - \frac{1}{2}y = 1$$ **Step 1:** Equations are in standard form.
$$-4x + 10y = -20$$

$$10\left(\frac{1}{5}x - \frac{1}{2}y\right) = 10 \cdot 1 \longrightarrow 2x - 5y = 10$$ **Step 2:** Clear fractions.
$$-4x + 10y = -20$$

$$2x - 5y = 10 \xrightarrow{\text{Multiply by 2.}} 4x - 10y = 20$$ **Step 3:** Multiply the first equation by 2.
$$\underline{-4x + 10y = -20 \longrightarrow -4x + 10y = -20}$$

$$0 = 0$$ **Step 4:** Add the equations.

Notice that both variables were eliminated. The system of equations is reduced to the identity $0 = 0$. Therefore, the two original equations are equivalent and the system is dependent. The solution set consists of an infinite number of ordered pairs (x, y) that fall on the common line of intersection $-4x + 10y = -20$, or equivalently $\frac{1}{5}x - \frac{1}{2}y = 1$. The solution set can be written in set notation as

$$\{(x, y) \mid -4x + 10y = -20\} \quad \text{or} \quad \left\{(x, y) \;\middle|\; \frac{1}{5}x - \frac{1}{2}y = 1\right\}$$

Skill Practice Answers

3. $(0, 5)$

Skill Practice

4. Solve by the addition method.

$$3x + y = 4$$

$$x = -\frac{1}{3}y + \frac{4}{3}$$

Example 5 Solving an Inconsistent System

Solve the system by using the addition method.

$$2y = -3x + 4$$

$$120x + 80y = 40$$

Solution:

$$2y = -3x + 4 \xrightarrow{\text{Standard form}} 3x + 2y = 4$$
$$120x + 80y = 40 \qquad\qquad 120x + 80y = 40$$

Step 1: Write the equations in standard form.

Step 2: There are no decimals or fractions.

$$3x + 2y = 4 \xrightarrow{\text{Multiply by } -40.} -120x - 80y = -160$$
$$120x + 80y = 40 \qquad\qquad \underline{120x + 80y = 40}$$
$$ 0 = -120$$

Step 3: Multiply the top equation by -40.

Step 4: Add the equations.

The equations reduce to a contradiction, indicating that the system has no solution. The system is inconsistent. The two equations represent parallel lines, as shown in Figure 3-8.

There is no solution.
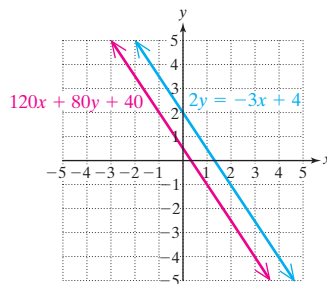
Figure 3-8

Skill Practice

5. Solve by the addition method.

$$18 + 10x = 6y$$

$$5x - 3y = 9$$

| Section 3.3 | **Practice Exercises** |

Study Skills Exercise

1. Instructors differ in what they emphasize on tests. For example, test material may come from the textbook, notes, handouts, or homework. What does your instructor emphasize?

Review Exercises

For Exercises 2–4, use the slope-intercept form of the lines to determine the number of solutions for the system of equations.

2. $y = \dfrac{1}{2}x - 4$

$y = \dfrac{1}{2}x + 1$

3. $y = 2.32x - 8.1$

$y = 1.46x - 8.1$

4. $4x = y + 7$

$-2y = -8x + 14$

Concept 1: The Addition Method

For Exercises 5–14, solve the system by the addition method.

5. $3x - y = -1$

$-3x + 4y = -14$

6. $5x - 2y = 15$

$3x + 2y = -7$

7. $2x + 3y = 3$

$-10x + 2y = -32$

8. $2x - 5y = 7$

$3x - 10y = 13$

9. $3x + 7y = -20$

$-5x + 3y = -84$

10. $6x - 9y = -15$

$5x - 2y = -40$

11. $3x = 10y + 13$

$7y = 4x - 11$

12. $-5x = 6y - 4$

$5y = 1 - 3x$

13. $1.2x - 0.6y = 3$

$0.8x - 1.4y = 3$

14. $1.8x + 0.8y = 1.4$

$1.2x + 0.6y = 1.2$

Concept 2: Solving Inconsistent Systems and Dependent Systems

For Exercises 15–22, solve the systems.

15. $3x - 2y = 1$

$-6x + 4y = -2$

16. $3x - y = 4$

$6x - 2y = 8$

17. $6y = 14 - 4x$

$0.2x = -0.3y - 0.7$

18. $2x = 4 - y$

$-0.1y = 0.2x - 0.2$

19. $12x - 4y = 2$

$0.6x = 0.1 + 0.2y$

20. $10x - 15y = 5$

$0.3y = 0.2x - 0.1$

21. $\dfrac{1}{2}x + y = \dfrac{7}{6}$

$x + 2y = 4.5$

22. $0.2x - 0.1y = -1.2$

$x - \dfrac{1}{2}y = 3$

Mixed Exercises

23. Describe a situation in which you would prefer to use the substitution method over the addition method.

24. If you used the addition method to solve the given system, would it be easier to eliminate the x- or y-variable? Explain.

$$3x - 5y = 4$$
$$7x + 10y = 31$$

For Exercises 25–50, solve by using either the addition method or the substitution method.

25. $2x - 4y = 8$
   $y = 2x + 1$

26. $8x + 6y = -8$
   $x = 6y - 10$

27. $2x + 5y = 9$
   $4x - 7y = -16$

28. $x + 5y = 7$
   $2x + 7y = 8$

29. $2x - y = 8$
   $x - y = 4$

30. $y = \dfrac{1}{2}x - 3$
   $4x + y = -3$

31. $0.4x - 0.6y = 0.5$
   $0.2x - 0.3y = 0.7$

32. $0.3x + 0.6y = 0.7$
   $0.2x + 0.4y = 0.5$

33. $\dfrac{1}{4}x - \dfrac{1}{6}y = -2$
   $-\dfrac{1}{6}x + \dfrac{1}{5}y = 4$

34. $\dfrac{1}{3}x + \dfrac{1}{5}y = 7$
   $\dfrac{1}{6}x - \dfrac{2}{5}y = -4$

35. $\dfrac{1}{3}x - \dfrac{1}{2}y = 0$
   $x = \dfrac{3}{2}y$

36. $\dfrac{2}{5}x - \dfrac{2}{3}y = 0$
   $y = \dfrac{3}{5}x$

37. $2(x + 2y) = 20 - y$
   $-7(x - y) = 16 + 3y$

38. $-3(x + y) = 10 - 4y$
   $4(x + 2y) = 50 + 3y$

39. $-4y = 10$
   $4x + 3 = 1$

40. $-9x = 15$
   $3y + 2 = 1$

41. $5x - 3y = 18$
   $-3x + 5y = 18$

42. $6x - 3y = -3$
   $4x + 5y = -9$

43. $3x - 2 = \dfrac{1}{3}(11 + 5y)$
   $x + \dfrac{2}{3}(2y - 3) = -2$

44. $2(2y + 3) - 2x = 1 - x$
   $x + y = \dfrac{1}{5}(7 + y)$

45. $\dfrac{1}{4}x + \dfrac{1}{2}y = \dfrac{11}{4}$
   $\dfrac{2}{3}x + \dfrac{1}{3}y = \dfrac{7}{3}$

46. $\dfrac{1}{10}x - \dfrac{1}{2}y = -\dfrac{8}{5}$
   $x + \dfrac{1}{4}y = -\dfrac{11}{2}$

47. $4x + y = -2$
   $5x - y = -7$

48. $4y = 8x + 20$
   $8x = 24$

49. $4x = 3y$
   $y = \dfrac{4}{3}x + 2$

50. $4x - 2y = 6$
   $x = \dfrac{1}{2}y + \dfrac{3}{2}$

Expanding Your Skills

For Exercises 51–54, use the addition method first to solve for x. Then repeat the addition method again, using the original system of equations, this time solving for y.

51. $6x - 2y = 5$
$5x + 3y = -2$

52. $5x - 4y = 7$
$-2x + 6y = 5$

53. $4x + 7y = 8$
$-5x - 2y = 3$

54. $2x - 4y = 9$
$-5x + 3y = 1$

Section 3.4 Applications of Systems of Linear Equations in Two Variables

Concepts

1. Applications Involving Cost
2. Applications Involving Mixtures
3. Applications Involving Principal and Interest
4. Applications Involving Distance, Rate, and Time
5. Applications Involving Geometry

1. Applications Involving Cost

In Chapter 1 we solved numerous application problems using equations that contained one variable. However, when an application has more than one unknown, sometimes it is more convenient to use multiple variables. In this section, we will solve applications containing two unknowns. When two variables are present, the goal is to set up a system of two independent equations.

Example 1 Solving a Cost Application

At an amusement park, five hot dogs and one drink cost \$16. Two hot dogs and three drinks cost \$9. Find the cost per hot dog and the cost per drink.

Solution:

Let h represent the cost per hot dog.   Label the variables.

Let d represent the cost per drink.

$\begin{pmatrix} \text{Cost of 5} \\ \text{hot dogs} \end{pmatrix} + \begin{pmatrix} \text{cost of 1} \\ \text{drink} \end{pmatrix} = \$16 \longrightarrow 5h + d = 16$ Write two equations.

$\begin{pmatrix} \text{Cost of 2} \\ \text{hot dogs} \end{pmatrix} + \begin{pmatrix} \text{cost of 3} \\ \text{drinks} \end{pmatrix} = \$9 \longrightarrow 2h + 3d = 9$

This system can be solved by either the substitution method or the addition method. We will solve by using the substitution method. The d-variable in the first equation is the easiest variable to isolate.

$5h + d = 16 \longrightarrow d = -5h + 16$ Solve for d in the first equation.

$2h + 3d = 9$

$2h + 3(-5h + 16) = 9$ Substitute the quantity $-5h + 16$ for d in the *second* equation.

$2h - 15h + 48 = 9$ Clear parentheses.

$-13h + 48 = 9$ Solve for h.

$-13h = -39$

$h = 3$

$d = -5(3) + 16 \longrightarrow d = 1$ Substitute $h = 3$ in the equation $d = -5h + 16$.

Because $h = 3$, the cost per hot dog is $3.00.

Because $d = 1$, the cost per drink is $1.00.

A word problem can be checked by verifying that the solution meets the conditions specified in the problem.

5 hot dogs + 1 drink $= 5(\$3.00) + 1(\$1.00) = \$16.00$ as expected

2 hot dogs + 3 drinks $= 2(\$3.00) + 3(\$1.00) = \$9.00$ as expected

Skill Practice

1. At the movie theater, Tom spent $7.75 on 3 soft drinks and 2 boxes of popcorn. Carly bought 5 soft drinks and 1 box of popcorn for total of $8.25. Use a system of equations to find the cost of a soft drink and the cost of a box of popcorn.

2. Applications Involving Mixtures

Example 2 Solving an Application Involving Chemistry

One brand of cleaner used to etch concrete is 25% acid. A stronger industrial-strength cleaner is 50% acid. How many gallons of each cleaner should be mixed to produce 20 gal of a 40% acid solution?

Solution:

Let x represent the amount of 25% acid cleaner.

Let y represent the amount of 50% acid cleaner.

	25% Acid	50% Acid	40% Acid
Number of gallons of solution	x	y	20
Number of gallons of pure acid	$0.25x$	$0.50y$	0.40(20), or 8

From the first row of the table, we have

$$\left(\begin{array}{c}\text{Amount of}\\ 25\% \text{ solution}\end{array}\right) + \left(\begin{array}{c}\text{amount of}\\ 50\% \text{ solution}\end{array}\right) = \left(\begin{array}{c}\text{total amount}\\ \text{of solution}\end{array}\right) \longrightarrow x + y = 20$$

From the second row of the table we have

$$\left(\begin{array}{c}\text{Amount of}\\ \text{pure acid in}\\ 25\% \text{ solution}\end{array}\right) + \left(\begin{array}{c}\text{amount of}\\ \text{pure acid in}\\ 50\% \text{ solution}\end{array}\right) = \left(\begin{array}{c}\text{amount of}\\ \text{pure acid in}\\ \text{resulting solution}\end{array}\right) \longrightarrow 0.25x + 0.50y = 8$$

$$\begin{array}{ll} x + \quad y = 20 & \longrightarrow \qquad x + \quad y = 20 \\ 0.25x + 0.50y = 8 & \longrightarrow \quad 25x + 50y = 800 \end{array} \qquad \text{Multiply by 100 to clear decimals.}$$

$$\begin{array}{lll} x + \quad y = 20 & \xrightarrow{\text{Multiply by } -25.} & -25x - 25y = -500 & \text{Create opposite} \\ 25x + \quad 50y = 800 & \longrightarrow & \underline{25x + 50y = 800} & \text{coefficients of } x. \\ & & 25y = 300 & \text{Add the equations} \\ & & y = 12 & \text{to eliminate } x. \end{array}$$

$$x + y = 20$$
$$x + 12 = 20$$
$$x = 8$$

Substitute $y = 12$ back into one of the original equations.

Therefore, 8 gal of 25% acid solution must be added to 12 gal of 50% acid solution to create 20 gal of a 40% acid solution.

Skill Practice

2. A pharmacist needs 8 ounces (oz) of a solution that is 50% saline. How many ounces of 60% saline solution and 20% saline solution must be mixed to obtain the mixture needed?

3. Applications Involving Principal and Interest

Example 3 Solving a Mixture Application Involving Finance

Serena invested money in two accounts: a savings account that yields 4.5% simple interest and a certificate of deposit that yields 7% simple interest. The amount invested at 7% was twice the amount invested at 4.5%. How much did Serena invest in each account if the total interest at the end of 1 year was $1017.50?

Solution:

Let x represent the amount invested in the savings account (the 4.5% account).

Let y represent the amount invested in the certificate of deposit (the 7% account).

	4.5% Account	7% Account	Total
Principal	x	y	
Interest	0.045x	0.07y	1017.50

Because the amount invested at 7% was twice the amount invested at 4.5%, we have

$$\begin{pmatrix} \text{Amount} \\ \text{invested} \\ \text{at 7\%} \end{pmatrix} = 2 \begin{pmatrix} \text{amount} \\ \text{invested} \\ \text{at 4.5\%} \end{pmatrix} \rightarrow y = 2x$$

From the second row of the table, we have

$$\begin{pmatrix} \text{Interest} \\ \text{earned from} \\ \text{4.5\% account} \end{pmatrix} + \begin{pmatrix} \text{interest} \\ \text{earned from} \\ \text{7\% account} \end{pmatrix} = \begin{pmatrix} \text{total} \\ \text{interest} \end{pmatrix} \rightarrow 0.045x + 0.07y = 1017.50$$

$$y = 2x$$
$$45x + 70y = 1,017,500$$

Multiply by 1000 to clear decimals.

Because the y-variable in the first equation is isolated, we will use the substitution method.

$$45x + 70(2x) = 1,017,500$$

Substitute the quantity $2x$ into the second equation.

$$45x + 140x = 1{,}017{,}500 \qquad \text{Solve for } x.$$
$$185x = 1{,}017{,}500$$
$$x = \frac{1{,}017{,}500}{185}$$
$$x = 5500$$

$$y = 2x$$
$$y = 2(5500) \qquad \text{Substitute } x = 5500 \text{ into the equation}$$
$$\qquad\qquad\qquad y = 2x \text{ to solve for } y.$$
$$y = 11{,}000$$

Because $x = 5500$, the amount invested in the savings account is $5500.

Because $y = 11{,}000$, the amount invested in the certificate of deposit is $11,000.

Check: $11,000 is twice $5500. Furthermore,

$$\begin{pmatrix} \text{Interest} \\ \text{earned from} \\ 4.5\% \text{ account} \end{pmatrix} + \begin{pmatrix} \text{interest} \\ \text{earned from} \\ 7\% \text{ account} \end{pmatrix} = \$5500(0.045) + \$11{,}000(0.07) = \$1017.50 \checkmark$$

Skill Practice

3. Seth invested money in two accounts, one paying 5% interest and the other paying 6% interest. The amount invested at 5% was $1000 more than the amount invested at 6%. He earned a total of $820 interest in 1 year. Use a system of equations to find the amount invested in each account.

4. Applications Involving Distance, Rate, and Time

Example 4 Solving a Distance, Rate, and Time Application

A plane flies 660 mi from Atlanta to Miami in 1.2 hr when traveling with a tailwind. The return flight against the same wind takes 1.5 hr. Find the speed of the plane in still air and the speed of the wind.

Solution:

Let p represent the speed of the plane in still air.

Let w represent the speed of the wind.

