EXTENDED MATHEMATICS
0580

REVISION

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First Edition
2011

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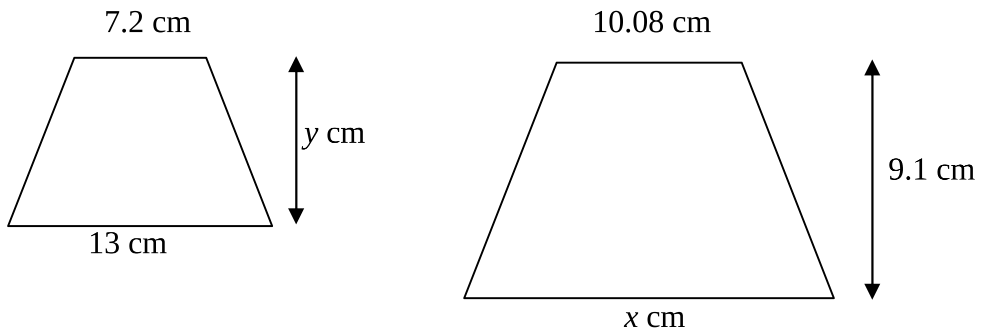
Revision Topic 3: Similar Shapes

Key Points to Remember:

* Shapes are **similar** if one shape is an **enlargement** of the other.
* When two shapes are similar, the **corresponding sides are in proportion** and **corresponding angles are equal**.
* The scale factor of enlargement is the ratio:

$$\frac{\text{length of a side on second shape}}{\text{length of corresponding side on first shape}}.$$

Example 1: **Simple example**
The two trapezia below are similar. Find the values of *x* and *y*.



The first step is to find the scale factor of the enlargement. This can be found by dividing the length of two corresponding sides. We can use the length of the two top sides to find the scale factor here.

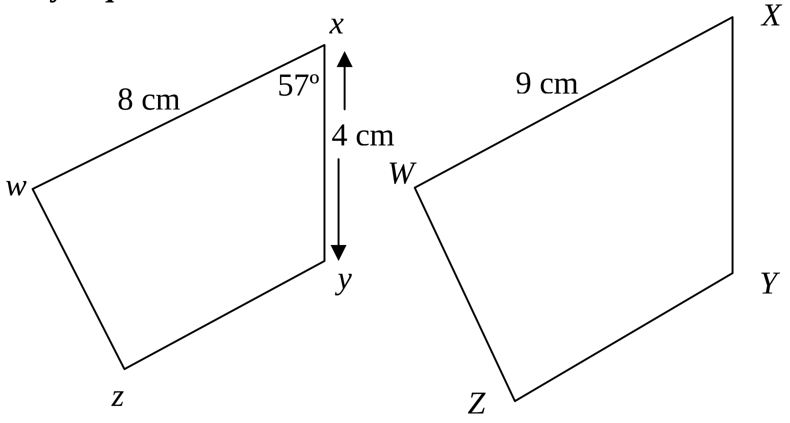
So, s.f. = $\frac{10.08}{7.2} = 1.4$.

The missing sides can be found by either **multiplying or dividing by the scale factor**.

To find *x*: **Multiply** 13 cm (the corresponding length on the first shape) by 1.4,
i.e. $x = 13 \times 1.4 = 18.2$ cm.
[You check that this answer is reasonable – as *x* is on the larger shape, *x* must be bigger than 13 cm]

To find y: **Divide** 9.1 cm (the corresponding length on the second shape) by 1.4,
i.e. $y = 9.1 \div 1.4 = 6.5$ cm.

Examination-style question:



The two shapes (wxyz and WXYZ) are mathematically similar.
a) Calculate the length of XY.
b) Find the size of angle YXW (marked θ on the diagram)

Similar Triangles

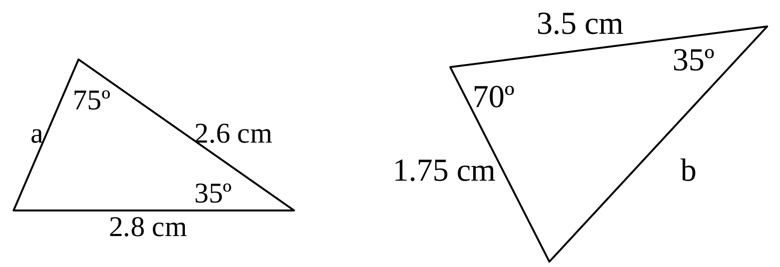
To show that two triangles are similar, you need to check that

Either they contain the same angles

Or corresponding lengths are in the same ratio.

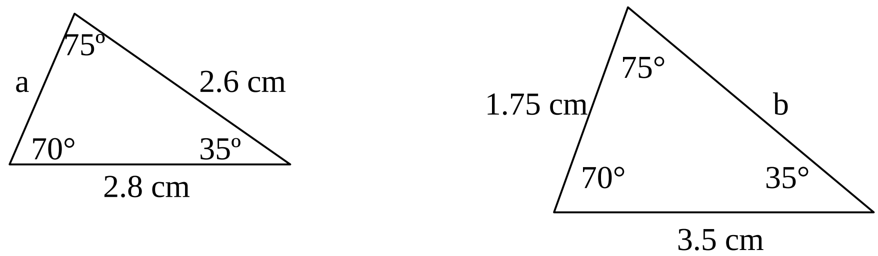
Example 2: Similar triangles

Explain why the two triangles below are similar. Calculate the lengths marked a and b.



Both triangles contain the same three angles (namely 75°, 70° and 35°). Therefore they are similar.

First the shapes must be redrawn so that angles that are equal are in the same position.

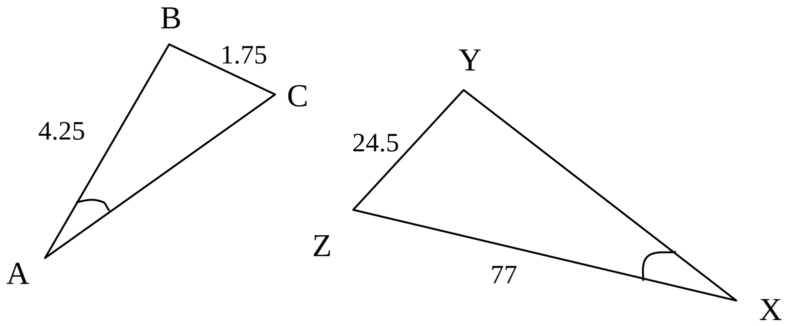


The scale factor for the enlargement is $3.5 \div 2.8 = 1.25$

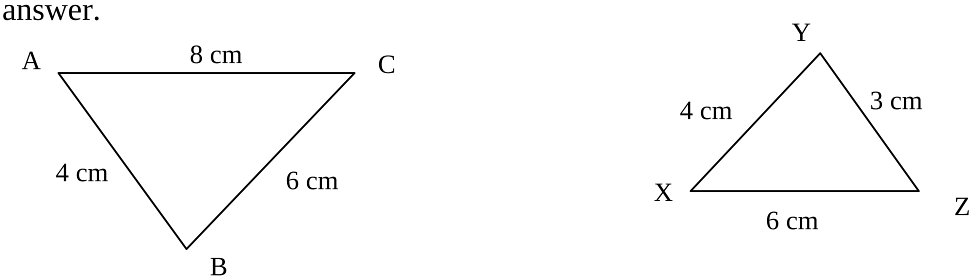
To find a: $a = 1.75 \div 1.25 = 1.4$ cm.

To find b: $b = 2.6 \times 1.25 = 3.25$ cm

Examination Question:
These triangles are similar.
They are not drawn accurately.
Angle A equals angle X.
Angle B equals angle Y.
Calculate the length of XY.



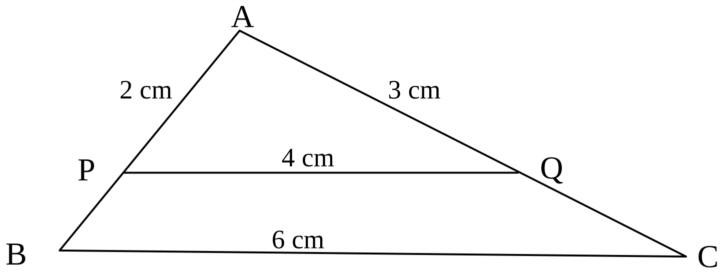
Examination Question:
State whether or not the triangles ABC and XYZ are similar. Show working to support your answer.



[Hint: As there are no angles shown, to see whether the triangles are similar or not you will need to calculate the scale factor based on each of the three pairs of corresponding sides– if all the scale factors are identical then the triangles are similar].

Example 3:

Show that triangles ABC and APQ are similar and calculate the length of AC and BP.



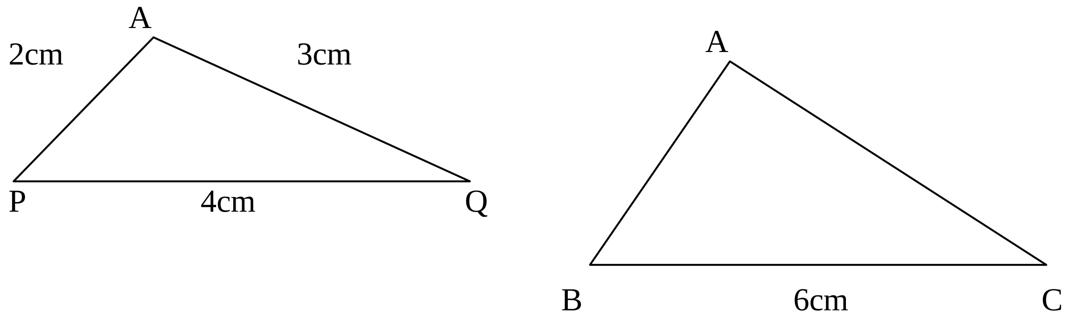
PQ and BC are parallel lines.

The triangles are similar if they contain the same angles. We can show this:

Angle A is common to both triangles ABC and APQ
Angle APQ = Angle ABC as they are corresponding angles (parallel lines)
Angle AQP = Angle ACB as they are corresponding angles.

So all three angles in the triangles are equal. So the triangles are parallel.

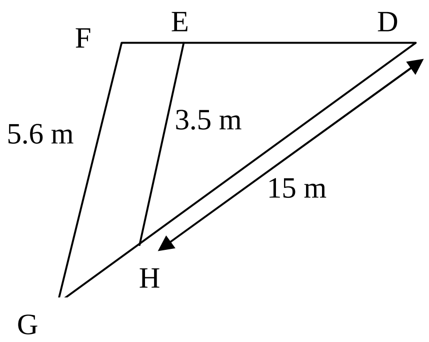
To calculate the lengths it is easiest to draw the two triangles separately:



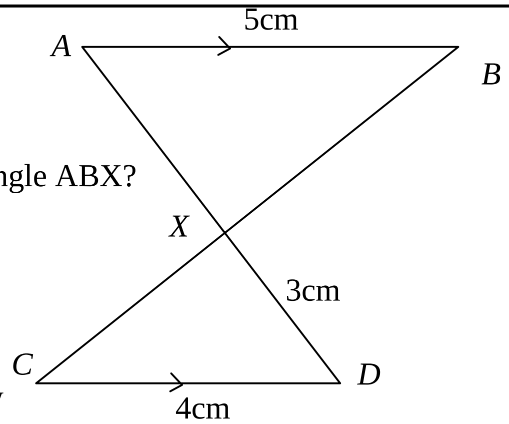
Using the bases of the two triangles we see that the scale factor is $6 \div 4 = 1.5$.
So: AC = $3 \times 1.5 = 4.5$ cm

Likewise, AB = $2 \times 1.5 = 3$ cm. So BP = AB – AP = 3cm – 2cm = 1 cm.

Examination Question:
In the diagram FG = 5.6 metres, EH = 3.5 metres and DH = 15 metres.
EH is parallel to FG.
FED and DHG are straight lines.
a) Explain why triangles DEH and DFG are similar.
b) Calculate the length of DG and GH.

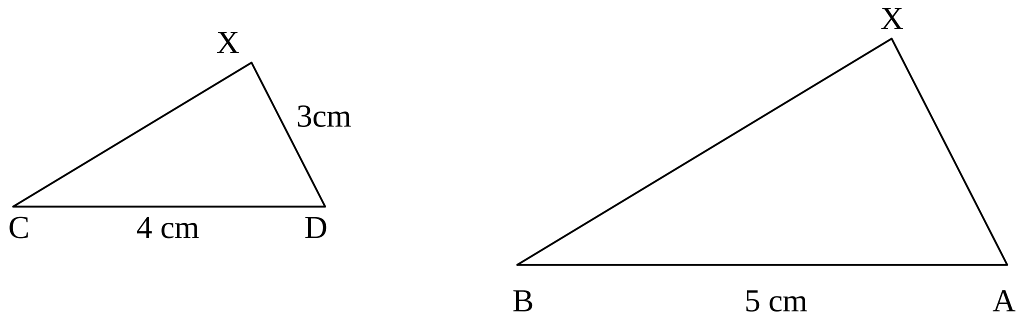


Example 4:
AB and CD are parallel lines.
AD and BC meet at X.
a) Prove that triangles ABX and DCX are similar.
b) Which side in triangle DCX corresponds to AX in triangle ABX?
c) Calculate the length of AX.



a) To show the triangles are similar we need to show they contain the same angles:
Angle AXB = Angle CXD (vertically opposite angles)
Angle BAX = Angle XDC (alternate angles)
So the third angles in each triangle must also be equal, i.e. Angle ABX = Angle XCD.
Therefore the triangles are similar.

b) It is simplest to now redraw the two triangles so that they are the same way up and with equal angles in corresponding positions:

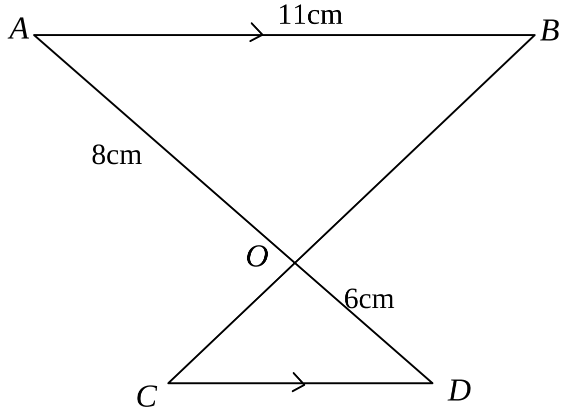


We see that AX corresponds to DX.

c) The scale factor is $5 \div 4 = 1.25$
So $AX = 3 \times 1.25 = 3.75$ cm.

Examination Question:
AB is parallel to CD.
Lines AD and BC intersect at point O.
AB = 11 cm. AO = 8 cm. OD = 6 cm.

Calculate the length of CD.



Revision Topic 4: Percentages

Percentages – Multipliers

The **multiplier** for an **increase of 6% is 1.06** (100% + 6% = 106% or 1.06).
To increase an amount by 6%, multiply it by 10.06.

The **multiplier** for a **decrease of 12% is 0.88** (100% - 12% = 88% or 0.88).
To decrease an amount by 12%, multiply it by 0.88.

For repeated percentage changes, multiply all the multipliers in turn.

Example:
A fast food restaurant decided to decrease by 18% the weight of packaging for their regular meals, which weighed 40 grams. Calculate the weight of the new packaging.

Method 1: Non-calculator paper
Find 18% of 40g: 10% of 40g = 4g
1% of 40g = 0.4g So 8% of 40g = 3.2g
So 18% of 40g = 4 + 3.2 = 7.2g
So new packaging would weigh 40 – 7.2 = 32.8g

Method 2: Calculator paper (Multipliers)
To decrease by 18%, the multiplier is 0.82 (100% - 18% = 82%).
So new weight is 0.82 × 40 = 32.8g

Example 2:
A manufacturer buys a machine for £56800. The machine is expected to **depreciate** by 12% in the first year and by 8.5% each future year. What will be the expected value of the machine after 5 years to the nearest £1?

Note: **Depreciate** means to lose value.

The multiplier for a decrease of 12% is 0.88.
The multiplier for a decrease of 8.5% is 0.915 (as 100% - 8.5% = 91.5% or 0.915).
So value of machine after 5 years is

$$56800 \times 0.88 \times \underbrace{0.915 \times 0.915 \times 0.915 \times 0.915}_{\text{multipliers for 2nd, 3rd, 4th and 5th years}} = 56800 \times 0.88 \times 0.915^4 = £35036 \text{ (to nearest £1)}$$

Examination Question
A shop is having a sale. Each day, prices are reduced by 20% of the price on the previous day.
Before the start of the sale, the price of a television is £450.
On the first day of the sale, the price is reduced by 20%.
Work out the price of the television on
a) the first day of the sale
b) the third day of the sale.

Examination Question:
A car was bought for £7600. It depreciated in value by 25% each year. What was the value of the car after 3 years?

Percentage Change

You need to know the following formulae

$$\text{Percentage increase (or decrease)} = \frac{\text{Increase (or decrease)}}{\text{Original amount}} \times 100$$

$$\text{Percentage profit (or loss)} = \frac{\text{Profit (or loss)}}{\text{Cost price}} \times 100$$

Example:
In 1990 a charity sold 2¼ million lottery tickets at 25p each. 80% of the money was kept by the charity.
a) Calculate the amount of money kept by the charity.

In 1991, the price of a lottery ticket fell by 20%.
Sales of lottery tickets increased by 20%.
80% of the money obtained was kept by the charity.

b) Calculate the percentage change in the amount of money kept by the charity.

a) Money made by selling tickets = 2 ¼ million × 25p = £562500
Money kept by charity = 80% of £562500 = 0.8 × £562500 = £450000.

b) In 1991 price of ticket = 80% of old price = 80% of 25p = 20p
In 1991, sales of lottery tickets = 120% of 1990 sales

$$= \frac{120}{100} \times 2250000 = 2700000$$

In 1991, money made by selling tickets = 2700000 × 20p = £540000
So amount to charity is 0.8 × £540000 = £432000

Therefore:

$$\text{Percentage decrease} = \frac{\text{decrease}}{\text{Original amount}} \times 100 = \frac{450000 - 432000}{450000} \times 100 = 4\%.$$

Examination Style Question:
Del 'boy' Rotter buys an old cottage for £84000. He spends £10400 on repairs and renovation then sells the cottage for £149000. Find his percentage profit to the nearest 1%.

Percentages- Compound Interest

Compound interest is interest paid on an amount and on the interest already received on that amount.

You can solve compound interest questions using the formula:

$$A = P\left(1 + \frac{R}{100}\right)^n$$

where: P is the amount invested initially;
R is the rate of interest (percentage per year)
n is the number of years invested;
A is the amount in the account at the end.

Example:
A building society pays compound interest at a fixed rate of 7% per annum. If £480 is invested in an account, what will be the value of the account after 3 years?

Method 1:
In first year: Interest paid is 7% of £480 = 0.07 × 480 = £33.60
So balance at end of 1st year is £480 + £33.60 = £513.60.

In second year: Interest paid is 7% of £513.60 = 0.07 × 513.60 = £35.952
So balance at end of 2nd year is £513.60 + £35.952 = £549.552

In third year: Interest paid is 7% of £549.552 = £38.46864
So balance at end of 3rd year is £549.552 + £38.46864 = £588.02 (to nearest 1p)

Method 2: Use of formula

After 3 years balance would be A = $P\left(1 + \frac{R}{100}\right)^n = 480\left(1 + \frac{7}{100}\right)^3 = 480 \times 1.07^3 = £588.02$.

Note: Sometimes you are asked for the total amount of interest that has been received. You get this by subtracting the initial amount invested from the final balance. Here the total interest received is £588.02 – 480 = £108.02.

Example:
Tony invests £500 in a bank account that pays 4% (compound) interest p.a. By what single number must £500 be multiplied by to get the amount in the account after 5 years?

The amount invested after 5 years is $P\left(1 + \frac{R}{100}\right)^n = 500 \times \left(1 + \frac{4}{100}\right)^5$.
So the required multiplier is $(1.04)^5$ = 1.21665 (to 5 decimal places).

Examination Question:
£500 is invested for 2 years at 6% per annum compound interest.
a Work out the total interest earned over the two years.

£250 is invested for three years at 7% per annum compound interest.

b By what single number must £250 be multiplied to obtain the total amount at the end of the 3 years?

Examination Question:
Nesta invests £508 in a bank account paying compound interest at a rate of 10% per annum.
Calculate the total amount in Nesta's bank account after 2 years.

Reverse Percentage Questions

Calculating the **original value** of something **before an increase or decrease took place** is called "calculating a reverse percentage".

Example 1:
Ella bought a pair of climbing boots for £45.60 in a sale that gave "20% off". What was the non-sale price of the boots.

Let the original price of the boots be 100%. The sale price then is 80%.
So 80% of the price = £45.60
So 1% of the price = £45.60 ÷ 80 = £0.57
So 100% of the price = £0.57 × 100 = £57.

So the non-sale price of the boots was £57.

Examination Question 1:
A clothes shop has a sale. All the original prices are reduced by 24% to give the sale price.
The sale price of a jacket is £36.86.
Work out the original price of the jacket.

Examination Question 2:
In a sale all prices are reduced by 16%. Alan buys a shaver in the sale for £21.
How much does he save by buying it in the sale?
[Hint: First work out the price before the sale]

Example 2:
The total price of a bike (including VAT at 17.5%) is £146.85.
Calculate the cost of the bike excluding VAT.

Let the cost of the bike before VAT be 100%. Then the cost including VAT would be 117.5%.
So 117.5% = £146.85.
Then 1% = £146.85 ÷ 117.5 = £1.249787…
So 100% = £1.249787 × 100 = £124.98 (to nearest 1p).

So the cost of the bike excluding VAT is £124.98.

Examination Question 3:
The price of a new television is £423. This price includes Value Added Tax (VAT) at 17.5%.

a) Work out the cost of the television **before** VAT was added.

By the end of each year, the value of a television has fallen by 12% of its value at the start of that year. The value of the television was £423 at the start of the first year.

b) Work out the value of the television at the end of the **third** year.

Examination Question 4:
The population of a town increased by 20% between 1981 and 1991. The population in 1991 was 43200.
What was the population in 1981?

[Hint: Let the population in 1981 be 100%].

Examination Question 5:
A tourist buys a stereo which costs £155.10, including VAT at 17.5%.
Tourists do not have to pay VAT. How much does the tourist pay?

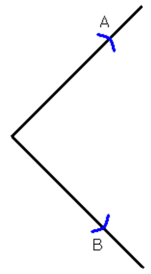
Revision Topic 6: Loci and Constructions

Constructions

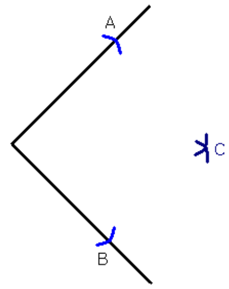
Bisecting an angle

N.B. To bisect an angle means to cut it in half.

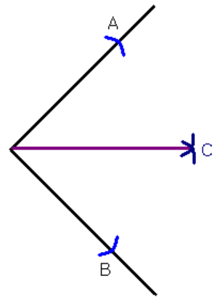
(1) Use your compasses to mark points A and B which are the same distance from the point (or vertex) of the angle.



(2) Without changing the settings of the compasses, put the compass point on A and draw an arc in the middle. Then put the compass point on B and draw an arc in the middle which crosses the first.

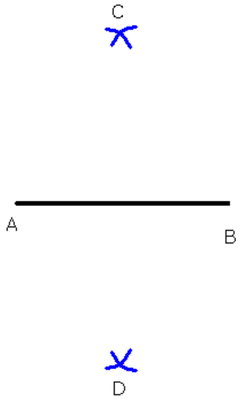


(3) This new point C will be in the middle of the angle. So we can draw a line from the point of the angle to C and we will have split the angle in two.



Perpendicular bisector of a line

A line which cuts a straight line in half at right angles is called a *perpendicular bisector*.



(1) Draw a line AB.

(2) Open your compasses to just over half the distance of the length AB.

(3) Put your compasses on A and draw an arc above and below the line. Then put your compasses on B and draw more arcs above and below the line. These arcs will cross over at two points C and D.

(4) Join up points C and D with a straight line. This is then the perpendicular bisector of the line AB.

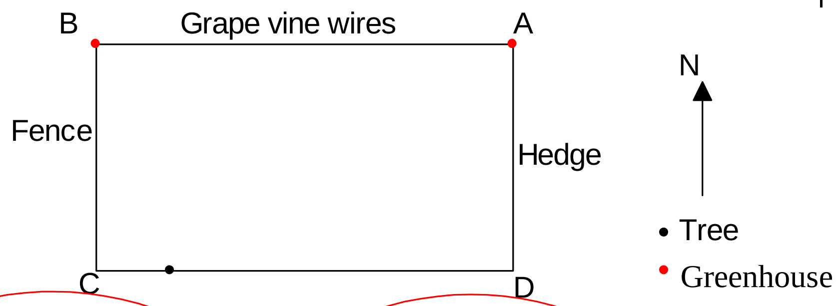
Constructions such as bisecting angles or lines can be used for making scale drawings where conditions have to be met.

Example

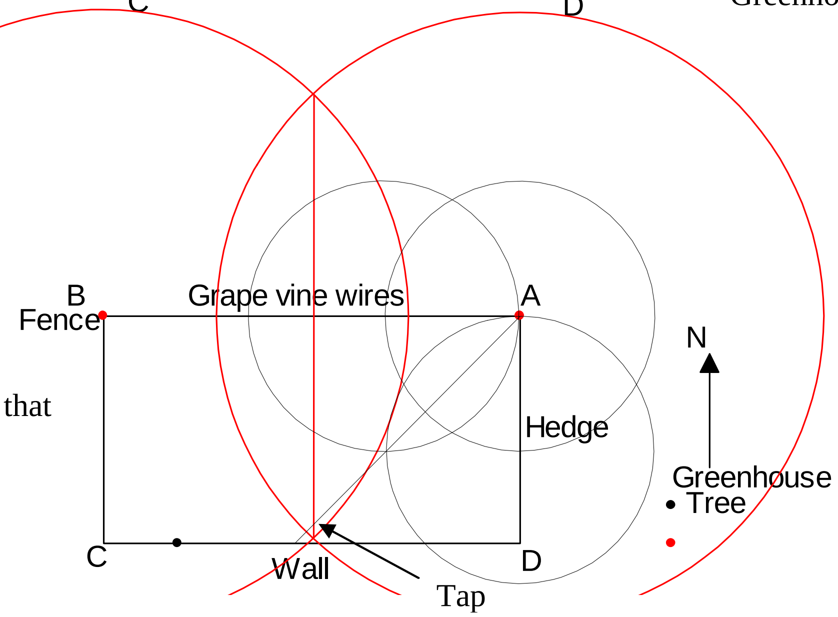
A water tap is to be put in a garden but it must meet these conditions:

1. It must be the same distance from the two greenhouses, A and B.
2. It must be the same distance from the grape line wires as from the hedge.

Where must the tap be placed?

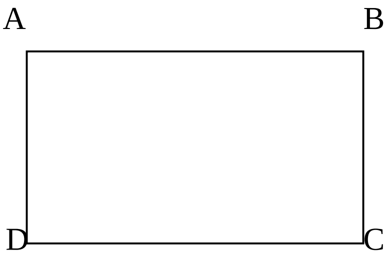


To meet condition 1 you draw the perpendicular bisector of the line BA.
This line is the locus of all points equidistant from B and A.
To meet condition 2 you bisect the angle BAD.
This line is the locus of all points that are equidistant from the lines BA and AD.



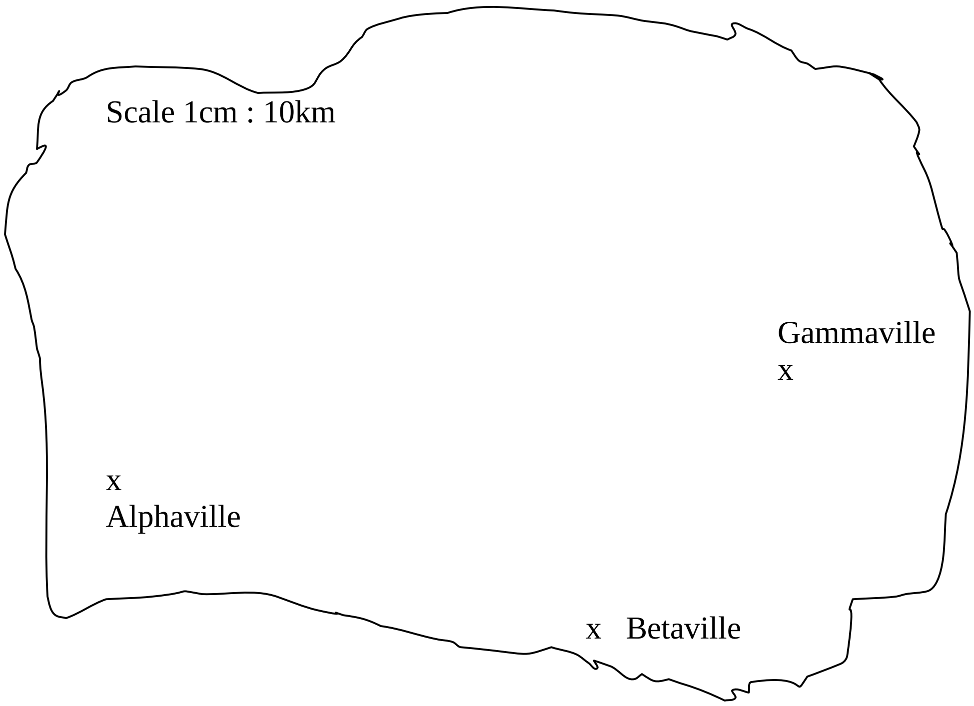
Where the two loci intersect Both conditions are met, so the tap must be at this point.

Examination Question 1:
The diagram shows a rectangular field ABCD.
The side AB is 80m long.
The side BC is 50m long.
Draw the diagram using a scale of 1cm to 10m.
Treasure is hidden in the field.



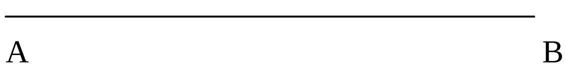
a) The treasure is equidistant from the sides AB and AD. Construct the locus of points for which this is true.
b) The treasure is 60m from corner C. Construct the locus of points for which this is true.
c) Mark with an X the position of the treasure.

Examination Question 2:
The map shows an island with three main towns, Alphaville, Betaville and Gammaville.
The map is drawn to a scale of 1 cm: 10 km.
A radio transmitter is to be installed.

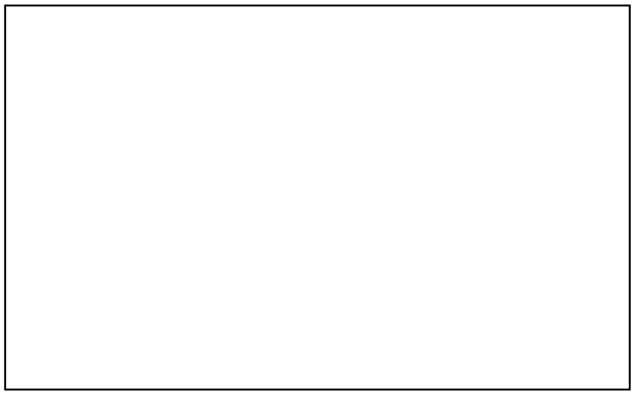


The transmitter must be equidistant from Alphaville and Betaville.
The transmitter must be between 35km and 50 km from Gammaville.
Mark on all the possible sites that the transmitter may be drawn.

Examination Question 3:
Draw the locus of all points that are 2.5 cm away from the line AB.



Examination Question 4:
On the diagram, draw the locus of the points outside the rectangle that are 3cm from the edge of this rectangle.



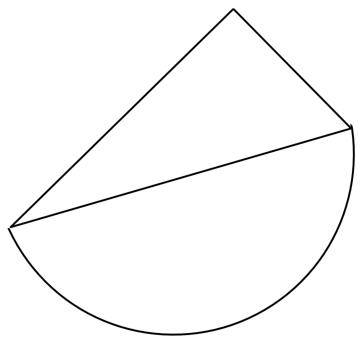
Points to remember

* To construct an angle of 90 degrees, draw a line and construct its perpendicular bisector. By bisecting this angle you can construct an angle of 45 degrees.

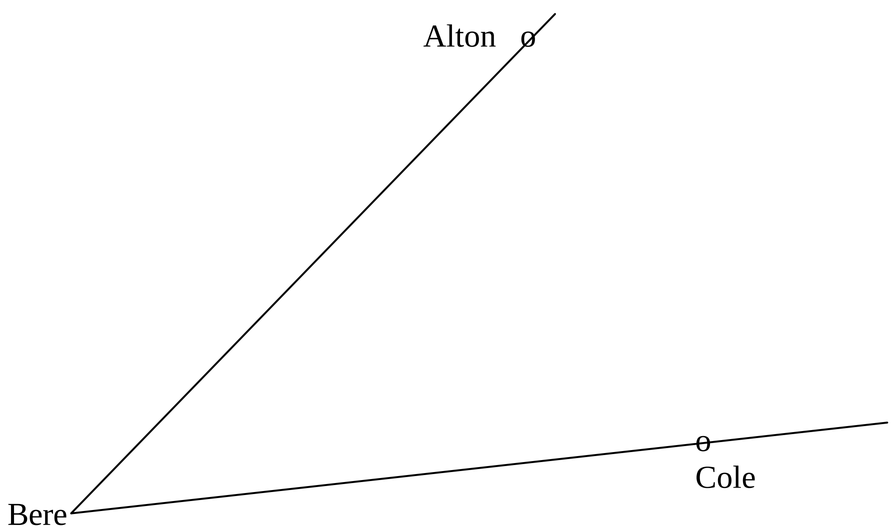
* To construct an angle of 60 degrees, construct an equilateral triangle. By bisecting this angle you can construct an angle of 30 degrees.

* When you are asked to construct something, do not rub out the construction lines.

Examination Question 5.
The diagram shows a penguin pool at a zoo. It consists of a right-angled triangle and a semi-circle. The scale is 1 cm to 1 m.
A safety fence is to be put up around the pool. The fence is always 2m from the pool. Draw accurately the position of the fence on the diagram.



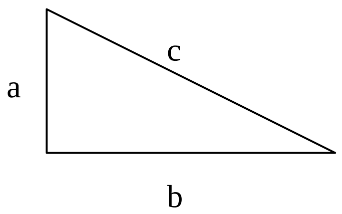
Examination Question 6:
Two straight roads are shown on the diagram.
A new gas pipe is to be laid from Bere equidistant from the two roads.
The diagram is drawn to a scale of 1cm to 1 km.
a) Construct the path of the gas pipe on the diagram.
b) The gas board needs to construct a site depot. The depot must be equidistant from Bere and Cole. The depot must be less that 3 km from Alton. Draw loci on the diagram to represent this information.
c) The depot must be nearer the road through Cole than the road through Alton. Mark on the diagram, with a cross, a possible location for the site depot that satisfies all these conditions.



Revision Topic 9: Pythagoras Theorem

Pythagoras' Theorem

Pythagoras' Theorem allows you to work out the length of sides in a **right-angled triangle**.

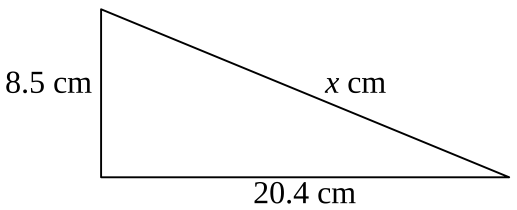


The side opposite the right-angle is called the *hypotenuse* – it is the *longest* side in the triangle.

If the sides are labelled a, b, c (with c being the length of the hypotenuse), then Pythagoras' theorem is

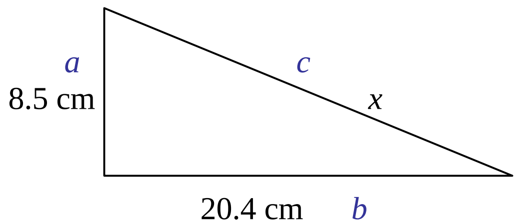
$$\boxed{a^2 + b^2 = c^2}$$

Example 1: Finding the length of the hypotenuse



Find the value of x.

We start by labelling the sides a, b, and c:
(c is the hypotenuse; a and b are the other two sides)



Write down Pythagoras' theorem: $a^2 + b^2 = c^2$

Put in the values of a, b, c: $8.5^2 + 20.4^2 = x^2$

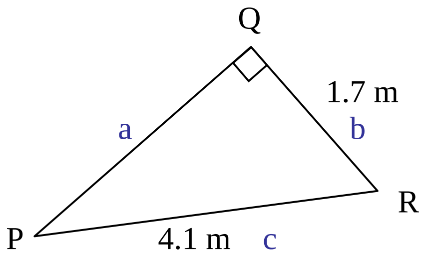
Work out the left-hand side: $72.25 + 416.16 = x^2$

$488.41 = x^2$

Square root: $x = \sqrt{488.41} = 22.1$ cm

Finally we should check the answer seems reasonable. The hypotenuse is the longest side in a right-angled triangle. As our answer is bigger than 20.4 cm, it seems reasonable.

Example 2: Finding the length of a shorter side



Find the length of PQ.

We label the triangle a, b, c (c must be the hypotenuse).

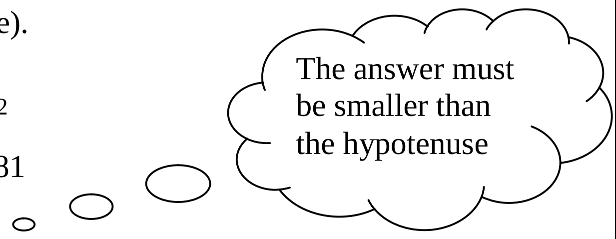
Pythagoras' theorem: $a^2 + b^2 = c^2$

Substitute in the numbers: $a^2 + 1.7^2 = 4.1^2$

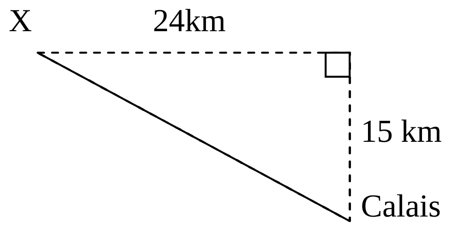
Work out the squares: $a^2 + 2.89 = 16.81$

Subtract 2.89 from both sides: $a^2 = 13.92$

Square root: $a = 3.73$ m (2 decimal places)

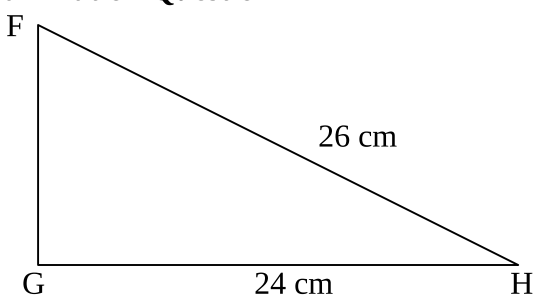


Examination Question 1:
The diagram shows the position of a ferry sailing between Folkestone and Calais.



The ferry is at X.
The position of the ferry from Calais is given as:
North of Calais 15km,
West of Calais 24km.
Calculate the distance of the ferry from Calais.
Give your answer to one decimal place.

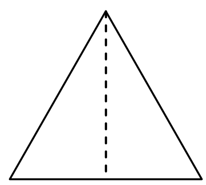
Examination Question 2



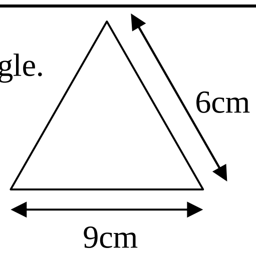
In the diagram, triangle FGH is right-angled at G.
GH = 24cm and FH = 26 cm.

Calculate the length of FG.

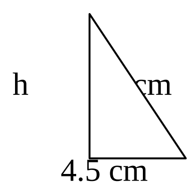
Pythagoras' Theorem in Isosceles Triangles
An isosceles triangle can be split into 2 right-angled triangles:
It is therefore possible to use Pythagoras theorem to find lengths in isosceles triangles.



Example:
Find the area of this triangle.



Split the triangle into two:



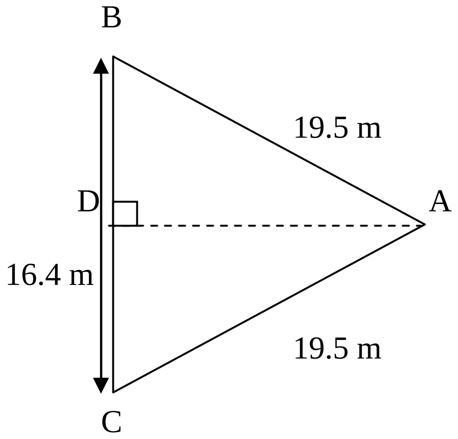
We can find the height of the triangle:

$$h^2 + 4.5^2 = 6^2$$
$$h^2 + 20.25 = 36$$
$$h^2 = 15.75$$

So… h = 3.97 cm

Therefore the area is … $\frac{1}{2} \times b \times h = \frac{1}{2} \times 9 \times 3.97 = 17.9\,\text{cm}^2$

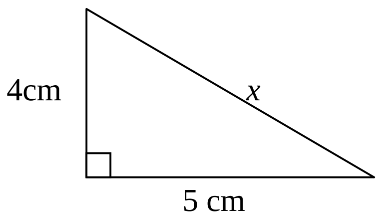
Examination Question 3



AB = 19.5 cm, AC = 19.5 cm and BC = 16.4 cm.
Angle ADB = 90˚.
BDC is a straight line.
Calculate the length of AD. Give your answer in **centimetres**, correct to 1 decimal place.

Note: Pythagoras' theorem could occur on the non-calculator paper.

Example for non-calculator paper



Find the value of x.

Pythagoras' theorem: $a^2 + b^2 = c^2$

$4^2 + 5^2 = x^2$

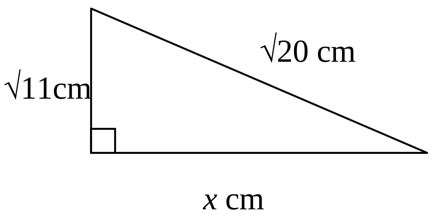
$16 + 25 = x^2$

$41 = x^2$

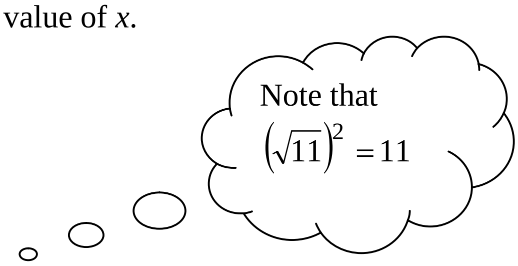
So, $x = \sqrt{41}$ cm

Since you do not have a calculator, leave the answer as a square root. Do not try to estimate the square root of 41 unless you are told to do so.

Example 2: Non-calculator paper



Find the value of *x*.



Pythagoras' theorem: $a^2 + b^2 = c^2$

$\left(\sqrt{11}\right)^2 + x^2 = \left(\sqrt{20}\right)^2$

$11 + x^2 = 20$

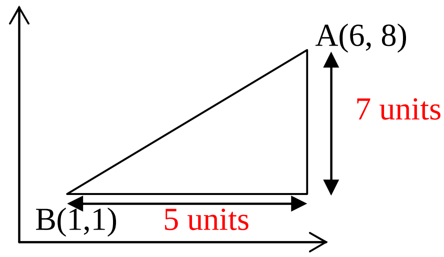
$x^2 = 9$

So, $x = 3$ cm

Application of Pythagoras' Theorem: Finding the distance between two points

Example: The coordinates of the points A and B are (6, 8) and (1, 1). Work out the length of AB.

Sketch



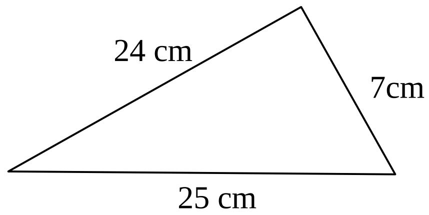
A sketch showing the positions of points A and B is an important first step.

$$a^2 + b^2 = c^2$$
$$7^2 + 5^2 = c^2$$
$$49 + 25 = c^2$$
$$74 = c^2$$

So the length AB is… $\sqrt{74} = 8.6$ units

Other uses of Pythagoras' Theorem

Example



Prove that the triangle is right-angled.

Solution

Pythagoras' theorem only works in right-angled triangles, so we need to show that the triangle above satisfies Pythagoras' theorem.

The longest side is 25 cm so this would have to be the hypotenuse, *c*. The two shortest sides, *a* and *b*, are 7 cm and 24 cm.

Pythagoras: $a^2 + b^2 = c^2$

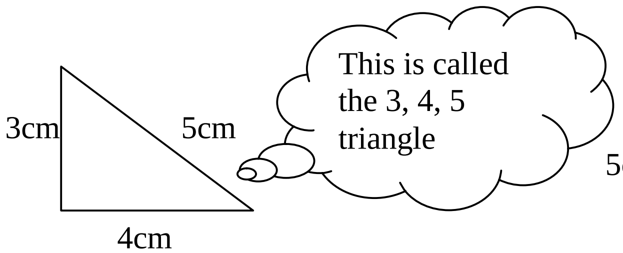
Left-hand side: $a^2 + b^2 = 7^2 + 24^2 = 625$

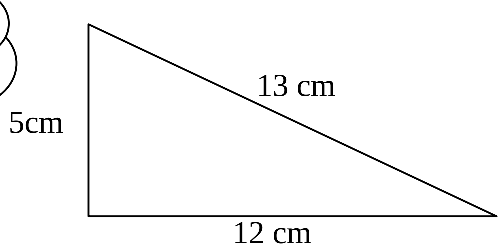
Right-hand side: $c^2 = 25^2 = 625$

So the right-hand side is equal to the left-hand side, so Pythagoras works in this triangle. Therefore the triangle is right-angled.

Special triangles - right-angled triangles with whole number sides

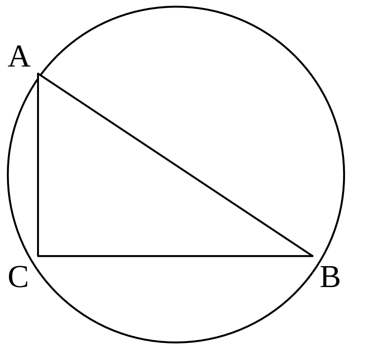
The triangle at the top of this page is special as it is a right-angled triangle that has sides which are all whole numbers. Two other common right-angled triangles with sides that are whole numbers are:





You can use these basic triangles to get other right-angled triangles by multiplying all the sides by the same number. For example, a triangle with lengths 6cm, 8cm and 10cm would be right-angled (as its sides are double those in the 3, 4, 5 triangle).

Right-angled triangles in semi-circles

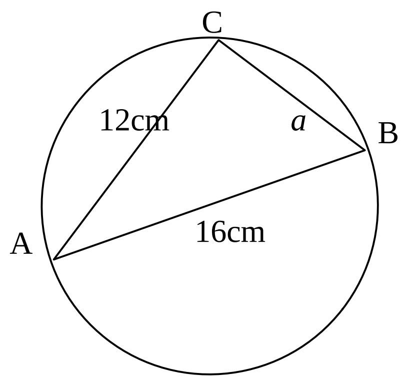


Look at the triangle shown here.

If AB is a *diameter* of the circle, then triangle ACB has a *right-angle* at C (this is one of the circle theorems – an angle in a semi-circle is 90 degrees).

Example

The side AB is the diameter of a circle. Find the length marked *a*. Give your answer to 1 decimal place.



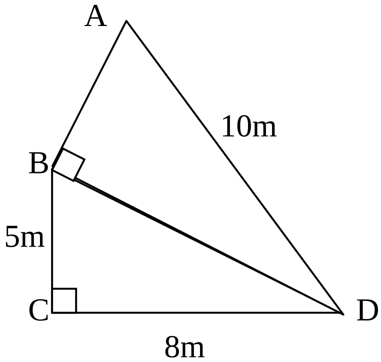
Solution

As AB is a diameter of a circle, angle ACB is a right angle. Therefore the hypotenuse is 16 cm.

Pythagoras:

$$a^2 + b^2 = c^2$$
$$12^2 + a^2 = 16^2$$
$$144 + a^2 = 256$$
$$a^2 = 256 - 144 = 112$$
$$a = \sqrt{112} = 10.6 \text{ cm (to 1 dp)}$$

Further examination question



Find length AB. Give your answer correct to 3 significant figures.

(Hint: You will need to find length BD first, using the lower triangle).

Revision Topic 10: Standard Form

Writing Numbers in Standard Form

Standard (index) form is a convenient way to write large numbers (like 230,000,000,000) and small numbers (like 0.000 000 06).

Standard form notation is $a \times 10^n$ where:

- a is a number between 1 and 10, and
- n is an integer (i.e. a positive or negative whole number).

(1) Writing large numbers in standard form:

The power of 10 is one less than the number of digits (to the left of the decimal point):

e.g. 230,000,000,000 = 2.3×10^{11} (there are 12 digits, so the power of 10 is 11)

Put a decimal point after first digit

e.g.2 7,265,000 = 7.265×10^6 (there are 7 digits).

e.g.3 $8.13 \times 10^5 = 813000$ (there must be 6 digits altogether)

e.g 4 $1.074 \times 10^7 = 10{,}740{,}000$ (there must be 8 digits altogether).

(2) Writing small numbers in standard form:

The power of 10 is negative for small numbers. You get the power by counting to see which decimal place the first non-zero digit occurs in):

e.g.1 0.000 000 06 = 6×10^{-8} (the first non-zero digit is in the 8th decimal place)

e.g.2 0.0045 = 4.5×10^{-3} (the first non-zero digit occurs in the 3rd decimal place)

e.g.3 $7.135 \times 10^{-5} = 0.00007135$ (the first non-zero digit will be the 5th decimal place)

e.g.4 $5.09 \times 10^{-2} = 0.0509$ (the first non-zero digit will be the 2nd decimal place).

Examination Style Question 1

The Earth's volume is about 1,080,000,000,000 km^3. Write this number in standard form.

Examination Style Question 2

The approximate area of the Australian desert is 1.47×10^6 square miles. Write this number in ordinary form.

Examination Style Question 3

The weight of a bee hummingbird is 0.0016 kg. Write this number in standard form.

Calculations involving numbers in standard form using a calculator

Your calculator has a key which can be used to enter a number written in standard form.
It is marked: EXP

Worked examination question:

The area of the Earth covered by the sea is 362,000,000 km^2.

(a) Write 362,000,000 in standard form.

The surface area A km^2 of the Earth may be found using the formula

$$A = 4\pi r^2$$

where r km is the radius of the Earth, r = 6.38×10^3.

(b) Calculate the surface area of the Earth. Give your answer in standard form correct to 3 significant figures.

(c) Calculate the percentage of the Earth's surface that is covered by the sea. Give your answer correct to 2 significant figures.

Solution:

(a) 362,000,000 = 3.62×10^8

(b) $A = 4\pi r^2 = 4 \times \pi \times (6.38 \times 10^3)^2$

Use the x^2 key

Use pi button

Use the EXP key

The answer is 511506575 = 512000000 (to 3 s.f.)

$= 5.12 \times 10^8$

(c) The percentage is $\dfrac{3.62 \times 10^8}{5.12 \times 10^8} \times 100 = 70.7....\% = 71\%$ (correct to 2 sf).

Examination Question:

The time taken for light to reach Earth from the edge of the known universe is 14,000,000,000 years.

a) Write 14,000,000,000 in standard form.

Light travels at the speed of 9.46×10^{12} km/year.

b) Calculate the distance, in kilometres, from the edge of the known universe to Earth. Give your answer in standard form.

(Remember that Distance = Speed × Time)

Examination Question:

The mass of a neutron is 1.675×10^{-24} grams. Calculate the total mass of 1500 neutrons. Give your answer in standard form.

Calculations involving numbers in standard form without a calculator

<u>Problems involving addition and subtraction</u>:
To add and subtract numbers in standard form without a calculator, you must first change them to ordinary numbers.

Example:

If x = 4×10^5 and y = 8.2×10^6 work out (a) x + y (b) y – x **giving your answers in standard form**.

Solution:
First write both x and y as ordinary numbers:

x = 400000 (as x must have 5 + 1 = 6 digits altogether)
y = 8200000 (as y has 6 + 1 = 7 digits altogether).

So x + y = 400000 + 8200000 = 8600000
= 8.6×10^6

$$\begin{array}{r} 8200000 \\ \underline{+400000} \\ 8600000 \end{array}$$

Likewise y – x = 8200000 – 4000000 = 7800000
= 7.8×10^6

$$\begin{array}{r} 8200000 \\ \underline{-400000} \\ 7800000 \end{array}$$

Examination Question:

Work out $4\times10^8 - 4\times10^6$. Give your answer in standard form

Examination Style Question

Work out $4.57\times10^{-4} - 6.3\times10^{-5}$. Give your answer as an ordinary number.

Problems involving multiplication and division:
To multiply and divide numbers in standard form without a calculator, there is no need to change them to ordinary numbers first.

Example
If x = 4×10^5, y = 8×10^6 and z = 5×10^{-2} work out (a) x × y (b) y ÷ z (c) x ÷ z **giving your answers in standard form**

Solution
(a) x × y = $4\times10^5\times8\times10^6 = (4\times8)\times(10^5\times10^6)$ (i.e. reorder so that you multiply together the numbers and the powers of 10)

$= 32\times10^{11}$ (remember you **add the powers** together when you multiply indices)

But this is not yet in standard form as 32 does not lie between 1 and 10. We write 32 in standard form as 3.2×10^1, so that

x × y = $3.2\times10^1\times10^{11} = 3.2\times10^{12}$

(b) y ÷ z = $\dfrac{8\times10^6}{5\times10^{-2}} = \dfrac{8}{5}\times\dfrac{10^6}{10^{-2}}$ (reorder so that you divide the numbers and the powers of 10)

$= 1.6\times10^8$ (remember that you subtract the powers when you divide indices and 6 - -2 = 8)

This number is in standard form.

(c) x ÷ z = $\dfrac{4\times10^5}{5\times10^{-2}} = \dfrac{4}{5}\times\dfrac{10^5}{10^{-2}} = 0.8\times10^7$

This number is not yet in standard form as 0.8 is not between 1 and 10. So we write 0.8 in standard form as 8×10^{-1}.

So x ÷ z = $(8\times10^{-1})\times10^7 = 8\times10^6$ (adding the powers of 10 together).

Examination Question
A = 3×10^4 B = 5×10^{-2}
(a) Calculate A × B. Give your answer in standard form.
(b) Calculate A ÷ B. Give your answer in standard form.

Examination Style Question

Carry out these calculations without a calculator, giving your answers in standard form.

(a) $(5 \times 10^8) \times (6 \times 10^7)$ (b) $(3 \times 10^8) \div (6 \times 10^{11})$

Examination Question (Non-calculator paper)

$$p = 8 \times 10^3$$

$$q = 2 \times 10^4$$

(a) Find the value of p × q. Give your answer in standard form.

(b) Find the value of p + q. Give your answer in standard form.

Examination Question (Non-calculator paper)

(a) Write 84,000,000 in standard form.

(b) Work out $\dfrac{84000000}{4 \times 10^{12}}$. Give your answer in standard form.

Mixed Questions

Calculator paper examination question

420000 carrot seeds weight 1 gram.
Each carrot seed weighs the same.
(a) Write the number 420000 in standard form.
(b) Calculate the weight in grams of one carrot seed. Give your answer in standard form correct to 2 significant figures.

Non-calculator paper examination question

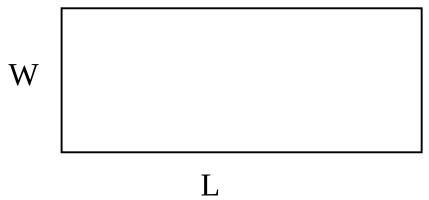
(a) i) Write the number 5.01×10^4 as an ordinary number.
ii) Write the number 0.0009 in standard form.
(b) Multiply 4×10^3 by 6×10^5. Give your answer in standard form.

Revision Topic 12: Area and Volume

Area of simple shapes

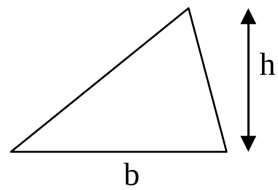
You need to **learn ALL** of the following area formulae:

Rectangle



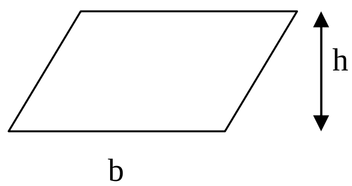
Area = length × width

Triangle



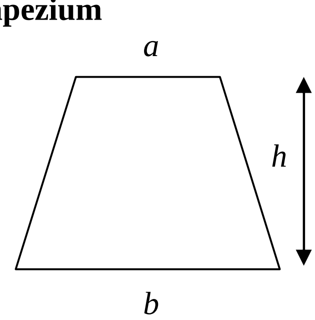
Area = $\dfrac{\text{base} \times \text{height}}{2}$ = ½ b×h

Parallelogram



Area = base × height = bh

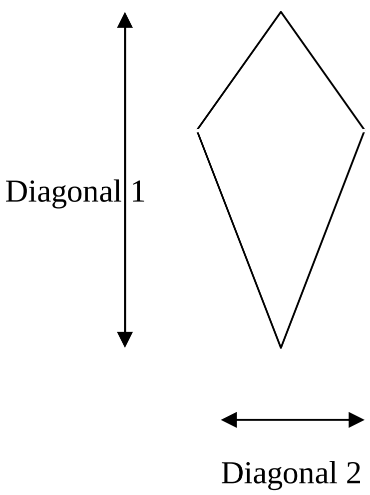
Trapezium



Area = $\dfrac{\text{sum of parallel sides}}{2} \times \text{height}$

$= \dfrac{1}{2}(a+b)h$

You also need to know how to find the area of a **kite**:

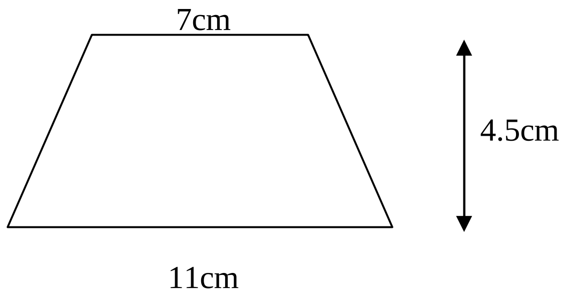


You can find the area of a kite by splitting it into triangles.