The speed of the plane *with* the wind: (Plane's still airspeed) + (wind speed): $p + w$

The speed of the plane *against* the wind: (Plane's still airspeed) − (wind speed): $p - w$

Set up a chart to organize the given information:

	Distance	Rate	Time
With a tailwind	660	$p + w$	1.2
Against a head wind	660	$p - w$	1.5

Skill Practice Answers

3. $8000 invested at 5% and $7000 invested at 6%

Two equations can be found by using the relationship $d = rt$ (distance = rate · time).

$$\begin{pmatrix} \text{Distance} \\ \text{with} \\ \text{wind} \end{pmatrix} = \begin{pmatrix} \text{speed} \\ \text{with} \\ \text{wind} \end{pmatrix}\begin{pmatrix} \text{time} \\ \text{with} \\ \text{wind} \end{pmatrix} \longrightarrow 660 = (p + w)(1.2)$$

$$\begin{pmatrix} \text{Distance} \\ \text{against} \\ \text{wind} \end{pmatrix} = \begin{pmatrix} \text{speed} \\ \text{against} \\ \text{wind} \end{pmatrix}\begin{pmatrix} \text{time} \\ \text{against} \\ \text{wind} \end{pmatrix} \longrightarrow 660 = (p - w)(1.5)$$

$660 = (p + w)(1.2)$ Notice that the first equation may be *divided* by 1.2
$660 = (p - w)(1.5)$ and still leave integer coefficients. Similarly, the second equation may be simplified by dividing by 1.5.

$660 = (p + w)(1.2) \xrightarrow{\text{Divide by 1.2.}} \dfrac{660}{1.2} = \dfrac{(p + w)\cancel{1.2}}{\cancel{1.2}} \longrightarrow 550 = p + w$

$660 = (p - w)(1.5) \xrightarrow{\text{Divide by 1.5.}} \dfrac{660}{1.5} = \dfrac{(p - w)\cancel{1.5}}{\cancel{1.5}} \longrightarrow 440 = p - w$

$550 = p + w$

$\underline{440 = p - w}$

$990 = 2p$ Add the equations.

$p = 495$

$550 = (495) + w$ Substitute $p = 495$ into the equation $550 = p + w$.

$55 = w$ Solve for w.

The speed of the plane in still air is 495 mph, and the speed of the wind is 55 mph.

Skill Practice

4. A plane flies 1200 mi from Orlando to New York in 2 hr with a tailwind. The return flight against the same wind takes 2.5 hr. Find the speed of the plane in still air and the speed of the wind.

5. Applications Involving Geometry

Example 5   Solving a Geometry Application

The sum of the two acute angles in a right triangle is 90°. The measure of one angle is 6° less than 2 times the measure of the other angle. Find the measure of each angle.

Solution:

Let x represent the measure of one acute angle.

Let y represent the measure of the other acute angle.
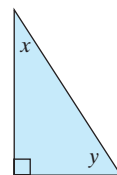
Skill Practice Answers

4. Speed of plane: 540 mph; speed of wind: 60 mph

The sum of the two acute angles is 90°: $x + y = 90$

One angle is 6° less than 2 times the other angle: $x = 2y - 6$

$x + y = 90$ Because one variable is already isolated, we will use the substitution method.
$x = 2y - 6$

$(2y - 6) + y = 90$ Substitute $x = 2y - 6$ into the first equation.

$3y - 6 = 90$

$3y = 96$

$y = 32$

$x = 2y - 6$ To find x, substitute $y = 32$ into the equation $x = 2y - 6$.

$x = 2(32) - 6$

$x = 64 - 6$

$x = 58$

The two acute angles in the triangle measure 32° and 58°.

Skill Practice

5. Two angles are supplementary. The measure of one angle is 16° less than 3 times the measure of the other. Use a system of equations to find the measures of the angles.

Skill Practice Answers

5. 49° and 131°

Section 3.4	Practice Exercises

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Study Skills Exercise

1. Make up a practice test for yourself. Use examples or exercises from the text. Be sure to cover each concept that was presented.

Review Exercises

2. State three methods that can be used to solve a system of linear equations in two variables.

For Exercises 3–6, state which method you would prefer to use to solve the system. Then solve the system.

3. $y = 9 - 2x$
$3x - y = 16$

4. $7x - y = -25$
$2x + 5y = 14$

5. $5x + 2y = 6$
$-2x - y = 3$

6. $x = 5y - 2$
$-3x + 7y = 14$

Concept 1: Applications Involving Cost

7. The local community college theater put on a production of *Chicago*. There were 186 tickets sold, some for $16 (nonstudent price) and others for $12 (student price). If the receipts for one performance totaled $2640, how many of each type of ticket were sold?

8. John and Ariana bought school supplies. John spent $10.65 on 4 notebooks and 5 pens. Ariana spent $7.50 on 3 notebooks and 3 pens. What is the cost of 1 notebook and what is the cost of 1 pen?

9. Joe bought lunch for his fellow office workers on Monday. He spent $7.35 on 3 hamburgers and 2 fish sandwiches. Corey bought lunch on Tuesday and spent $7.15 for 4 hamburgers and 1 fish sandwich. What is the price of 1 hamburger, and what is the price of 1 fish sandwich?

10. A group of four golfers pays $150 to play a round of golf. Of these four, one is a member of the club and three are nonmembers. Another group of golfers consists of two members and one nonmember and pays a total of $75. What is the cost for a member to play a round of golf, and what is the cost for a nonmember?

11. Meesha has a pocket full of change consisting of dimes and quarters. The total value is $3.15. There are 7 more quarters than dimes. How many of each coin are there?

12. Crystal has several dimes and quarters in her purse, totaling $2.70. There is 1 less dime than there are quarters. How many of each coin are there?

13. A coin collection consists of 50¢ pieces and $1 coins. If there are 21 coins worth $15.50, how many 50¢ pieces and $1 coins are there?

14. Suzy has a piggy bank consisting of nickels and dimes. If there are 30 coins worth $1.90, how many nickels and dimes are in the bank?

Concept 2: Applications Involving Mixtures

15. A jar of one face cream contains 18% moisturizer, and another type contains 24% moisturizer. How many ounces of each should be combined to get 12 oz of a cream that is 22% moisturizer?

16. A chemistry student wants to mix an 18% acid solution with a 45% acid solution to get 16 L of a 36% acid solution. How many liters of the 18% solution and how many liters of the 45% solution should be mixed?

17. How much pure bleach must be combined with a solution that is 4% bleach to make 12 oz of a 12% bleach solution?

18. A fruit punch that contains 25% fruit juice is combined with a fruit drink that contains 10% fruit juice. How many ounces of each should be used to make 48 oz of a mixture that is 15% fruit juice?

Concept 3: Applications Involving Principal and Interest

19. Alina invested $27,000 in two accounts: one that pays 2% simple interest and one that pays 3% simple interest. At the end of the first year, her total return was $685. How much was invested in each account?

20. Didi invested a total of $12,000 into two accounts paying 7.5% and 6% simple interest. If her total return at the end of the first year was $840, how much did she invest in each account?

21. A credit union offers 5.5% simple interest on a certificate of deposit (CD) and 3.5% simple interest on a savings account. If Mr. Sorkin invested $200 more in the CD than in the savings account and the total interest after the first year was $245, how much was invested in each account?

22. Jody invested $5000 less in an account paying 4% simple interest than she did in an account paying 3% simple interest. At the end of the first year, the total interest from both accounts was $675. Find the amount invested in each account.

## Concept 4: Applications Involving Distance, Rate, and Time

23. It takes a boat 2 hr to go 16 mi downstream with the current and 4 hr to return against the current. Find the speed of the boat in still water and the speed of the current.

24. The Gulf Stream is a warm ocean current that extends from the eastern side of the Gulf of Mexico up through the Florida Straits and along the southeastern coast of the United States to Cape Hatteras, North Carolina. A boat travels with the current 100 mi from Miami, Florida, to Freeport, Bahamas, in 2.5 hr. The return trip against the same current takes $3\frac{1}{3}$ hr. Find the speed of the boat in still water and the speed of the current.

25. A plane flew 720 mi in 3 hr with the wind. It would take 4 hr to travel the same distance against the wind. What is the rate of the plane in still air and the rate of wind?

26. Nikki and Tatiana rollerblade in opposite directions. Tatiana averages 2 mph faster than Nikki. If they began at the same place and ended up 20 mi apart after 2 hr, how fast did each of them travel?

Concept 5: Applications Involving Geometry

For Exercises 27–32, solve the applications involving geometry. If necessary, refer to the geometry formulas listed in the inside front cover of the text.

27. In a right triangle, one acute angle measures 6° more than 3 times the other. If the sum of the measures of the two acute angles must equal 90°, find the measures of the acute angles.

28. An isosceles triangle has two angles of the same measure (see figure). If the angle represented by y measures 3° less than the angle x, find the measures of all angles of the triangle. (Recall that the sum of the measures of the angles of a triangle is 180°.)

29. Two angles are supplementary. One angle measures 2° less than 3 times the other. What are the measures of the two angles?

30. The measure of one angle is 5 times the measure of another. If the two angles are supplementary, find the measures of the angles.

31. One angle measures 3° more than twice another. If the two angles are complementary, find the measures of the angles.

32. Two angles are complementary. One angle measures 15° more than 2 times the measure of the other. What are the measures of the two angles?
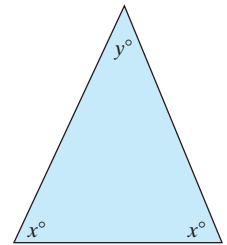
Mixed Exercises

33. How much pure gold (24K) must be mixed with 60% gold to get 20 grams of 75% gold?

34. Two trains leave the depot at the same time, one traveling north and the other traveling south. The speed of one train is 15 mph slower than the other. If after 2 hr the distance between the trains is 190 miles, find the speed of each train.

35. There are two types of tickets sold at the Canadian Formula One Grand Prix race. The price of 6 grandstand tickets and 2 general admissions tickets costs $2330. The price of 4 grandstand tickets and 4 general admission tickets cost $2020. What is the price of each type of ticket?

36. A granola mix contains 5% nuts. How many ounces of nuts must be added to get 25 oz of granola with 24% nuts?

37. A bank offers two accounts, a money market account at 2% simple interest and a regular savings account at 1.3% interest. If Svetlana deposits $3000 between the two accounts and receives $51.25 total interest in the first year, how much did she invest in each account?

38. A rectangle has the perimeter of 42 m. The length is 1 m longer than the width. Find the dimensions of the rectangle.

39. Kyle rode his bike for one-half hour. He got a flat tire and had to walk for 1 hr to get home. He rides his bike 2.5 mph faster than he walks. If the distance he traveled was 6.5 miles, what was his speed riding and what was his speed walking?

40. A basketball player scored 19 points by shooting two-point and three-point baskets. If she made a total of eight baskets, how many of each type did she make?

41. In a right triangle, the measure of one acute angle is one-fourth the measure of the other. Find the measures of the acute angles.

42. Angelo invested $8000 in two accounts: one that pays 3% and one that pays 1.8%. At the end of the first year, his total interest earned was $222. How much did he deposit in the account that pays 3%?

Expanding Your Skills

For Exercises 43–46, solve the business applications.

43. The *demand* for a certain printer cartridge is related to the price. In general, the higher the price x, the lower the demand y. The *supply* for the printer cartridges is also related to price. The supply and demand for the printer cartridges depend on the price according to the equations

$y_d = -10x + 500$ where x is the price per cartridge in dollars and y_d is the demand measured in 1000s of cartridges

$y_s = \dfrac{20}{3}x$ where x is the price per cartridge in dollars and y_s is the supply measured in 1000s of cartridges

Find the price at which the supply and demand are in equilibrium (supply = demand), and confirm your answer with the graph.

Supply and Demand of Printer Cartridges Versus Price
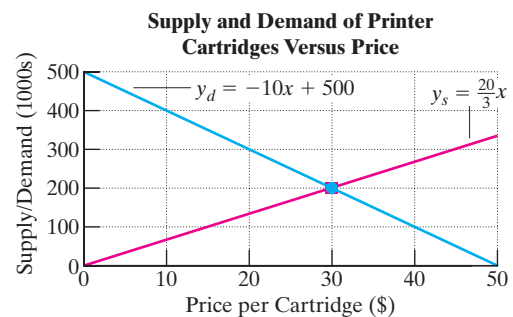
44. The supply and demand for a pack of note cards depend on the price according to the equations

$y_d = -130x + 660$ where x is the price per pack in dollars and y_d is the demand in 1000s of note cards

$y_s = 90x$ where x is the price per pack in dollars and y_s is the supply measured in 1000s of note cards

Find the price at which the supply and demand are in equilibrium (supply = demand).

45. A rental car company rents a compact car for $20 a day, plus $0.25 per mile. A midsize car rents for $30 a day, plus $0.20 per mile.

     a. Write a linear equation representing the cost to rent the compact car.

     b. Write a linear equation representing the cost to rent a midsize car.

     c. Find the number of miles at which the cost to rent either car would be the same.

46. One phone company charges $0.15 per minute for long-distance calls. A second company charges only $0.10 per minute for long-distance calls, but adds a monthly fee of $4.95.

     a. Write a linear equation representing the cost for the first company.

     b. Write a linear equation representing the cost for the second company.

     c. Find the number of minutes of long-distance calling for which the total bill from either company would be the same.

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Systems of Linear Equations in Three Variables and Applications

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Section 3.5

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1. Solutions to Systems of Linear Equations in Three Variables

In Sections 3.1–3.3, we solved systems of linear equations in two variables. In this section, we will expand the discussion to solving systems involving three variables.

A **linear equation in three variables** can be written in the form $Ax + By + Cz = D$, where A, B, and C are not all zero. For example, the equation $2x + 3y + z = 6$ is a linear equation in three variables. Solutions to this equation are **ordered triples** of the form (x, y, z) that satisfy the equation. Some solutions to the equation $2x + 3y + z = 6$ are

Concepts

1. Solutions to Systems of Linear Equations in Three Variables
2. Solving Systems of Linear Equations in Three Variables
3. Applications of Linear Equations in Three Variables

Solution:	Check:
$(1, 1, 1) \longrightarrow$	$2(1) + 3(1) + (1) = 6$ ✔ True
$(2, 0, 2) \longrightarrow$	$2(2) + 3(0) + (2) = 6$ ✔ True
$(0, 1, 3) \longrightarrow$	$2(0) + 3(1) + (3) = 6$ ✔ True

Infinitely many ordered triples serve as solutions to the equation $2x + 3y + z = 6$.

The set of all ordered triples that are solutions to a linear equation in three variables may be represented graphically by a plane in space. Figure 3-9 shows a portion of the plane $2x + 3y + z = 6$ in a 3-dimensional coordinate system.

A solution to a system of linear equations in three variables is an ordered triple that satisfies *each* equation. Geometrically, a solution is a point of intersection of the planes represented by the equations in the system.

A system of linear equations in three variables may have *one unique solution, infinitely many solutions,* or *no solution.*

Figure 3-9

One unique solution (planes intersect at one point)
- The system is consistent.
- The system is independent.

No solution (the three planes do not all intersect)
- The system is inconsistent.
- The system is independent.

Infinitely many solutions (planes intersect at infinitely many points)
- The system is consistent.
- The system is dependent.
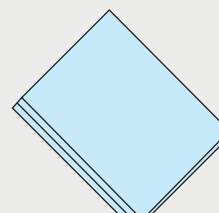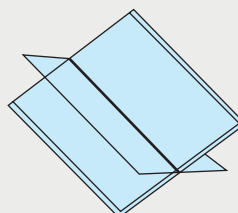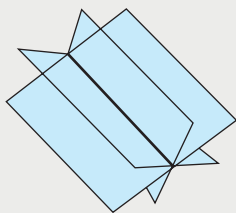
2. Solving Systems of Linear Equations in Three Variables

To solve a system involving three variables, the goal is to eliminate one variable. This reduces the system to two equations in two variables. One strategy for eliminating a variable is to pair up the original equations two at a time.

Solving a System of Three Linear Equations in Three Variables

1. Write each equation in standard form $Ax + By + Cz = D$.

2. Choose a pair of equations, and eliminate one of the variables by using the addition method.

3. Choose a different pair of equations and eliminate the *same* variable.

4. Once steps 2 and 3 are complete, you should have two equations in two variables. Solve this system by using the methods from Sections 3.2 and 3.3.

5. Substitute the values of the variables found in step 4 into any of the three original equations that contain the third variable. Solve for the third variable.

6. Check the ordered triple in each of the original equations.

Example 1 Solving a System of Linear Equations in Three Variables

Solve the system.

$$2x + y - 3z = -7$$
$$3x - 2y + z = 11$$
$$-2x - 3y - 2z = 3$$

Solution:

A $\quad 2x + y - 3z = -7$

B $\quad 3x - 2y + z = 11$

C $\quad -2x - 3y - 2z = 3$

Step 1: The equations are already in standard form.

- It is often helpful to label the equations.
- The *y*-variable can be easily eliminated from equations A and B and from equations A and C. This is accomplished by creating opposite coefficients for the *y*-terms and then adding the equations.

Step 2:   Eliminate the y-variable from equations $\boxed{\text{A}}$ and $\boxed{\text{B}}$.

$\boxed{\text{A}}$ $2x + y - 3z = -7$ $\xrightarrow{\text{Multiply by 2.}}$ $4x + 2y - 6z = -14$

$\boxed{\text{B}}$ $3x - 2y + z = 11$ $\longrightarrow$ $\underline{3x - 2y + z = 11}$

$\qquad\qquad\qquad\qquad\qquad\qquad\quad 7x \qquad - 5z = -3\,\boxed{\text{D}}$

Step 3:   Eliminate the y-variable again, this time from equations $\boxed{\text{A}}$ and $\boxed{\text{C}}$.