Alternatively, you can find the area of a kite by multiplying the two diagonals together and then dividing by 2.

Area of a kite = $\dfrac{1}{2} \times \text{product of diagonals}$

Example 1:
Find the area of the following trapezium.



To find the area,	you add the parallel sides	7 + 11 = 18
	you multiply by the height	18 × 4.5 = 81
	you divide by 2	81 ÷ 2 = 40.5 cm^2

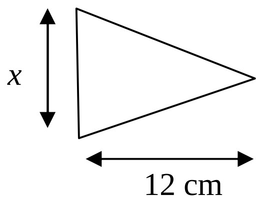
Alternatively, you can use the formula:

$$\text{Area} = \frac{1}{2}(a+b)h = \frac{1}{2}(7+11)\times 4.5 = \frac{1}{2}\times 18\times 4.5 = 40.5\ \text{cm}^2$$

Note: In all area questions it is important to show your **method** and to give **units**.

Example 2:

If the area of this triangle is 24cm^2, find *x*.

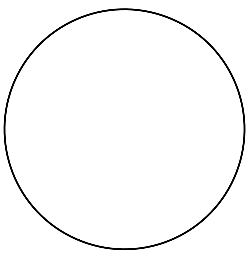


The formula for the area of a triangle is $A = \frac{1}{2}b \times h$.

Here the triangle has been rotated – the height is 12cm and we need to find the base, x.

So…	$A = \frac{1}{2}b \times h$
	24 = ½ × x × 12
Double both sides…	48 = x × 12
So…	x = 4 cm.

Circles



You need to **learn** the following:

Circumference of a circle = πD
Area of a circle = πr^2

where D is the diameter of the circle and r is the radius ($D = 2\times r$).

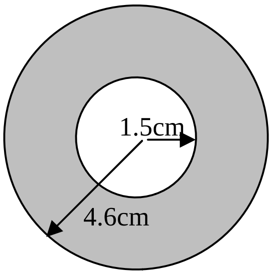
Example 3:
A circle has a circumference of 50cm. Calculate the radius of the circle.

Solution

The formula for the circumference is…	$C = \pi D$
So….	$50 = \pi \times D$
Therefore…	$D = 50 \div \pi$
So…	$D = 15.915$ cm

The radius of the circle is $15.915 \div 2 = 7.96$ cm

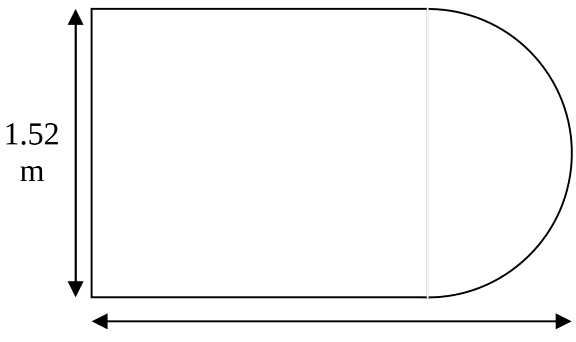
Example 4:
Calculate the percentage of the diagram below that is shaded.



The area of the larger circle is $\pi r^2 = \pi \times 4.6^2 = 66.48\,\text{cm}^2$
The area of the smaller circle is $\pi r^2 = \pi \times 1.5^2 = 7.07\,\text{cm}^2$
So the area shaded is $66.48 - 7.07 = 59.41\ \text{cm}^2$

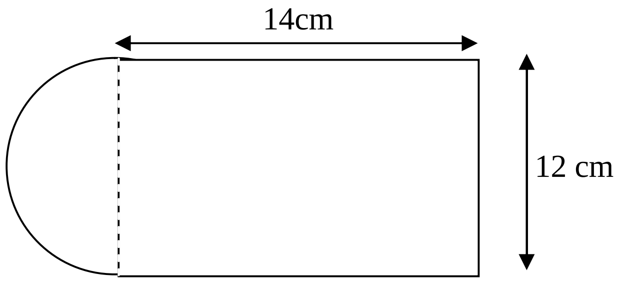
Therefore the percentage shaded is $\frac{59.41}{66.48} \times 100 = 89\%$ (to the nearest whole number).

Examination Question 1



A mat is made in the shape of a rectangle with a semicircle added at one end.
The width of the mat is 1.52 metres.
The length of the mat is 1.86 metres.
Calculate the area of the mat, giving your answer correct to 2 decimal places.

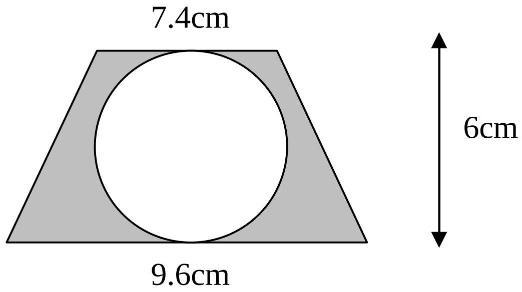
Examination Question 2



The diagram shows a shape, made from a semi-circle and a rectangle.
The diameter of the semi-circle is 12cm.
The length of the rectangle is 14cm.

Calculate the perimeter of the shape. Give your answer to 3 significant figures.

Examination style question 3

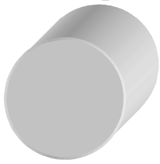


Calculate the shaded area in the above diagram.

Volume of prisms

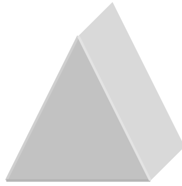
A **prism** is a three dimensional shape with a cross-section that is the same all the way through the shape.

These are all examples of prisms:

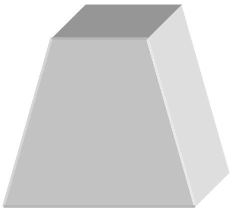


Cylinder
(circular cross-section)

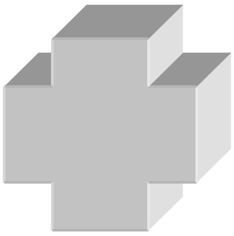
Cuboid
(rectangular cross-section)



Triangular prism



Prism with a trapezium-shaped cross-section

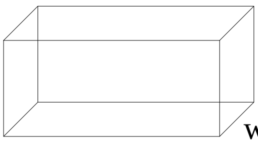


Prism with a cross-shaped cross-section

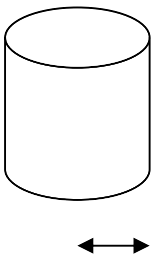
At grade B/C level, you need to be able to work out the volume of *any* prism. The formula is

$$\boxed{\text{Volume of prism} = \text{cross-sectional area} \times \text{length}}$$

Cuboids and cylinders have their own formulae, which are special cases of the general formula above:

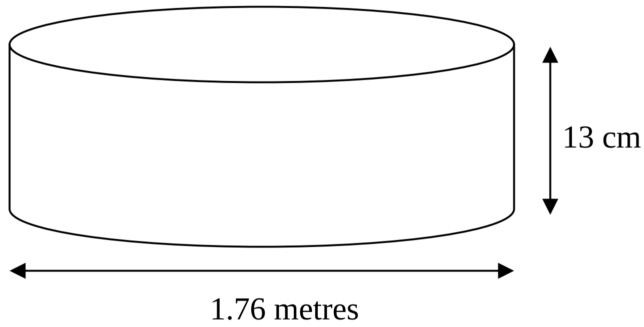


$$\boxed{\text{Volume of a cuboid} = \text{length} \times \text{width} \times \text{height}}$$



$$\boxed{\text{Volume of a cylinder} = \pi r^2 h}$$

Worked examination question 1:



A cylinder is 13 cm high.
The diameter of the base is 1.76 m.
Calculate the volume in cm^3 of the cylinder. Give your answer correct to 3 significant figures.

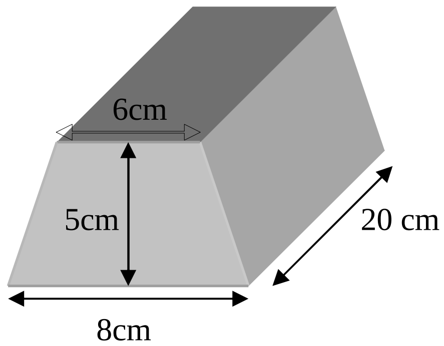
The formula for the volume of a cylinder is $\pi r^2 h$ (**remember this**!!).
The diameter is 1.76 m or 176 cm (we want lengths in cm as volume must be given in cm^3).
So the radius is 176 ÷ 2 = 88 cm.
As the height is 13 cm, the volume must be

$$\pi \times 88^2 \times 13 = 316270 \text{ cm}^3 \quad \text{or} \quad 316000 \text{ cm}^3 \text{ (to 3 sf).}$$

Density
Some questions on volume relate to *density*. You need to know the formula:

$$\text{Density} = \frac{\text{mass}}{\text{volume}} \qquad \text{or} \qquad \text{mass} = \text{density} \times \text{volume}$$

Worked examination question 2:



The diagram shows a prism.
The cross-section of the prism is a trapezium.
The lengths of the parallel sides of the trapezium are 8 cm and 6 cm.
The distance between the parallel sides of the trapezium is 5 cm.
The length of the prism is 20 cm.

(a) Work out the volume of the prism.
(b) The prism is made out of gold. Gold has a density of 19.3 grams per cm^3. Work out the mass of the prism. Give your answer in kilograms.

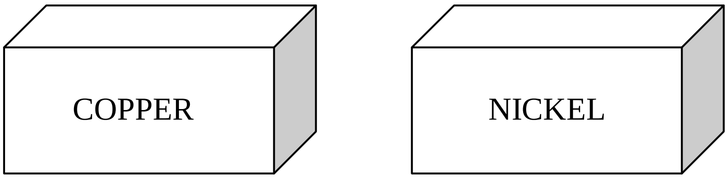
(a) To find the volume, we must first find the cross-sectional area (i.e the area of the trapezium).

Area of trapezium = $\frac{1}{2}(a+b)h = \frac{1}{2}(6+8)\times 5 = \frac{1}{2}\times 14\times 5 = 35\,\text{cm}^2$.

Volume of prism = cross-sectional area × length = 35 × 20 = 700 cm^3.

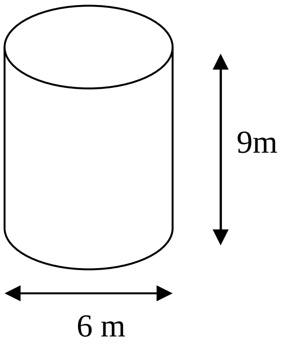
(b) Mass = density × volume = 19.3 × 700 = 13510 grams.
The answer must be in kilograms, so 13510g = 13.51 kg.

Examination Question 1:



These two metal blocks each have a volume of $0.5m^3$.
The density of the copper block is 8900 kg per m^3.
The density of the nickel block is 8800kg per m^3.
Calculate the difference in the masses of the blocks.

Examination Question 2:
On a farm, wheat grain is stored in a cylindrical tank.
The cylindrical tank has an internal diameter of 6 metres and a height of 9 metres.

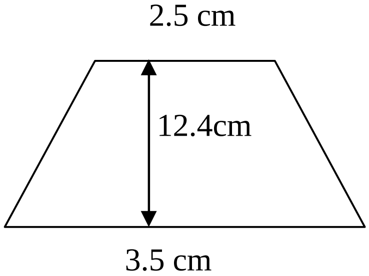


a) Calculate the volume, in m^3, of the tank. Give your answer correct to 2 decimal places.

1 m^3 of wheat grain weights 0.766 tonnes.

b) Calculate the weight, in tonnes, of the weight grain in the storage tank when it is full.

Examination Question 3:
A gold bar is in the shape of a prism of length 10cm.
It has a cross-section in the shape of this trapezium.
Calculate the volume of the bar.



Revision Topic 13: Statistics 1

Averages

There are three common types of average: the *mean*, *median* and *mode*.
The *mode* (or *modal value*) is the data value (or values) that occurs the *most* often.
The *median* is the *middle* value, once all the data has been written in order of size. If there are n values in a list, the median is in position $\frac{n+1}{2}$.
The mean is found by *adding* all the data values together and *dividing* by the total frequency.

When to use each type of average

- The mode is particularly useful for non-numerical data (such as eye colour or make of car) – it is the only average that can be found for data that is not numerical. It is also frequently used for sizes of clothes.
- The mean is used when the data are reasonably symmetrical with no anomalous (or outlying values). It is the most commonly used average.
- The median is useful when that data are skewed or when there are anomalous values in the data.

Grade C example for the mean

A class took a test. The mean mark for the 20 boys in the class was 17.4. The mean mark for the 10 girls in the class was 13.8.
(a) Calculate the mean mark for the whole class.

5 pupils in another class took the test. Their marks, written in order, were 1, 2, 3, 4 and x.
The mean of these 5 marks is twice the median of these marks.
(b) Calculate the value of x.

Solution

(a) To find the mean mark for the whole class we divide the total of all the class's marks by 30 (since 30 is the number of pupils in the class).

The total of the boys marks is $17.4 \times 20 = 348$ (as mean = $\frac{\text{total}}{\text{frequency}}$
so total = mean × total frequency)
The total of the girls marks is $13.8 \times 10 = 138$.
Therefore the total for the whole class is $348 + 138 = 486$
So… the mean mark for the class is $486 \div 30 = 16.2$.

(b) As the marks are written in order, the median mark is 3 (i.e. the middle mark).
So the mean of all 5 marks must be 6 (as it is twice the median).
Therefore the total of all 5 marks must be $6 \times 5 = 30$.
The sum of the first four numbers is $1 + 2 + 3 + 4 = 10$.
So, the value of x must be $30 – 10 = 20$.

Examination question 1:
27 boys and 34 girls took the same test.
The mean mark of the boys was 76. The mean mark of the girls was 82.

Calculate the mean mark of all these students. Give your answer correct to 1 decimal place.

Examination question 2 (non-calculator paper):
A shop employs 8 men and 2 women.
The mean weekly wage of the 10 employees is £396. The mean weekly wage of the 8 men is £400.
Calculate the mean weekly wage of the 2 women.

Finding the mean, median and mode from a table

Example 1: Frequency table
The table shows the boot sizes of players on a rugby team.

Boot size	Frequency
8	3
9	5
10	6
11	1

Find a) the mode b) the median, c) the mean boot size.

Solution

a) The mode is the boot size that occurs most often. The modal size is therefore size 10.

b) The total number of rugby players is 15 (add up the frequency column).

The median is the value in position $\frac{n+1}{2} = \frac{15+1}{2} = 8$.

The first three players have size 8; the next 5 have size 9. Therefore the 8th player has size 9 boots.

c) To find the mean, we add an extra column to the table:

Boot size, *x*	**Frequency, *f***	**Boot size × freq, $x \times f$**
8	3	8 × 3 = 24
9	5	9 × 5 = 45
10	6	10 × 6 = 60
11	1	11 × 1 = 11
TOTAL	**15**	**140**

This is the total of the boot sizes of all 15 players

You find the mean by dividing the total of the x × f column divided by the total of the frequency column:

Mean = 140 ÷ 15 = 9.33 (to 3 sf).

Examination question
20 students took part in a competition. The frequency table shows information about the points they scored.

Points scored	Frequency
1	9
2	4
3	7

Work out the mean number of points scored by the 20 students.

Example 2: Grouped frequency table

The table shows the masses of a group of children.

(a) Calculate an estimate of the mean mass.
(b) Find the modal interval.
(c) Find the interval that contains the median.

Mass (kg)	Frequency
40 – 50	3
50 – 60	10
60 – 70	6
70 - 80	12

Note: We cannot find the exact value of the mean from a grouped frequency table as we do not know the actual values of the data are not known. We can estimate the mean if we assume that all the values in each class are equal to the mid-point of that class.

(a) We add 2 further columns to the table:

The middle of the interval 40-50 is 45

Mass (kg)	Frequency, *f*	Midpoint, *x*	*x* × *f*
40 – 50	3	45	3×45 = 135
50 – 60	10	55	550
60 – 70	6	65	390
70 - 80	12	75	900
Total	**31**		**1975**

The mean is the total of the x × f column divided by the total of the frequency column:
1975 ÷ 31 = 63.7 kg (to 1 dp)

Note: We should check that our mean mass is sensible (i.e. lies within the range of the data).

(b) The interval that occurs the most often is 70-80 kg. This is the modal interval.

(c) The median is in position $\frac{n+1}{2} = \frac{31+1}{2} = 16$.

There are 13 children with weights below 60 kg and 19 children below 70 kg. Therefore the median weight lies within the 60 – 70 kg interval.

Examination question:

75 boys took part in a darts competition. Each boy threw darts until he hit the centre of the dartboard. The numbers of darts thrown by the boys are grouped in this frequency table.

Number of darts thrown	Frequency		
1 to 5	10		
6 to 10	17		
11 to 15	12		
16 to 20	4		
21 to 25	12		
26 to 30	20		

(a) Work out the class interval which contains the median.

(b) Work out an estimate for the mean number of darts thrown by each boy.

Cumulative frequency tables and curves

Cumulative frequency curves can be used to find the *median* for grouped tables. They can also be used to find the *interquartile range*.

The *interquartile range* is a measure of spread. It tells you how variable the data are. The interquartile range (IQR) is found using the formula

IQR = upper quartile – lower quartile

The interquartile range is a better measure of spread than the range because it is less effected by extreme values in the data.

Example:
A secretary weighed a sample of letters to be posted.

This interval is short for 50 – 60 g

Mass (g)	20 -	30 -	40 -	50 -	60 -	70 -	80 – 90
Number of students	2	4	12	7	8	17	3

Draw a cumulative frequency graph for the data
Use your graph to find the median weight of a letter and the interquartile range of the weights.

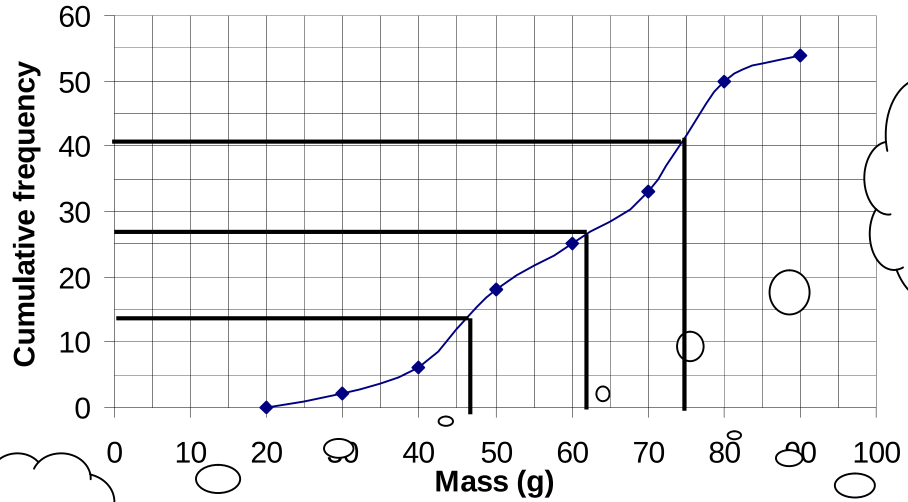
Solution:
We first need to work out the *cumulative frequencies* – these are a *running total* of the frequencies.

Mass (g)	Frequency	Cumulative frequency
20 – 30	2	2
30 – 40	4	6
40 – 50	12	18
50 – 60	7	25
60 – 70	8	33
70 – 80	17	50
80 – 90	4	54

There are 18 letters so far

There are 54 letters below 90g

We plot the cumulative frequency graph by plotting the cumulative frequencies on the vertical axis and the masses on the horizontal axis. *It is important that the cumulative frequencies are plotted above the* ***endpoint*** *of each interval.* So we plot the points (30, 2), (40, 6), (50, 18), …, (90, 54). As no letter weighed less than 20g, we can also plot the point (20, 0).



The median is about 62g

L.Q. is about 47g

U.Q. is about 75g

The total number of letters examined was 54. The median will be approximately the 54 ÷ 2 = 27th letter. We draw a line across from 27 on the vertical axis and then find the median on the horizontal axis. We see that the median is about 62g.

The lower quartile will be the $\frac{1}{4} \times 54 = 13.5th$ value. From the horizontal scale we find that the lower quartile is about 47g.

The upper quartile is the $\frac{3}{4} \times 54 = 40.5th$ value. This is about 75g.

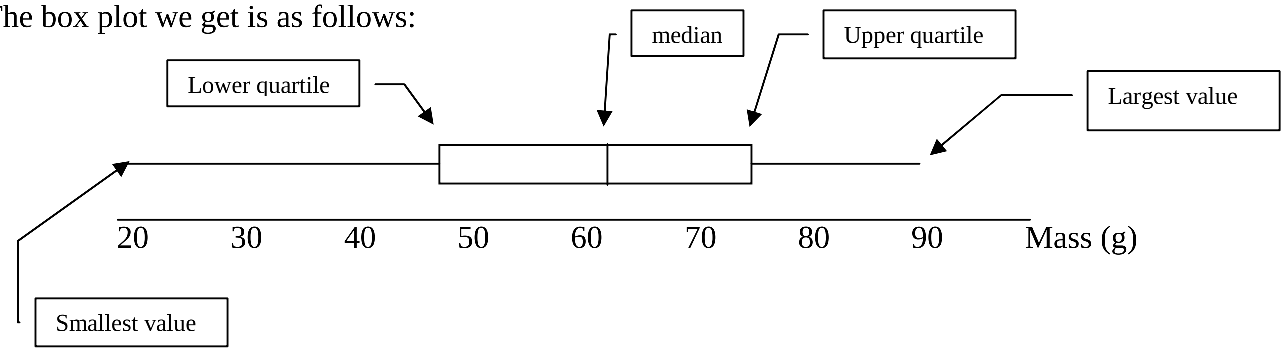
Therefore the interquartile range is U.Q – L.Q. = 75 – 47 = 28g.

Note: We can represent the data in the above example as a *box-and-whisker plot*. A box plot is a simple diagram that is based on 5 measurements:

- The lowest value
- The lower quartile
- The median
- The upper quartile
- The largest value.

In the example above we don't know the exact values of the lightest and heaviest letters. However we do know that no letter weighed less than 20g and no letter weighed more than 90g. So we take the lowest and largest values as 20g and 90g respectively.

The box plot we get is as follows:



Examination question

At a supermarket, members of staff recorded the lengths of time that 80 customers had to wait in the queues at the checkouts.

The waiting times are grouped in the frequency table below.

Waiting time (t seconds)	Frequency
$0 < t \le 50$	4
$50 < t \le 100$	7
$100 < t \le 150$	10
$150 < t \le 200$	16
$200 < t \le 250$	30
$250 < t \le 300$	13

(a) Complete the cumulative frequency table below.

Waiting time (t seconds)	Cumulative frequency
$0 < t \le 50$	
$50 < t \le 100$	
$100 < t \le 150$	
$150 < t \le 200$	
$200 < t \le 250$	
$250 < t \le 300$	

(b) On the grid below, draw a cumulative frequency graph for this data.

(c) Use your graph to work out an estimate for

the median waiting time,

the number of these customers who had to wait more than three minutes.

Examination Question
The grouped frequency table gives information about the weekly rainfall (d) in millimetres at Heathrow airport in 1995.

Weekly rainfall (d) in mm	**Number of weeks**		
$0 \leq d < 10$	20		
$10 \leq d < 20$	18		
$20 \leq d < 30$	6		
$30 \leq d < 40$	4		
$40 \leq d < 50$	2		
$50 \leq d < 60$	2		

a) Copy the table and complete it to calculate an estimate for the mean weekly rainfall.
b) Write down the probability that the rainfall in any week in 1995, chosen at random, was greater than or equal to 20mm and less than 40mm.
c) Copy and complete this cumulative frequency table for the data.

Weekly rainfall (d) in mm	**Cumulative frequency**
$0 \leq d < 10$	
$10 \leq d < 20$	
$20 \leq d < 30$	
$30 \leq d < 40$	
$40 \leq d < 50$	
$50 \leq d < 60$	

d) Draw a cumulative frequency graph to show the data.
e) Use your cumulative frequency graph to estimate the median weekly rainfall and the interquartile range.

Revision Topic 14: Algebra

Indices:

At Grade B and C levels, you should be familiar with the following rules of indices:

$y^a \times y^b = y^{a+b}$	i.e. add powers when multiplying;
$y^a \div y^b = y^{a-b}$	i.e. subtract powers when dividing;
$(y^a)^b = y^{ab}$	i.e. when you have a power of a power, multiply powers together.
$y^0 = 1$	i.e. anything to the power 0 is 1.

Example:

Simplify each of the following, if possible:

a) $a^4 \times a^2 \times a^3$ b) $\dfrac{c^4 \times c}{c^2}$ c) $(4x)^0$ d) $\left(e^3\right)^2$ e) $a^3 \times b^2$

Solution:

a) $a^4 \times a^2 \times a^3 = a^{4+2+3} = a^9$ (add powers together).

b) $\dfrac{c^4 \times c}{c^2} = \dfrac{c^4 \times c^1}{c^2} = \dfrac{c^5}{c^2}$ simplifying top

$= c^3$ subtracting powers

c) $(4x)^0 = 1$ (anything to the power 0 is 1)

d) $\left(e^3\right)^2 = e^6$ (multiply powers together)

e) $a^3 \times b^2$ this cannot be simplified as the base numbers (a and b) are different letters.

Example 2:

Work out the value of each of the following:

a) 3^4 b) 7^0 c) $2^9 \div 2^6$

Solution:

a) $3^4 = 3 \times 3 \times 3 \times 3 = 81$

b) $7^0 = 1$ (anything to the power 0 is 1)

c) $2^9 \div 2^6 = 2^{9-6} = 2^3 = 2 \times 2 \times 2 = 8$

Example 3:

Simplify: a) $4x \times 3x^4$ b) $\dfrac{15y^7}{3y^2}$ c) $6a^3b^2 \times 4a^2b^4$

Solution:

a) $4x \times 3x^4 = 4 \times 3 \times x \times x^4 = 12x^5$ (i.e. multiply together the numbers and add the powers)

b) $\dfrac{15y^7}{3y^2} = \dfrac{15}{3} y^{7-2} = 5y^5$ (i.e. divide numbers and subtract powers)

c) $6a^3b^2 \times 4a^2b^4 = (6 \times 4) \times (a^3 \times a^2) \times (b^2 \times b^4) = 24a^5b^6$.

Examination Style Question

Find the value of x in each of the following:

a) $7^6 \times 7^3 = 7^x$ b) $7^6 \div 7^3 = 7^x$ c) $(7^6)^3 = 7^x$ d) $7^0 = x$

Examination Style Question 2:

Simplify fully each of these expressions. Leave your answers in power form.

a) $3^2 \times 3^3$ b) $4^{-2} \times 4^5$ c) $5^6 \div 5^3$ d) $9^4 \div 9^{-2}$ e) $\dfrac{2^3 \times 2}{2^6}$.

Examination Style Question 3:

Simplify each of the following expressions.

a) $2a^3 \times 3a$ b) $6x^8 \div 3x^2$ c) $4x^3y^2 \times 5x^2y^2$

Expanding brackets

Expanding out a single bracket: You can remove a single bracket by multiplying everything inside the bracket by the number, or expression, on the outside.

Example

Expand the following brackets:

a) $6(7d - 4)$ b) $y(8y - 2x + 1)$ c) $5x(2x^2 + 3x - 2)$ d) $2xy(4x - y)$

Solution:

a) $6(7d - 4)$: Multiply both the $7d$ and the 4 by the number on the outside:

We get $6(7d - 4) = 42d - 24$

b) $y(8y - 2x + 1)$: Multiply everything in the bracket by y:

We get: $y(8y - 2x + 1) = 8y^2 - 2xy + y$

c) $5x(2x^2 + 3x - 2)$: Multiply all the terms inside the bracket by $5x$:

We get $5x(2x^2 + 3x - 2) = 10x^3 + 15x^2 - 10x$

d) $2xy(4x - y)$: Multiply the $4x$ and y by $2xy$:

This gives: $2xy(4x - y) = 8x^2y - 2xy^2$.

You need to take care when there is a minus sign in front of a bracket.

Example 2:

Expand and simplify:

a) $-4(2x - 3y)$ b) $6x(3x + 2) - 3(5x - 2)$

Solution:

a) Here we multiply everything in the bracket by -4.

This gives: $-4(2x - 3y) = -8x + 12y$

$-4 \times -3y = +12y$
as two minuses
multiply to
make a plus!

b) If we multiply out the first bracket we get: $6x(3x + 2) = 18x^2 + 12x$

If we multiply out the second bracket, we get: $- 3(5x - 2) = -15x + 6$.

Putting it all together: $6x(3x + 2) - 3(5x - 2) = 18x^2 + 12x - 15x + 6 = 18x^2 - 3x + 6$.

Examination Question:

a) Multiply out: $t^2(t^3 - t^4)$.

b) Multiply out and simplify: $3(2a + 6) - 2(3a - 6)$

c) Simplify: $\dfrac{12a^2b}{4ab}$.

Expanding out double brackets: When there is a pair of brackets multiplied together, you need to multiply everything in the first bracket by everything in the second.

Example: Multiply out the following brackets:

a) $(3x - 2)(x + 4)$ b) $(2x - 3y)(2x - 4)$ c) $(2x + 3y)^2$

Solution:

a) $(3x - 2)(x + 4)$

We can expand these brackets directly, multiplying everything in the first bracket by the terms in the second bracket. This gives:

$$(3x - 2)(x + 4) = 3x^2 + 12x - 2x - 8 = 3x^2 + 10x - 8.$$

Alternatively, you can draw a grid to help expand the brackets:

×	$3x$	-2
x	$3x^2$	$-2x$
4	$12x$	-8

Adding the numbers inside the grid gives:

$(3x - 2)(x + 4)$

$= 3x^2 + 12x - 2x - 8$

$= 3x^2 + 10x - 8.$

b) $(2x - 3y)(2x - 4)$

The grid for this would look like:

×	$2x$	$-3y$
$2x$	$4x^2$	$-6xy$
-4	$-8x$	$+12y$

Adding the numbers inside the grid gives:

$(2x - 3y)(2x - 4) = 4x^2 - 6xy - 8x + 12y$

This cannot be simplified!

c) $(2x + 3y)^2$

To square a bracket, you multiply it by itself! Drawing a grid:

×	$2x$	$3y$
$2x$	$4x^2$	$6xy$
$3y$	$6xy$	$9y^2$

Adding the expressions inside the grid:

$(2x + 3y)(2x + 3y) = 4x^2 + 6xy + 6xy + 9y^2$

$= 4x^2 + 12xy + 9y^2$

Examination Questions:

Expand and simplify:

1) $(2x - y)(3x + 4y)$
2) $(2x - 5)(x + 3)$
3) $(3x - y)^2$

Factorising

Factorising is the reverse of multiplying out brackets, i.e. when you factorise an expression you need to put brackets back into an expression.

Common factors: Some expressions can be factorised by finding common factors.

Example: Factorise the following expressions.
a) $12e + 18$ b) $2xy + 5x$ c) $x^2 - 6x$ d) $4x^2 + 2x$ e) $10x^2y - 15xy^2$.

a) We look for a common factor of 12e and 18. We notice that 6 goes into both of them. We therefore write 6 outside a bracket:

$$12e + 18 = 6(2e + 3)$$

6 goes into 12 twice

6 goes into 18 3 times

b) We notice that x appears in both $2xy$ and $5x$. This can be taken outside a bracket:

$$2xy + 5x = 2\times x\times y + 5\times x = x(2\times y + 5) = x(2y + 5)$$

c) As x^2 is $x\times x$, both x^2 and $6x$ have x as a factor.

So, $x^2 - 6x = x\times x - 6x = x(x - 6)$

d) Looking at the number parts, we notice that 2 is a common factor of both 4 and 2.

$$4x^2 + 2x = 2(2x^2 + x).$$

This hasn't been completely factorised yet, as both $2x^2$ and x also contain an x. We therefore can an x outside the bracket.

$$4x^2 + 2x = 2(2x^2 + x) = 2x(2x + 1).$$

x is $1x$, so when x is taken outside the bracket, we are left with 1 inside.

e) $10x^2y - 15xy^2$: Looking at the numbers, we see that both 10 and 15 have 5 as a factor. Both terms also have an x and a y in common. We can therefore factorise by writing $5xy$ in front of a bracket.

$$10x^2y - 15xy^2 = \underline{5}\times 2\times \underline{x}\times x\times \underline{y} - \underline{5}\times 3\times \underline{x}\times y\times \underline{y} = 5xy(2x - 3y)$$

Note: You can check your answers by expanding out the brackets.

Examination Question
Factorise completely: (a) $x^2 - 3x$ (b) $2p^2q + pq^2$.

Examination Question 2:
a) Expand and simplify: $5(3x + 1) - 2(4x - 3)$.
b) Expand and simplify: $(x + 5)(x - 3)$.
c) Factorise completely: $6a^2 - 9ab$.

Factorising quadratics

Simple quadratics like $x^2 + 3x + 2$ or $x^2 - 7x + 12$ can often be factorised into two brackets.

General steps for factorising $x^2 + bx + c$	**Example**: Factorise $x^2 + 9x + 18$
Step 1: Find two numbers that multiply to make c and add to make b.	Step 1: Find two numbers that multiply to make 18 and add to give 9. These numbers are 6 and 3.
Step 2: Write these two numbers in the brackets: $(x \quad)(x \quad)$	Step 2: The factorised expression is $(x + 9)(x + 3)$

Example 2: Factorise $x^2 - 7x + 12$

We need to find two numbers that multiply to make 12 and add to give -7. These numbers are -3 and -4.
So the answer is $(x - 3)(x - 4)$.

Example 3: a) Factorise $x^2 - 8x - 20$
b) Solve $x^2 - 8x - 20 = 0$

a) We have to find two numbers that multiply to make -20 and add to give -8. These are -10 and 2.
The factorised expressions is $(x - 10)(x + 2)$.

b) To solve the equation $x^2 - 8x - 20 = 0$ we use our factorised expression: $(x - 10)(x + 2) = 0$.
We have two brackets that multiply together to make 0. The only way this can happen is if one of the brackets is 0.
If the first bracket is 0, then $x - 10 = 0$, i.e. $x = 10$.
If the second bracket is 0, then $x + 2 = 0$, i.e. $x = -2$.
So the solutions are $x = 10$ and $x = -2$.

Examination question
Factorise $x^2 + 4x - 12$.
Hence or otherwise solve $x^2 + 4x - 12 = 0$.

Examination question:
a) Factorise $2x + 8y$.
b) Factorise completely $3ac^2 – 6ac$.
c) Factorise $x^2 - 9x + 18$.

Examination question:
a) Expand and simplify $(x + 5)(x – 3)$.
b) Factorise $x^2 - 5x - 14$.
c) Solve $x^2 - 5x - 14 = 0$.

Difference of two squares
When you expand out the brackets for $(x + a)(x – a)$ you get $x^2 + ax – ax – a^2$ which simplifies to $x^2 – a^2$.

The result $x^2 – a^2 = (x + a)(x – a)$ is called the difference of two squares result.

Examples:
1) $y^2 – 16 = y^2 – 4^2 = (y + 4)(y – 4)$.
2) $z^2 – 25 = z^2 – 5^2 = (z + 5)(z – 5)$.
3) $9x^2 – 4y^2 = (3x)^2 – (2y)^2 = (3x + 2y)(3x – 2y)$.

Examination question:
a) Factorise $x^2 – y^2$.
b) Use your answer to a) to work out the EXACT answer to
$123456789^2 – 123456788^2$.

Revision Topic 15: Algebra 2

Sequences

A **linear sequence** is one which goes up by the same amount each time.

E.g. 5, 8, 11, 14, 17, … is a linear sequence – it goes up in 3's.

E.g. 2 13, 11, 9, 7, 5, …is a linear sequence – it goes down in 2's.

nth term formula

The terms of a sequence can often be generated by a formula.

Example: The formula for the nth term of a sequence is 4n + 5.

a) Find the first 4 terms of the sequence.

b) Which term of the sequence is 49.

a) To get the first 4 terms of the sequence you have to replace *n* by 1, 2, 3, and 4.

The 1^{st} term of the sequence is $4 \times 1 + 5 = 9$

The 2^{nd} term of the sequence is $4 \times 2 + 5 = 13$

The 3^{rd} term of the sequence is $4 \times 3 + 5 = 17$

The 4^{th} term of the sequence is $4 \times 4 + 5 = 21$.

b) To see which term of the sequence is 49 we solve:

$4n + 5 = 49$

i.e $4n = 44$

i.e. $n = 11$

The 11^{th} term of the sequence is 49.

Finding the nth term of a formula

The formula for the nth term of a linear sequence is:

Difference between terms × n + The term in the sequence that would be before the 1^{st} term.

Example: Find an expression for the *n*th number in the sequence 7, 12, 17, 22, 27.

Solution: The sequence goes up in 5's.

The term before 7 in the sequence would be 2.

The nth term of the sequence is therefore 5n + 2.

Example 2: Find the formula for the nth term of the sequence 19, 16, 13, 10, …

Solution: The sequence goes down in 3's. The difference between terms is -3.

The term before 19 in the sequence would be 22.

The nth term of the sequence is therefore -3n + 22

Example 3: Find the formula for the nth term of the sequence: 1, 2.5, 4, 5.5, …

Solution: The sequence goes up in 1.5's. The previous term in the sequence would be -0.5. So the formula for the nth term is: 1.5n - 0.5.

Examination Question: Here are the first 4 terms of a number sequence: 3, 7, 11, 15.
a) Write down the next two terms of the sequence.
b) Write down an expression in terms of *n* for the *n*th term of the sequence.

Examination Question 2:
Marco writes down a number sequence.
He starts at 120.
Each time he subtracts 12 to get the next number in the sequence.
a) Write down the first 5 numbers in the sequence.
b) Write down an expression for the nth number in the sequence.

Simultaneous Equations

The equations $5x + 2y = 11$ and $3x - 4y = 4$ are known as simultaneous equations – there are two equations and two variables (x and y).
We solve simultaneous equations by trying to get the same number of x's or y's in the equations.

Example 1: Solve $5x + 2y = 11$ and $3x - 4y = 4$.

$5x + 2y = 11$ $3x - 4y = 4$	To make the number of y's the same we can multiply the top equation by 2.
$10x + 4y = 22$ $3x - 4y = 4$	The number of y's is the **same** but the signs are **different**. To eliminate the y's the equations must be **added**.
$13x = 26$ i.e. $\boldsymbol{x = 2}$	
Substitute x = 2 into 5x + 2y = 11 $5\times2 + 2y = 11$ $10 + 2y = 11$ $2y = 1$ $\mathbf{y = 0.5}$	To find the value of y, substitute $x = 2$ into one of the original equations.
Check: $3\times2 - 4\times0.5 = 6 - 2 = 4$ as required.	You can check the solution by substituting x = 2 and y = 0.5 into 3x – 4y = 4.

Example 2: Solve the simultaneous equations x + 3y = 13 and 4x + 2y = 2.

x + 3y = 13 4x + 2y = 2	To make the number of x's the same we can multiply the top equation by 4.
4x + 12y = 52 4x + 2y = 2	The number of x's is the **same** and the signs are **the same**. To eliminate the x's the equations must be **subtracted**.
10*y* = 50 i.e. ***y* = 5**	
Substitute y = 5 into x + 3y = 13 x + 3×5 = 13 x + 15 = 13 **x = -2**	To find the value of x, substitute *y* = 5 into one of the original equations.
Check: 4×-2 + 2×5 = -8 + 10 = 2 as required.	You can check the solution by substituting x = -2 and y = 5 into 4x + 2y = 2.

Example 3: Solve the simultaneous equations 2x + 3y = 7 and 3x – 2y = 17.

2x + 3y = 7 3x - 2y = 17	To make the number of y's the same we can multiply the top equation by 2 and the second equation by 3
4x + 6y = 14 9x - 6y = 51	The y's can be eliminated by adding the equations.
13*x* = 65 i.e. ***x* = 5**	
Substitute x = 5 into 2x + 3y = 7 2×5 + 3y = 7 10 + 3y = 7 3y = -3 **y = -1**	To find the value of y, substitute *x* = 5 into one of the original equations.
Check: 3×5 - 2×-1 = 15 – (-2) = 17 as required.	You can check the solution by substituting x = 5 and y = -1 into 3x - 2y = 17.

Examination question 1:
Solve the simultaneous equations: 3x + y = 13
2x – 3y = 16

Examination question 2:
Solve the simultaneous equations: 2x + 5y = -1 and 6x – y = 5

Examination question 3:

Solve the simultaneous equations: $x + 8y = 5$

$3x - 4y = 8$

Simultaneous Equations: Problems

Example: Two groups visited Waterworld. The first group of four adults and three children paid a total of £38 for their tickets. The second group of five adults and two children paid £40.50 for their tickets. What are the charges for adult and child tickets at Waterworld?

Solution: Let a be the cost for an adult and c the cost for a child.

Group 1: $4a + 3c = 38$
Group 2: $5a + 2c = 40.50$

Multiply top equation by 2 and bottom equation by 3: $8a + 6c = 76$

$15a + 6c = 121.50$

Subtract the bottom equation from the top equation: $7a = 45.50$

So, $a = 6.50$

From the equation for group 1: $4\times6.50 + 3c = 38$

$3c = 12$

$c = 4$

So adults pay £6.50 and children £4.

Examination Question:

Mrs Rogers bought 3 blouses and 2 scarfs. She paid £26.
Miss Summers bought 4 blouses and 1 scarf. She paid £28.
The cost of a blouse was x pounds. The cost of a scarf was y pounds.

a) Use the information to write down two equations in x and y.

b) Solve these equations to find the cost of one blouse.

Revision Topic 17: Probability

Estimating probabilities: Relative frequency

Probabilities can be estimated from experiments. The relative frequency is found using the formula:

$$\frac{\text{number of times event occurs}}{\text{total number of trials}}.$$

Example:
This data shows the colour of cars passing a factory gate one morning.
Estimate the probability that, at a random time, a car passing will be red.
If 400 cars pass the factory gates the following morning, estimate the number of cars that will be red.

Colour	**Frequency**
Red	68
Black	14
Yellow	2
Green	34
Blue	52
Grey	35
Other	23
Total	**228**

Solution

The relative frequency of red cars is $\frac{68}{228} = 0.30$ (to 2 d.p.)
So the probability of a red car is 0.30 or 30%.

The following morning 400 cars pass the factory gate. The number of red cars will be approximately 30% of 400 = 0.3 × 400 = 120.

Examination Style Question
The sides of a six-sided spinner are numbered from 1 to 6. The table shows the results for 100 spins.

Number on spinner	1	2	3	4	5	6
Frequency	27	18	17	15	16	7

a) What is the relative frequency of getting a 1?
b) Do you think the spinner is fair? Give a reason for your answer.
c) The spinner is spun 3000 times. Estimate the number of times the result will be a 4.

Probability diagrams

Example:

Kevin has a spinner in the shape of a regular pentagon, and a normal dice.
The five sections of the spinner are labelled 1, 2, 3, 4, 5.
Kevin spins the spinner once and rolls the dice once.
He records the outcomes.

a) Show all the outcomes in a probability diagram.
b) Find the probability that both of the numbers are prime;
c) Find the probability that the difference between the two numbers is 1.

Solution:

a) The outcomes can be shown in a table:

		Number on spinner				
		1	2	3	4	5
Number on dice	1	1,1	1,2	1,3	1,4	1,5
	2	2,1	**2,2**	**2,3**	2,4	**2,5**
	3	3,1	**3,2**	**3,3**	3,4	**3,5**
	4	4,1	4,2	4,3	4,4	4,5
	5	5,1	**5,2**	**5,3**	5,4	**5,5**
	6	6,1	6,2	6,3	6,4	6,5

b) The outcomes for which both numbers are prime are shown in bold in the table above. There are 30 outcomes in the table and 9 have both numbers prime. The probability therefore is $\frac{9}{30} = \frac{3}{10}$.

c) The outcomes for which the difference between the numbers is 1 are shaded. 9 outcomes have been shaded so the probability is also $\frac{9}{30} = \frac{3}{10}$.

Examination Question:

Jack has two fair dice.
One of the dice has 6 faces numbered from 1 to 6. The other dice has 4 faces numbered from 1 to 4.
Jack is going to throw the two dice. He will add the scores together tom get the total.
Work out the probability that he will get
a) a total of 7,
b) a total of less than 5.

Hint: Draw a diagram as in the example above.

Mutually Exclusive Events

Mutually exclusive events cannot occur at the same time.
When A and B are mutually exclusive events: P(A **OR** B) = P(A) + P(B).

Example:
A box contains red, green, blue and yellow counters.
The table shows the probability of getting each colour.

Colour	Red	Green	Blue	Yellow
Probability	0.4	0.25	0.25	0.1

A counter is taken from the box at random.
What is the probability of getting a red or blue counter?

Solution
P(red **or** blue) = P(red) + P(blue)
= 0.4 + 0.25
= 0.65

Examination Style Question:
A bag contains some red, some white and some blue counters. A counter is picked at random.
The probability that it will be red is 0.2.
The probability that it will be white is 0.3.

a) What is the probability that a counter picked at random will be either red or white?
b) What is the probability that a counter picked at random will be either red or blue?

Examination Style Question:
Some red, white and blue cubes are numbered 1 and 2.
The table shows the probabilities of obtaining each colour and number when a cube is taken at random.

	Red	**White**	**Blue**
1	0.1	0.3	0
2	0.3	0.1	0.2

A cube is taken at random.
a) What is the probability of taking a red cube?
b) What is the probability of taking a cube numbered 2?
c) State whether or not the following pairs of events are mutually exclusive. Give a reason for your answer.

(i) Taking a cube numbered 1 and taking a blue cube.
(ii) Taking a cube numbered 2 and taking a blue cube.

Independent Events

The outcomes of independent events do not influence each other.
When A and B are independent events: P(A **AND** B) = P(A) × P(B).

Example (examination question):
Nikki and Romana both try to score a goal in netball.
The probability that Nikki will score a goal on the first try is 0.65.
The probability that Ramana will score a goal on the first try is 0.8.

a) Work out the probability that Nikki and Ramana will both score a goal on their first tries.
b) Work out the probability that neither Nikki nor Ramana will score a goal on their first tries.

a) P(Nikki scores AND Ramana scores) = P(Nikki scores) × P(Ramana scores)
= 0.65 × 0.8
= 0.52.

b) P(Nikki doesn't score AND Ramana doesn't score) = P(Nikki doesn't) × P(Ramana doesn't)
= 0.35 × 0.2
= 0.07.

Examination Style Question
Samantha takes examinations in maths and English.
The probability that she passes maths is 0.7.
The probability that she passes English is 0.8.
The results in each subject are independent of each other.
Calculate the probability that

a) Samantha passes both subjects;
b) Samantha passes maths and fails English.

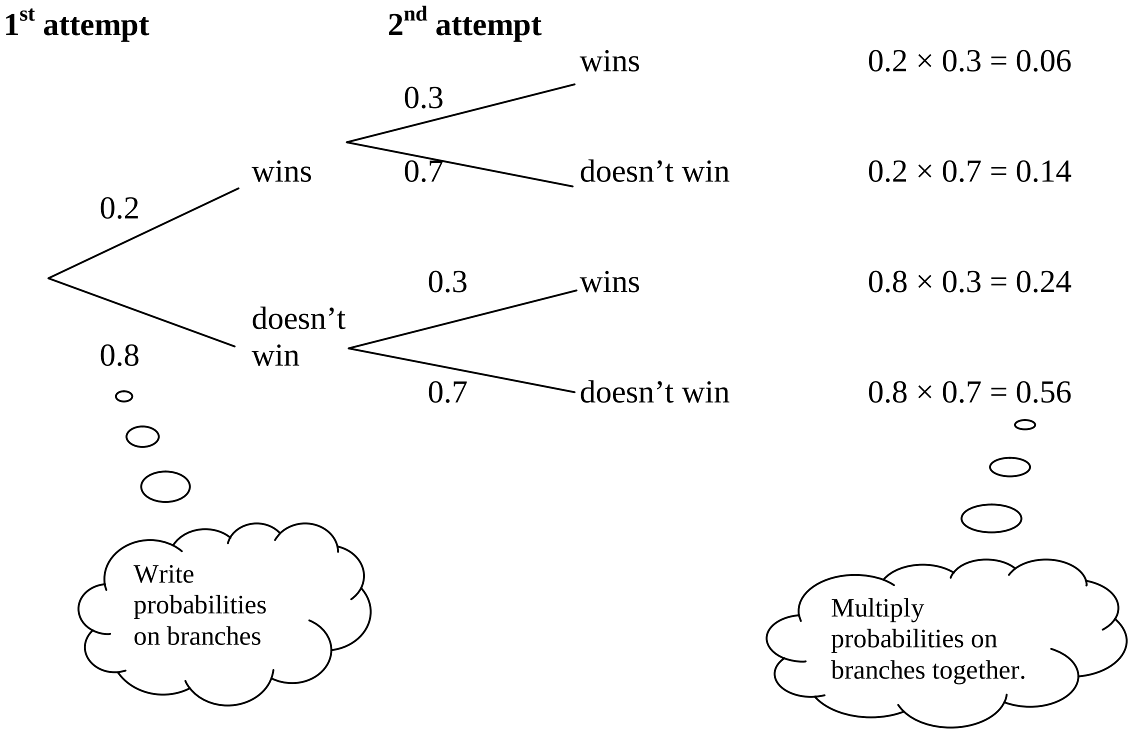
Tree Diagrams

A tree diagram is a way of calculating probabilities when two events are combined.

Example (worked examination question):
Helen tries to win a coconut at the fair. She throws a ball at a coconut. If she knocks the coconut off the standard she wins the coconut. Helen has two throws.
The probability that she wins a coconut with her first throw is 0.2.
The probability that she will win a coconut with her second throw is 0.3.
Work out the probability that, with her two throws, Helen will win

a) 2 coconuts;
b) exactly 1 coconut.

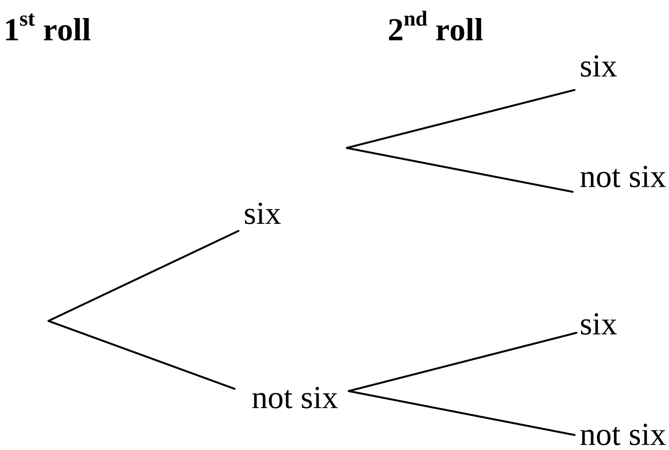
Solution:
We can draw a tree diagram to answer this question.



a) From the diagram, the probability that Helen will win 2 coconuts is 0.06.
b) P(Helen wins exactly 1 coconut) = 0.14 + 0.24 = 0.38.

Examination Question
Tina has a biased dice.
When she rolls it, the probability that she will get a six is 0.09.
Tina is going to roll the biased dice dice.
Complete the tree diagram shown.
Work out the probability that she will get:

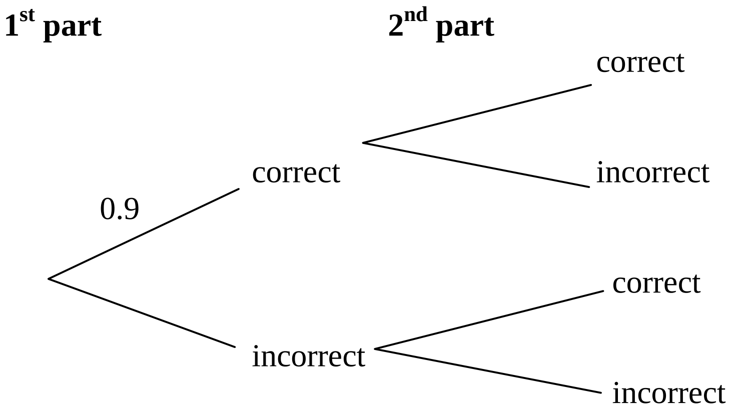
a) two sixes,
b) exactly one six.



Examination Question

A machine makes two parts which fit together to make a tool. The probability that the first part will be made correctly is 0.9. The probability that the second part will be made correctly is 0.95.

a) Complete the tree diagram below giving the missing probabilities.



b) Use the tree diagram to work out the probability that both parts will be made correctly.

Examination Question:

The probability of a person having brown eyes is ¼.

The probability of a person having blue eyes is 1/3.

Two people are chosen at random.

Work out the probability that

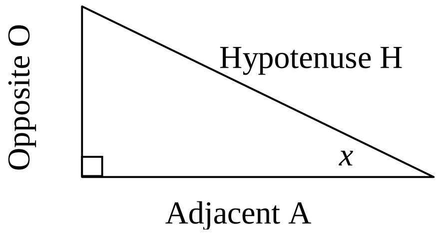
a) both people will have brown eyes;
b) one person will have blue eyes and the other person will have brown eyes.

Revision Topic 18: Trigonometry

Trigonometry connects the length of sides and angles in *right-angled* triangles.

Some important terms

In a right-angled triangle, the side opposite the right angle is called the *hypotenuse*. The hypotenuse is the longest side in a right-angled triangle.
The side opposite the angle of interest is called the *opposite*.
The third side (which is next to the angle of interest) is called the *adjacent*.



Sin, cos and tan

Three important formulae connect the lengths of O, A and H with the angle x:

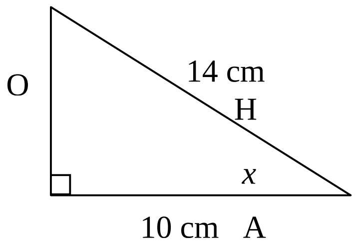
$$\sin x = \frac{O}{H} \qquad \cos x = \frac{A}{H} \qquad \tan x = \frac{O}{A}.$$

You need to learn these formulae! It sometimes helps to remember them as SOHCAHTOA.

Finding an angle

Trigonometry can be used to find angles in right-angled triangles.

Example 1:



Start by labelling the triangle H, O and A.
We then have to decide whether to use sin, cos or tan

SOHCAHTOA

Since we know the values of A and H, we use cos:

$$\cos x = \frac{A}{H}$$

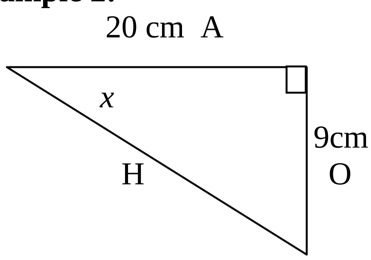
$$\cos x = \frac{10}{14} = 0.7143$$

To find the angle x, you have to press SHIFT cos 0.7143.
You get:

$x = 44.4^\circ$

Note: Write down several calculator digits in your working.

Example 2:



We start by labelling the triangle.
This time we will need to use tan as we know O and A:

SOHCAHTOA

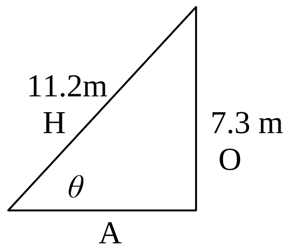
$$\tan x = \frac{O}{A}$$

$$\tan x = \frac{9}{20} = 0.45$$

To find angle x, you press SHIFT tan 0.45.
You get:

$x = 24.2^\circ$

Example 3:



The triangle is labelled H, O and A.
Because we know O and H, we need to use sin this time:

SOHCAHTOA

So,

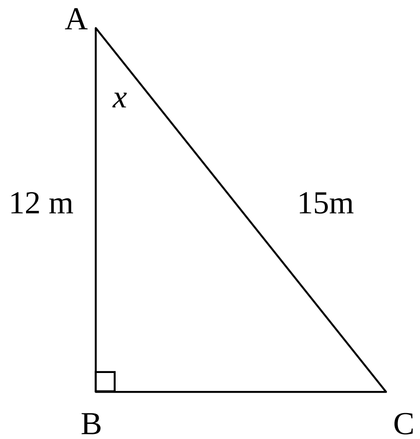
$$\sin\theta = \frac{O}{H}$$

$$\sin\theta = \frac{7.3}{11.2} = 0.6518$$

To recover angle θ, you press SHIFT sin 0.6518.
The answer is:

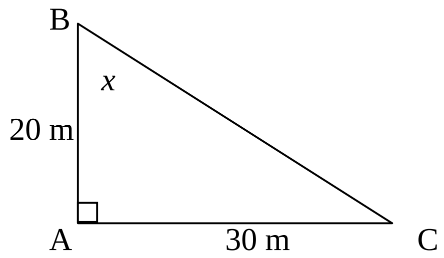
$\theta = 40.7^\circ$.