$\boxed{\text{A}}$ $2x + y - 3z = -7$ $\xrightarrow{\text{Multiply by 3.}}$ $6x + 3y - 9z = -21$

$\boxed{\text{C}}$ $-2x - 3y - 2z = 3$ $\longrightarrow$ $\underline{-2x - 3y - 2z = 3}$

$\qquad\qquad\qquad\qquad\qquad\qquad\quad 4x \qquad - 11z = -18\,\boxed{\text{E}}$

Step 4:   Now equations $\boxed{\text{D}}$ and $\boxed{\text{E}}$ can be paired up to form a linear system in two variables. Solve this system.

$\boxed{\text{D}}$ $7x - 5z = -3$ $\xrightarrow{\text{Multiply by }-4.}$ $-28x + 20z = 12$

$\boxed{\text{E}}$ $4x - 11z = -18$ $\xrightarrow{\text{Multiply by 7.}}$ $\underline{28x - 77z = -126}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad -57z = -114$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad z = 2$

> **TIP:** It is important to note that in steps 2 and 3, the *same* variable is eliminated.

Once one variable has been found, substitute this value into either equation in the two-variable system, that is, either equation $\boxed{\text{D}}$ or $\boxed{\text{E}}$.

$\boxed{\text{D}}$ $7x - 5z = -3$

$\qquad 7x - 5(2) = -3$ $\qquad$ Substitute $z = 2$ into equation $\boxed{\text{D}}$.

$\qquad 7x - 10 = -3$

$\qquad\qquad 7x = 7$

$\qquad\qquad\quad x = 1$

$\boxed{\text{A}}$ $2x + y - 3z = -7$ $\qquad$ **Step 5:**   Now that two variables are known, substitute these values for x and z into any of the original three equations to find the remaining variable y.

$\qquad 2(1) + y - 3(2) = -7$

$\qquad\qquad 2 + y - 6 = -7$

$\qquad\qquad\qquad y - 4 = -7$ $\qquad\qquad\qquad$ Substitute $x = 1$ and $z = 2$ into equation $\boxed{\text{A}}$.

$\qquad\qquad\qquad\qquad y = -3$

The solution is $(1, -3, 2)$. $\qquad$ **Step 6:**   Check the ordered triple in the three original equations.

$\quad$ <u>Check:</u> $\quad 2x + y - 3z = -7 \rightarrow 2(1) + (-3) - 3(2) = -7$ ✔ True

$\qquad\qquad\qquad 3x - 2y + z = 11 \rightarrow 3(1) - 2(-3) + (2) = 11$ ✔ True

$\qquad\qquad\quad -2x - 3y - 2z = 3 \rightarrow -2(1) - 3(-3) - 2(2) = 3$ ✔ True

Skill Practice

1. Solve the system.

$$x + 2y + z = 1$$
$$3x - y + 2z = 13$$
$$2x + 3y - z = -8$$

Example 2 **Applying Systems of Linear Equations in Three Variables**

In a triangle, the smallest angle measures 10° more than one-half of the largest angle. The middle angle measures 12° more than the smallest angle. Find the measure of each angle.

Solution:

Let x represent the measure of the smallest angle.

Let y represent the measure of the middle angle.

Let z represent the measure of the largest angle.
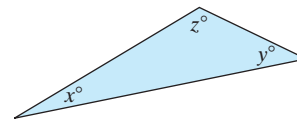
To solve for three variables, we need to establish three independent relationships among x, y, and z.

$\boxed{A}$ $x = \dfrac{z}{2} + 10$	The smallest angle measures 10° more than one-half the measure of the largest angle.
$\boxed{B}$ $y = x + 12$	The middle angle measures 12° more than the measure of the smallest angle.
$\boxed{C}$ $x + y + z = 180$	The sum of the interior angles of a triangle measures 180°.

Clear fractions and write each equation in standard form.

Standard Form

$\boxed{A}$ $x \quad = \dfrac{z}{2} + 10$ $\xrightarrow{\text{Multiply by 2.}}$ $2x = z + 20 \longrightarrow$ $2x \quad - z = \quad 20$

$\boxed{B}$ $\quad y \quad = x + 12$ $\xrightarrow{\hspace{3cm}}$ $-x + y \quad = \quad 12$

$\boxed{C}$ $x + y + z = \quad 180$ $\xrightarrow{\hspace{3cm}}$ $x + y + z = 180$

Notice equation $\boxed{B}$ is missing the z-variable. Therefore, we can eliminate z again by pairing up equations $\boxed{A}$ and $\boxed{C}$.

$\boxed{A}$ $2x \quad - z = \quad 20$
$\boxed{C}$ $\underline{\;x + y + z = 180\;}$
$\quad\;\; 3x + y \quad\;\; = 200$ $\boxed{D}$

$\boxed{B}$ $-x + y = 12$ $\xrightarrow{\text{Multiply by }-1.}$ $x - y = -12$ Pair up equations $\boxed{B}$ and

$\boxed{D}$ $3x + y = 200$ $\xrightarrow{\hspace{2cm}}$ $\underline{3x + y = \;\; 200}$ $\boxed{D}$ to form a system of two variables.

$\qquad\qquad\qquad\qquad\qquad\qquad\quad 4x \quad\;\; = \;\; 188$

$\qquad\qquad\qquad\qquad\qquad\qquad\quad\; x = 47$ Solve for x.

From equation $\boxed{B}$ we have $-x + y = 12 \longrightarrow \quad -47 + y = 12 \rightarrow y = 59$

From equation $\boxed{C}$ we have $x + y + z = 180 \rightarrow 47 + 59 + z = 180 \rightarrow z = 74$

The smallest angle measures $47°$, the middle angle measures $59°$, and the largest angle measures $74°$.

Skill Practice

2. The perimeter of a triangle is 30 in. The shortest side is 4 in. shorter than the longest side. The longest side is 6 in. less than the sum of the other two sides. Find the length of each side.

Example 3 Solving a Dependent System of Linear Equations

Solve the system. If there is not a unique solution, label the system as either dependent or inconsistent.

$$\boxed{A} \quad 3x + \ y - \ z = 8$$
$$\boxed{B} \quad 2x - \ y + 2z = 3$$
$$\boxed{C} \quad \ x + 2y - 3z = 5$$

Solution:

The first step is to make a decision regarding the variable to eliminate. The y-variable is particularly easy to eliminate because the coefficients of y in equations $\boxed{A}$ and $\boxed{B}$ are already opposites. The y-variable can be eliminated from equations $\boxed{B}$ and $\boxed{C}$ by multiplying equation $\boxed{B}$ by 2.

$$\boxed{A} \quad 3x + y - \ z = 8$$
$$\boxed{B} \quad 2x - y + 2z = 3$$
$$\overline{\quad 5x \qquad + \ z = 11 \quad \boxed{D}}$$

Pair up equations $\boxed{A}$ and $\boxed{B}$ to eliminate y.

$$\boxed{B} \quad 2x - \ y + 2z = 3 \xrightarrow{\text{Multiply by 2.}} 4x - 2y + 4z = \ 6$$
$$\boxed{C} \quad \ x + 2y - 3z = 5 \longrightarrow \quad \underline{\ x + 2y - 3z = \ 5}$$
$$5x \qquad + \ z = 11 \quad \boxed{E}$$

Pair up equations $\boxed{B}$ and $\boxed{C}$ to eliminate y.

Because equations $\boxed{D}$ and $\boxed{E}$ are equivalent equations, it appears that this is a dependent system. By eliminating variables we obtain the identity $0 = 0$.

$$\boxed{D} \quad 5x + z = 11 \xrightarrow{\text{Multiply by } -1.} -5x - z = -11$$
$$\boxed{E} \quad 5x + z = 11 \longrightarrow \quad \underline{\ 5x + z = \ \ 11}$$
$$0 = \ \ 0$$

The result $0 = 0$ indicates that there are infinitely many solutions and that the system is dependent.

Skill Practice

3. Solve the system. If the system does not have a unique solution, identify the system as dependent or inconsistent.

$$x + \ y + \ z = \ \ 8$$
$$2x - \ y + \ z = \ \ 6$$
$$-5x - 2y - 4z = -30$$

Skill Practice Answers

2. 8 in., 10 in., and 12 in.
3. Dependent system

| Example 4 | Solving an Inconsistent System of Linear Equations |

Solve the system. If there is not a unique solution, identify the system as either dependent or inconsistent.

$$2x + 3y - 7z = 4$$
$$-4x - 6y + 14z = 1$$
$$5x + y - 3z = 6$$

Solution:

We will eliminate the x-variable.

$$\boxed{A} \quad 2x + 3y - 7z = 4 \quad \xrightarrow{\text{Multiply by 2.}} \quad 4x + 6y - 14z = 8$$
$$\boxed{B} \quad -4x - 6y + 14z = 1 \quad \xrightarrow{} \quad \underline{-4x - 6y + 14z = 1}$$
$$\boxed{C} \quad 5x + y - 3z = 6 \qquad\qquad\qquad\qquad\quad 0 = 9 \quad \text{(contradiction)}$$

The result $0 = 9$ is a contradiction, indicating that the system has no solution. The system is inconsistent.

| Skill Practice |

4. Solve the system. If the system does not have a unique solution, identify the system as dependent or inconsistent.

$$x - 2y + z = 5$$
$$x - 3y + 2z = -7$$
$$-2x + 4y - 2z = 6$$

3. Applications of Linear Equations in Three Variables

| Example 5 | Applying Systems of Linear Equations to Nutrition |

Doctors have become increasingly concerned about the sodium intake in the U.S. diet. Recommendations by the American Medical Association indicate that most individuals should not exceed 2400 mg of sodium per day.

Liz ate 1 slice of pizza, 1 serving of ice cream, and 1 glass of soda for a total of 1030 mg of sodium. David ate 3 slices of pizza, no ice cream, and 2 glasses of soda for a total of 2420 mg of sodium. Melinda ate 2 slices of pizza, 1 serving of ice cream, and 2 glasses of soda for a total of 1910 mg of sodium. How much sodium is in one serving of each item?

Solution:

Let x represent the sodium content of 1 slice of pizza.

Let y represent the sodium content of 1 serving of ice cream.

Let z represent the sodium content of 1 glass of soda.

From Liz's meal we have: $\boxed{A} \quad x + y + z = 1030$

From David's meal we have: $\boxed{B} \quad 3x \qquad + 2z = 2420$

From Melinda's meal we have: $\boxed{C} \quad 2x +$

Equation $\boxed{B}$ is missing the y-variable. Eliminating y from equations $\boxed{A}$ and $\boxed{C}$, we have

$$\boxed{A}\quad x + y + z = 1030 \xrightarrow{\text{Multiply by } -1.} -x - y - z = -1030$$

$$\boxed{C}\quad 2x + y + 2z = 1910 \longrightarrow \underline{\quad 2x + y + 2z = \quad 1910}$$

$$\boxed{D}\quad x \qquad + z = \qquad 880$$

Solve the system formed by equations $\boxed{B}$ and $\boxed{D}$.

$$\boxed{B}\quad 3x + 2z = 2420 \longrightarrow 3x + 2z = \quad 2420$$

$$\boxed{D}\quad x + z = \quad 880 \xrightarrow{\text{Multiply by } -2.} \underline{-2x - 2z = -1760}$$

$$x \qquad = \qquad 660$$

From equation $\boxed{D}$ we have $x + z = 880 \longrightarrow 660 + z = 880 \longrightarrow z = 220$

From equation $\boxed{A}$ we have $x + y + z = 1030 \longrightarrow 660 + y + 220 = 1030 \longrightarrow y = 150$

Therefore, 1 slice of pizza has 660 mg of sodium, 1 serving of ice cream has 150 mg of sodium, and 1 glass of soda has 220 mg of sodium.

Skill Practice

5. Annette, Barb, and Carlita work in a clothing shop. One day the three had combined sales of $1480. Annette sold $120 more than Barb. Barb and Carlita combined sold $280 more than Annette. How much did each person sell?

Skill Practice Answers

5. Annette sold $600, Barb sold $480, and Carlita sold $400.

Section 3.5   Practice Exercises

Study Skills Exercises

1. Look back over your notes for this chapter. Have you highlighted the important topics? Have you underlined the key terms? Have you indicated the places where you are having trouble? If you find that you have problems with a particular topic, write a question that you can ask your instructor either in class or in the office.

2. Define the key terms.

   a. **Linear equation in three variables**   b. **Ordered triple**

Review Exercises

For Exercises 3–4, solve the systems by using two methods: (**a**) the substitution method and (**b**) the addition method.

3. $\begin{aligned} 3x + y &= 4 \\ 4x + y &= 5 \end{aligned}$

4. $\begin{aligned} 2x - 5y &= 3 \\ -4x + 10y &= 3 \end{aligned}$

5. Two cars leave Kansas City at the same time. One travels east and one travels west. After 3 hr the cars are 369 mi apart. If one car travels 7 mph slower than the other, find the speed of each car.

Concept 1: Solutions to Systems of Linear Equations in Three Variables

6. How many solutions are possible when solving a system of three equations with three variables?

7. Which of the following points are solutions to the system?

$$(2, 1, 7), (3, -10, -6), (4, 0, 2)$$

$$2x - y + z = 10$$
$$4x + 2y - 3z = 10$$
$$x - 3y + 2z = 8$$

8. Which of the following points are solutions to the system?

$$(1, 1, 3), (0, 0, 4), (4, 2, 1)$$

$$-3x - 3y - 6z = -24$$
$$-9x - 6y + 3z = -45$$
$$9x + 3y - 9z = 33$$

9. Which of the following points are solutions to the system?

$$(12, 2, -2), (4, 2, 1), (1, 1, 1)$$

$$-x - y - 4z = -6$$
$$x - 3y + z = -1$$
$$4x + y - z = 4$$

10. Which of the following points are solutions to the system?

$$(0, 4, 3), (3, 6, 10), (3, 3, 1)$$

$$x + 2y - z = 5$$
$$x - 3y + z = -5$$
$$-2x + y - z = -4$$

Concept 2: Solving Systems of Linear Equations in Three Variables

For Exercises 11–24, solve the system of equations.

11.
$$2x + y - 3z = -12$$
$$3x - 2y - z = 3$$
$$-x + 5y + 2z = -3$$

12.
$$-3x - 2y + 4z = -15$$
$$2x + 5y - 3z = 3$$
$$4x - y + 7z = 15$$

13.
$$x - 3y - 4z = -7$$
$$5x + 2y + 2z = -1$$
$$4x - y - 5z = -6$$

14.
$$6x - 5y + z = 7$$
$$5x + 3y + 2z = 0$$
$$-2x + y - 3z = 11$$

15.
$$-3x + y - z = 8$$
$$-4x + 2y + 3z = -3$$
$$2x + 3y - 2z = -1$$

16.
$$2x + 3y + 3z = 15$$
$$3x - 6y - 6z = -23$$
$$-9x - 3y + 6z = 8$$

17.
$$4x + 2z = 12 + 3y$$
$$2y = 3x + 3z - 5$$
$$y = 2x + 7z + 8$$

18.
$$y = 2x + z + 1$$
$$-3x - 1 = -2y + 2z$$
$$5x + 3z = 16 - 3y$$

19.
$$x + y + z = 6$$
$$-x + y - z = -2$$
$$2x + 3y + z = 11$$

20.
$$x - y - z = -11$$
$$x + y - z = 15$$
$$2x - y + z = -9$$

21.
$$2x - 3y + 2z = -1$$
$$x + 2y = -4$$
$$x + z = 1$$

22.
$$x + y + z = 2$$
$$2x - z = 5$$
$$3y + z = 2$$

23.
$$4x + 9y = 8$$
$$8x + 6z = -1$$
$$6y + 6z = -1$$

24.
$$3x + 2z = 11$$
$$y - 7z = 4$$
$$x - 6y = 1$$

Concept 3: Applications of Linear Equations in Three Variables

25. A triangle has one angle that measures 5° more than twice the smallest angle, and the largest angle measures 11° less than 3 times the measure of the smallest angle. Find the measures of the three angles.

26. The largest angle of a triangle measures 4° less than 5 times the measure of the smallest angle. The middle angle measures twice that of the smallest angle. Find the measures of the three angles.

27. The perimeter of a triangle is 55 cm. The measure of the shortest side is 8 cm less than the middle side. The measure of the longest side is 1 cm less than the sum of the other two sides. Find the lengths of the sides.

28. The perimeter of a triangle is 5 ft. The longest side of the triangle measures 20 in. more than the shortest side. The middle side is 3 times the measure of the shortest side. Find the lengths of the three sides in *inches*.

29. A movie theater charges $7 for adults, $5 for children under age 17, and $4 for seniors over age 60. For one showing of *Batman* the theater sold 222 tickets and took in $1383. If twice as many adult tickets were sold as the total of children and senior tickets, how many tickets of each kind were sold?

30. Goofie Golf has 18 holes that are par 3, par 4, or par 5. Most of the holes are par 4. In fact, there are 3 times as many par 4s as par 3s. There are 3 more par 5s than par 3s. How many of each type are there?

31. Combining peanuts, pecans, and cashews makes a party mixture of nuts. If the amount of peanuts equals the amount of pecans and cashews combined, and if there are twice as many cashews as pecans, how many ounces of each nut is used to make 48 oz of party mixture?

32. Souvenir hats, T-shirts, and jackets are sold at a rock concert. Three hats, two T-shirts, and one jacket cost $140. Two hats, two T-shirts, and two jackets cost $170. One hat, three T-shirts, and two jackets cost $180. Find the prices of the individual items.