Examination Question 1:



Calculate the angle marked x.
Give your answer correct to one decimal place.

Examination Question 2:

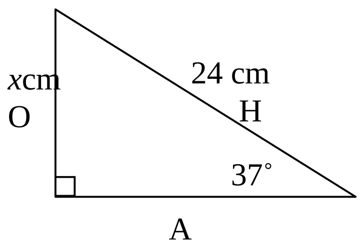


Work out the size of the angle marked x.

Finding a side

Trigonometry can also be used to find sides in right-angled triangles if you know one of the angles.

Example 1:



Start by labelling the triangle H, O and A.
We then have to decide whether to use sin, cos or tan

SOHCAHTOA

Since we want to find O and we know H, we use sin:

$$\sin 37 = \frac{O}{H}$$

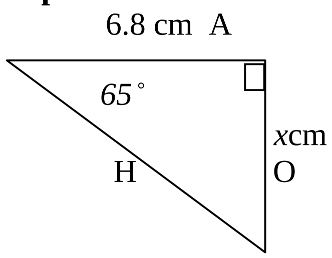
$$\sin 37 = \frac{x}{24}$$

$$0.6018 = \frac{x}{24}$$

So, $x = 24 \times 0.6018 = 14.4$ cm.

Remember to put a unit on the answer!!

Example 2:



We start by labelling the triangle.
This time we will need to use tan as we know A and want O:

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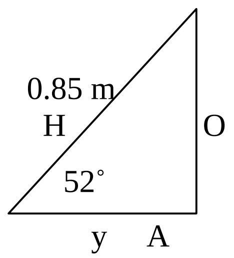
$$\tan 65 = \frac{O}{A}$$

$$\tan 65 = \frac{x}{6.8}$$

$$2.1445 = \frac{x}{6.8}$$

So, $x = 2.1445 \times 6.8 = 14.6$ cm.

Example 3:



The triangle is labelled H, O and A.
Because we know H and want A, we need to use cos this time:

SOHCAHTOA

So,

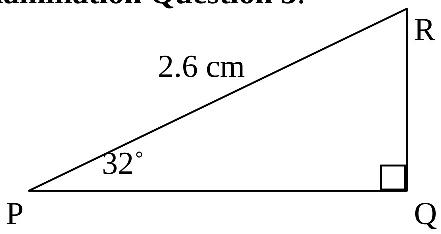
$$\cos 52 = \frac{A}{H}$$

$$\cos 52 = \frac{y}{0.85}$$

$$0.61566 = \frac{y}{0.85}$$

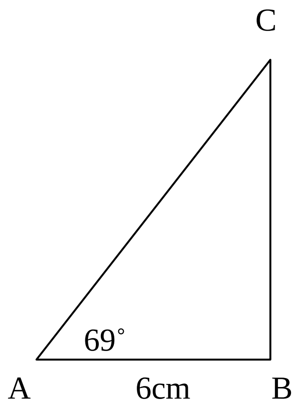
So,
$y = 0.85 \times 0.61566 = 0.523$ m.

Examination Question 3:



Angle Q = 90 degrees, angle P= 32 degrees and PR = 2.6 m. Calculate the length of QR. Give your answer in metres to 3 significant figures.

Examination Question 4:
Triangle ABC is shown on the right.
a) Calculate the length of the side BC.
b) Calculate the length of the side AC.



Finding the length of sides (continued)

Some side lengths are harder to calculate – they appear on the denominator of the fractions. Consider the following example:

Example:
Find the length x cm in this triangle:

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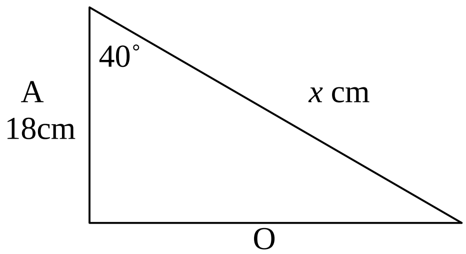


The triangle is labelled H, O and A as normal. Since we know A and want to find H we use cos.

$$\cos 40 = \frac{A}{H} \quad \text{so} \quad \cos 40 = \frac{18}{x} \quad \text{so} \quad 0.7660 = \frac{18}{x}$$

The side we wish to find appears on the denominator. There are two methods that we could use to find the value of x:

Method 1:

Write both sides as fractions: $\frac{0.7660}{1} = \frac{18}{x}$

Invert both sides: $\frac{1}{0.7660} = \frac{x}{18}$

Work out the left hand side: $1.305 = \frac{x}{18}$

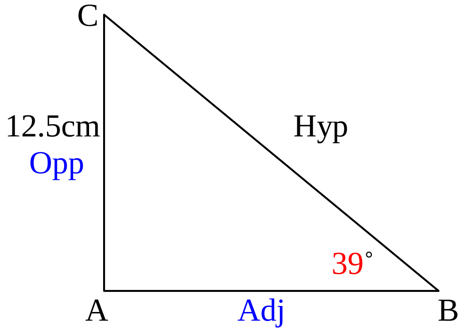
Therefore $x = 18 \times 1.305 = 23.5$ cm

Method 2:
Rearrange the equation by first getting rid of the fraction (multiply by x): $x \times 0.7660 = 18$
Get x on its own by dividing by 0.7660:

$$x = \frac{18}{0.7660} = 23.5 \text{ cm.}$$

We can check our answer – since we are finding the hypotenuse, it should be the longest side in the triangle.

Example 2:



Calculate the length of AB.

We need to use tan: SOHCAHTOA

$$\tan 39 = \frac{\text{Opp}}{\text{Adj}}$$

$$\tan 39 = \frac{12.5}{x}$$

$$0.8098 = \frac{12.5}{x}$$

As *x* appears on the bottom, we need to use one of the methods used in the previous example.

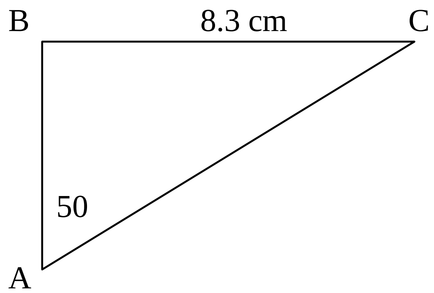
Eg:

$$\frac{1}{0.8098} = \frac{x}{12.5}$$

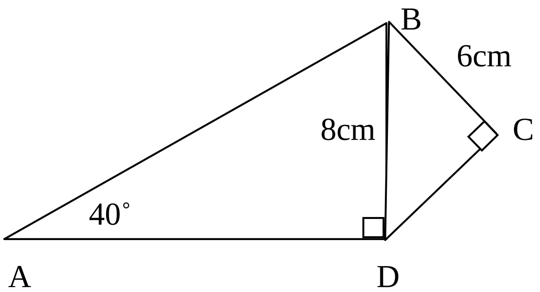
$$\text{So, } x = 12.5 \times \frac{1}{0.8098} = 15.4\text{cm}$$

Examination Question 5:

The triangle ABC has a right angle at B.
Angle BAC = 50 degrees and BC = 8.3 cm.
Calculate the length of AC.



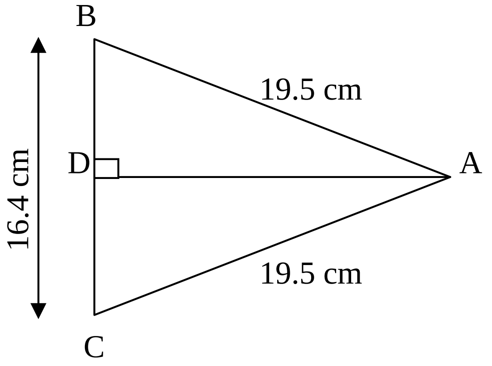
Examination Question 6:



ABCD is a quadrilateral. Angle BDA = 90˚, angle BCD = 90˚, angle BAD = 40˚.
BC = 6 cm, BD = 8cm.
a) Calculate the length of DC. Give your answer correct to 3 significant figures. [Hint: use Pythagoras' theorem!]
b) Calculate the size of angle DBC. Give your answer correct to 3 significant figures.
c) Calculate the length of AB. Give your answer correct to 3 significant figures.

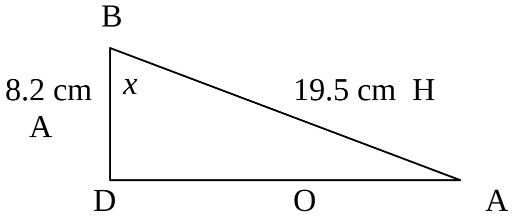
Trigonometry: Harder Questions:

Example:



AB = 19.5 cm, AC = 19.5 cm and BC = 16.4 cm.
Angle ADB = 90 degrees.
BDC is a straight line.
Calculate the size of angle ABC. Give your answer correct to 1 decimal place.

Because triangle ABC is isosceles we can just consider the top triangle BDA. Length BD would be half of 16.4 cm, i.e. 8.2 cm.

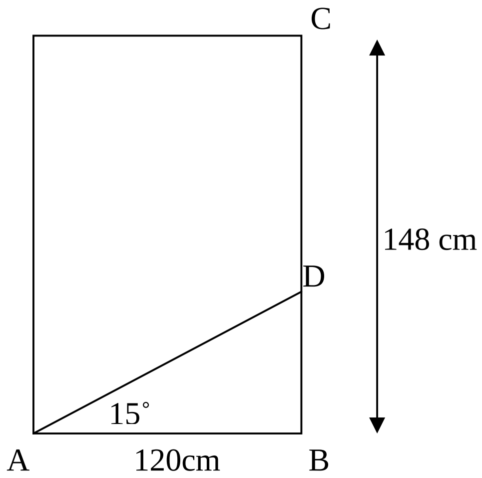


$$\cos x = \frac{A}{H}$$

$$\cos x = \frac{8.2}{19.5} = 0.4205$$

$$x = \text{SHIFT} \cos 0.4205 = 65.1^{\circ}$$

Examination Question 7:



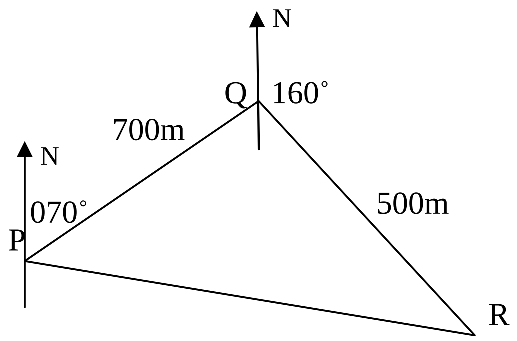
AB and BC are two sides of a rectangle.
AB = 120 cm and BC = 148 cm.
D is a point on BC.
Angle BAD = 15 degrees.

Work out the length of CD. Give your answer to the nearest centimetre.

Trigonometry and Bearings

Recall that bearings measure direction. They are angles that are measured clockwise from a north line. Bearings have three digits.

Example:
The diagram shows the path of a jet-ski from P to Q to R.

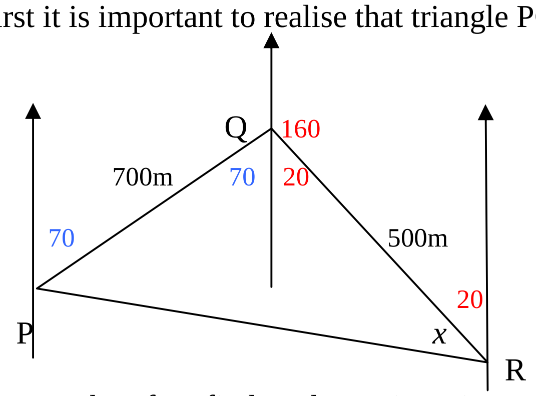


Q is 700m from P on a bearing of 070 degrees.

R is 500m from Q on a bearing of 160˚.

Calculate the bearing of P from R.

First it is important to realise that triangle PQR is a right-angled triangle (with the right angle at Q):



The blue angles are alternate (Z angles) so are equal.
The red angles add up to 180 degrees as they make a straight line.
Therefore angle Q is a right angle.

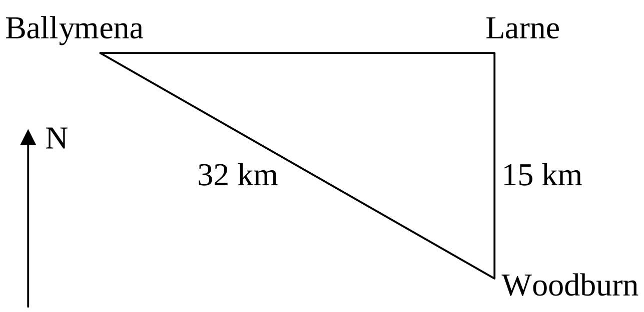
We can therefore find angle x using trigonometry:

$$\tan x = \frac{O}{A} = \frac{700}{500} = 1.4$$

$$x = \text{SHIFT } \tan 1.4 = 54.46^{\circ}$$

The angle of P from R is the clockwise angle measured from R to P. It is
360˚ - 20 - 54.46 = 285.54˚.
So the bearing is 286˚ (to nearest degree).

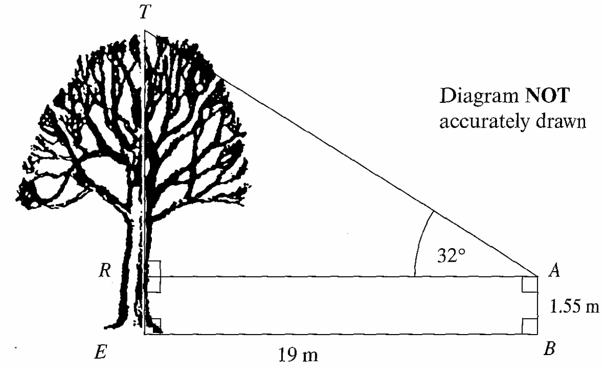
Examination Question



Ballymena is due west of Larne.
Woodburn is 15 km due south of Larne.
Ballymena is 32 km from Woodburn.
a) Calculate the distance of Larne from Ballymena, correct to 1 decimal place.
b) Calculate the bearing of Ballymena from Woodburn.

Angle of elevation

Worked examination question



Abbi is standing on level ground, at B, a distance of 19 metres away from the foot E of a tree TE. She measures the angle of elevation of the top of the tree at a height of 1.55 metres above the ground as 32˚.
Calculate the height TE of the tree. Give your answer correct to 3 significant figures.

Solution:
First find the length TR using trigonometry, specifically using tan:

$$\tan 32 = \frac{O}{A} = \frac{TR}{19}$$

$$0.6249 = \frac{TR}{19}$$

$$TR = 19 \times 0.6249 = 11.87 \text{ metres}$$

So, to get the height of the tree you need to add Abbi's height to this distance.
Therefore, height of tree = 11.87 + 1.55 = 13.4 metres (to 3 s.f.)

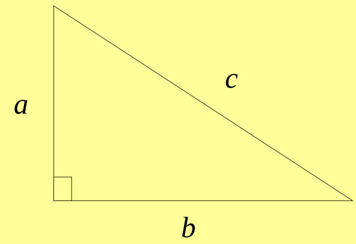
Revision Topic 20: Sine and Cosine Rules and 3D Trigonometry

The objectives of this unit are to:
* use the sine and cosine rules to find the length of sides and angles in triangles;
* to use the formula for the area of a triangles;
* to solve problems involving the sine and cosine rules;
* to solve problems involving trigonometry in 3 dimensions.

Brief recap of Grade B and C material:

Pythagoras' Theorem:
This theorem, which connects the lengths of the sides in right-angled triangles, states that:

$$a^2 + b^2 = c^2$$

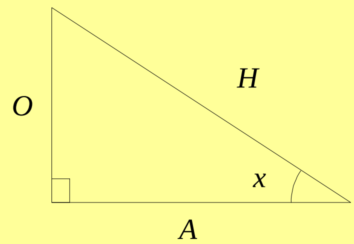


where c is the length of the hypotenuse (i.e. the side opposite the right-angle) and a and b are the lengths of the other two sides.

Note that the hypotenuse is the longest side in a right-angled triangle.

Trigonometry
The following formulae link the sides and angles in right-angled triangles:

$$\sin x = \frac{O}{H}$$
$$\cos x = \frac{A}{H}$$
$$\tan x = \frac{O}{A}$$



where H is the length of the **hypotenuse**;
O is the length of the side **opposite** the angle;
A is the length of the side **adjacent** to the angle.

These formulae are often remembered using the acronym SOHCAHTOA or by using mnemonics. Here is a commonly used mnemonic:
Silly Old Harry Couldn't Answer His Test On Algebra

When finding angles, remember that you need to use the SHIFT key.

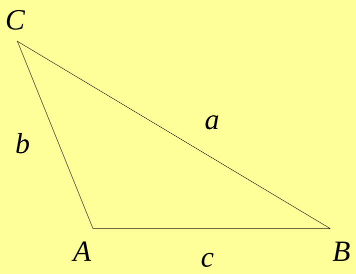
Further notes, examples and examination questions relating to Pythagoras' theorem and trigonometry are contained in separate revision booklets.

Sometimes you need to calculate lengths and angles in triangles which do not contain any right-angles. This is when the sine and cosine rules are useful.

Labelling a triangle

To use the sine and cosine rules, you need to understand the convention for labelling sides and angles in any triangle.

Consider a general triangle:

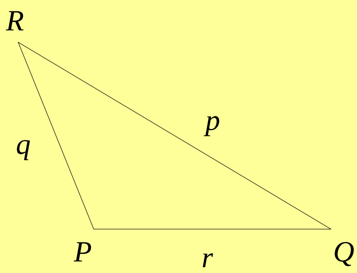


Triangles are named after their vertices - the above triangle is called triangle ABC.
The three angles are commonly referred to as angles A, B and C.
The length of the sides are given lower case letters:

Side a is the side opposite angle A. It is sometimes referred to as side BC.
Side b is the side opposite angle B. It is equivalently called side AC.
Side c is the side opposite angle C. It is also known as side AB.

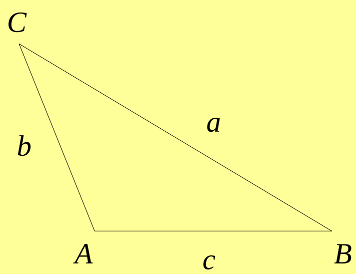
A triangle doesn't have to be labelled using the letters A, B and C.

For example, consider the triangle PQR below:



Sine Rule

The sine rule connects the length of sides and angles in any triangle ABC:



It states that:

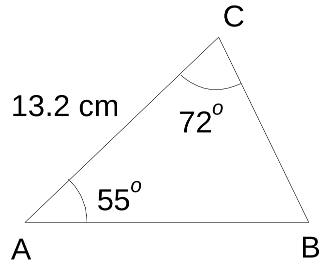
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

An alternative version of the formula is often used when finding the

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Example: Finding the length of a side

The diagram shows triangle ABC.
Calculate the length of side AB.

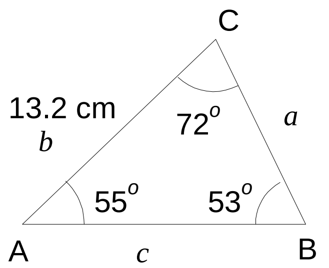


Solution:

To find the length of a side using the sine rule, follow these steps:

Step 1: Label the triangle using the conventions outlined earlier.
Step 2: Look to see whether any additional information can be added to the diagram (for example, can you deduce the length of any other angles?)
Step 3: Substitute information from the diagram into the sine rule formula.
Step 4: Delete the unnecessary part of the formula.
Step 5: Rearrange and then work out the length of the required side.

In our example, we begin by labelling the sides and by working out the size of the 3rd angle (using the fact that the sum of the angles in any triangle is 180°.



Substituting into the formula $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$, we get:

$$\frac{a}{\sin 55} = \frac{13.2}{\sin 53} = \frac{c}{\sin 72}$$

As we want to calculate the length c and as the middle part of the formula is completely known, we delete the first part of the formula:

$$\frac{13.2}{\sin 53} = \frac{c}{\sin 72}$$

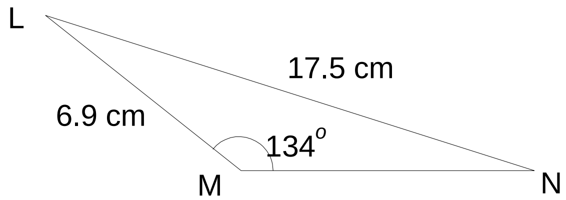
Rearranging this formula (by multiplying by sin72) gives:

$$\frac{13.2 \times \sin 72}{\sin 53} = c$$

i.e. c = 15.7 cm (to 1 decimal place).

Example: Finding the length of an angle

The diagram shows triangle LMN.
Calculate the size of angle LNM.



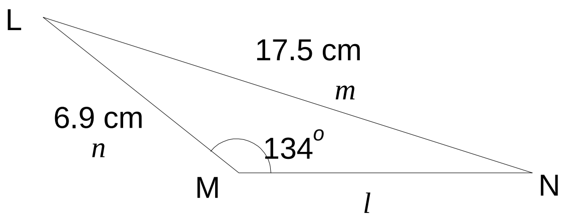
Solution:

Step 1: Label the triangle using the conventions outlined earlier.

Step 2: Substitute information from the diagram into the sine rule formula $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

Step 3: Delete the unnecessary part of the formula.
Step 4: Rearrange and then work out the size of the required angle.

In our example, the labelled triangle looks like:



The cosine rule formula (adjusted for our lettering) is:

$$\frac{\sin L}{l} = \frac{\sin M}{m} = \frac{\sin N}{n}$$

Substituting into this gives:

$$\frac{\sin L}{l} = \frac{\sin 134}{17.5} = \frac{\sin N}{6.9}$$

We want to find angle N and we know the middle part of the formula completely. We therefore delete the first part of the formula, leaving

$$\frac{\sin 134}{17.5} = \frac{\sin N}{6.9}$$

If we multiply by 6.9, we get: $\sin N = \frac{6.9 \times \sin 134}{17.5} = 0.2836$.

So angle N = SHIFT sin 0.2836 = 16.5° (to 1 d.p.)

Examination style question

In triangle ABC, angle ABC = 65°, angle ACB = 38°, BC = 15 cm.

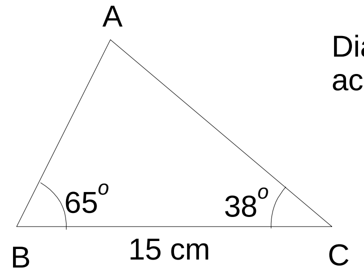


Diagram NOT
accurately drawn

Work out the length of AB.

Examination style question:

In triangle ABC, angle BAC = 115°, AC = 5 cm and BC = 9 cm.
Calculate the size of angle ABC.

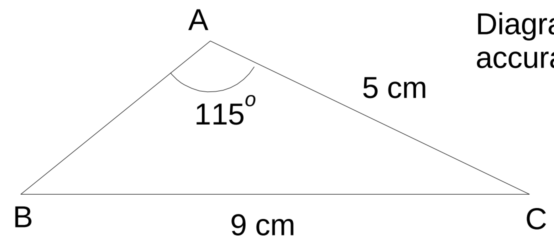
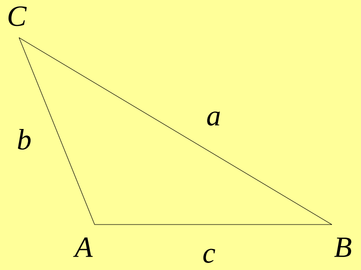


Diagram NOT
accurately drawn

Cosine Rule

The cosine rule also connects the length of sides and angles in any triangle ABC:



It states that:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Equivalently, we also have these formulae:

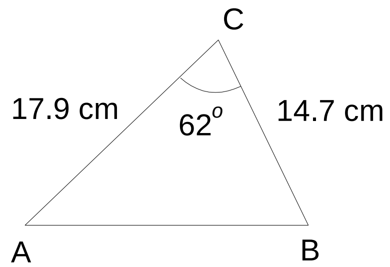
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

You need to be familiar with the structure of these formulae. In particular note that the letter that appears as the subject of the formula also appears as the angle.

Note that the cosine rule can be considered as an extension of Pythagoras' theorem.

Example: Finding the length of a side

The diagram shows triangle ABC.
Calculate the length of side AB.

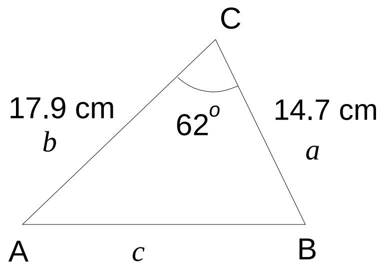


Solution:

To find the length of a side using the cosine rule, follow these steps:

<u>Step 1</u>: Label the triangle using the conventions outlined earlier.
<u>Step 2</u>: Write down the appropriate version of the cosine rule formula and substitute information from the diagram into it.
<u>Step 3</u>: Work out the length of the required side.

Our labelled diagram here is:



As we wish to find the length of *c*, we need the formula with *c* as the subject:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Substitute in: $c^2 = 14.7^2 + 17.9^2 - 2 \times 14.7 \times 17.9 \times \cos 62$

Typing the right hand side into a calculator (in one long string and pressing = only at the end) gives:

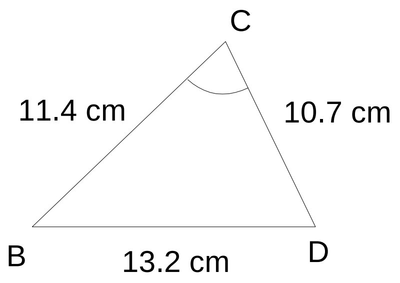
$$c^2 = 289.436$$

Square rooting gives

$$c = 17.0 \text{ cm}$$

Example: Finding the length of an angle

The diagram shows triangle BCD.
Calculate the length of angle BDC.

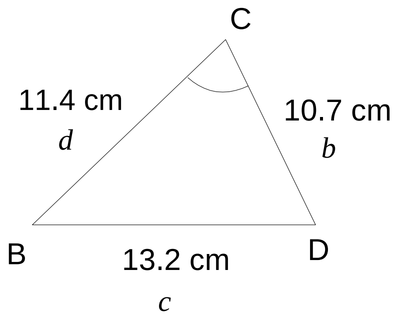


Solution:

To find the length of a side using the cosine rule, follow these steps:

Step 1: Label the triangle using the conventions outlined earlier.
Step 2: Write down the appropriate version of the cosine rule formula and substitute information from the diagram into it.
Step 3: Rearrange and work out the length of the required angle.

The labelled diagram here looks like:



We want to find angle *D*. We therefore need to write down a version of the cosine rule formula that contains angle *D*.
The subject of the appropriate formula would therefore be d^2:

$$d^2 = b^2 + c^2 - 2bc\cos D$$

Substituting into this formula gives:

$11.4^2 = 10.7^2 + 13.2^2 - 2 \times 10.7 \times 13.2 \times \cos D$
$129.96 = 114.49 + 174.24 - 282.48 \times \cos D$ (work out the squares of the terms)
$129.96 = 288.73 - 282.48\cos D$ (add together the first two numbers on the right)
$-158.77 = -282.48\cos D$ (subtract 288.73 from both sides)

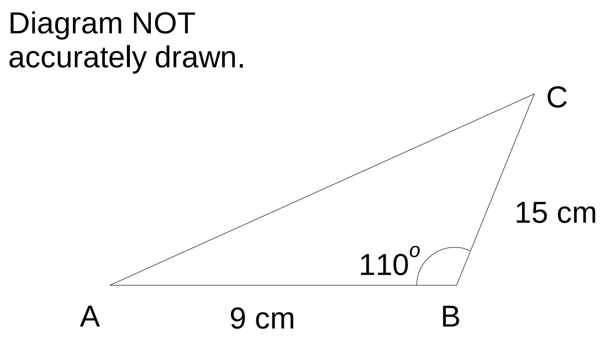
Therefore

$$\cos D = 0.56206$$

i.e. D = SHIFT cos 0.56206 = 55.8°

Worked Examination Question

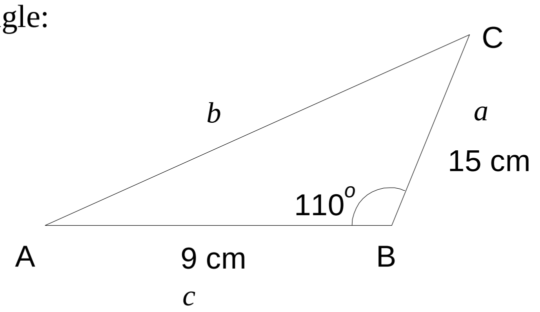
In triangle ABC, AB = 9 cm, BC = 15 cm and angle ABC = 110°.



Calculate the perimeter of the triangle.
Give your answer correct to the nearest cm.

Solution:
In order to calculate the perimeter, we need to work out the length of the third side.

Labelling the triangle:



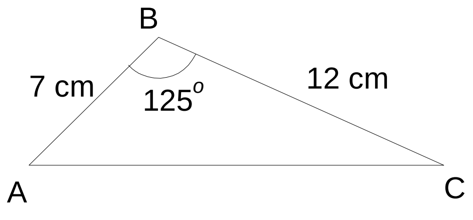
Using the cosine rule:

$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$b^2 = 15^2 + 9^2 - 2\times 15\times 9\times \cos 110$$
$$b^2 = 398.345$$

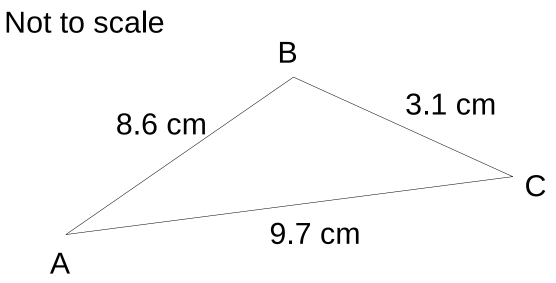
i.e. b = 20 cm (to nearest cm)

So perimeter = 20 + 15 + 9 = 44 cm (to nearest cm)

Past examination question (Edexcel):
In triangle ABC, AB = 7cm, BC = 12 cm and angle ABC = 125°.
Calculate the length of AC.



Past examination question (SEG):
The diagram shows triangle ABC. AB = 8.6 cm, BC = 3.1 cm and AC = 9.7 cm.



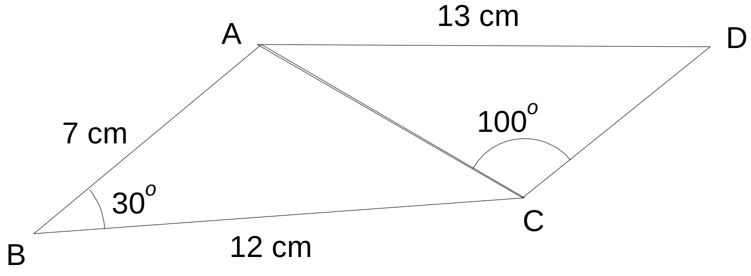
Calculate angle ABC.

When do you use the sine rule and when do you use the cosine rule?

In an examination, you will need to decide whether to use the sine rule or the cosine rule. It is helpful to remember that you will need to use the sine rule unless

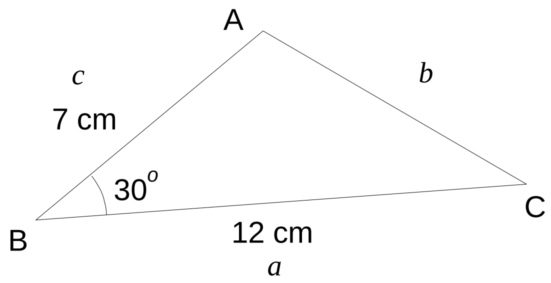
1) you are told all three sides, in which you can use the cosine rule to find any angle;
2) you are given 2 sides and the angle in between, in which case you can find the final side using the cosine rule.

Worked example:



Calculate
a) the length AC;
b) angle ADC.

Solution
Consider first triangle ABC. In this triangle we know two sides and the **included** angle (i.e. the angle in between). We can therefore use the cosine rule to find the third side, AC.

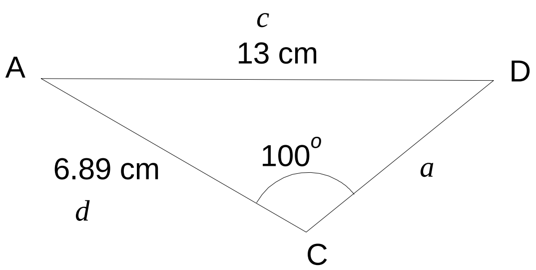


Using the cosine rule:

$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$b^2 = 7^2 + 12^2 - 2 \times 7 \times 12 \times \cos 30$$
$$b^2 = 47.5077$$
$$b = 6.89\ldots \text{ cm}$$

Therefore AC = 6.9 cm (to 1 d.p.)

(b) To find angle ADC, we now consider triangle ADC:



We are not in either of the situations where the cosine rule can be used, so here we will be using the sine rule:

$$\frac{\sin D}{d} = \frac{\sin A}{a} = \frac{\sin C}{c}$$

Substituting into this formula gives:

$$\frac{\sin D}{6.89} = \cancel{\frac{\sin A}{a}} = \frac{\sin 100}{13}$$

Rearranging gives:

$$\sin D = \frac{6.89 \times \sin 100}{13} = 0.5219$$

So $D = 31.5°$

Examination Question (Edexcel June 2001)

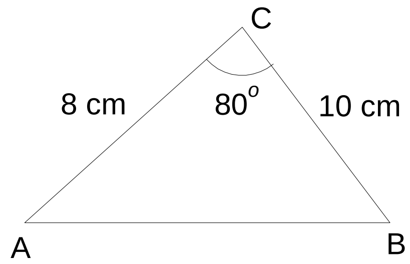


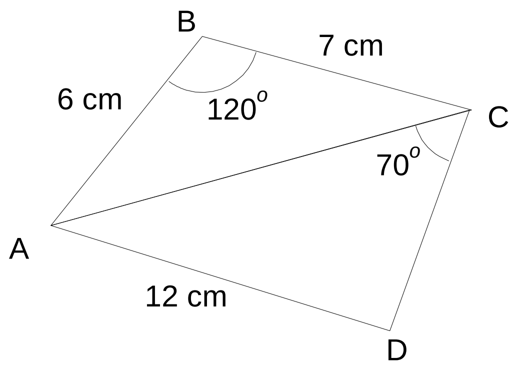
Diagram NOT accurately drawn.

a) Calculate the length of AB. Give your answer in centimetres correct to 3 significant figures.
b) Calculate the size of angle ABC. Give your answer correct to 3 significant figures.

Examination Question (Edexcel November 1998)

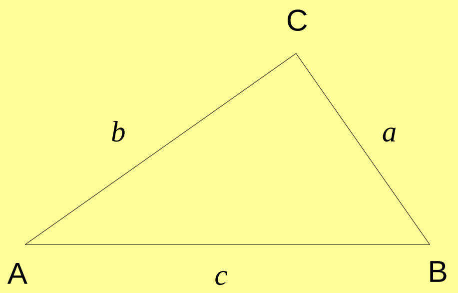
In the quadrilateral ABCD, AB = 6 cm, BC = 7 cm, AD = 12 cm, angle ABC = 120°, angle ACD = 70°.

Calculate the size of angle ADC. Give your answer correct to 3 significant figures.



Area of a triangle

The area of a triangle can be found using this alternative formula:



$$\text{Area of a triangle} = \frac{1}{2}ab\sin C$$

Alternative versions are:

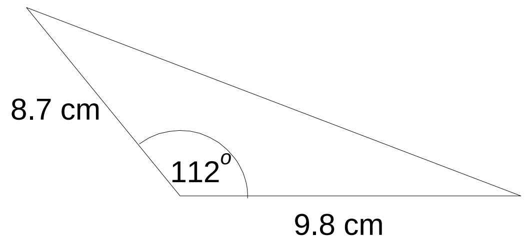
$$\text{Area} = \frac{1}{2}ac\sin B \qquad \text{or} \qquad \text{Area} = \frac{1}{2}bc\sin A$$

This can equivalently be thought of as

Area = ½ × product of two sides × sine of the included angle.

Example:

Find the area of the triangle:



We can use the above formula to find the area of this triangle as we have two sides and the included angle (i.e. the angle in between the given sides):

$$\text{Area} = \frac{1}{2} \times 8.7 \times 9.8 \times \sin 112 = 39.5\text{cm}^2 \quad \text{(to 3 s.f.)}$$

Worked examination question

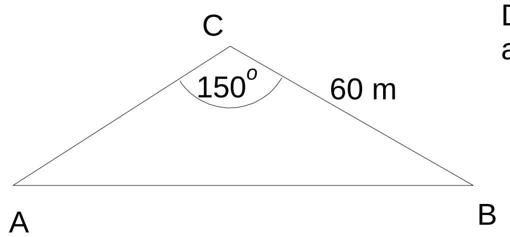
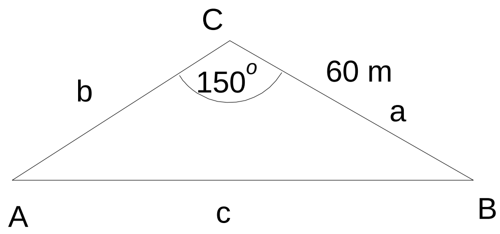


Diagram NOT accurately drawn.

Angle ACB = 150°.
BC = 60 m.
The area of triangle is 450 m^2.
Calculate the perimeter of triangle ABC. Give your answer correct to 3 significant figures.

We can use the formula for the area of a triangle to find the length of AC:



Area = $\frac{1}{2}ab\sin C$

So $\quad 450 = \frac{1}{2} \times 60 \times b \times \sin 150$

$450 = 30b\sin 150$

$450 = 15b \quad$ as $\sin 150 = 0.5$

So $\quad b = 30$ m.

To find the perimeter, we also need the length AB. We can use the cosine rule:

$c^2 = a^2 + b^2 - 2ab\cos C$

$c^2 = 30^2 + 60^2 - 2 \times 30 \times 60 \cos 150$

$c^2 = 7617.69...$

So $\quad c = 87.3$ m

Therefore the perimeter is 30 + 60 + 87.3 = 177.3 = 177 m^2 to 3 SF.

Examination question (June 2004)

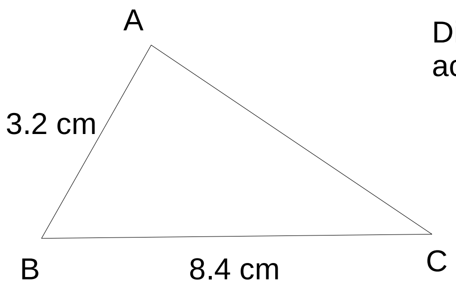
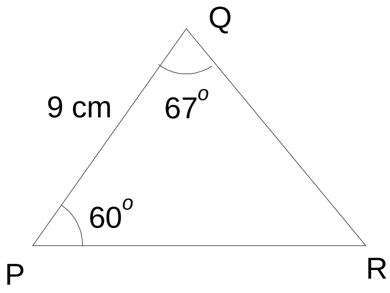


Diagram NOT accurately drawn.

AB = 3.2 cm. BC = 8.4 cm. The area of triangle ABC is 10 cm^2.
Calculate the perimeter of triangle ABC.
Give your answer correct to 3 SF.

Examination style question



In triangle PQR, PQ = 9 cm, angle PQR = 67° and angle QPR = 60°.
Calculate the area of triangle PQR.

Problem style questions

Questions are often set involving bearings or angles of elevation.
If a diagram has not been drawn in the question, you will need to begin by sketching a diagram to illustrate the situation.
There will usually be several steps required in order to get to the solution.

Worked examination question (Edexcel June 2006)
The diagram shows a vertical tower DC on horizontal ground ABC. ABC is a straight line.
The angle of elevation of D from A is 28°.
The angle of elevation of D from B is 54°.
AB = 25 m.

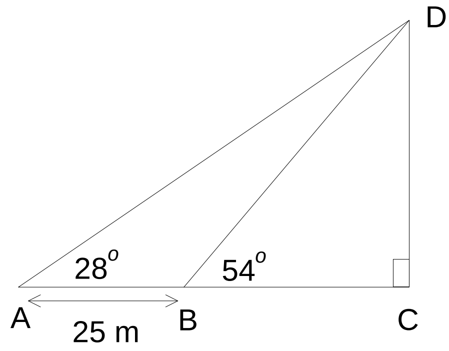


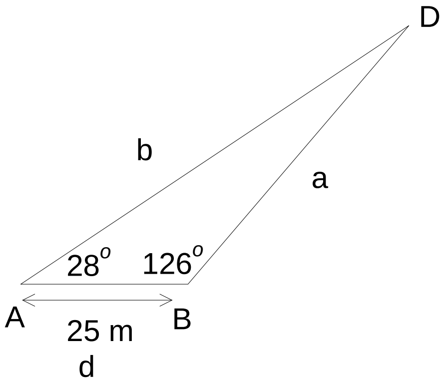
Diagram NOT accurately drawn.

Calculate the height of the tower. Give your answer to 3 significant figures.

Solution:

Step 1: Use triangle ADB to find the length DB.
Step 2: Use triangle DBC to find the height DC.

Step 1: From the original diagram, we can deduce that angle ABD = 126° and angle ADB = 26°.

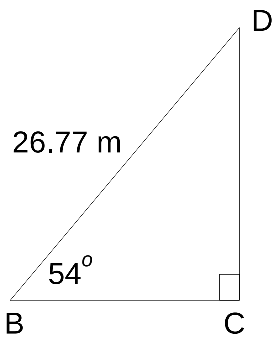


Using the sine rule:

$$\frac{a}{\sin 28} = \cancel{\frac{b}{\sin 126}} = \frac{25}{\sin 26}$$

So $a = \dfrac{25\sin 28}{\sin 26} = 26.77m$

Step 2:



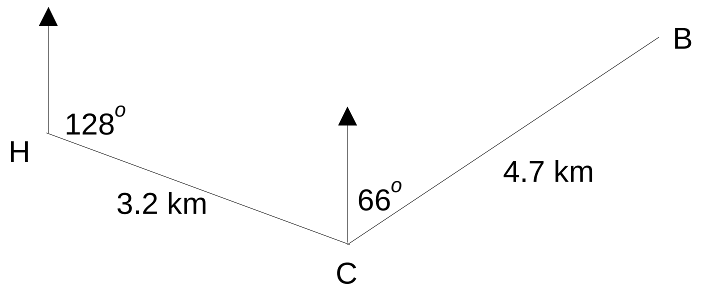
Using trigonometry for right-angled triangles:

$$\sin 54 = \frac{CD}{26.77}$$
$$CD = 26.77 \times \sin 54$$
$$CD = 21.7\,\text{m}$$

So the tower is 21.7 m tall.

Examination question (NEAB)

A helicopter leaves a heliport H and its measuring instruments show that it flies 3.2 km on a bearing of 128° to a checkpoint C. It then flies 4.7 km on a bearing of 066° to its base B.



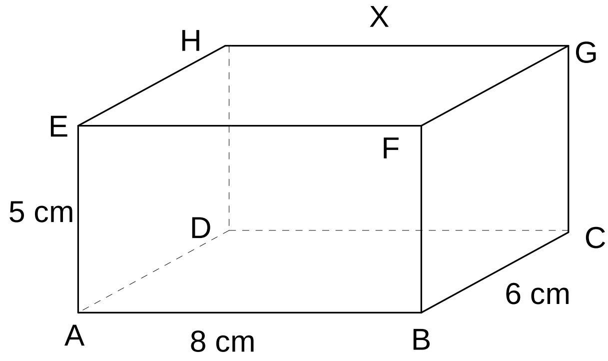
a) Show that angle HCB is 118°.

b) Calculate the direct distance from the heliport H to the base.

3D trigonometry

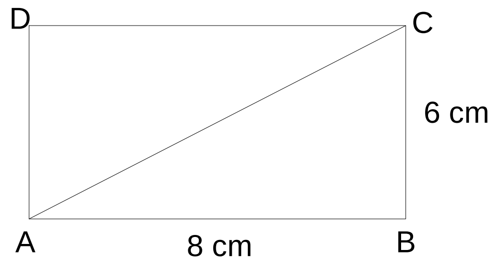
Grade A/ A* questions often involve you finding distances and angles in 3 dimensional objects. The key to these questions is to identify and draw the relevant 2 dimensional triangle.

Example 1:
ABCDEFGH is a cuboid with dimensions 8cm, 6cm and 5cm (as shown in the diagram). X is the midpoint of side HG.



a) Find length AC.

AC is length diagonally across the base of the cuboid.
* We start by sketching the base ABCD and we mark on the length we want to find:

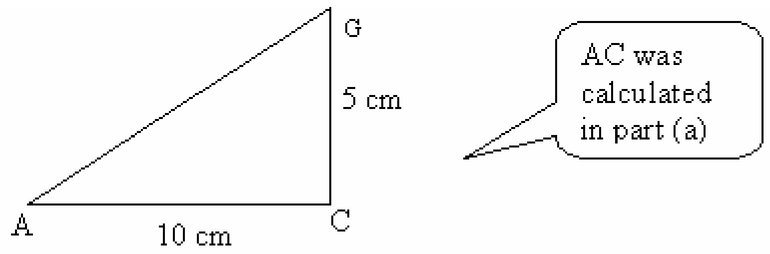


* We identify a relevant right-angled triangle (here obviously triangle ABC).
* We can then use Pythagoras' theorem to find AC:

$AC^2 = AB^2 + BC^2$
$AC^2 = 6^2 + 8^2$
$AC^2 = 100$
So $AC = 10$ cm

b) Find the length AG

* We begin by identifying a relevant right-angled triangle. Here we use triangle AGC (we use this triangle because C is vertically below G). We mark on the diagram all the lengths we know:

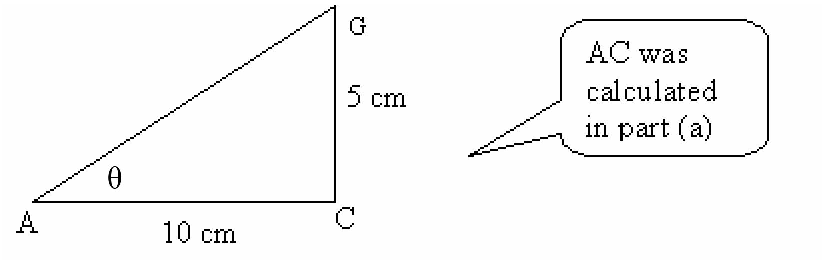


We now use Pythagoras' theorem to find AG:

$$AG^2 = AC^2 + CG^2$$
$$AG^2 = 10^2 + 5^2$$
$$AG^2 = 125$$
So $\quad AG = 11.2$ cm

c) Find angle GAC.

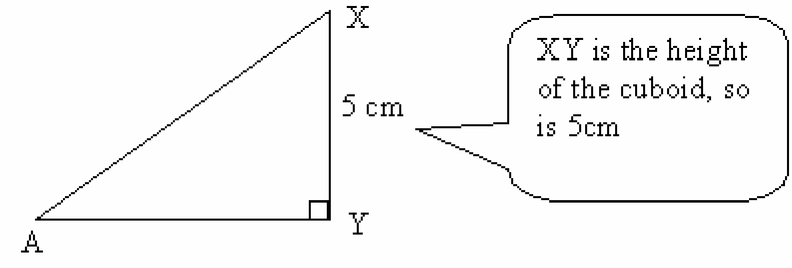
The letters mentioned in the name of the angle tell you which triangle to draw (i.e. triangle GAC). This is the triangle drawn in part (b).
We can use trigonometry to find angle GAC.



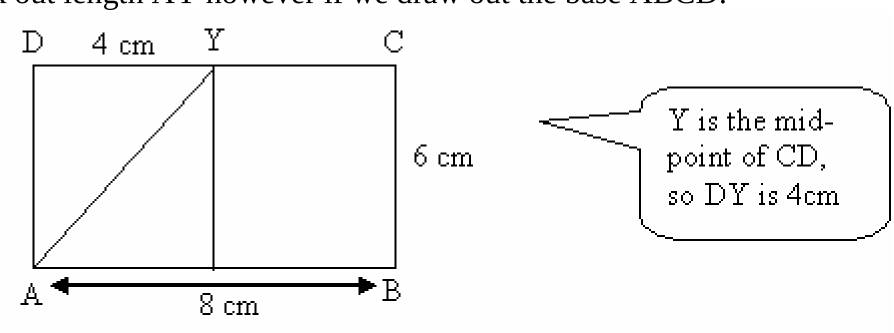
$$\tan\theta = \frac{5}{10}$$
i.e $\quad \theta = 26.6°$

d) Find the length AX.

* AX is a diagonal length across the cuboid. **Let Y be the point vertically below X**. We draw triangle AXY:

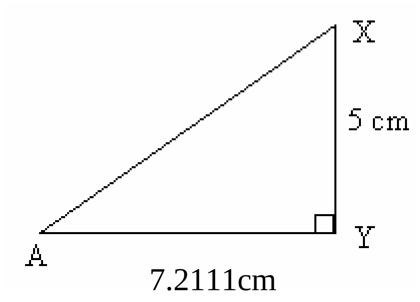


* There is not yet enough information in the diagram to find length AX.
* We can work out length AY however if we draw out the base ABCD:



We can use Pythagoras' theorem to find AY:
$$AY^2 = AD^2 + DY^2$$
$$AY^2 = 6^2 + 4^2$$
$$AY^2 = 52$$
So $\quad AY = 7.2111$ cm (note that we don't round too early).

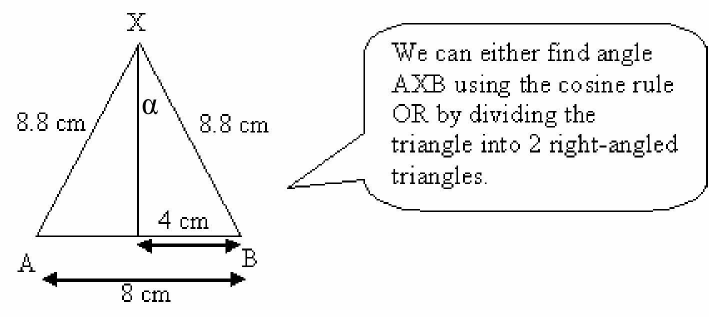
Now we can find AX from triangle AXY:



$AX^2 = AY^2 + XY^2$
$AX^2 = 7.2111^2 + 5^2$
$AX^2 = 77$
So $AY = 8.775$ cm = 8.8 cm (to 1 DP)

e) Find angle AXB

We begin by drawing out triangle AXB (i.e. the triangle with the same letters as the angle we want). This is an isosceles triangle as AX = XB.



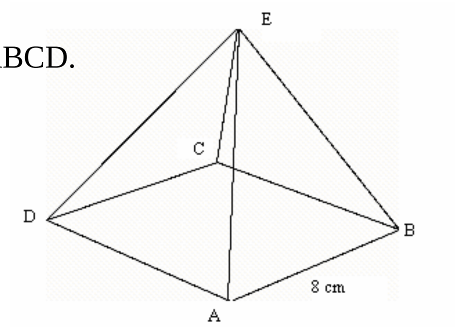
We begin by finding α:

$$\sin\alpha = \frac{4}{8.8}$$
$$\alpha = 27°$$

So angle AXB = 54°

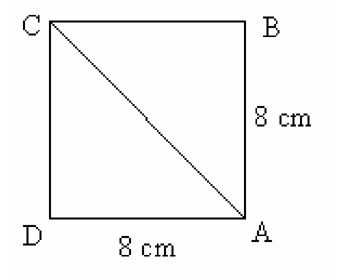
Example 2:

ABCDE is a square-based pyramid. The length AB is 8cm.
Point E is vertically above point X, the centre point of square ABCD.
The height of the pyramid, EX, is 7cm.



a) Calculate length AC.

* AC lies along the base of the pyramid.
We therefore begin by drawing out the base.

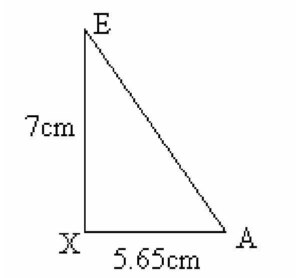


Triangle ABC is a right-angled triangle. We can therefore use Pythagoras' theorem to find AC:

$AC^2 = 8^2 + 8^2$
$AC^2 = 128$
$AC = 11.3$ cm

b) Calculate length AE.

* The point below E is point X. We therefore draw out triangle AEX.



Note: AX is half of AC, i.e. 5.65 cm.
We can find AE using Pythagoras' theorem:

$AE^2 = 7^2 + 5.65^2$
$AE^2 = 81$
$AE = 9.00$ cm

c) Calculate angle EAC.

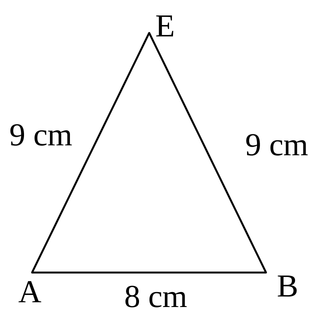
* This angle is the same as angle EAX.
We can calculate this angle using the diagram in (b).
Using trigonometry

$$\tan(\angle EAX) = \frac{7}{5.65}$$
$$\angle EAX = 51.1^\circ$$

So angle EAC = 51.1°

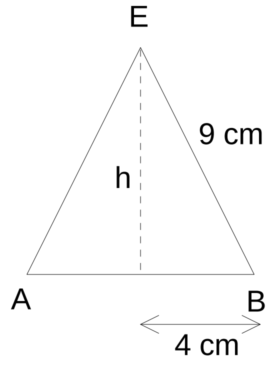
d) Calculate the area of face AEB.

* We begin by drawing out triangle AEB, an isosceles triangle (AE = EB).



* One way to find the area of triangle AEB would be to find the height of this triangle (by splitting it into 2 right-angled triangles) and then using the formula ½ b×h. Alternatively, we could find angle AEB (for example using the cosine rule) and then using the formula: area = ½ *ab*sin*C*.

If we use the first method, then we must begin by finding the height h of the triangle.
Using Pythagoras' theorem:



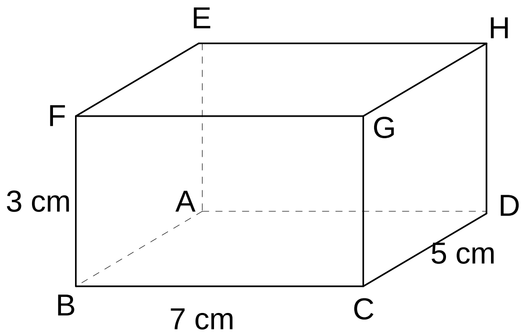
$$h^2 + 4^2 = 9^2$$
$$h^2 + 16 = 81$$
$$h^2 = 65$$
$$h = 8.06 \text{ cm}$$

Therefore the area is

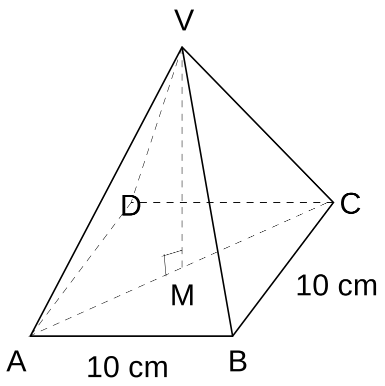
$$\frac{1}{2} \times 8 \times 8.06 = 32.2 \, \text{cm}^2$$

Examination question (Edexcel November 2004)

The diagram represents a cuboid ABCDEFGH.
CD = 5 cm, BC = 7 cm, BF = 3 cm.
a) Calculate the length of AG. Give your answer correct to 3 significant figures.
b) Calculate the size of the angle between AG and the face ABCD. Give your answer correct to 3 significant figures.



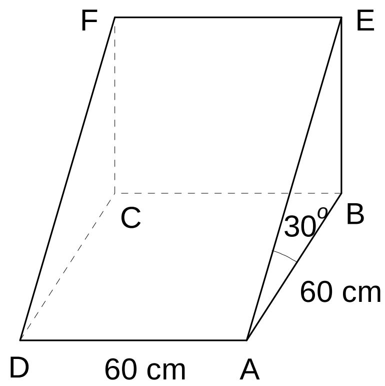
Examination question (Edexcel November 2005)



The diagram shows a pyramid.
The base, ABCD, is a horizontal square of side 10 cm.
The vertex V is vertically above the midpoint, M, of the base.
VM = 12 cm.

Calculate the size of angle VAM.

Examination question (Edexcel)



The diagram represents a prism.
AEFD is a rectangle.
ABCD is a square.
EF and FC are perpendicular to plane ABCD.
AB = AD = 60 cm.
Angle ABE = 90 degrees.
Angle BAE = 30 degrees.

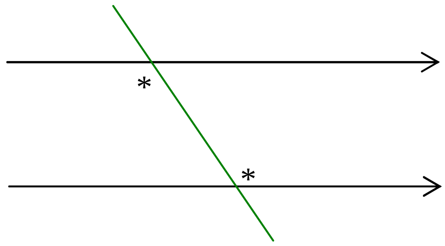
Calculate the size of the angle that the line DE makes with the plane ABCD.