33. In 2002, Baylor University in Waco, Texas, had twice as many students as Vanderbilt University in Nashville, Tennessee. Pace University in New York City had 2800 more students than Vanderbilt University. If the enrollment for all three schools totaled 27,200, find the enrollment for each school.

34. Annie and Maria traveled overseas for seven days and stayed in three different hotels in three different cities: Stockholm, Sweden; Oslo, Norway; and Paris, France.

The total bill for all seven nights (not including tax) was $1040. The total tax was $106. The nightly cost (excluding tax) to stay at the hotel in Paris was $80 more than the nightly cost (excluding tax) to stay in Oslo. Find the cost per night for each hotel excluding tax.

City	Number of Nights	Cost/Night ($)	Tax Rate
Paris, France	1	x	8%
Stockholm, Sweden	4	y	11%
Oslo, Norway	2	z	10%

Mixed Exercises

For Exercises 35–44, solve the system. If there is not a unique solution, label the system as either dependent or inconsistent.

35. $2x + y + 3z = 2$
    $x - y + 2z = -4$
    $x + 3y - z = 1$

36. $x + y = z$
    $3(x - y) + 6z = 1 - y$
    $7x + 3(y + 1) = 7 - z$

37. $6x - 2y + 2z = 2$
    $4x + 8y - 2z = 5$
    $-2x - 4y + z = -2$

38. $3x + 2y + z = 3$
    $x - 3y + z = 4$
    $-6x - 4y - 2z = 1$

39. $\frac{1}{2}x + \frac{2}{3}y = \frac{5}{2}$
    $\frac{1}{5}x - \frac{1}{2}z = -\frac{3}{10}$
    $\frac{1}{3}y - \frac{1}{4}z = \frac{3}{4}$

40. $\frac{1}{2}x + \frac{1}{4}y + z = 3$
    $\frac{1}{8}x + \frac{1}{4}y + \frac{1}{4}z = \frac{9}{8}$
    $x - y - \frac{2}{3}z = \frac{1}{3}$

41. $2x + y = 3(z - 1)$
$3x - 2(y - 2z) = 1$
$2(2x - 3z) = -6 - 2y$

42. $2x + y = -3$
$2y + 16z = -10$
$-7x - 3y + 4z = 8$

43. $-0.1y + 0.2z = 0.2$
$0.1x + 0.1y + 0.1z = 0.2$
$-0.1x + 0.3z = 0.2$

44. $0.1x - 0.2y = 0$
$0.3y + 0.1z = -0.1$
$0.4x - 0.1z = 1.2$

Expanding Your Skills

The systems in Exercises 45–48 are called homogeneous systems because each system has $(0, 0, 0)$ as a solution. However, if a system is dependent, it will have infinitely many more solutions. For each system determine whether $(0, 0, 0)$ is the only solution or if the system is dependent.

45. $2x - 4y + 8z = 0$
$-x - 3y + z = 0$
$x - 2y + 5z = 0$

46. $2x - 4y + z = 0$
$x - 3y - z = 0$
$3x - y + 2z = 0$

47. $4x - 2y - 3z = 0$
$-8x - y + z = 0$
$2x - y - \frac{3}{2}z = 0$

48. $5x + y = 0$
$4y - z = 0$
$5x + 5y - z = 0$

Section 3.6 　 Solving Systems of Linear Equations by Using Matrices

Concepts

1. Introduction to Matrices
2. Solving Systems of Linear Equations by Using the Gauss-Jordan Method

1. Introduction to Matrices

In Sections 3.2, 3.3, and 3.5, we solved systems of linear equations by using the substitution method and the addition method. We now present a third method called the Gauss-Jordan method that uses matrices to solve a linear system.

A **matrix** is a rectangular array of numbers (the plural of *matrix* is *matrices*). The rows of a matrix are read horizontally, and the columns of a matrix are read vertically. Every number or entry within a matrix is called an element of the matrix.

The **order of a matrix** is determined by the number of rows and number of columns. A matrix with m rows and n columns is an $m \times n$ (read as "m by n") matrix. Notice that with the order of a matrix, the number of rows is given first, followed by the number of columns.

Example 1 　 Determining the Order of a Matrix

Determine the order of each matrix.

a. $\begin{bmatrix} 2 & -4 & 1 \\ 5 & \pi & \sqrt{7} \end{bmatrix}$ 　 **b.** $\begin{bmatrix} 1.9 \\ 0 \\ 7.2 \\ -6.1 \end{bmatrix}$ 　 **c.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 　 **d.** $\begin{bmatrix} a & b & c \end{bmatrix}$

Solution:

a. This matrix has two rows and three columns. Therefore, it is a 2×3 matrix.

b. This matrix has four rows and one column. Therefore, it is a 4×1 matrix. A matrix with one column is called a **column matrix**.

c. This matrix has three rows and three columns. Therefore, it is a 3×3 matrix. A matrix with the same number of rows and columns is called a **square matrix**.

d. This matrix has one row and three columns. Therefore, it is a 1×3 matrix. A matrix with one row is called a **row matrix**.

Skill Practice   Determine the order of the matrix.

1. $\begin{bmatrix} -5 & 2 \\ 1 & 3 \\ 8 & 9 \end{bmatrix}$ **2.** $[4 \ -8]$ **3.** $\begin{bmatrix} 5 \\ 10 \\ 15 \end{bmatrix}$ **4.** $\begin{bmatrix} 2 & -0.5 \\ -1 & 6 \end{bmatrix}$

A matrix can be used to represent a system of linear equations written in standard form. To do so, we extract the coefficients of the variable terms and the constants within the equation. For example, consider the system

$$2x - y = 5$$
$$x + 2y = -5$$

The matrix **A** is called the **coefficient matrix**.

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

If we extract both the coefficients and the constants from the equations, we can construct the **augmented matrix** of the system:

$$\begin{bmatrix} 2 & -1 & | & 5 \\ 1 & 2 & | & -5 \end{bmatrix}$$

A vertical bar is inserted into an augmented matrix to designate the position of the equal signs.

Example 2   **Writing the Augmented Matrix of a System of Linear Equations**

Write the augmented matrix for each linear system.

a. $-3x - 4y = 3$
$\quad\ 2x + 4y = 2$

b. $2x \qquad - 3z = 14$
$\qquad 2y + z = 2$
$\quad\ x + y \qquad = 4$

Solution:

a. $\begin{bmatrix} -3 & -4 & | & 3 \\ 2 & 4 & | & 2 \end{bmatrix}$

TIP: Notice that zeros are inserted to denote the coefficient of each missing term.

b. $\begin{bmatrix} 2 & 0 & -3 & | & 14 \\ 0 & 2 & 1 & | & 2 \\ 1 & 1 & 0 & | & 4 \end{bmatrix}$

Skill Practice   Write the augmented matrix for the system.

5. $-x + y = 4$
$\quad\ 2x - y = 1$

6. $2x - y + z = 14$
$\quad -3x + 4y \qquad = 8$
$\qquad\ x - y + 5z = 0$

Skill Practice Answers

1. 3×2 **2.** 1×2
3. 3×1 **4.** 2×2
5. $\begin{bmatrix} -1 & 1 & | & 4 \\ 2 & -1 & | & 1 \end{bmatrix}$
6. $\begin{bmatrix} 2 & -1 & 1 & | & 14 \\ -3 & 4 & 0 & | & 8 \\ 1 & -1 & 5 & | & 0 \end{bmatrix}$

Example 3   Writing a Linear System from an Augmented Matrix

Write a system of linear equations represented by each augmented matrix.

a. $\begin{bmatrix} 2 & -5 & | & -8 \\ 4 & 1 & | & 6 \end{bmatrix}$

b. $\begin{bmatrix} 2 & -1 & 3 & | & 14 \\ 1 & 1 & -2 & | & -5 \\ 3 & 1 & -1 & | & 2 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 0 & 0 & | & 4 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$

Solution:

a. $2x - 5y = -8$
   $4x + y = 6$

b. $2x - y + 3z = 14$
   $x + y - 2z = -5$
   $3x + y - z = 2$

c. $x + 0y + 0z = 4$ $x = 4$
   $0x + y + 0z = -1$ or $y = -1$
   $0x + 0y + z = 0$ $z = 0$

Skill Practice   Write a system of linear equations represented by each augmented matrix.

7. $\begin{bmatrix} 2 & 3 & | & 5 \\ -1 & 8 & | & 1 \end{bmatrix}$

8. $\begin{bmatrix} -3 & 2 & 1 & | & 4 \\ 14 & 1 & 0 & | & 20 \\ -8 & 3 & 5 & | & 6 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 0 & 0 & | & -5 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$

2. Solving Systems of Linear Equations by Using the Gauss-Jordan Method

We know that interchanging two equations results in an equivalent system of linear equations. Interchanging two rows in an augmented matrix results in an equivalent augmented matrix. Similarly, because each row in an augmented matrix represents a linear equation, we can perform the following elementary row operations that result in an equivalent augmented matrix.

Elementary Row Operations

The following *elementary row operations* performed on an augmented matrix produce an equivalent augmented matrix:

1. Interchange two rows.
2. Multiply every element in a row by a nonzero real number.
3. Add a multiple of one row to another row.

Skill Practice Answers

7. $2x + 3y = 5$
   $-x + 8y = 1$
8. $-3x + 2y + z = 4$
   $14x + y = 20$
   $-8x + 3y + 5z = 6$
9. $x = -5, y = 2, z = 0$

When we are solving a system of linear equations by any method, the goal is to write a series of simpler but equivalent systems of equations until the solution is obvious. The *Gauss-Jordan method* uses a series of elementary row operations performed on the augmented matrix to produce a simpler augmented matrix. In

particular, we want to produce an augmented matrix that has 1s along the diagonal of the matrix of coefficients and 0s for the remaining entries in the matrix of coefficients. A matrix written in this way is said to be written in **reduced row echelon form**. For example, the augmented matrix from Example 3(c) is written in reduced row echelon form.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The solution to the corresponding system of equations is easily recognized as $x = 4$, $y = -1$, and $z = 0$.

Similarly, matrix **B** represents a solution of $x = a$ and $y = b$.

$$\mathbf{B} = \left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & b \end{array}\right]$$

Example 4 Solving a System of Linear Equations by Using the Gauss-Jordan Method

Solve by using the Gauss-Jordan method.

$$2x - y = 5$$
$$x + 2y = -5$$

Solution:

$$\left[\begin{array}{cc|c} 2 & -1 & 5 \\ 1 & 2 & -5 \end{array}\right]$$ Set up the augmented matrix.

$$\xrightarrow{R_1 \Leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & 2 & -5 \\ 2 & -1 & 5 \end{array}\right]$$ Switch row 1 and row 2 to get a 1 in the upper left position.

$$\xrightarrow{-2R_1 + R_2 \Rightarrow R_2} \left[\begin{array}{cc|c} 1 & 2 & -5 \\ 0 & -5 & 15 \end{array}\right]$$ Multiply row 1 by -2 and add the result to row 2. This produces an entry of 0 below the upper left position.

$$\xrightarrow{-\frac{1}{5}R_2 \Rightarrow R_2} \left[\begin{array}{cc|c} 1 & 2 & -5 \\ 0 & 1 & -3 \end{array}\right]$$ Multiply row 2 by $-\frac{1}{5}$ to produce a 1 along the diagonal in the second row.

$$\xrightarrow{-2R_2 + R_1 \Rightarrow R_1} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -3 \end{array}\right]$$ Multiply row 2 by -2 and add the result to row 1. This produces a 0 in the first row, second column.

The matrix **C** is in reduced row echelon form. From the augmented matrix, we have $x = 1$ and $y = -3$. The solution to the system is $(1, -3)$.

$$\mathbf{C} = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -3 \end{array}\right]$$

Skill Practice

10. Solve by using the Gauss-Jordan method.

$$x - 2y = -21$$
$$2x + y = -2$$

Skill Practice Answers

10. $(-5, 8)$

The order in which we manipulate the elements of an augmented matrix to produce reduced row echelon form was demonstrated in Example 4. In general, the order is as follows.

- First produce a 1 in the first row, first column. Then use the first row to obtain 0s in the first column below this element.
- Next, if possible, produce a 1 in the second row, second column. Use the second row to obtain 0s above and below this element.
- Next, if possible, produce a 1 in the third row, third column. Use the third row to obtain 0s above and below this element.
- The process continues until reduced row echelon form is obtained.

Example 5 **Solving a System of Linear Equations by Using the Gauss-Jordan Method**

Solve by using the Gauss-Jordan method.

$$x \qquad\qquad = -y + 5$$
$$-2x \qquad + 2z = \quad y - 10$$
$$3x + 6y + 7z = 14$$

Solution:

First write each equation in the system in standard form.

$$x \qquad\qquad = -y + 5 \longrightarrow \quad x + \quad y \qquad = \quad 5$$
$$-2x + \qquad 2z = y - 10 \longrightarrow -2x - \quad y + 2z = -10$$
$$3x + 6y + 7z = 14 \qquad \longrightarrow \quad 3x + 6y + 7z = \quad 14$$

$$\begin{bmatrix} 1 & 1 & 0 & 5 \\ -2 & -1 & 2 & -10 \\ 3 & 6 & 7 & 14 \end{bmatrix}$$ Set up the augmented matrix.

$$2R_1 + R_2 \Rightarrow R_2 \longrightarrow \begin{bmatrix} 1 & 1 & 0 & 5 \\ 0 & 1 & 2 & 0 \\ 0 & 3 & 7 & -1 \end{bmatrix}$$ Multiply row 1 by 2 and add the result to row 2. Multiply row 1 by -3 and add the result to row 3.
$$-3R_1 + R_3 \Rightarrow R_3 \longrightarrow$$

$$-1R_2 + R_1 \Rightarrow R_1 \longrightarrow \begin{bmatrix} 1 & 0 & -2 & 5 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$ Multiply row 2 by -1 and add the result to row 1. Multiply row 2 by -3 and add the result to row 3.
$$-3R_2 + R_3 \Rightarrow R_3 \longrightarrow$$

$$2R_3 + R_1 \Rightarrow R_1 \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$ Multiply row 3 by 2 and add the result to row 1. Multiply row 3 by -2 and add the result to row 2.
$$-2R_3 + R_2 \Rightarrow R_2 \longrightarrow$$

From the reduced row echelon form of the matrix, we have $x = 3$, $y = 2$, and $z = -1$. The solution to the system is $(3, 2, -1)$.

Skill Practice Solve by using the Gauss-Jordan method.

11. $x + \ y + \ z = 2$

$\quad x - \ y + \ z = 4$

$\quad x + 4y + 2z = 1$

It is particularly easy to recognize a dependent or inconsistent system of equations from the reduced row echelon form of an augmented matrix. This is demonstrated in Examples 6 and 7.

Example 6 **Solving a Dependent System of Equations by Using the Gauss-Jordan Method**

Solve by using the Gauss-Jordan method.

$$x - \ 3y = 4$$

$$\frac{1}{2}x - \frac{3}{2}y = 2$$

Solution:

$$\begin{bmatrix} 1 & -3 & | & 4 \\ \frac{1}{2} & -\frac{3}{2} & | & 2 \end{bmatrix} \qquad \text{Set up the augmented matrix.}$$

$\xrightarrow{-\frac{1}{2}R_1 + R_2 \Rightarrow R_2}$ $\begin{bmatrix} 1 & -3 & | & 4 \\ 0 & 0 & | & 0 \end{bmatrix}$ Multiply row 1 by $-\frac{1}{2}$ and add the result to row 2.

The second row of the augmented matrix represents the equation $0 = 0$; hence, the system is dependent. The solution is $\{(x, y)\,|\,x - 3y = 4\}$.

Skill Practice Solve by using the Gauss-Jordan method.

12. $4x - 6y = 16$

$\quad 6x - 9y = 24$

Example 7 **Solving an Inconsistent System of Equations by Using the Gauss-Jordan Method**

Solve by using the Gauss-Jordan method.

$$x + 3y = 2$$

$$-3x - 9y = 1$$

Solution:

$$\begin{bmatrix} 1 & 3 & | & 2 \\ -3 & -9 & | & 1 \end{bmatrix} \qquad \text{Set up the augmented matrix.}$$

$\xrightarrow{3R_1 + R_2 \Rightarrow R_2}$ $\begin{bmatrix} 1 & 3 & | & 2 \\ 0 & 0 & | & 7 \end{bmatrix}$ Multiply row 1 by 3 and add the result to row 2.

The second row of the augmented matrix represents the contradiction $0 = 7$; hence, the system is inconsistent. There is no solution.