Revision Topic 19: Angles

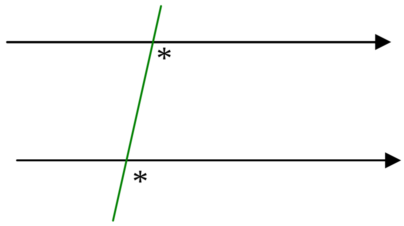
Angles and parallel lines

Recall the following types of angle:

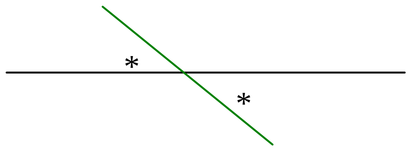
Alternate angles (or Z angles) are equal:



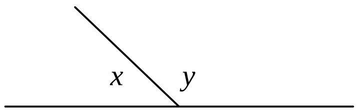
Corresponding angles (or F angles) are equal:



Vertically opposite angles are equal:

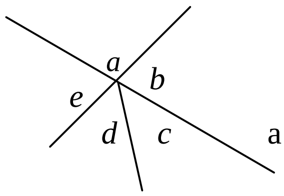


Angles on a straight line add up to 180°:



$x + y = 180$

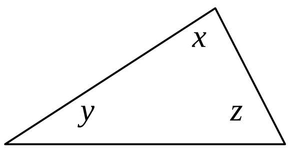
Angles around a point add up to 360°:



$a + b + c + d + e = 360°$

Angles in triangles

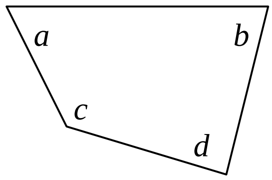
The three angles inside a triangle add up to 180°



$x + y + z = 180°$

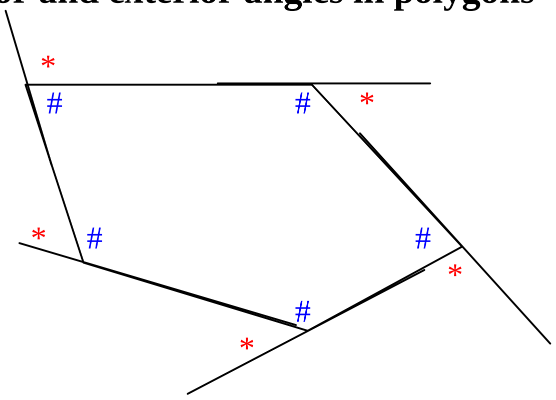
Angles in quadrilaterals

The four angles inside a quadrilateral add up to 360°.



$a + b + c + d = 360°$

Interior and exterior angles in polygons



The angles marked # are interior angles.

The angles marked * are exterior angles.

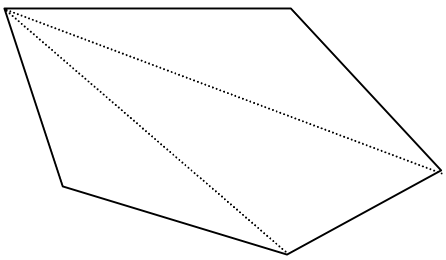
The interior and exterior angle at a corner of the shape add up to 180 degrees.

Sum of interior angles

We can find the sum of the (interior) angles for any polygon by splitting the polygon into non-overlapping triangles.

Example: Pentagon

A pentagon can be divided into 3 non-overlapping triangles.



As the triangles in each triangle add up to 180˚, the angles in a pentagon must add up to $3\times180 = 540^\circ$.

The same principle applies for any polygon:

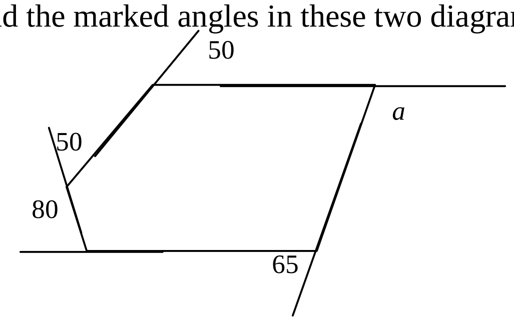
Polygon	**Number of sides**	**Number of non-overlapping triangles**	**Sum of interior angles**
Triangle	3	1	$1 \times 180 = 180^\circ$
Quadrilateral	4	2	$2 \times 180 = 360^\circ$
Pentagon	5	3	$3 \times 180 = 540^\circ$
Hexagon	6	4	$4 \times 180 = 720^\circ$
Heptagon	7	5	$5 \times 180 = 900^\circ$
Octagon	8	6	$6 \times 180 = 1080^\circ$
General polygon	n	$n - 2$	$(n-2) \times 180 = 180(n - 2)$

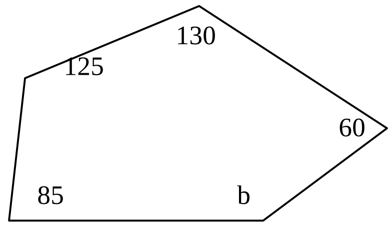
Sum of exterior angles

The exterior angles for *any* polygon add up to 360 degrees.

Example:

Find the marked angles in these two diagrams:





The first diagram above concerns exterior angles. The sum of exterior angles in any shape is 360 degrees. Therefore angle a is 360 – (50 + 65 + 80 + 50) = 115˚.

The second diagram above concerns interior angles. The interior angles of a pentagon add up to 540˚. Therefore angle b is 540 – (85 + 125 + 130 + 60) = 140˚.

Regular Polygons

In a regular polygon all the interior angles are the same. Likewise all the exterior angles are the same.

Because the exterior angles add up to 360˚, each of them must be 360 ÷ n (where n is the number of sides).

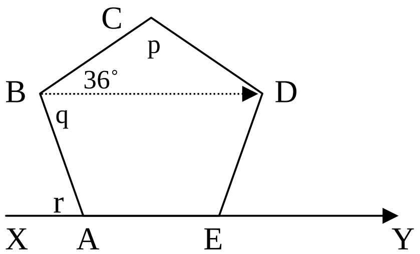
Since each interior and exterior angle adds up to 180˚, the interior angles in a regular polygon with n sides must be $180 - \frac{360}{n}$.

Example:
a) A regular polygon has 9 sides. Find the size of an interior angle.
b) A regular polygon has an exterior angle of 20˚. Show that the sum of the interior angles is 2880˚.

Solution:
a) Each exterior angle must be 360 ÷ 9 = 40˚. So the interior angles must be 180 – 40 = 140˚.

b) The exterior angle is 20˚. The number of sides must be 360 ÷ 20 = 18.
The interior angles must each be 180 – 20 = 160˚.
As there are 18 sides, sum of interior angles = 18 × 160 = 2880˚.

Examination Question

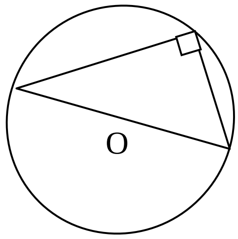


The diagram shows a regular pentagon ABCDE.
BD is parallel to XY.
Angle CBD = 36˚.
Work out the size of angles p, q and r.

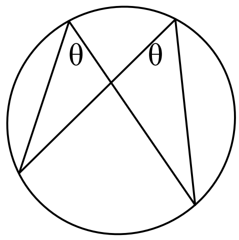
Angles in Circles

Circle Theorems

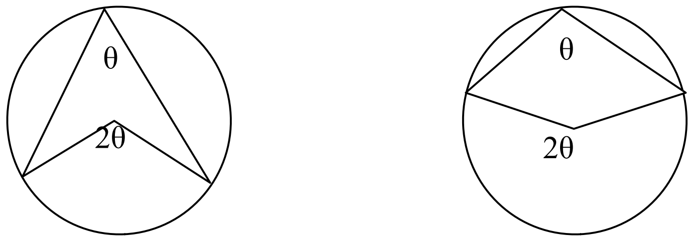
Theorem 1: The angle in a semi-circle is a right angle.



Theorem 2: Angles at the circumference are equal.



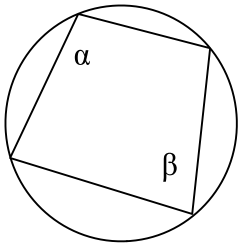
Theorem 3: Angle at the centre is twice the angle at the circumference



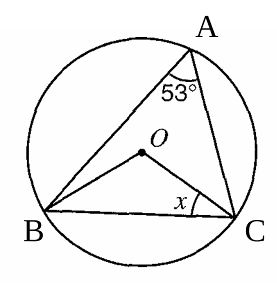
A quadrilateral whose vertices **ALL** lie on the circumference of a circle is called a **cyclic quadrilateral**.

Theorem 4: Opposite angles of a cyclic quadrilateral add up to 180º.

$\alpha + \beta = 180^\circ$



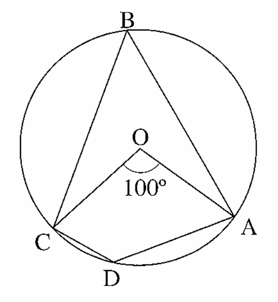
Example:



Angle BOC = 2 × 53˚ = 106˚ (the angle at the centre is twice the angle at the circumference).

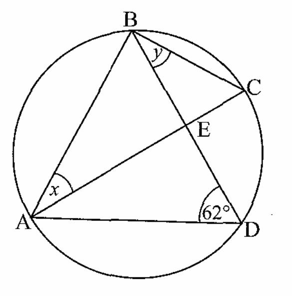
Triangle BOC is isosceles (as OB and OC are equal since both radii of the circle).
Therefore x = (180 – 106)/2 = 37˚.

Examination Question



In the diagram, O is the centre of the circle.
Angle COA = 100˚.
Calculate
a) angle CBA;
b) angle CDA.

Examination Style Question:



The diagram shows a circle with diameter AC.

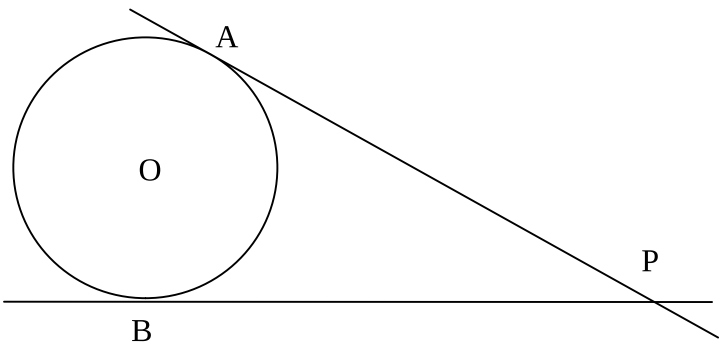
a) i) What can you say about angles ADB and ACB?
   ii) What is angle ABC? Explain your answer.
   iii) Explain why x = 28˚.

b) If AD = BD, find *y*.

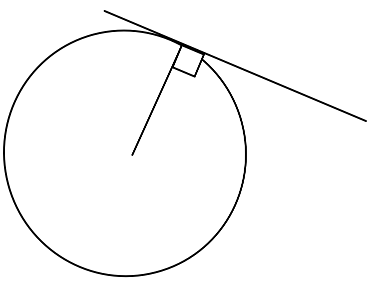
Tangents to circles

The tangent to a circle at a point is a line that just touches the circle at that point.

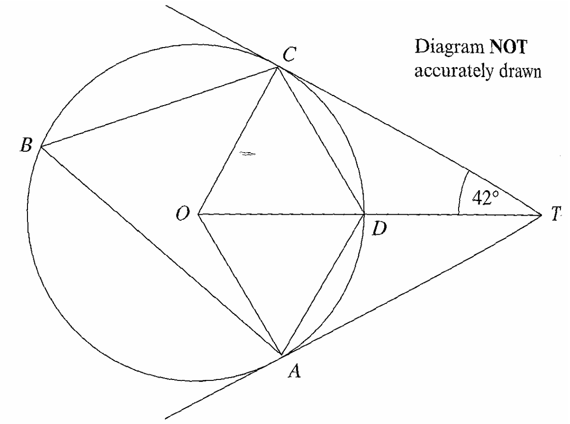
Theorem 5: Two tangents are drawn to a circle. One tangent touches the circle at A and the other touches the circle at B. If the tangents cross over at P, then PA = PB. If O is the centre of the circle then PO bisects angle AOB.



Theorem 6: The angle between a tangent and a radius is 90˚.



Examination Question

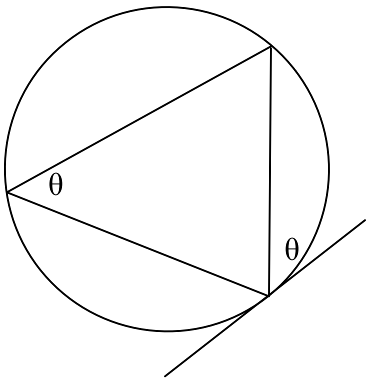


A, B, C and D are points on the circumference of a circle.
TA and TC are tangents to the circle.
The centre of the circle is at O.
ODT is a straight line.
Angle OTC = 42˚.

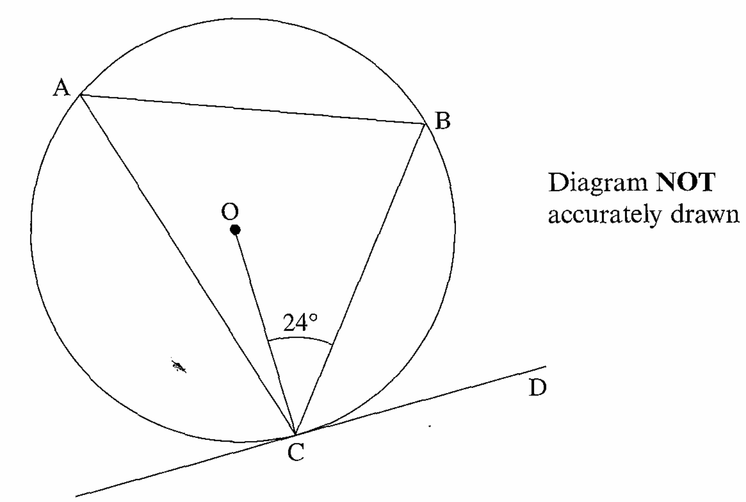
a) Write down the size of angle OCT. Give a reason for your answer.
b) Calculate the size of angle COT. Give reasons for your answer.

Grade A circle theorem: The Alternate Segment Theorem

The angle between a tangent and a chord is equal to any angle on the circumference that stands on that chord.



Examination Question



O is the centre of a circle.
CD is a tangent to the circle.
Angle OCB is 24˚.
a) Find the size of angle BCD. Give reasons for your answer.
b) Find the size of angle CAB. Give a reason for your answer.

Topic 22: Straight Line Graphs: Perpendicular Lines

The objectives of this unit are to:
* recap the equation of a straight line and to recap finding the equation of parallel lines;
* to understand the relationship between the gradients of perpendicular lines;
* to calculate the equation of a line perpendicular to a given line.

Recap: Grade B and C content

Gradient and parallel lines

The equation of a straight line has the form $y = mx + c$, where m is the gradient of the line and c is the y-interecpt.

Parallel lines have the same gradient.

The gradient of a line passing through two points is found using the formula:

$$\text{gradient} = \frac{\text{change in y coordinates}}{\text{change in x coordinates}}.$$

Example: $y = 2x + 5$
This is already written in the form $y = mx + c$.
So the gradient is 2 and the y-intercept is 5.

Example 2: $x + y = 7$
To find the gradient and y-intercept of this line, we need to rearrange to the form y = mx + c.
We get:
$y = -1x + 7$.
So gradient is -1 and y-intercept is 7.

Example 3: $3x + 2y = 12$
Rearranging: $2y = 12 – 3x$
$y = 6 – 1.5x$.
So gradient is -1.5 and the y-intercept is 6.

Example 4: Find the equation of the line parallel to $y = 3x – 1$ that passes through the point (0, 5).

Solution: As the line is parallel to $y = 3x – 1$, it must have the same gradient, i.e. 3.
As our line must pass through (0, 5), the y-intercept is 5.
So the required equation is $y = 3x + 5$.

Example 5: Find the equation of the line parallel to $y = 8 – 2x$ passing through the point (3, 7).

Solution: A parallel line has the same gradient i.e. -2.
The equation of the parallel line therefore is $y = -2x + c$.
In order to find c, we can use the coordinates of the point that we wish our line to pass through.
Substituting $x = 3$, $y = 7$ gives:
$7 = -2\times3 + c$
$c = 7 + 6 = 13$.
So the equation is $y = -2x + 13$.

Examination question (Edexcel March 2003)
Find the gradient of the straight line with equation $5y = 3 - 2x$.

Examination question (Edexcel June 2004)
A straight line has equation $y = 2(3 - 4x)$.
Find the gradient of the straight line.

Examination question (Edexcel November 2004)
The straight line L_1 has equation $y = 2x + 3$.
The straight line L_2 is parallel to the straight line L_1.
The straight line L_2 passes through the point (3, 2).
Find an equation of the straight line L_2.

Example:
Find the equation of the line passing through the points (1, 5) and (5, -3).

Solution: The y coordinates have gone from 5 to -3, a change of -8.
The *x* coordinates have gone from 1 to 5, a change of 4.
So gradient is $\frac{-8}{4} = -2$.
The equation of the line therefore has the form $y = -2x + c$.
If we substitute in the coordinates of one of our points, for example $x = 1$, $y = 5$, we get:
$5 = -2 + c$
$c = 7$.
So the line has equation $y = -2x + 7$.

Examination question (Edexcel January 2005)
A straight line passes through the points (0, 5) and (3, 17).
Find the equation of the straight line.

Perpendicular lines

Two lines are perpendicular is their gradients multiply to give -1, i.e. the gradient of one line is the negative reciprocal of the gradient of the other.

So if one line has gradient m_1, then the gradient of a perpendicular line must be $\frac{-1}{m_1}$.

Example: Which two of the lines below are perpendicular to the line $y = 4 - 2x$.

$y = -2x + 5$
$y = 0.5x + 4$
$2y = x + 5$
$x + 2y = 7$

Solution:
The gradient of the line $y = 4 - 2x$ is -2.

A perpendicular line must therefore have gradient $\frac{-1}{-2} = 0.5$.

We now look to see which of the four lines given have a gradient of 0.5 by rearranging their equations to the form $y = mx + c$.

Line	Rearranged form	Gradient
$y = -2x + 5$	-	-2
$y = 0.5x + 4$	-	0.5
$2y = x + 5$	$y = \frac{1}{2}x + 2.5$	½
$x + 2y = 7$	$y = 3.5 - \frac{1}{2}x$	-½

The lines perpendicular to $y = 4 - 2x$ therefore are $y = 0.5x + 4$ and $2y = x + 5$

Example 2: Find the equation of the line perpendicular to $3y = 5 - 2x$ passing through the point (4, 10).

Solution:

The equation $3y = 5 - 2x$ is equivalent to $y = \frac{5}{3} - \frac{2}{3}x$.

The gradient is therefore $-\frac{2}{3}$.

A perpendicular line has gradient $\frac{3}{2}$ (i.e the negative reciprocal).

So our equation has the form $y = \frac{3}{2}x + c$

Substitute in $x = 4$ and $y = 10$:

$$10 = \frac{3}{2} \times 4 + c$$

i.e. c = 4

So the perpendicular line has equation $y = \frac{3}{2}x + 4$.

Examination question (Edexcel November 2003)

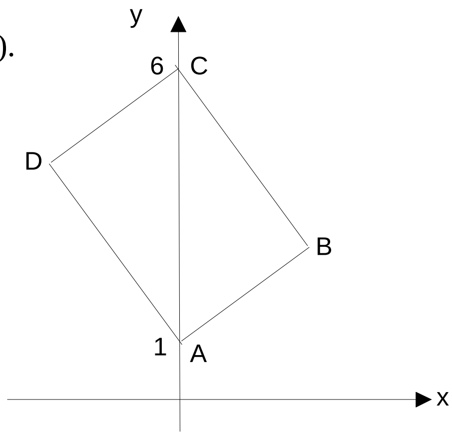
A straight line, L, passes through the point with coordinates (4, 7) and is perpendicular to the line with equation $y = 2x + 3$.

Find an equation of the straight line L.

Examination question (Edexcel June 2004)

ABCD is a rectangle. A is the point (0, 1). C is the point (0, 6).

The equation of the straight line through A and B is $y = 2x + 1$.



a) Find the equation of the straight line through D and C.

b) Find the equation of the straight line through B and C.

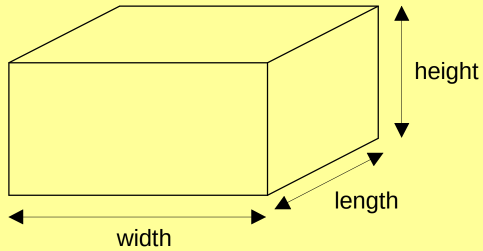
Further Volume and Surface Area

Objectives
* To find the volume and surface area of spheres, cones, pyramids and cylinders.
* To solve problems involving volume and surface area of spheres, cones, pyramids and cylinders.

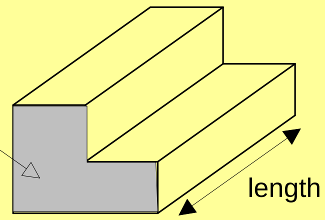
Section 1: Volume

Recap from grade B and C work

- Volume of **cuboid** = length × width × height

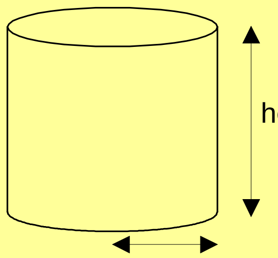


- Volume of **prism** = cross-sectional area × length



- Volume of cylinder = $\pi r^2 h$,

where r is the radius and h is the height of the cylinder.



Example:
A cuboid measures 15 cm by 12 cm by 8 cm. Find the capacity of the cuboid.
Give your answers in litres.

Solution:
Volume = 15 × 12 × 8 = 1440 cm^3.
As 1 litre = 1000 cm^3, the capacity of the cuboid = 1.44 litres.

Example 2:
A cylinder has a volume of 965 cm^3. If the height of the cylinder is 16 cm, find the radius.
Give your answer to 2 significant figures.

Solution:
Substitute the information from the question into the formula for the volume of a cylinder:

Volume of cylinder = $\pi r^2 h$

$965 = \pi \times r^2 \times 16$

$965 = \pi \times 16 \times r^2$

$965 = 50.26548 \times r^2$

$19.198 = r^2$

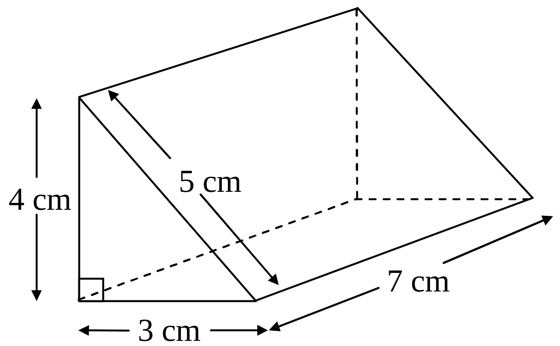
$4.38156 = r$

So the radius of the cylinder is 4.4 cm (to 2 SF)

Past examination question (Edexcel November 2003)
A can of drink has the shape of a cylinder.
The can has a radius of 4 cm and a height of 15 cm.
Calculate the volume of the cylinder.
Give your answer correct to three significant figures.

Past examination question (Edexcel November 2004)

Diagram **NOT** accurately drawn

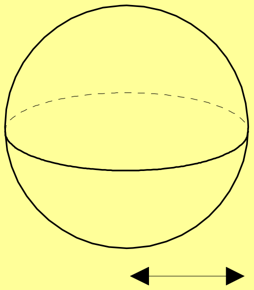


Calculate the volume of the triangular prism.

Volume of a sphere

Volume of a **sphere** = $\frac{4}{3}\pi r^3$

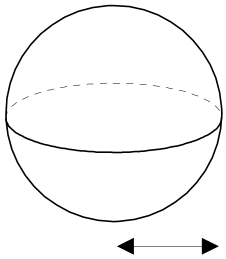
(This formula is given on the GCSE formula sheet).



A **hemisphere** is half a sphere.

Example

The radius of a sphere is 6.7 cm. Find the volume.



Solution:
Substitute r = 6.7 cm into the formula

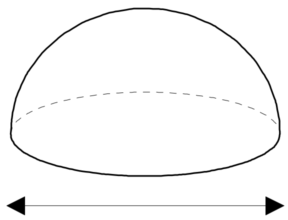
Volume = $\frac{4}{3}\pi r^3$

V = $\frac{4}{3} \times \pi \times 6.7^3$

V = 1259.833 (remember to use the cube button on your calculator)

V = 1260 cm^3 (to 3 SF)

Example 2:
Find the volume of the hemisphere shown in the diagram.



Solution:
The diameter of the hemisphere is 18.4 cm.
Therefore the radius is 9.2 cm.

Volume of the hemisphere = $\frac{1}{2} \times$ volume of sphere

$= \frac{1}{2} \times \frac{4}{3}\pi r^3$

$= \frac{1}{2} \times \frac{4}{3} \times \pi \times 9.2^3$

$= \frac{1}{2} \times 3261.76$

= 1630 cm^3 (to 3 SF)

Example 3:
A sphere has a volume of 86.5 cm^3. Find the radius of the sphere.

Solution:

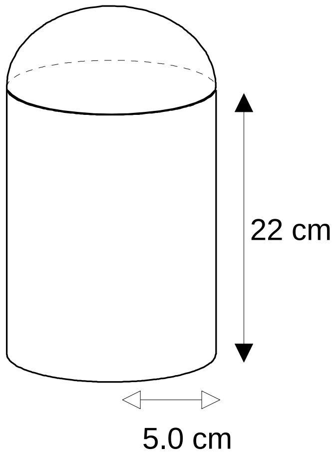
Substitute into the formula for the volume of a sphere: Volume = $\frac{4}{3}\pi r^3$

$$86.5 = \frac{4}{3}\pi r^3$$

So $86.5 = 4.18879 r^3$
i.e. $20.65035 = r^3$
So $r = 2.74$ cm (to 3 SF) (cube rooting)

The sphere has radius 2.74 cm.

Examination style question
The object shown is made up from a cylinder and a hemisphere.
The cylinder has radius 5.0 cm and height 22 cm.
Find the volume of the object.



Solution:

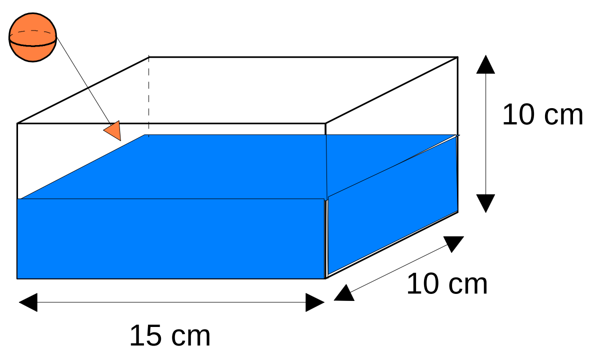
$$\text{Volume of cylinder} = \pi r^2 h$$
$$= \pi \times 5^2 \times 22$$
$$= 1728 \text{ cm}^3 \text{ (to nearest whole number)}$$

The hemisphere must also have radius 5 cm.

$$\text{Volume of the hemisphere} = \frac{1}{2} \times \text{volume of sphere}$$
$$= \frac{1}{2} \times \frac{4}{3}\pi r^3$$
$$= \frac{1}{2} \times \frac{4}{3} \times \pi \times 5^3$$
$$= 262 \text{ cm}^3$$

Therefore total volume of the object = 1728 + 262 = 1990 cm^3.

Problem style example
A tank measures 15 cm by 10 cm by 10 cm
The tank is half-full of water.



A solid metal sphere with radius 2 cm is placed into the tank.
Assuming that the sphere sinks to the bottom of the tank, calculate the amount by which the water level in the tank rises.

Solution
As the sphere will be completely submerged, it will displace its volume of water.

$$\text{Volume of sphere} = \frac{4}{3}\pi r^3 = \frac{4}{3}\times\pi\times 2^3 = 33.51 \text{ cm}^3.$$

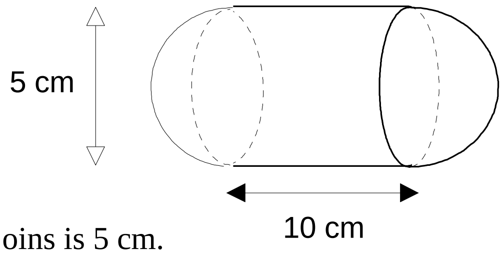
Therefore the water displaced is 33.51 cm^3.

The water displaced has the form of a cuboid with measurements 15 cm by 10 cm by *h* cm, where *h* is the height by which the water level rises.