Skill Practice

13. Solve by using the Gauss-Jordan method.

$$6x + 10y = 1$$
$$15x + 25y = 3$$

Calculator Connections

Many graphing calculators have a matrix editor in which the user defines the order of the matrix and then enters the elements of the matrix. For example, the 2×3 matrix

$$\mathbf{D} = \begin{bmatrix} 2 & -3 & -13 \\ 3 & 1 & 8 \end{bmatrix}$$

is entered as shown.

```
NAMES MATH EDIT
1█[A]
2:[B]
3:[C]
4:[D]
5:[E]
6:[F]
7↓[G]
```

```
MATRIX[A]  2 ×3
[ 2      -3      -13   ]
[ 3       1       8    ]
```
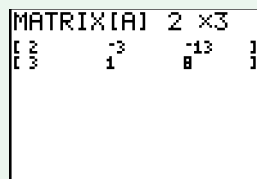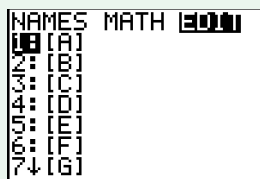
Once an augmented matrix has been entered into a graphing calculator, a *rref* function can be used to transform the matrix into reduced row echelon form.

```
rref([A])
          [[1 0 1],
           [0 1 5]]
```
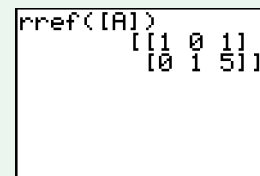
Section 3.6   Practice Exercises

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Study Skills Exercises

1. Prepare a one-page summary sheet with the most important information that you need for the next test. On the day of the test, look at this sheet several times to refresh your memory, instead of trying to memorize new information.

2. Define the key terms.

   a. Matrix   **b. Order of a matrix**   **c. Column matrix**   **d. Square matrix**

   e. Row matrix   **f. Coefficient matrix**   **g. Augmented matrix**   **h. Reduced row echelon form**

Review Exercises

For Exercises 3–5, solve the system by using any method.

3. $x - 6y = 9$
$x + 2y = 13$

4. $x + y - z = 8$
$x - 2y + z = 3$
$x + 3y + 2z = 7$

5. $2x - y + z = -4$
$-x + y + 3z = -7$
$x + 3y - 4z = 22$

Concept 1: Introduction to Matrices

For Exercises 6–14, **(a)** determine the order of each matrix and **(b)** determine if the matrix is a row matrix, a column matrix, a square matrix, or none of these.

6. $\begin{bmatrix} 4 \\ 5 \\ -3 \\ 0 \end{bmatrix}$

7. $\begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$

8. $\begin{bmatrix} -9 & 4 & 3 \\ -1 & -8 & 4 \\ 5 & 8 & 7 \end{bmatrix}$

9. $\begin{bmatrix} 3 & -9 \\ -1 & -3 \end{bmatrix}$

10. $\begin{bmatrix} 4 & -7 \end{bmatrix}$

11. $\begin{bmatrix} 0 & -8 & 11 & 5 \end{bmatrix}$

12. $\begin{bmatrix} 5 & -8.1 & 4.2 & 0 \\ 4.3 & -9 & 18 & 3 \end{bmatrix}$

13. $\begin{bmatrix} \frac{1}{3} & \frac{3}{4} & 6 \\ -2 & 1 & -\frac{7}{8} \end{bmatrix}$

14. $\begin{bmatrix} 5 & 1 \\ -1 & 2 \\ 0 & 7 \end{bmatrix}$

For Exercises 15–19, set up the augmented matrix.

15. $x - 2y = -1$
$2x + y = -7$

16. $x - 3y = 3$
$2x - 5y = 4$

17. $y = 2x - 1$
$y = -3x + 7$

18. $x - 2y = 5 - z$
$2x + 6y + 3z = -2$
$3x - y - 2z = 1$

19. $5x - 17 = -2z$
$8x + 6z = 26 + y$
$8x + 3y - 12z = 24$

For Exercises 20–23, write a system of linear equations represented by the augmented matrix.

20. $\left[\begin{array}{cc|c} 4 & 3 & 6 \\ 12 & 5 & -6 \end{array}\right]$

21. $\left[\begin{array}{cc|c} -2 & 5 & -15 \\ -7 & 15 & -45 \end{array}\right]$

22. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 7 \end{array}\right]$

23. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0.5 \\ 0 & 1 & 0 & 6.1 \\ 0 & 0 & 1 & 3.9 \end{array}\right]$

Concept 2: Solving Systems of Linear Equations by Using the Gauss-Jordan Method

24. Given the matrix **E**

$$\mathbf{E} = \left[\begin{array}{cc|c} 3 & -2 & 8 \\ 9 & -1 & 7 \end{array}\right]$$

   a. What is the element in the second row and third column?

   b. What is the element in the first row and second column?

25. Given the matrix **F**

$$\mathbf{F} = \left[\begin{array}{cc|c} 1 & 8 & 0 \\ 12 & -13 & -2 \end{array}\right]$$

   a. What is the element in the second row and second column?

   b. What is the element in the first row and third column?

26. Given the matrix **Z**

$$\mathbf{Z} = \left[\begin{array}{cc|c} 2 & 1 & 11 \\ 2 & -1 & 1 \end{array} \right]$$

write the matrix obtained by multiplying the elements in the first row by $\frac{1}{2}$.

27. Given the matrix **J**

$$\mathbf{J} = \left[\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 3 & -6 \end{array} \right]$$

write the matrix obtained by multiplying the elements in the second row by $\frac{1}{3}$.

28. Given the matrix **K**

$$\mathbf{K} = \left[\begin{array}{cc|c} 5 & 2 & 1 \\ 1 & -4 & 3 \end{array} \right]$$

write the matrix obtained by interchanging rows 1 and 2.

29. Given the matrix **L**

$$\mathbf{L} = \left[\begin{array}{cc|c} 9 & 6 & 13 \\ -7 & 2 & 19 \end{array} \right]$$

write the matrix obtained by interchanging rows 1 and 2.

30. Given the matrix **M**

$$\mathbf{M} = \left[\begin{array}{cc|c} 1 & 5 & 2 \\ -3 & -4 & -1 \end{array} \right]$$

write the matrix obtained by multiplying the first row by 3 and adding the result to row 2.

31. Given the matrix **N**

$$\mathbf{N} = \left[\begin{array}{cc|c} 1 & 3 & -5 \\ -2 & 2 & 12 \end{array} \right]$$

write the matrix obtained by multiplying the first row by 2 and adding the result to row 2.

32. Given the matrix **R**

$$\mathbf{R} = \left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 4 & 1 & -5 & 6 \\ -2 & 0 & -3 & 10 \end{array} \right]$$

a. Write the matrix obtained by multiplying the first row by -4 and adding the result to row 2.

b. Using the matrix obtained from part (a), write the matrix obtained by multiplying the first row by 2 and adding the result to row 3.

33. Given the matrix **S**

$$\mathbf{S} = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 10 \\ 5 & 1 & -4 & 3 \\ -3 & 4 & 5 & 2 \end{array} \right]$$

a. Write the matrix obtained by multiplying the first row by -5 and adding the result to row 2.

b. Using the matrix obtained from part (a), write the matrix obtained by multiplying the first row by 3 and adding the result to row 3.

For Exercises 34–49, solve the systems by using the Gauss-Jordan method.

34. $x - 2y = -1$
$2x + y = -7$

35. $x - 3y = 3$
$2x - 5y = 4$

36. $x + 3y = 6$
$-4x - 9y = 3$

37. $2x - 3y = -2$
$x + 2y = 13$

38. $x + 3y = 3$
$4x + 12y = 12$

39. $2x + 5y = 1$
$-4x - 10y = -2$

40. $x - y = 4$
$2x + y = 5$

41. $2x - y = 0$
$x + y = 3$

42. $x + 3y = -1$
$-3x - 6y = 12$

43. $x + y = 4$
$2x - 4y = -4$

44. $3x + y = -4$
$-6x - 2y = 3$

45. $2x + y = 4$
$6x + 3y = -1$

46. $x + y + z = 6$ **47.** $2x - 3y - 2z = 11$ **48.** $x - 2y\quad = 5 - z$ **49.** $5x\quad - 10z = 15$

$\quad\quad x - y + z = 2$ $\quad\quad x + 3y + 8z = 1$ $\quad\quad 2x + 6y + 3z = -10$ $\quad\quad x - y + 6z = 23$

$\quad\quad x + y - z = 0$ $\quad\quad 3x - y + 14z = -2$ $\quad\quad 3x - y - 2z = 5$ $\quad\quad x + 3y - 12z = 13$

For Exercises 50–53, use the augmented matrices **A**, **B**, **C**, and **D** to answer true or false.

$$\mathbf{A} = \begin{bmatrix} 6 & -4 & | & 2 \\ 5 & -2 & | & 7 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 5 & -2 & | & 7 \\ 6 & -4 & | & 2 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & -\frac{2}{3} & | & \frac{1}{3} \\ 5 & -2 & | & 7 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 5 & -2 & | & 7 \\ -12 & 8 & | & -4 \end{bmatrix}$$

50. The matrix **A** is a 2×3 matrix.   **51.** Matrix **B** is equivalent to matrix **A**.

52. Matrix **A** is equivalent to matrix **C**.   **53.** Matrix **B** is equivalent to matrix **D**.

54. What does the notation $R_2 \Leftrightarrow R_1$ mean when one is performing the Gauss-Jordan method?

55. What does the notation $2R_3 \Rightarrow R_3$ mean when one is performing the Gauss-Jordan method?

56. What does the notation $-3R_1 + R_2 \Rightarrow R_2$ mean when one is performing the Gauss-Jordan method?

57. What does the notation $4R_2 + R_3 \Rightarrow R_3$ mean when one is performing the Gauss-Jordan method?

Graphing Calculator Exercises

For Exercises 58–63, use the matrix features on a graphing calculator to express each augmented matrix in reduced row echelon form. Compare your results to the solution you obtained in the indicated exercise.

58. $\begin{bmatrix} 1 & -2 & | & -1 \\ 2 & 1 & | & -7 \end{bmatrix}$ **59.** $\begin{bmatrix} 1 & -3 & | & 3 \\ 2 & -5 & | & 4 \end{bmatrix}$ **60.** $\begin{bmatrix} 1 & 3 & | & 6 \\ -4 & -9 & | & 3 \end{bmatrix}$

Compare with Exercise 34.   Compare with Exercise 35.   Compare with Exercise 36.

61. $\begin{bmatrix} 2 & -3 & | & -2 \\ 1 & 2 & | & 13 \end{bmatrix}$ **62.** $\begin{bmatrix} 1 & 1 & 1 & | & 6 \\ 1 & -1 & 1 & | & 2 \\ 1 & 1 & -1 & | & 0 \end{bmatrix}$ **63.** $\begin{bmatrix} 2 & -3 & -2 & | & 11 \\ 1 & 3 & 8 & | & 1 \\ 3 & -1 & 14 & | & -2 \end{bmatrix}$

Compare with Exercise 37.   Compare with Exercise 46.   Compare with Exercise 47.

Section 3.7 Determinants and Cramer's Rule

1. Introduction to Determinants

Associated with every square matrix is a real number called the **determinant** of the matrix. A determinant of a square matrix $\mathbf{A}$, denoted $\det \mathbf{A}$, is written by enclosing the elements of the matrix within two vertical bars. For example,

$$\text{If} \quad \mathbf{A} = \begin{bmatrix} 2 & -1 \\ 6 & 0 \end{bmatrix} \quad \text{then} \quad \det \mathbf{A} = \begin{vmatrix} 2 & -1 \\ 6 & 0 \end{vmatrix}$$

$$\text{If} \quad \mathbf{B} = \begin{bmatrix} 0 & -5 & 1 \\ 4 & 0 & \frac{1}{2} \\ -2 & 10 & 1 \end{bmatrix} \quad \text{then} \quad \det \mathbf{B} = \begin{vmatrix} 0 & -5 & 1 \\ 4 & 0 & \frac{1}{2} \\ -2 & 10 & 1 \end{vmatrix}$$

Determinants have many applications in mathematics, including solving systems of linear equations, finding the area of a triangle, determining whether three points are collinear, and finding an equation of a line between two points.

The determinant of a 2×2 matrix is defined as follows:

> **Determinant of a 2 × 2 Matrix**
>
> The determinant of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the real number $ad - bc$. It is written as
>
> $$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example 1 Evaluating a 2 × 2 Determinant

Evaluate the determinants.

a. $\begin{vmatrix} 6 & -2 \\ 5 & \frac{1}{3} \end{vmatrix}$ **b.** $\begin{vmatrix} 2 & -11 \\ 0 & 0 \end{vmatrix}$

Solution:

a. $\begin{vmatrix} 6 & -2 \\ 5 & \frac{1}{3} \end{vmatrix}$ For this determinant, $a = 6$, $b = -2$, $c = 5$, and $d = \frac{1}{3}$.

$$ad - bc = (6)\left(\frac{1}{3}\right) - (-2)(5)$$

$$= 2 + 10$$

$$= 12$$

b. $\begin{vmatrix} 2 & -11 \\ 0 & 0 \end{vmatrix}$ For this determinant, $a = 2$, $b = -11$, $c = 0$, $d = 0$.

$$ad - bc = (2)(0) - (-11)(0)$$

$$= 0 - 0$$

$$= 0$$

TIP: Example 1(b) illustrates that the value of a determinant having a row of all zeros is 0. The same is true for a determinant having a column of all zeros.

> **Skill Practice** Evaluate the determinants.
>
> **1a.** $\begin{vmatrix} 2 & 8 \\ -1 & 5 \end{vmatrix}$ **b.** $\begin{vmatrix} -6 & 0 \\ 4 & 0 \end{vmatrix}$

2. Determinant of a 3 × 3 Matrix

To find the determinant of a 3×3 matrix, we first need to define the **minor** of an element of the matrix. For any element of a 3×3 matrix, the minor of that element is the determinant of the 2×2 matrix obtained by deleting the row and column in which the element resides. For example, consider the matrix

$$\begin{bmatrix} 5 & -1 & 6 \\ 0 & -7 & 1 \\ 4 & 2 & 6 \end{bmatrix}$$

The minor of the element 5 is found by deleting the first row and first column and then evaluating the determinant of the remaining 2×2 matrix:

$$\begin{bmatrix} 5 & -1 & 6 \\ 0 & -7 & 1 \\ 4 & 2 & 6 \end{bmatrix}$$ Now evaluate the determinant: $\begin{vmatrix} -7 & 1 \\ 2 & 6 \end{vmatrix} = (-7)(6) - (1)(2)$

$$= -44$$

For this matrix, the minor for the element 5 is -44.

To find the minor of the element -7, delete the second row and second column, and then evaluate the determinant of the remaining 2×2 matrix.

$$\begin{bmatrix} 5 & -1 & 6 \\ 0 & -7 & 1 \\ 4 & 2 & 6 \end{bmatrix}$$ Now evaluate the determinant: $\begin{vmatrix} 5 & 6 \\ 4 & 6 \end{vmatrix} = (5)(6) - (6)(4) = 6$

For this matrix, the minor for the element -7 is 6.

Example 2 Determining the Minor for Elements in a 3 × 3 Matrix

Find the minor for each element in the first column of the matrix.

$$\begin{bmatrix} 3 & 4 & -1 \\ 2 & -4 & 5 \\ 0 & 1 & -6 \end{bmatrix}$$

Solution:

For 3: $\begin{bmatrix} 3 & 4 & -1 \\ 2 & -4 & 5 \\ 0 & 1 & -6 \end{bmatrix}$ The minor is: $\begin{vmatrix} -4 & 5 \\ 1 & -6 \end{vmatrix} = (-4)(-6) - (5)(1) = 19$

Skill Practice Answers
1a. 18   **b.** 0

For 2: $\begin{bmatrix} 3 & 4 & -1 \\ 2 & -4 & 5 \\ 0 & 1 & -6 \end{bmatrix}$ The minor is: $\begin{vmatrix} 4 & -1 \\ 1 & -6 \end{vmatrix} = (4)(-6) - (-1)(1) = -23$

For 0: $\begin{bmatrix} 3 & 4 & -1 \\ 2 & -4 & 5 \\ 0 & 1 & -6 \end{bmatrix}$ The minor is: $\begin{vmatrix} 4 & -1 \\ -4 & 5 \end{vmatrix} = (4)(5) - (-1)(-4) = 16$

Skill Practice

2. Find the minor for the element 3.

$$\begin{bmatrix} -1 & 8 & -6 \\ \frac{1}{2} & 3 & 2 \\ 5 & 7 & 4 \end{bmatrix}$$

The determinant of a 3×3 matrix is defined as follows.

Definition of a Determinant of a 3 × 3 Matrix

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \cdot \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \cdot \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \cdot \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

From this definition, we see that the determinant of a 3×3 matrix can be written as

$$a_1 \cdot (\text{minor of } a_1) - a_2 \cdot (\text{minor of } a_2) + a_3 \cdot (\text{minor of } a_3)$$

Evaluating determinants in this way is called *expanding minors*.

Example 3   Evaluating a 3 × 3 Determinant

Evaluate the determinant. $\begin{vmatrix} 2 & 4 & 2 \\ 1 & -3 & 0 \\ -5 & 5 & -1 \end{vmatrix}$

Solution:

$$\begin{vmatrix} 2 & 4 & 2 \\ 1 & -3 & 0 \\ -5 & 5 & -1 \end{vmatrix} = 2 \cdot \begin{vmatrix} -3 & 0 \\ 5 & -1 \end{vmatrix} - (1) \cdot \begin{vmatrix} 4 & 2 \\ 5 & -1 \end{vmatrix} + (-5) \cdot \begin{vmatrix} 4 & 2 \\ -3 & 0 \end{vmatrix}$$

$$= 2[(-3)(-1) - (0)(5)] - 1[(4)(-1) - (2)(5)] - 5[(4)(0) - (2)(-3)]$$

$$= 2(3) - 1(-14) - 5(6)$$

$$= 6 + 14 - 30$$

$$= -10$$

Skill Practice Answers

2. $\begin{vmatrix} -1 & -6 \\ 5 & 4 \end{vmatrix} = 26$

Skill Practice

3. Evaluate the determinant.

$$\begin{vmatrix} -2 & 4 & 9 \\ 5 & -1 & 2 \\ 1 & 1 & 6 \end{vmatrix}$$

Although we defined the determinant of a matrix by expanding the minors of the elements in the first column, *any row or column may be used.* However, we must choose the correct sign to apply to each term in the expansion. The following array of signs is helpful.

$$\begin{array}{ccc} + & - & + \\ - & + & - \\ + & - & + \end{array}$$

The signs alternate for each row and column, beginning with + in the first row, first column.

> **TIP:** There is another method to determine the signs for each term of the expansion. For the a_{ij} element, multiply the term by $(-1)^{i+j}$.

Example 4 Evaluating a 3 × 3 Determinant

Evaluate the determinant, by expanding minors about the elements in the second row.

$$\begin{vmatrix} 2 & 4 & 2 \\ 1 & -3 & 0 \\ -5 & 5 & -1 \end{vmatrix}$$

Solution:

Signs obtained from the array of signs

$$\begin{vmatrix} 2 & 4 & 2 \\ 1 & -3 & 0 \\ -5 & 5 & -1 \end{vmatrix} = -(1) \cdot \begin{vmatrix} 4 & 2 \\ 5 & -1 \end{vmatrix} + (-3) \cdot \begin{vmatrix} 2 & 2 \\ -5 & -1 \end{vmatrix} - (0) \cdot \begin{vmatrix} 2 & 4 \\ -5 & 5 \end{vmatrix}$$

$$= -1[(4)(-1) - (2)(5)] - 3[(2)(-1) - (2)(-5)] - 0$$

$$= -1(-14) - 3(8)$$

$$= 14 - 24$$

$$= -10$$

> **TIP:** Notice that the value of the determinant obtained in Examples 3 and 4 is the same.

Skill Practice

4. Evaluate the determinant.

$$\begin{vmatrix} 4 & -1 & 2 \\ 3 & 6 & -8 \\ 0 & \frac{1}{2} & 5 \end{vmatrix}$$

In Example 4, the third term in the expansion of minors was zero because the element 0 when multiplied by its minor is zero. To simplify the arithmetic in evaluating a determinant of a 3 × 3 matrix, expand about the row or column that has the most 0 elements.

Skill Practice Answers

3. -42
4. 154

Calculator Connections

The determinant of a matrix can be evaluated on a graphing calculator. First use the matrix editor to enter the elements of the matrix. Then use a *det* function to evaluate the determinant. The determinant from Examples 3 and 4 is evaluated below.

```
MATRIX[A] 3 ×3
[ 2    4    2  ]
[ 1   -3    0  ]
[ -5   5   -1  ]
```

```
det([A])
                -10
```

---

3. Cramer's Rule

In Sections 3.2, 3.3, and 3.6, we learned three methods to solve a system of linear equations: the substitution method, the addition method, and the Gauss-Jordan method. In this section, we will learn another method called **Cramer's rule** to solve a system of linear equations.

Cramer's Rule for a 2 × 2 System of Linear Equations

The solution to the system
$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

is given by
$$x = \frac{\mathbf{D}_x}{\mathbf{D}} \quad \text{and} \quad y = \frac{\mathbf{D}_y}{\mathbf{D}}$$

where $\mathbf{D} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ (and $\mathbf{D} \neq 0$) $\mathbf{D}_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ $\mathbf{D}_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

Example 5   Using Cramer's Rule to Solve a 2 × 2 System of Linear Equations

Solve the system by using Cramer's rule.

$$3x - 5y = 11$$
$$-x + 3y = -5$$

Solution:

For this system: $a_1 = 3$ $b_1 = -5$ $c_1 = 11$

$a_2 = -1$ $b_2 = 3$ $c_2 = -5$

$$\mathbf{D} = \begin{vmatrix} 3 & -5 \\ -1 & 3 \end{vmatrix} = (3)(3) - (-5)(-1) = 9 - 5 = 4$$

$$\mathbf{D}_x = \begin{vmatrix} 11 & -5 \\ -5 & 3 \end{vmatrix} = (11)(3) - (-5)(-5) = 33 - 25 = 8$$

$$\mathbf{D}_y = \begin{vmatrix} 3 & 11 \\ -1 & -5 \end{vmatrix} = (3)(-5) - (11)(-1) = -15 + 11 = -4$$

Therefore, $\quad x = \dfrac{\mathbf{D}_x}{\mathbf{D}} = \dfrac{8}{4} = 2 \qquad y = \dfrac{\mathbf{D}_y}{\mathbf{D}} = \dfrac{-4}{4} = -1$

The solution is $(2, -1)$. <u>Check</u>: $\quad 3x - 5y = 11 \longrightarrow 3(2) - 5(-1) = 11 \checkmark$

$\qquad\qquad\qquad\qquad\qquad\qquad -x + 3y = -5 \longrightarrow -(2) + 3(-1) = -5 \checkmark$

Skill Practice

5. Solve using Cramer's rule.

$$2x + y = 5$$
$$-x - 3y = 5$$

TIP: Here are some memory tips to help you remember Cramer's rule.

Coefficients of
x-terms *y*-terms
↓   ↓

1. The determinant **D** is the determinant of the coefficients of *x* and *y*.

$$\mathbf{D} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

x-coefficients
replaced by c_1 and c_2
↓

2. The determinant $\mathbf{D}_x$ has the column of *x*-term coefficients replaced by c_1 and c_2.

$$\mathbf{D}_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$$

y-coefficients
replaced by c_1 and c_2
↓

3. The determinant $\mathbf{D}_y$ has the column of *y*-term coefficients replaced by c_1 and c_2.

$$\mathbf{D}_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

It is important to note that the linear equations must be written in standard form to apply Cramer's rule.

Example 6 Using Cramer's Rule to Solve a 2 × 2 System of Linear Equations

Solve the system by using Cramer's rule.

$$-16y = -40x - 7$$
$$40y = 24x + 27$$

Skill Practice Answers

5. $(4, -3)$

Solution:

$$-16y = -40x - 7 \quad\longrightarrow\quad 40x - 16y = -7 \qquad \text{Rewrite each equation}$$
$$40y = 24x + 27 \quad\longrightarrow\quad -24x + 40y = 27 \qquad \text{in standard form.}$$

For this system:

$$a_1 = 40 \qquad b_1 = -16 \qquad c_1 = -7$$
$$a_2 = -24 \qquad b_2 = 40 \qquad c_2 = 27$$

$$\mathbf{D} = \begin{vmatrix} 40 & -16 \\ -24 & 40 \end{vmatrix} = (40)(40) - (-16)(-24) = 1216$$

$$\mathbf{D}_x = \begin{vmatrix} -7 & -16 \\ 27 & 40 \end{vmatrix} = (-7)(40) - (-16)(27) = 152$$

$$\mathbf{D}_y = \begin{vmatrix} 40 & -7 \\ -24 & 27 \end{vmatrix} = (40)(27) - (-7)(-24) = 912$$

Therefore, $x = \dfrac{\mathbf{D}_x}{\mathbf{D}} = \dfrac{152}{1216} = \dfrac{1}{8}$ $y = \dfrac{\mathbf{D}_y}{\mathbf{D}} = \dfrac{912}{1216} = \dfrac{3}{4}$

The solution $\left(\frac{1}{8}, \frac{3}{4}\right)$ checks in the original equations.

> **Skill Practice**

6. Solve using Cramer's rule.

$$9x - 12y = -8$$
$$18x + 30y = -7$$

Cramer's rule can be used to solve a 3×3 system of linear equations by using a similar pattern of determinants.

Cramer's Rule for a 3 × 3 System of Linear Equations

The solution to the system

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

is given by

$$x = \frac{\mathbf{D}_x}{\mathbf{D}} \qquad y = \frac{\mathbf{D}_y}{\mathbf{D}} \qquad \text{and} \qquad z = \frac{\mathbf{D}_z}{\mathbf{D}}$$

where $\mathbf{D} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ (and $\mathbf{D} \neq 0$) $\mathbf{D}_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$

$\mathbf{D}_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ $\mathbf{D}_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

Skill Practice Answers

6. $\left(-\dfrac{2}{3}, \dfrac{1}{6}\right)$

Example 7 Using Cramer's Rule to Solve a 3 × 3 System of Linear Equations

Solve the system by using Cramer's rule.

$$x - 2y + 4z = 3$$
$$x - 4y + 3z = -5$$
$$x + 3y - 2z = 6$$

Solution:

$$D = \begin{vmatrix} 1 & -2 & 4 \\ 1 & -4 & 3 \\ 1 & 3 & -2 \end{vmatrix} = 1 \cdot \begin{vmatrix} -4 & 3 \\ 3 & -2 \end{vmatrix} - 1 \cdot \begin{vmatrix} -2 & 4 \\ 3 & -2 \end{vmatrix} + 1 \cdot \begin{vmatrix} -2 & 4 \\ -4 & 3 \end{vmatrix}$$

$$= 1(-1) - 1(-8) + 1(10)$$

$$= 17$$

$$D_x = \begin{vmatrix} 3 & -2 & 4 \\ -5 & -4 & 3 \\ 6 & 3 & -2 \end{vmatrix} = 3 \cdot \begin{vmatrix} -4 & 3 \\ 3 & -2 \end{vmatrix} - (-5) \cdot \begin{vmatrix} -2 & 4 \\ 3 & -2 \end{vmatrix} + 6 \cdot \begin{vmatrix} -2 & 4 \\ -4 & 3 \end{vmatrix}$$

$$= 3(-1) + 5(-8) + 6(10)$$

$$= 17$$

> **TIP:** In Example 7, we expanded the determinants about the first column.

$$D_y = \begin{vmatrix} 1 & 3 & 4 \\ 1 & -5 & 3 \\ 1 & 6 & -2 \end{vmatrix} = 1 \cdot \begin{vmatrix} -5 & 3 \\ 6 & -2 \end{vmatrix} - 1 \cdot \begin{vmatrix} 3 & 4 \\ 6 & -2 \end{vmatrix} + 1 \cdot \begin{vmatrix} 3 & 4 \\ -5 & 3 \end{vmatrix}$$

$$= 1(-8) - 1(-30) + 1(29)$$

$$= 51$$

$$D_z = \begin{vmatrix} 1 & -2 & 3 \\ 1 & -4 & -5 \\ 1 & 3 & 6 \end{vmatrix} = 1 \cdot \begin{vmatrix} -4 & -5 \\ 3 & 6 \end{vmatrix} - 1 \cdot \begin{vmatrix} -2 & 3 \\ 3 & 6 \end{vmatrix} + 1 \cdot \begin{vmatrix} -2 & 3 \\ -4 & -5 \end{vmatrix}$$

$$= 1(-9) - 1(-21) + 1(22)$$

$$= 34$$

Hence

$$x = \frac{D_x}{D} = \frac{17}{17} = 1 \qquad y = \frac{D_y}{D} = \frac{51}{17} = 3 \qquad \text{and} \qquad z = \frac{D_z}{D} = \frac{34}{17} = 2$$

The solution is $(1, 3, 2)$.

$$\underline{\text{Check:}} \quad x - 2y + 4z = 3 \qquad (1) - 2(3) + 4(2) = 3 \quad ✔$$
$$x - 4y + 3z = -5 \qquad (1) - 4(3) + 3(2) = -5 \quad ✔$$
$$x + 3y - 2z = 6 \qquad (1) + 3(3) - 2(2) = 6 \quad ✔$$

Skill Practice

7. Solve using Cramer's rule.

$$x + 3y - 3z = -14$$
$$x - 4y + z = 2$$
$$x + y + 2z = 6$$

Skill Practice Answers

7. $(-2, 0, 4)$

Cramer's rule may seem cumbersome for solving a 3×3 system of linear equations. However, it provides convenient formulas that can be programmed into a computer or calculator to solve for x, y, and z. Cramer's rule can also be extended to solve a 4×4 system of linear equations, a 5×5 system of linear equations, and in general an $n \times n$ system of linear equations.

It is important to remember that Cramer's rule does not apply if $\mathbf{D} = 0$. In such a case, the system of equations is either dependent or inconsistent, and another method must be used to analyze the system.

Example 8 Analyzing a Dependent System of Equations

Solve the system. Use Cramer's rule if possible.

$$2x - 3y = 6$$
$$-6x + 9y = -18$$

Solution:

$$\mathbf{D} = \begin{vmatrix} 2 & -3 \\ -6 & 9 \end{vmatrix} = (2)(9) - (-3)(-6) = 18 - 18 = 0$$

Because $\mathbf{D} = 0$, Cramer's rule does not apply. Using the addition method to solve the system, we have

$$\begin{array}{ccc} & \text{Multiply by 3.} & \\ 2x - 3y = 6 & \longrightarrow & 6x - 9y = 18 \\ -6x + 9y = -18 & \longrightarrow & \underline{-6x + 9y = -18} \\ & & 0 = 0 \quad \text{The system is dependent.} \end{array}$$

The solution is $\{(x, y) \mid 2x - 3y = 6\}$.

Skill Practice

8. Solve. Use Cramer's rule if possible.

$$x - 6y = 2$$
$$2x - 12y = -2$$

Skill Practice Answers

8. No solution; Inconsistent system

Section 3.7 Practice Exercises

Study Skills Exercise

1. Define the key terms.

    a. Determinant   **b. Minor**   **c. Cramer's rule**

Concept 1: Introduction to Determinants

For Exercises 2–7, evaluate the determinant of the 2 × 2 matrix.

2. $\begin{vmatrix} -3 & 1 \\ 5 & 2 \end{vmatrix}$

3. $\begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix}$

4. $\begin{vmatrix} -2 & 2 \\ -3 & -5 \end{vmatrix}$

5. $\begin{vmatrix} 5 & -1 \\ 1 & 0 \end{vmatrix}$

6. $\begin{vmatrix} \frac{1}{2} & 3 \\ -2 & 4 \end{vmatrix}$

7. $\begin{vmatrix} -3 & \frac{1}{4} \\ 8 & -2 \end{vmatrix}$

Concept 2: Determinant of a 3 × 3 Matrix

For Exercises 8–11, evaluate the minor corresponding to the given element from matrix **A**.

$$\mathbf{A} = \begin{bmatrix} 4 & -1 & 8 \\ 2 & 6 & 0 \\ -7 & 5 & 3 \end{bmatrix}$$

8. 4   **9.** −1   **10.** 2   **11.** 3

For Exercises 12–15, evaluate the minor corresponding to the given element from matrix **B**.

$$\mathbf{B} = \begin{bmatrix} -2 & 6 & 0 \\ 4 & -2 & 1 \\ 5 & 9 & -1 \end{bmatrix}$$

12. 6   **13.** 5   **14.** 1   **15.** 0

16. Construct the sign array for a 3 × 3 matrix.

17. Evaluate the determinant of matrix **B**, using expansion by minors.

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 2 \\ 3 & -1 & 2 \\ 3 & 2 & -2 \end{bmatrix}$$

    a. About the first column

    b. About the second row

18. Evaluate the determinant of matrix **C**, using expansion by minors.

$$\mathbf{C} = \begin{bmatrix} 4 & 1 & 3 \\ 2 & -2 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

    a. About the first row

    b. About the second column

19. When evaluating the determinant of a 3 × 3 matrix, explain the advantage of being able to choose any row or column about which to expand minors.

For Exercises 20–25, evaluate the determinants.

20. $\begin{vmatrix} 8 & 2 & -4 \\ 4 & 0 & 2 \\ 3 & 0 & -1 \end{vmatrix}$

21. $\begin{vmatrix} 5 & 2 & 1 \\ 3 & -6 & 0 \\ -2 & 8 & 0 \end{vmatrix}$

22. $\begin{vmatrix} -2 & 1 & 3 \\ 1 & 4 & 4 \\ 1 & 0 & 2 \end{vmatrix}$

23. $\begin{vmatrix} 3 & 2 & 1 \\ 1 & -1 & 2 \\ 1 & 0 & 4 \end{vmatrix}$

24. $\begin{vmatrix} -5 & 4 & 2 \\ 0 & 0 & 0 \\ 3 & -1 & 5 \end{vmatrix}$

25. $\begin{vmatrix} 0 & 5 & -8 \\ 0 & -4 & 1 \\ 0 & 3 & 6 \end{vmatrix}$

For Exercises 26–31, evaluate the determinants.

26. $\begin{vmatrix} x & 3 \\ y & -2 \end{vmatrix}$

27. $\begin{vmatrix} a & 2 \\ b & 8 \end{vmatrix}$

28. $\begin{vmatrix} a & 5 & -1 \\ b & -3 & 0 \\ c & 3 & 4 \end{vmatrix}$

29. $\begin{vmatrix} x & 0 & 3 \\ y & -2 & 6 \\ z & -1 & 1 \end{vmatrix}$

30. $\begin{vmatrix} p & 0 & q \\ r & 0 & s \\ t & 0 & u \end{vmatrix}$

31. $\begin{vmatrix} f & e & 0 \\ d & c & 0 \\ b & a & 0 \end{vmatrix}$

Concept 3: Cramer's Rule

For Exercises 32–34, evaluate the determinants represented by **D**, **D**$_x$, and **D**$_y$.