So $\quad 15 \times 10 \times h = 33.51$
i.e. $\quad h = 0.22$ cm

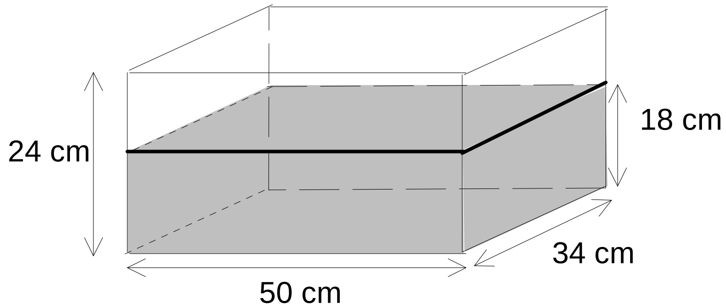
The water rises by 0.22 cm.

Examination question (Edexcel November 1998)
A solid plastic toy is made in the shape of a cylinder which is joined to a hemisphere at both ends.



The diameter of the toy at the joins is 5 cm.
The length of the cylindrical part of the toy is 10 cm.
Calculate the volume of plastic needed to make the toy. Give your answer correct to three significant figures.

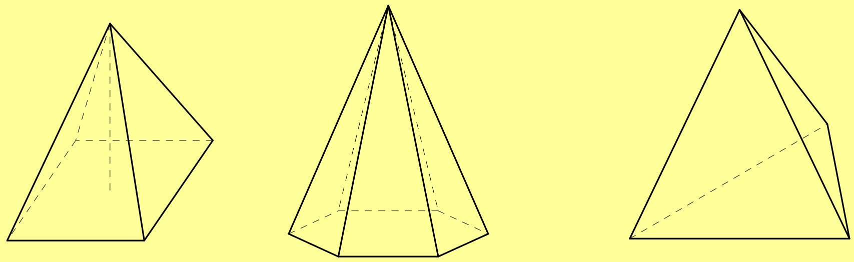
Examination question (Problem style) (AQA June 2004)
A water tank is 50 cm long, 34 cm wide and 24 cm high.
It contains water to a depth of 18 cm.



Four identical spheres are placed in the tank and are fully submerged.
The water level rises by 4.5cm.
Calculate the radius of the spheres.

Volume of a pyramid

Pyramids come in a range of shapes. They can have bases which are any shape e.g. triangular, square, rectangular, circular etc.

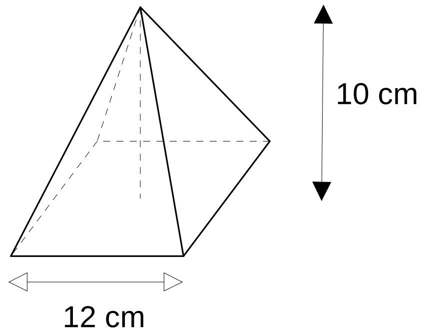


The volume of any **pyramid** can be found using the formula:

$$\text{Volume of } \textbf{pyramid} = \frac{1}{3} \times \text{base area} \times \text{height}$$

(This formula is NOT given to you in the exam – you will need to learn it!)

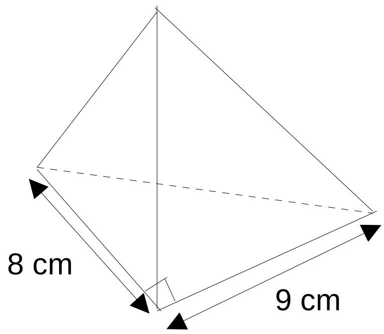
Example: (non-calculator paper)
The pyramid shown has a square base.
The square has sides of length 12 cm.
The height of the pyramid is 10 cm.
Find the volume.



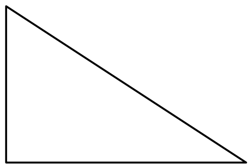
Solution:
The area of the square base is 12 × 12 = 144 cm^2
So, the volume of the pyramid is:

$$\text{Volume} = \frac{1}{3} \times 144 \times 10$$
$$= 48 \times 10$$
$$= 480 \text{ cm}^3.$$

Example 2:
The diagram shows a triangular-based pyramid.
The base of the pyramid is a right-angled triangle.
The volume of the pyramid is 325 cm^3.
Find the height of the pyramid.



Solution:
The base of the pyramid is as shown:



The area of the base is $\frac{1}{2} \times 9 \times 8 = 36 \text{ cm}^2$.
Substitute information into the formula for the volume of a pyramid.

$$\text{Volume of pyramid} = \frac{1}{3} \times \text{base area} \times \text{height}$$

$$325 = \frac{1}{3} \times 36 \times \text{height}$$
$$325 = 12 \times \text{height}.$$

So, height = 325 ÷ 12 = 27.08 cm (to 4 SF).

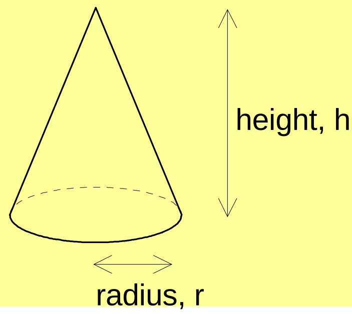
Volume of a cone

A cone is a pyramid with a circular base.

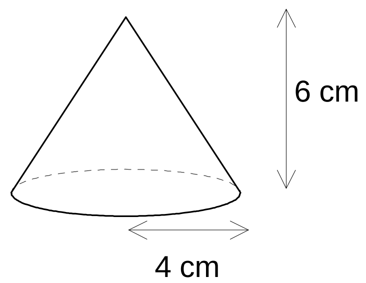
The formula for the volume of a cone is:

$$\text{Volume of cone} = \frac{1}{3}\pi r^2 h$$

where r is the radius of the cone and h is the height of the cone.



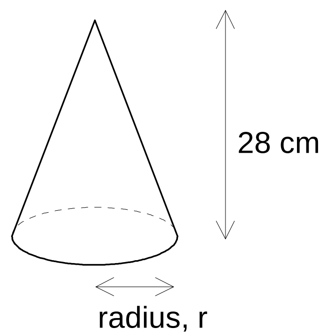
Example 1 (non-calculator paper)
The base of a cone has a radius of 4 cm.
The height of the cone is 6 cm.
Find the volume of the cone.
Leave your answer in terms of π.



Solution:
Substitute the information into the formula for the volume of a cone:

$$\text{Volume of cone} = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3} \times \pi \times 4^2 \times 6$$
$$= 2 \times \pi \times 16 \qquad \text{(start by finding 1/3 of 6)}$$
$$\text{volume} = 32\pi \text{ cm}^3.$$

Example 2:
A cone has a volume of 1650 cm^3.
The cone has a height of 28 cm.
Find the radius of the cone.
Give your answer correct to 2 significant figures.



Solution:
Substitute information into the formula:

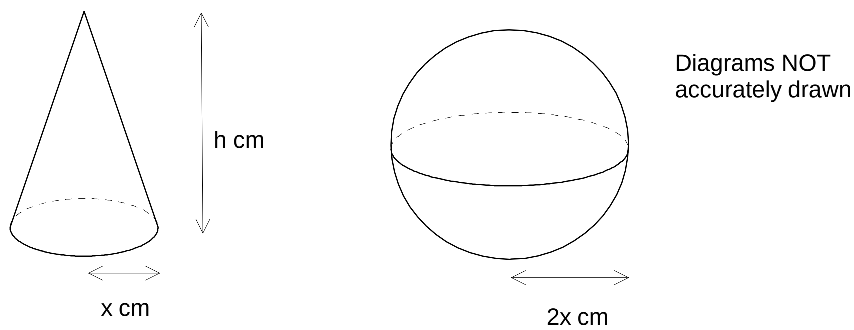
$$\text{Volume of cone} = \frac{1}{3}\pi r^2 h$$
$$1650 = \frac{1}{3} \times \pi \times r^2 \times 28$$
$$1650 = 29.32153r^2 \qquad \left(\text{evaluating } \frac{1}{3} \times \pi \times 28\right)$$
$$r^2 = 56.2726$$

i.e. $r = 7.5$ cm (to 2 SF)

The radius of the cone is therefore 7.5 cm.

Problem solving: Worked examination question (Edexcel June 2005 – non calculator paper)

The radius of the base of a cone is x cm and its height is h cm.
The radius of a sphere is $2x$ cm.



The volume of the cone and the volume of the sphere are equal.
Express h in terms of x.
Give your answer in its simplest form.

Solution:

The volume of the cone is $\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi x^2 h$

The volume of the sphere is $\frac{4}{3}\pi r^3 = \frac{4}{3}\pi(2x)^3$ (note: the brackets around $2x$ are important)

$= \frac{4}{3}\pi \times 8x^3$ (cubing both 2 and x)

$= \frac{32}{3}\pi x^3$

As the sphere and the cone have the same volume, we can form an equation:

$\frac{1}{3}\pi x^2 h = \frac{32}{3}\pi x^3$

$\pi x^2 h = 32\pi x^3$ (multiplying both sides by 3)

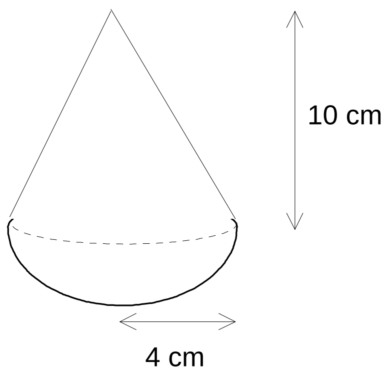
$x^2 h = 32x^3$ (dividing both sides by π)

$h = 32x$ (diving both sides by x^2)

Past examination question (Edexcel 1997)

A child's toy is made out of plastic.
The toy is solid.
The top of the toy is a cone of height 10 cm and base radius 4 cm.
The bottom of the toy is a hemisphere of radius 4 cm.

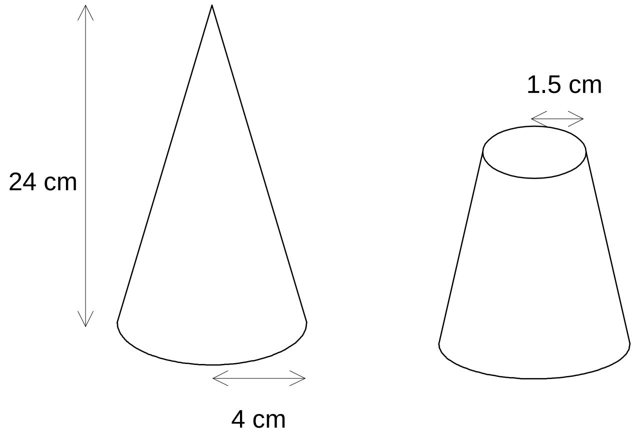
Calculate the volume of plastic needed to make the toy.



Volume of a frustrum

A frustrum is a cone with a smaller cone sliced off the top.

Examination style question
The diagram shows a large cone of height 24 cm and base radius 4 m.



A small cone of radius 1.5 cm is cut off the top leaving a frustrum.
Calculate the volume of the frustrum.

Solution:

The volume of the large cone is: $\frac{1}{3} \times \pi \times 4^2 \times 24 = 402.12 \text{ cm}^3$

To find the volume of the small cone, we need its height.

The radius of the small cone is $\frac{1.5}{4} = \frac{3}{8}$ of the radius of the large cone.

Therefore the height of the small cone is $\frac{3}{8}$ of the height of the large cone, i.e. the small cone has height $\frac{3}{8} \times 24 = 9 \text{ cm}$

So the volume of the small cone is $\frac{1}{3} \times \pi \times 1.5^2 \times 9 = 21.21 \text{ cm}^3$

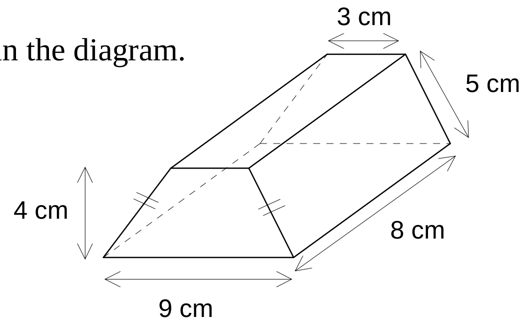
The volume of the frustrum is 402.12 – 21.21 = 381 cm^3 (to 3F)

Section 2: Surface Area

Recap: Grade B and C

You should be familiar with finding the surface area of prisms (such as cuboids, triangular prisms, etc). The surface area of a prism is found by adding together the area of each face.

Examination style question
Find the total surface area of the solid prism shown in the diagram.
The cross-section is an isosceles trapezium.



Solution:
The prism has six faces – two are trapeziums and 4 are rectangles.

The area of the front and back faces are:

$$\frac{1}{2}(3+9)\times 4 = 6\times 4 = 24\,\text{cm}^2$$

The formula for the area of a trapezium is:

$$\frac{1}{2}\times(\text{sum of parallel sides})\times\text{height}$$

The two sides faces each have an area equal to

$5 \times 8 = 40$ cm^2

The area of the top face is

$3 \times 8 = 24$ cm^2

The area of the base is

$9 \times 8 = 72$ cm^2

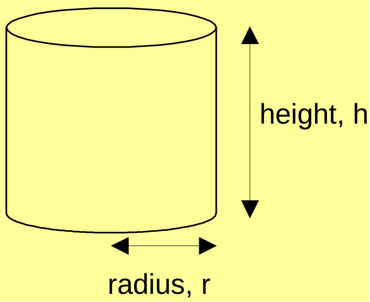
So the total surface area is $24 + 24 + 40 + 40 + 24 + 72 = 224$ cm^2.

Surface area of cylinders, spheres, cones and pyramids

Cylinders
A solid cylinder has 3 faces – a circular face at either end and a curved face around the middle:

$$\text{Surface area of a cylinder} = \underbrace{2\pi rh}_{\text{curved surface area}} + \underbrace{2\pi r^2}_{\text{area of top and bottom}}$$

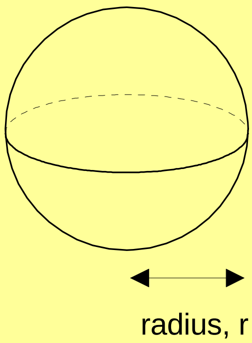
(This formula is not on the formula sheet).



Sphere
A sphere has a single curved face.

$$\text{Surface area of a sphere} = 4\pi r^2$$

(This formula is on the formula sheet)



Cone

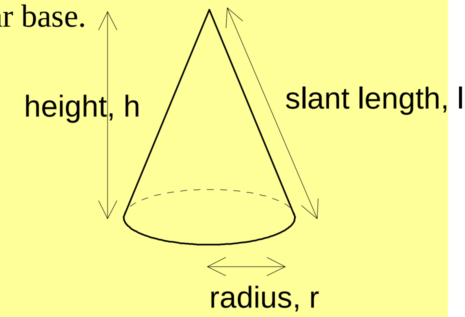
A solid cone has two surfaces – the curved surface and the circular base.
The formula for the curved surface area is:

curved surface area = πrl

where l is the slant length.

The values of l, r and h are related by Pythagoras' theorem:

$h^2 + r^2 = l^2$.

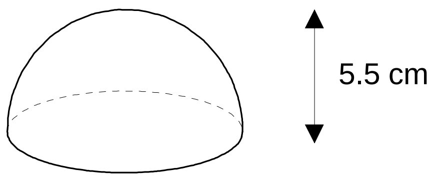


Pyramid

There is no general formula for the total surface area of a pyramid. Just take each face in turn and use the relevant formula for finding the area of that face's shape.

Worked example 1:

Find the total surface area of the solid hemisphere shown.



Solution:

The hemisphere has a radius of 5.5 cm.
It has 2 surfaces – a circular base and a curved surface.

The area of the circular base is $\pi r^2 = \pi \times 5.5^2 = 95.033\,\text{cm}^2$

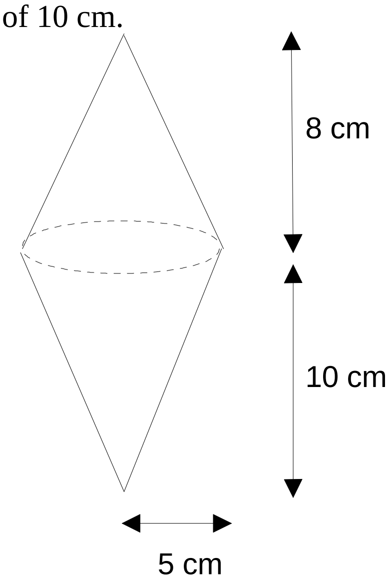
The area of the curved surface is $\frac{1}{2} \times \underbrace{4\pi r^2}_{\text{formula for surface area of a whole sphere}} = \frac{1}{2} \times 4 \times \pi \times 5.5^2 = 190.066\,\text{cm}^2$

So, total surface area = 285 cm^2 (to 3 SF)

Worked example 2

The diagram shows an object made from two cones, one on top of the other.
The top cone has a height of 8 cm and the bottom cone has a height of 10 cm.
Both cones have a radius of 5 cm.

Find the total surface area of the object.



Solution:

The formula for the curved surface area of a cone is: πrl.

We can find the slant length, l, for each cone using Pythagoras' theorem – we know the radius and the height of each cone.

<u>Top cone</u>:

$l^2 = 5^2 + 8^2 = 25 + 64 = 89$

$l = \sqrt{89} = 9.434cm$

Therefore,

Curved surface area = $\pi \times 5 \times 9.434 = 148.2cm$

Bottom cone:

$$l^2 = 5^2 + 10^2 = 25 + 100 = 125$$

$$l = \sqrt{125} = 11.180cm$$

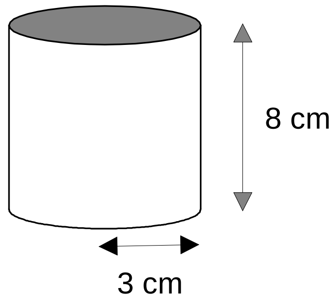
Therefore,

Curved surface area = $\pi \times 5 \times 11.180 = 175.6cm$

So total surface area is 324cm^2 (to 3SF)

Worked example 3: (non-calculator)

A cylinder is made from metal.
It has a base but no lid.
The height of the cylinder is 8 cm.
The radius of the cylinder is 3 cm.



Find the amount of metal required to make the cylinder.
Leave your answer in terms of π .

Solution:

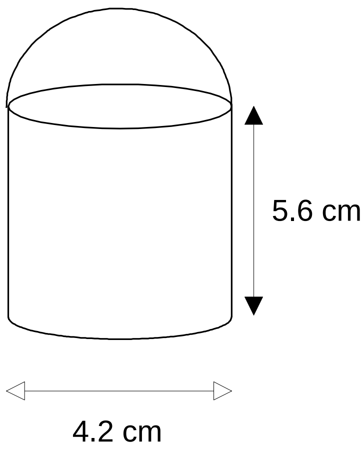
The area of the base is $\pi r^2 = \pi \times 3^2 = 9\pi$

The curved surface area is $2\pi rh = 2 \times \pi \times 3 \times 8 = 48\pi$

So the area of metal required = $9\pi + 48\pi = 57\pi$ cm^2

Examination style question 1:

A solid object is formed by joining a hemisphere to a cylinder.
Both the hemisphere and the cylinder have a diameter of 4.2 cm.
The cylinder has a height of 5.6 cm.



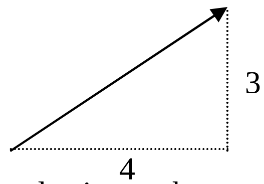
Calculate the total surface area of the whole object.
Give your answer to 3 SF.

Examination style question 2:
A sphere has a volume of 356 cm^3.
Calculate the surface area of the sphere.

Vectors

Vectors on grids:

* The vector $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ represents a line going 4 units to the right and 3 units up.



* The length of a vector (sometimes called the **magnitude**) can be found using pythagoras' thm. For example, the length of the above vector is 5.

* $\begin{pmatrix} -2 \\ -4 \end{pmatrix}$ is a vector 2 units to the left and 4 units down. Its length is $\sqrt{(-2)^2 + (-4)^2} = \sqrt{20}$ = 4.47 (to 2 dp).

* Vectors can be added, subtracted and multiplied by a scalar (number):

e.g. $\begin{pmatrix} 2 \\ -4 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$; $\quad \begin{pmatrix} 3 \\ 7 \end{pmatrix} - \begin{pmatrix} -1 \\ 6 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ $\quad 4\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ 8 \end{pmatrix}$.

* The notation $\overrightarrow{AB}$ (or **AB**) represents the vector needed to go from point *A* to point *B*. For example, if *A* is (4, 5) and *B* is (7, 2) then $\overrightarrow{AB} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$ (this can be found by subtracting *A*'s coordinates from *B*'s).

Worked Examination Question [June 1998 Paper 6]

A is the point (2, 3) and *B* is the point (-2, 0).
a) Find $\overrightarrow{AB}$ as a column vector.

C is the point such that $\overrightarrow{BC} = \begin{pmatrix} 4 \\ 9 \end{pmatrix}$.

b) Write down the coordinates of the point *C*.
X is the mid-point of *AB*. *O* is the origin.
c) Find $\overrightarrow{OX}$ as a column vector.

Solution:

a) To get from A to B, we move 4 units left and 3 units down. So $\overrightarrow{AB} = \begin{pmatrix} -4 \\ -3 \end{pmatrix}$

b) Since $\overrightarrow{BC} = \begin{pmatrix} 4 \\ 9 \end{pmatrix}$, we know that to move from *B* to *C* we move 4 units right and 9 units up. *B* is the point (-2, 0) so *C* is the point (2 , 9).

c) A is the point (2, 3) and B is (-2, 0)
Since *X* is the mid-point of AB, we find the coordinates of *X* by finding the average of the two x-coordinates and the average of the two y-coordinates.

So *X* is the point $\left(\frac{2 + (-2)}{2}, \frac{3 + 0}{2} \right) = (0, 1.5)$

Worked Examination question (November 1999 Paper 6)

$\mathbf{p} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

a) Write down as a column vector … $2\mathbf{p} + \mathbf{q}$ and $\mathbf{p} - 2\mathbf{q}$.

A is the point (15, 15) and O is the point (0, 0). The vector $\overrightarrow{OA}$ can be written in the form c**p** + d**q**, where c and d are scalars.

b) Using part (a), or otherwise, find the values of c and d.

Solution:

a) $2\mathbf{p} + \mathbf{q} = 2\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$

$\mathbf{p} - 2\mathbf{q} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - 2\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$

b) $\overrightarrow{OA} = \begin{pmatrix} 15 \\ 15 \end{pmatrix}$

So we need to find values c and d such that $\begin{pmatrix} 15 \\ 15 \end{pmatrix} = c\begin{pmatrix} 2 \\ 1 \end{pmatrix} + d\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Reading across the top line: $15 = 2c + d$ (1)
Reading across the bottom line: $15 = c - 2d$ (2)

We can solve these simultaneous equations by multiplying the top equation by 2:

$30 = 4c + 2d$
$15 = c - 2d$

Adding these equations gives $45 = 5c$
So $c = 9$

Therefore from equation (1): $15 = 18 + d$
So $d = -3$

Examination Question 1 (June 1999 Paper 5)

A is the point (0, 4). $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

a) Find the coordinates of B.

C is the point (3, 4). BD is a diagonal of the parallelogram ABCD.

b) Express $\overrightarrow{BD}$ as a column vector.

c) $\overrightarrow{CE} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$. Calculate the length of AE.

Examination Question 2 (June 2001 Paper 5)

A is the point (2, 3) and B is the point (-2, 0).

a) i) Write $\overrightarrow{AB}$ as a column vector.

ii) Find the length of the vector $\overrightarrow{AB}$.

D is the point such that $\overrightarrow{BD}$ is parallel to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and the length of $\overrightarrow{AD}$ is equal to the length of $\overrightarrow{AB}$.

O is the point (0, 0).

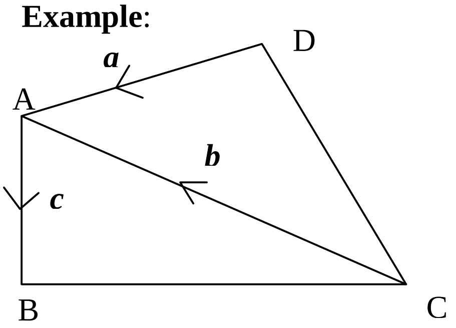
b) Find $\overrightarrow{OD}$ as a column vector.

C is the point such that ABCD is a rhombus. AC is a diagonal of the rhombus.

c) Find the coordinates of C.

Vector Geometry

Example:



Using the information in the diagram, find in terms of **a**, **b** and **c**:

a) $\overrightarrow{DC}$

b) $\overrightarrow{BC}$

c) $\overrightarrow{DB}$

Solution

a) To find an expression for $\overrightarrow{DC}$ we look for a route that takes us from D to C.

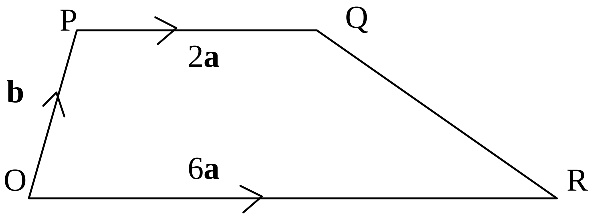
$\overrightarrow{DC} = \overrightarrow{DA} + \overrightarrow{AC} = \mathbf{a} - \mathbf{b}$ (the vector $\overrightarrow{AC}$ goes in the opposite direction to vector **b** and so is negative)

b) Likewise, $\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC} = -\mathbf{c} - \mathbf{b}$ (the vector $\overrightarrow{BA}$ goes in the opposite direction to vector **c** and so is negative)

c) $\overrightarrow{DB} = \overrightarrow{DA} + \overrightarrow{AB} = \mathbf{a} + \mathbf{c}$

Note: The position vector of a point, is the vector from the origin to that point. So the position vector of A is the vector $\overrightarrow{OA}$. If A is the point (a, b) then the position vector of A is $\begin{pmatrix} a \\ b \end{pmatrix}$.

Worked Examination Question (June 2000 Paper 5):



OPQR is a trapezium. PQ is parallel to OR.

$\overrightarrow{OP} = \mathbf{b}$, $\overrightarrow{PQ} = 2\mathbf{a}$, $\overrightarrow{OR} = 6\mathbf{a}$.

M is the mid-point of PQ

N is the mid-point of OR.

a) Find $\overrightarrow{OM}$ and $\overrightarrow{MN}$ in terms of **a** and **b**.

b) X is the mid-point of MN. Find, in terms of **a** and **b**, the vector $\overrightarrow{OX}$.

a) $\overrightarrow{OM} = \overrightarrow{OP} + \overrightarrow{PM} = \mathbf{b} + \mathbf{a}$ ($\overrightarrow{PM}$ is half of the vector $\overrightarrow{PQ}$, i.e. **a**).

$\overrightarrow{MN} = \overrightarrow{MP} + \overrightarrow{PO} + \overrightarrow{ON} = -\mathbf{a} - \mathbf{b} + 3\mathbf{a} = 2\mathbf{a} - \mathbf{b}$ ($\overrightarrow{ON}$ is half of $\overrightarrow{OR}$)

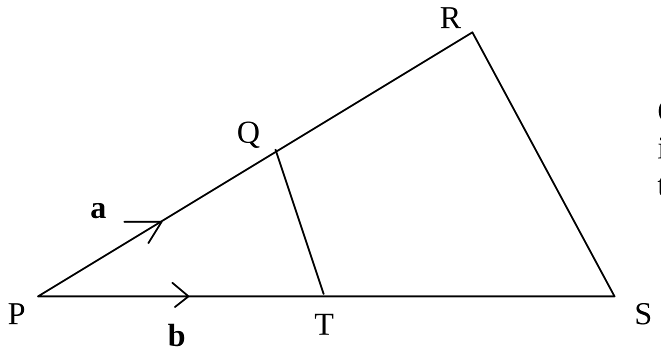
b) Using the answers to (a) we see that:

$$\overrightarrow{OX} = \overrightarrow{OM} + \tfrac{1}{2}\overrightarrow{MN} = \mathbf{b} + \mathbf{a} + \tfrac{1}{2}(2\mathbf{a} - \mathbf{b})$$

$$= \mathbf{b} + \mathbf{a} + \mathbf{a} - \tfrac{1}{2}\mathbf{b}$$

$$= 2\mathbf{a} + \tfrac{1}{2}\mathbf{b}$$

Further examination question (June 1997 Paper 5)



Q is the mid-point of the side PR and T is the mid-point of the side PS of triangle PRS.
$\overrightarrow{PQ} = \mathbf{a}$ and $\overrightarrow{PT} = \mathbf{b}$.

(a) Write down, in terms of **a** and **b**, the vectors
(i) $\overrightarrow{QT}$ (ii) $\overrightarrow{PR}$ (iii) $\overrightarrow{RS}$.

(b) Write down one geometrical fact about QT and RS that could be deduced from your answers to (a).

EXTENDED MATHEMATICS
0580

REVISION

Compiled & Edited by:

Dr. Tayeb Abdul Rahman

First Edition
2011

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