32. $x - 4y = 2$
$3x + 2y = 1$

33. $4x + 6y = 9$
$-2x + y = 12$

34. $-3x + 8y = -10$
$5x + 5y = -13$

For Exercises 35–40, solve the system by using Cramer's rule.

35. $2x + y = 3$
$x - 4y = 6$

36. $2x - y = -1$
$3x + y = 6$

37. $x - 4y = 8$
$3x + 7y = 5$

38. $7x + 3y = 4$
$5x - 4y = 9$

39. $4x - 3y = 5$
$2x + 5y = 7$

40. $2x + 3y = 4$
$6x - 12y = -5$

41. When does Cramer's rule not apply in solving a system of equations?

42. How can a system be solved if Cramer's rule does not apply?

For Exercises 43–48, solve the system of equations by using Cramer's rule, if possible. If not possible, use another method.

43. $4x - 2y = 3$
$-2x + y = 1$

44. $6x - 6y = 5$
$x - y = 8$

45. $4x + y = 0$
$x - 7y = 0$

46. $-3x - 2y = 0$
$-x + 5y = 0$

47. $x + 5y = 3$
$2x + 10y = 6$

48. $-2x - 10y = -4$
$x + 5y = 2$

For Exercises 49–54, solve for the indicated variable by using Cramer's rule.

49. $2x - y + 3z = 9$
$x + 4y + 4z = 5$ for x
$3x + 2y + 2z = 5$

50. $x + 2y + 3z = 8$
$2x - 3y + z = 5$ for y
$3x - 4y + 2z = 9$

51. $3x - 2y + 2z = 5$
$6x + 3y - 4z = -1$ for z
$3x - y + 2z = 4$

52. $4x + 4y - 3z = 3$
$8x + 2y + 3y = 0$ for x
$4x - 4y + 6z = -3$

53. $5x + 6z = 5$
$-2x + y = -6$ for y
$3y - z = 3$

54. $8x + y = 1$
$7y + z = 0$ for y
$x - 3z = -2$

For Exercises 55–58, solve the system by using Cramer's rule, if possible.

55. $\begin{aligned} x &= 3 \\ -x + 3y &= 3 \\ y + 2z &= 4 \end{aligned}$

56. $\begin{aligned} 4x \quad\; + z &= 7 \\ y \quad\;\; &= 2 \\ x \quad\;\; + z &= 4 \end{aligned}$

57. $\begin{aligned} x + y + \;\; 8z &= 3 \\ 2x + y + 11z &= 4 \\ x \quad\; + \;\; 3z &= 0 \end{aligned}$

58. $\begin{aligned} -8x + y + z &= 6 \\ 2x - y + z &= 3 \\ 3x \quad\; - z &= 0 \end{aligned}$

Expanding Your Skills

For Exercises 59–62, solve the equation.

59. $\begin{vmatrix} 6 & x \\ 2 & -4 \end{vmatrix} = 14$

60. $\begin{vmatrix} y & -2 \\ 8 & 7 \end{vmatrix} = 30$

61. $\begin{vmatrix} 3 & 1 & 0 \\ 0 & 4 & -2 \\ 1 & 0 & w \end{vmatrix} = 10$

62. $\begin{vmatrix} -1 & 0 & 2 \\ 4 & t & 0 \\ 0 & -5 & 3 \end{vmatrix} = -4$

For Exercises 63–64, evaluate the determinant by using expansion by minors about the first column.

63. $\begin{vmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 4 \\ -2 & 0 & 0 & 1 \\ 4 & -1 & -2 & 0 \end{vmatrix}$

64. $\begin{vmatrix} 5 & 2 & 0 & 0 \\ 0 & 4 & -1 & 1 \\ -1 & 0 & 3 & 0 \\ 0 & -2 & 1 & 0 \end{vmatrix}$

For Exercises 65–66, refer to the following system of four variables.

$$\begin{aligned} x + y + z + w &= 0 \\ 2x \quad\;\; - z + w &= 5 \\ 2x + y \quad\;\; - w &= 0 \\ y + z \quad\;\; &= -1 \end{aligned}$$

65. a. Evaluate the determinant **D**.

    b. Evaluate the determinant $\mathbf{D}_x$.

    c. Solve for x by computing $\dfrac{\mathbf{D}_x}{\mathbf{D}}$.

66. a. Evaluate the determinant $\mathbf{D}_y$.

    b. Solve for y by computing $\dfrac{\mathbf{D}_y}{\mathbf{D}}$.

Chapter 3   SUMMARY

Section 3.1   Solving Systems of Equations by Graphing

Key Concepts

A **system of linear equations** in two variables can be solved by graphing.

A **solution to a system of linear equations** is an ordered pair that satisfies each equation in the system. Graphically, this represents a point of intersection of the lines.

There may be one solution, infinitely many solutions, or no solution.

One solution   Infinitely many   No solution
Consistent   solutions   Inconsistent
Independent   Consistent   Independent
                      Dependent

A system of equations is **consistent** if there is at least one solution. A system is **inconsistent** if there is no solution.

A linear system in x and y is **dependent** if two equations represent the same line. The solution set is the set of all points on the line.

If two linear equations represent different lines, then the system of equations is **independent**.
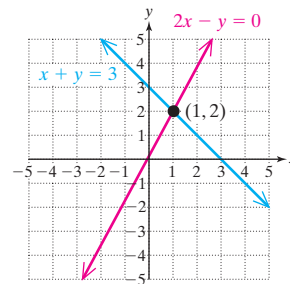
Examples

Example 1

Solve by graphing. $x + y = 3$
$$2x - y = 0$$

Write each equation in $y = mx + b$ form to graph.

$y = -x + 3$

$y = 2x$

The solution is the point of intersection $(1, 2)$.

Section 3.2 Solving Systems of Equations by Using the Substitution Method

Key Concepts

Substitution Method

1. Isolate one of the variables.
2. Substitute the quantity found in step 1 into the *other* equation.
3. Solve the resulting equation.
4. Substitute the value from step 3 back into the equation from step 1 to solve for the remaining variable.
5. Check the ordered pair in both equations, and write the answer as an ordered pair.

Examples

Example 1

$2y = -6x + 14$

$2x + y = 5 \xrightarrow{\text{Isolate a variable.}} y = -2x + 5$

Substitute

$2(-2x + 5) = -6x + 14$

$-4x + 10 = -6x + 14$

$2x + 10 = 14$

$2x = 4$

$x = 2$

$y = -2x + 5$ Now solve for y.

$y = -2(2) + 5$

$y = 1$

The solution is (2, 1) and checks in both equations.

Section 3.3 Solving Systems of Equations by Using the Addition Method

Key Concepts

Examples

Addition Method

1. Write both equations in standard form $Ax + By = C$.
2. Clear fractions or decimals (optional).
3. Multiply one or both equations by nonzero constants to create opposite coefficients for one of the variables.
4. Add the equations from step 3 to eliminate one variable.
5. Solve for the remaining variable.
6. Substitute the known value from step 5 back into one of the original equations to solve for the other variable.
7. Check the ordered pair in both equations.

A system is **consistent** if there is at least one solution. A system is **inconsistent** if there is no solution. An inconsistent system is detected by a contradiction (such as $0 = 5$).

A system is **independent** if the two equations represent different lines. A system is **dependent** if the two equations represent the same line. This produces infinitely many solutions. A dependent system is detected by an identity (such as $0 = 0$).

Example 1

$$3x - 4y = 18 \xrightarrow{\text{Mult. by 3.}} 9x - 12y = 54$$
$$-5x - 3y = -1 \xrightarrow{\text{Mult. by }-4.} \underline{20x + 12y = 4}$$
$$29x = 58$$
$$x = 2$$

$$3(2) - 4y = 18$$
$$6 - 4y = 18$$
$$-4y = 12$$
$$y = -3$$

The solution is $(2, -3)$ and checks in both equations.

Example 2

$$2x + y = 3 \xrightarrow{\text{Mult. by 2.}} 4x + 2y = 6$$
$$-4x - 2y = 1 \xrightarrow{\phantom{\text{Mult. by 2.}}} \underline{-4x - 2y = 1}$$
$$0 = 7$$
$$\text{Contradiction.}$$

There is no solution. The system is inconsistent.

Example 3

$$x + 3y = 1 \xrightarrow{\text{Mult. by }-2.} -2x - 6y = -2$$
$$2x + 6y = 2 \xrightarrow{\phantom{\text{Mult. by }-2.}} \underline{2x + 6y = 2}$$
$$0 = 0 \quad \text{Identity.}$$

There are infinitely many solutions. The system is dependent.

| Section 3.4 | **Applications of Systems of Linear Equations in Two Variables** |

Key Concepts

Solve application problems by using systems of linear equations in two variables.

- Cost applications
- Mixture applications
- Applications involving principal and interest
- Applications involving distance, rate, and time
- Geometry applications

Steps to Solve Applications:

1. Label two variables.
2. Construct two equations in words.
3. Write two equations.
4. Solve the system.
5. Write the answer in words.

Examples

Example 1

Mercedes invested $1500 more in a certificate of deposit that pays 6.5% simple interest than she did in a savings account that pays 4% simple interest. If her total interest at the end of 1 year is $622.50, find the amount she invested in the 6.5% account.

Let x represent the amount of money invested at 6.5%. Let y represent the amount of money invested at 4%.

$$\begin{pmatrix} \text{Amount invested} \\ \text{at } 6.5\% \end{pmatrix} = \begin{pmatrix} \text{amount invested} \\ \text{at } 4\% \end{pmatrix} + \$1500$$

$$\begin{pmatrix} \text{Interest earned} \\ \text{from } 6.5\% \\ \text{account} \end{pmatrix} + \begin{pmatrix} \text{interest earned} \\ \text{from } 4\% \\ \text{account} \end{pmatrix} = \$622.50$$

$$x = y + 1500$$

$$0.065x + 0.04y = 622.50$$

Using substitution gives

$$0.065(y + 1500) + 0.04y = 622.50$$

$$0.065y + 97.5 + 0.04y = 622.50$$

$$0.105y = 525$$

$$y = 5000$$

$$x = 5000 + 1500 = 6500$$

Mercedes invested $6500 at 6.5% and $5000 at 4%.

| Section 3.5 | **Systems of Linear Equations in Three Variables and Applications** |

Key Concepts

A **linear equation in three variables** can be written in the form $Ax + By + Cz = D$, where A, B, and C are not all zero. The graph of a linear equation in three variables is a plane in space.

    A solution to a system of linear equations in three variables is an **ordered triple** that satisfies each equation. Graphically, a solution is a point of intersection among three planes.

    A system of linear equations in three variables may have one unique solution, infinitely many solutions (dependent system), or no solution (inconsistent system).

Examples

Example 1

$\boxed{A}$ $x + 2y - z = 4$

$\boxed{B}$ $3x - y + z = 5$

$\boxed{C}$ $2x + 3y + 2z = 7$

$\boxed{A}$ and $\boxed{B}$ $x + 2y - z = 4$

$\phantom{\boxed{A} \text{ and } \boxed{B} xxx} 3x - y + z = 5$

$\phantom{\boxed{A} \text{ and } \boxed{B} xx} \underline{4x + y = 9} \; \boxed{D}$

$2 \cdot \boxed{A}$ and $\boxed{C}$ $2x + 4y - 2z = 8$

$\phantom{2 \cdot \boxed{A} \text{ and } \boxed{C} x} 2x + 3y + 2z = 7$

$\phantom{2 \cdot \boxed{A} \text{ and } \boxed{C} xx} \underline{4x + 7y = 15} \; \boxed{E}$

$\boxed{D}$ $4x + y = 9 \;\rightarrow\; -4x - y = -9$

$\boxed{E}$ $4x + 7y = 15 \;\rightarrow\; \underline{4x + 7y = 15}$

$\phantom{\boxed{E} xxxxxxxxxxxxxxxxxxxxx} 6y = 6$

$\phantom{\boxed{E} xxxxxxxxxxxxxxxxxxxxxxxx} y = 1$

Substitute $y = 1$ into either equation $\boxed{D}$ or $\boxed{E}$.

$\boxed{D}$ $4x + (1) = 9$

$\phantom{\boxed{D} xxx} 4x = 8$

$\phantom{\boxed{D} xxxx} x = 2$

Substitute $x = 2$ and $y = 1$ into equation $\boxed{A}$, $\boxed{B}$, or $\boxed{C}$.

$\boxed{A}$ $(2) + 2(1) - z = 4$

$\phantom{\boxed{A} xxxxxxxxxx} z = 0$

The solution is $(2, 1, 0)$.

| Section 3.6 | Solving Systems of Linear Equations by Using Matrices |

Key Concepts

A **matrix** is a rectangular array of numbers displayed in rows and columns. Every number or entry within a matrix is called an element of the matrix.

The **order of a matrix** is determined by the number of rows and number of columns. A matrix with m rows and n columns is an $m \times n$ matrix.

A system of equations written in standard form can be represented by an **augmented matrix** consisting of the coefficients of the terms of each equation in the system.

The Gauss-Jordan method can be used to solve a system of equations by using the following elementary row operations on an augmented matrix.

1. Interchange two rows.
2. Multiply every element in a row by a nonzero real number.
3. Add a multiple of one row to another row.

These operations are used to write the matrix in **reduced row echelon form**.

$$\begin{bmatrix} 1 & 0 & | & a \\ 0 & 1 & | & b \end{bmatrix}$$

which represents the solution, $x = a$ and $y = b$.

Examples

Example 1

$[1 \quad 2 \quad 5]$ is a 1×3 matrix (called a row matrix).

$\begin{bmatrix} -1 & 8 \\ 1 & 5 \end{bmatrix}$ is a 2×2 matrix (called a square matrix).

$\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ is a 2×1 matrix (called a column matrix).

Example 2

The augmented matrix for

$$\begin{aligned} 4x + y &= -12 \\ x - 2y &= 6 \end{aligned}$$

is $\begin{bmatrix} 4 & 1 & | & -12 \\ 1 & -2 & | & 6 \end{bmatrix}$

Example 3

Solve the system from Example 2 by using the Gauss-Jordan method.

$R_1 \Leftrightarrow R_2$ $\quad\begin{bmatrix} 1 & -2 & | & 6 \\ 4 & 1 & | & -12 \end{bmatrix}$

$\underrightarrow{-4R_1 + R_2 \Rightarrow R_2}$ $\quad\begin{bmatrix} 1 & -2 & | & 6 \\ 0 & 9 & | & -36 \end{bmatrix}$

$\underrightarrow{\frac{1}{9}R_2 \Rightarrow R_2}$ $\quad\begin{bmatrix} 1 & -2 & | & 6 \\ 0 & 1 & | & -4 \end{bmatrix}$

$\underrightarrow{2R_2 + R_1 \Rightarrow R_1}$ $\quad\begin{bmatrix} 1 & 0 & | & -2 \\ 0 & 1 & | & -4 \end{bmatrix}$

Solution:

$x = -2 \qquad$ and $\qquad y = -4$

Section 3.7   Determinants and Cramer's Rule

Key Concepts

The **determinant** of matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

is denoted $\det\mathbf{A} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

The determinant of a 2×2 matrix is

defined as $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

The determinant of a 3×3 matrix is defined by

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \cdot \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \cdot \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$$

$$+ a_3 \cdot \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

Cramer's rule can be used to solve a 2×2 system of linear equations.

$a_1 x + b_1 y = c_1$ is given by $x = \dfrac{\mathbf{D}_x}{\mathbf{D}}$

$a_2 x + b_2 y = c_2$

$$\text{and } y = \frac{\mathbf{D}_y}{\mathbf{D}}$$

where $\mathbf{D} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ (and $\mathbf{D} \neq 0$),

$\mathbf{D}_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ and $\mathbf{D}_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$.

Examples

Example 1

For $\mathbf{A} = \begin{bmatrix} 7 & -2 \\ 3 & 2 \end{bmatrix}$,

$$\det\mathbf{A} = \begin{vmatrix} 7 & -2 \\ 3 & 2 \end{vmatrix} = 7(2) - (-2)(3)$$

$$= 14 - (-6)$$

$$= 20$$

Example 2

For $\mathbf{B} = \begin{bmatrix} 3 & -1 & 4 \\ 0 & 5 & 1 \\ 6 & -2 & -3 \end{bmatrix}$,

$$\det\mathbf{B} = \begin{vmatrix} 3 & -1 & 4 \\ 0 & 5 & 1 \\ 6 & -2 & -3 \end{vmatrix}$$

$$= 3 \cdot \begin{vmatrix} 5 & 1 \\ -2 & -3 \end{vmatrix} - 0 \cdot \begin{vmatrix} -1 & 4 \\ -2 & -3 \end{vmatrix} + 6 \cdot \begin{vmatrix} -1 & 4 \\ 5 & 1 \end{vmatrix}$$

$$= 3(-15 + 2) - 0(3 + 8) + 6(-1 - 20)$$

$$= -39 - 0 + (-126)$$

$$= -165$$

Example 3

Solve $2x + 8y = 0$

$\qquad\quad x + 3y = 1$

$$\mathbf{D} = \begin{vmatrix} 2 & 8 \\ 1 & 3 \end{vmatrix} = -2, \quad \mathbf{D}_x = \begin{vmatrix} 0 & 8 \\ 1 & 3 \end{vmatrix} = -8,$$

$$\mathbf{D}_y = \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = 2$$

Therefore, $x = \dfrac{-8}{-2} = 4, \ y = \dfrac{2}{-2} = -1$

Chapter 3   Review Exercises

Section 3.1

1. Determine if the ordered pair is a solution to the system.

$$-5x - 7y = 4$$
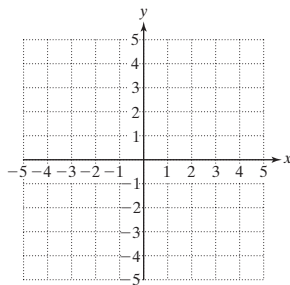
$$y = -\frac{1}{2}x - 1$$

a. $(2, 2)$ **b.** $(2, -2)$

For Exercises 2–4, answer true or false.

2. An inconsistent system has one solution.

3. Parallel lines form an inconsistent system.

4. Lines with different slopes intersect in one point.
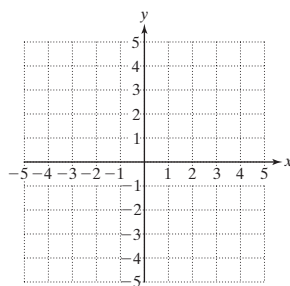
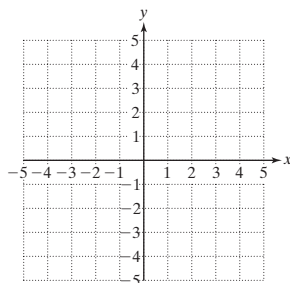For Exercises 5–7, solve the system by graphing.

5. $y = 2x + 7$

$y = -x - 5$

6. $6x + 2y = 4$

$3x = -y + 2$

7. $y = \frac{1}{2}x - 2$

$-4x + 8y = -8$

Section 3.2

For Exercises 8–11, solve the systems by using the substitution method.

8. $y = \frac{3}{4}x - 4$ **9.** $3x = 11y - 9$

$-x + 2y = -6$ $y = \frac{3}{11}x + \frac{6}{11}$

10. $4x + y = 7$ **11.** $6x + y = 5$

$x + \frac{1}{4}y = \frac{7}{4}$ $5x + y = 3$

Section 3.3

For Exercises 12–21, solve the systems by using the addition method.

12. $\frac{2}{5}x + \frac{3}{5}y = 1$ **13.** $4x + 3y = 5$

$x - \frac{2}{3}y = \frac{1}{3}$ $3x - 4y = 10$

14. $3x + 4y = 2$ **15.** $3x + y = 1$

$2x + 5y = -1$ $-x - \frac{1}{3}y = -\frac{1}{3}$

16. $2y = 3x - 8$ **17.** $3x + y = 16$

$-6x = -4y + 4$ $3(x + y) = y + 2x + 2$

18. $-(y + 4x) = 2x - 9$ **19.** $-(4x - 35) = 3y$

$-2x + 2y = -10$ $-(x - 15) = y$

20. $-0.4x + 0.3y = 1.8$

$0.6x - 0.2y = -1.2$

21. $0.02x - 0.01y = -0.11$

$0.01x + 0.04y = 0.26$

Section 3.4

22. Antonio invested twice as much money in an account paying 5% simple interest as he did in an account paying 3.5% simple interest. If his total interest at the end of 1 year is $303.75, find the amount he invested in the 5% account.

23. A school carnival sold tickets to ride on a Ferris wheel. The charge was $1.50 for adults and $1.00 for students. If 54 tickets were sold for a total of $70.50, how many of each type of ticket were sold?

24. How many liters of 20% saline solution must be mixed with 50% saline solution to produce 16 L of a 31.25% saline solution?

25. It takes a pilot $1\frac{3}{4}$ hr to travel with the wind to get from Jacksonville, Florida, to Myrtle Beach, South Carolina. Her return trip takes 2 hr flying against the wind. What is the speed of the wind and the speed of the plane in still air if the distance between Jacksonville and Myrtle Beach is 280 mi?

26. Two phone companies offer discount rates to students.

Company 1: $9.95 per month, plus $0.10 per minute for long-distance calls

Company 2: $12.95 per month, plus $0.08 per minute for long-distance calls

a. Write a linear equation describing the total cost, y, for x min of long-distance calls from Company 1.

b. Write a linear equation describing the total cost, y, for x min of long-distance calls from Company 2.

c. How many minutes of long-distance calls would result in equal cost for both offers?

27. Two angles are complementary. One angle measures 6° more than 5 times the measure of the other. What are the measures of the two angles?

Section 3.5

For Exercises 28–31, solve the systems of equations. If a system does not have a unique solution, label the system as either dependent or inconsistent.

28.
$$5x + 5y + 5z = 30$$
$$-x + y + z = 2$$
$$10x + 6y - 2z = 4$$

29.
$$5x + 3y - z = 5$$
$$x + 2y + z = 6$$
$$-x - 2y - z = 8$$

30.
$$x + y + z = 4$$
$$-x - 2y - 3z = -6$$
$$2x + 4y + 6z = 12$$

31.
$$3x + 4z = 5$$
$$2y + 3z = 2$$
$$2x - 5y = 8$$

32. The perimeter of a right triangle is 30 ft. One leg is 2 ft longer than twice the shortest leg. The hypotenuse is 2 ft less than 3 times the shortest leg. Find the lengths of the sides of this triangle.

33. Three pumps are working to drain a construction site. Working together, the pumps can pump 950 gal/hr of water. The slowest pump pumps 150 gal/hr less than the fastest pump. The fastest pump pumps 150 gal/hr less than the sum of the other two pumps. How many gallons can each pump drain per hour?

Section 3.6

For Exercises 34–37, determine the order of each matrix.

34. $\begin{bmatrix} 2 & 4 & -1 \\ 5 & 0 & -3 \\ -1 & 6 & 10 \end{bmatrix}$

35. $\begin{bmatrix} -5 & 6 \\ 9 & 2 \\ 0 & -3 \end{bmatrix}$

36. $\begin{bmatrix} 0 & 13 & -4 & 16 \end{bmatrix}$

37. $\begin{bmatrix} 7 \\ 12 \\ -4 \end{bmatrix}$

For Exercises 38–39, set up the augmented matrix.

38.
$$x + y = 3$$
$$x - y = -1$$

39.
$$x - y + z = 4$$
$$2x - y + 3z = 8$$
$$-2x + 2y - z = -9$$

For Exercises 40–41, write a corresponding system of equations from the augmented matrix.

40. $\begin{bmatrix} 1 & 0 & | & 9 \\ 0 & 1 & | & -3 \end{bmatrix}$ **41.** $\begin{bmatrix} 1 & 0 & 0 & | & -5 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & | & -8 \end{bmatrix}$

42. Given the matrix **C**

$$C = \begin{bmatrix} 1 & 3 & | & 1 \\ 4 & -1 & | & 6 \end{bmatrix}$$

   a. What is the element in the second row and first column?

   b. Write the matrix obtained by multiplying the first row by −4 and adding the result to row 2.

43. Given the matrix **D**

$$D = \begin{bmatrix} 1 & 2 & 0 & | & -3 \\ 4 & -1 & 1 & | & 0 \\ -3 & 2 & 2 & | & 5 \end{bmatrix}$$

   a. Write the matrix obtained by multiplying the first row by −4 and adding the result to row 2.

   b. Using the matrix obtained in part (a), write the matrix obtained by multiplying the first row by 3 and adding the result to row 3.

For Exercises 44–47, solve the system by using the Gauss-Jordan method.

44. $x + y = 3$
$x - y = -1$

45. $4x + 3y = 6$
$12x + 5y = -6$

46. $x - y + z = -4$
$2x + y - 2z = 9$
$x + 2y + z = 5$

47. $x - y + z = 4$
$2x - y + 3z = 8$
$-2x + 2y - z = -9$

Section 3.7

For Exercises 48–51, evaluate the determinant.

48. $\begin{vmatrix} 5 & -2 \\ 2 & -3 \end{vmatrix}$ **49.** $\begin{vmatrix} -6 & 1 \\ 0 & 10 \end{vmatrix}$

50. $\begin{vmatrix} \frac{1}{2} & 3 \\ 1 & 8 \end{vmatrix}$ **51.** $\begin{vmatrix} 9 & 3 \\ -2 & \frac{2}{3} \end{vmatrix}$

For Exercises 52–55, evaluate the minor corresponding to the given element from matrix **A**.

$$A = \begin{bmatrix} 8 & 2 & 0 \\ -1 & 4 & -2 \\ 3 & -3 & 6 \end{bmatrix}$$

52. 8   **53.** 2   **54.** −2   **55.** 4

For Exercises 56–59, evaluate the determinant.

56. $\begin{vmatrix} 2 & 1 & 0 \\ -4 & 3 & -1 \\ 3 & 0 & 1 \end{vmatrix}$ **57.** $\begin{vmatrix} 1 & 0 & 2 \\ 3 & -2 & 4 \\ 0 & 1 & 1 \end{vmatrix}$

58. $\begin{vmatrix} 4 & -2 & 0 \\ 9 & 5 & 4 \\ 1 & 2 & 0 \end{vmatrix}$ **59.** $\begin{vmatrix} -1 & 0 & 2 \\ 5 & -2 & 6 \\ 3 & 0 & -4 \end{vmatrix}$

For Exercises 60–65, solve the system using Cramer's rule.

60. $3x - 4y = 1$
$2x + 3y = 12$

61. $3x + 2y = 11$
$-x + y = 3$

62. $3x + 2y = 9$
$x - 3y = 1$

63. $-x + 3y = -8$
$2x - y = -2$

64. $2x + 3y - z = -7$
$x + 3z = 10$
$2y + z = -1$

65. $6y + 4z = -12$
$3x + 3y = 9$
$2x - 3z = 10$

For Exercises 66–67, solve the system of equations using Cramer's rule if possible. If not possible, use another method.

66. $2x - y = 1$
$4x - 2y = 2$

67. $x + y - 3z = -1$
$y - z = 6$
$-x + 2y = 1$

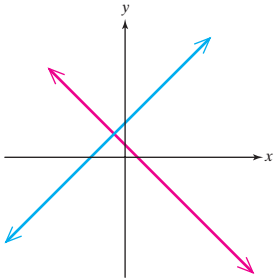## Chapter 3   **Test**

1. Determine if the ordered pair $\left(\frac{1}{4}, 2\right)$ is a solution to the system.
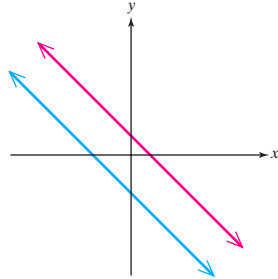
$$4x - 3y = -5$$
$$12x + 2y = 7$$

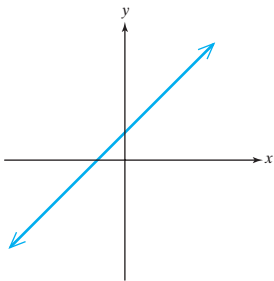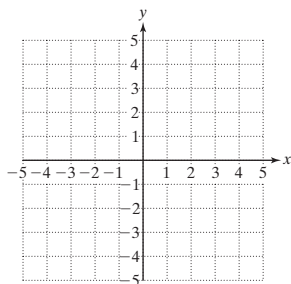Match each figure with the appropriate description.

2.

3.

4.

a. The system is consistent and dependent. There are infinitely many solutions.

b. The system is consistent and independent. There is one solution.

c. The system is inconsistent and independent. There are no solutions.

5. Solve the system by graphing.

$$4x - 2y = -4$$
$$3x + y = 7$$

6. Solve the system by using the substitution method.

$$3x + 5y = 13$$
$$y = x + 9$$

7. Solve the system by using the addition method.

$$6x + 8y = 5$$
$$3x - 2y = 1$$

For Exercises 8–14, solve the system of equations.

8. $7y = 5x - 21$
$9y + 2x = -27$

9. $3x - 5y = -7$
$-18x + 30y = 42$

10. $\dfrac{1}{5}x = \dfrac{1}{2}y + \dfrac{17}{5}$

$\dfrac{1}{4}(x + 2) = -\dfrac{1}{6}y$

11. $4x = 5 - 2y$
$y = -2x + 4$

12. $-0.03y + 0.06x = 0.3$
$0.4x - 2 = -0.5y$

13. $2x + 2y + 4z = -6$
$3x + y + 2z = 29$
$x - y - z = 44$

14. $2(x + z) = 6 + x - 3y$
$2x = 11 + y - z$
$x + 2(y + z) = 8$

15. How many liters of a 20% acid solution should be mixed with a 60% acid solution to produce 200 L of a 44% acid solution?

16. Two angles are complementary. Two times the measure of one angle is 60° less than the measure of the other. Find the measure of each angle.

17. Working together, Joanne, Kent, and Geoff can process 504 orders per day for their business. Kent can process 20 more orders per day than Joanne can process. Geoff can process 104 fewer orders per day than Kent and Joanne combined. Find the number of orders that each person can process per day.

18. Write an example of a 3×2 matrix.

19. Given the matrix **A**

$$\mathbf{A} = \left[\begin{array}{ccc|c} 1 & 2 & 1 & -3 \\ 4 & 0 & 1 & -2 \\ -5 & -6 & 3 & 0 \end{array} \right]$$

    a. Write the matrix obtained by multiplying the first row by -4 and adding the result to row 2.

    b. Using the matrix obtained in part (a), write the matrix obtained by multiplying the first row by 5 and adding the result to row 3.

For Exercises 20–21, solve by using the Gauss-Jordan method.

20. $5x - 4y = 34$
    $x - 2y = 8$

21. $x + y + z = 1$
    $2x + y = 0$
    $-2y - z = 5$

For Exercises 22–23, find the determinant of the matrix.

22. $\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$

23. $\begin{bmatrix} 0 & 5 & -2 \\ 0 & 0 & 2 \\ 2 & 3 & 1 \end{bmatrix}$

For Exercises 24–25, use Cramer's rule for solve for y.

24. $6x - 5y = 13$
    $-2x - 2y = 9$

25. $2x + 2z = 2$
    $5x + 3y = 4$
    $3y - 4z = 4$

26. Solve the system: $6x - 2y = 0$
                   $3x - y = 0$

Chapters 1–3   Cumulative Review Exercises

For Exercises 1–2, solve the equation.

1. $-5(2x - 1) - 2(3x + 1) = 7 - 2(8x + 1)$

2. $\dfrac{1}{2}(a - 2) - \dfrac{3}{4}(2a + 1) = -\dfrac{1}{6}$

3. Simplify the expression. $\dfrac{(6a^2b)^{-2}}{a^{-5}b^0}$

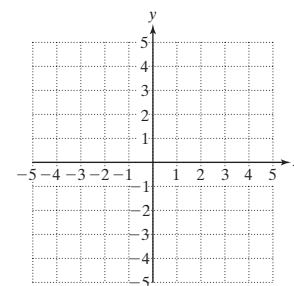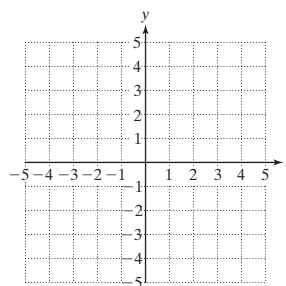**4.** Solve the inequality. Write the answer in interval notation.

$$-3y - 2(y + 1) < 5$$

5. Identify the slope and the x- and y-intercepts of the line $5x - 2y = 15$.

For Exercises 6–7, graph the lines.

6. $y = -\dfrac{1}{3}x - 4$

7. $x = -2$

8. Find the slope of the line passing through the points $(4, -10)$ and $(6, -10)$.

9. Find an equation for the line that passes through the points $(3, -8)$ and $(2, -4)$. Write the answer in slope-intercept form.

10. Solve the system by using the addition method.

$$2x - 3y = 6$$

$$\frac{1}{2}x - \frac{3}{4}y = 1$$

11. Solve the system by using the substitution method.

$$2x + y = 4$$

$$y = 3x - 1$$

12. A child's piggy bank contains 19 coins consisting of nickels, dimes, and quarters. The total amount of money in the bank is $3.05. If the number of quarters is 1 more than twice the number of nickels, find the number of each type of coin in the bank.

13. Two video clubs rent tapes according to the following fee schedules:

Club 1: $25 initiation fee plus $2.50 per tape
Club 2: $10 initiation fee plus $3.00 per tape

  a. Write a linear equation describing the total cost, y, of renting x tapes from club 1.

  b. Write a linear equation describing the total cost, y, of renting x tapes from club 2.

  c. How many tapes would have to be rented to make the cost for club 1 the same as the cost for club 2?

14. Solve the system.

$$3x + 2y + 3z = 3$$

$$4x - 5y + 7z = 1$$

$$2x + 3y - 2z = 6$$

15. Determine the order of the matrix.

$$\begin{bmatrix} 4 & 5 & 1 \\ -2 & 6 & 0 \end{bmatrix}$$

16. Write an example of a 2×4 matrix.

17. List at least two different row operations.

18. Solve the system by using the Gauss-Jordan method.

$$2x - 4y = -2$$

$$4x + y = 5$$

19. Find the determinant of matrix $\mathbf{C} = \begin{bmatrix} 8 & -2 \\ 3 & 1 \end{bmatrix}$.

20. Solve the system using Cramer's rule.

$$2x - 4y = -2$$

$$4x + y = 5$$